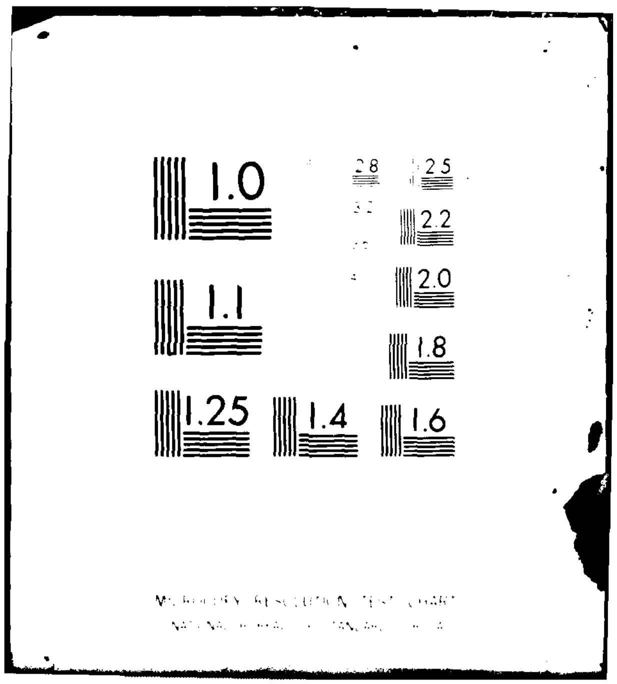
AD-A114 598 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
FINITE DATA LATTICE ALGORITHMS FOR INSTRUMENTAL VARIABLE RECURS--ETC(U)
MAR 82 V SOLO DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2347 NL

1 OF 1
ADA
114598

END
DATE
FILMED
06-82
DTIC


1.0
2.8
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

MRC Technical Summary Report # 2347

FINITE DATA LATTICE ALGORITHMS FOR INSTRUMENTAL VARIABLE RECURSIONS

Victor Solo

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

March 1982

(Received August 25, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

FINITE DATA LATTICE ALGORITHMS FOR INSTRUMENTAL VARIABLE RECURSIONS

by Victor Solo*,**

Technical Summary Report #2347

March 1982

ABSTRACT

The last few years have seen a rapid development of the so-called Lattice Algorithms for the fast solution of finite data least squares problems. While a fast algorithm has been given for finite data Instrumental Variable Recursions, as yet no finite data lattice schemes have been given. In this work a lattice algorithm is derived for a finite data Instrumental Variable Recursion and its use in both ARMA and ARMAX Time Series models is indicated.

AMS(MOS) Subject Classifications: 62M10, 93B30

Key Words: Time series, Signal processing, Instrumental Variables, Hankel Matrix, Recursive least squares, ARMA model, Fast Algorithm, Toeplitz matrix

Work Unit Number 4 - Statistics and Probability.

---

* This work was completed at the Mathematics Research Center, University of Wisconsin-Madison. The author is with the Department of Statistics, Harvard University, Cambridge, MA 02138.

** Permission to publish this abstract separately is granted.

Sponsored by the United States Army under Contract No. DAAG29-80-0041.

SIGNIFICANCE AND EXPLANATION

In the field of signal processing in Electrical Engineering one of the basic problems is the design of computationally fast algorithms for the solution of least squares problems. For example in the problem of adaptive equilization for a communication channel. A signal is transmitted over a communication channel and received in distorted form. It is then passed through a linear filter called an equalizer that estimates the original transmitted signal. Usually the one equalizer must deal with several communications channels and should have different parameters for each one. Since the parameter alteration must be done in real time the need for fast algorithms is apparent. The equalizer problem can be posed as a least squares problem. Actually fast algorithms for least squares problems have been recently given. However for some signal processing problems (and equalization problems) other estimation schemes must be used. This article provides a fast algorithm for one such scheme, an Instrumental Variable scheme.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

FINITE DATA LATTICE ALGORITHMS
FOR INSTRUMENTAL VARIABLE RECURSIONS

by Victor Solo*

0. Introduction

A fundamental position in Signal Processing and Time Series modelling is occupied by the Autoregressive (AR) model or all-pole linear model driven by white noise. The infinite data least squares problem (i.e. estimation of the AR parameters by minimizing an expected squared error) yields a set of Toeplitz equations for which the Levinson-Whittle-Robinson algorithm provides a fast solution. This algorithm can also be re-expressed as a lattice *algorithm* (LA): in this case it is Burg's algorithm (see Makhoul [3]). When only finite data is available the theoretical autocorrelations can be replaced by estimated ones but then the least squares nature of the solution is lost. Recently Lee et. al. [4], [5], have shown how a fast algorithm (a LA) may be constructed for a solution to the finite data least squares problem (i.e. minimization of a sum of squared errors): for a readable discussion see also Sartorius and Shensa [8] and Shensa [9].

For infinite data Instrumental Variable (IV) estimation of the AR parameters in a scalar autoregressive Moving Average (ARMA) model, Carayannis et. al [2] have provided a fast algorithm recently - it is an extension of the one of Levinson. For finite data IV parameter estimation in general, Ljung et. al [6] have recently given a fast algorithm but it is not a LA. In this

---

* This work was completed at the Mathematics Research Center, University of Wisconsin-Madison. The author is with the Department of Statistics, Harvard University, Cambridge, MA 02138.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

work a finite data LA is given for a general IV problem and its use with ARMA and ARMAX Time series models is indicated.

Section I contains a derivation of an infinite data LA for IV estimation of the AR parameters in a block identifiable ARMA model: this discussion which is much simpler than the finite data case reveals the basic idea used in all the derivations. In Section II an infinite data LA is derived for a multivariate model. In Section III a general infinite data IVLA is presented. In Section IV it is shown how this algorithm may be applied to an ARMAX (Autoregressive Moving Average Exogenous (i.e. input)) Time series model.

I. The ARMA model: infinite data IVLA.

Here we consider the estimation of the AR parameters of an ARMA Time Series model from the autocorrelations. Actually the scalar version of the lattice algorithm derived in this section is implicit in Carayannis et. al [2] article which itself was implicit in the work of Risannen [7] (see C below). However the present derivation is especially simple and provides a stochastic interpretation for various quantities appearing in the algorithm. Further, the arguments used here guide the more complex derivations in the finite data case of Section II below.

A. Preliminaries

It is well known that the AR parameters of a multivariate ARMA (p,q) $\underline{y}_t$ (dimension r) obey the equations

$$\underline{A}^p \underline{R}_p = -\underline{R}_{-q-1,-q-p} \tag{1a}$$

where, with $\underline{R}_j = E(\underline{y}_t \underline{y}'_{t+j})$, $\underline{R}_p$ is the nonsymmetric Toeplitz matrix

$$\underline{R}_p = \begin{bmatrix} \underline{R}_{-q} & \underline{R}_{-q-1} & \cdots & \underline{R}_{-q-p+1} \\ \underline{R}_{-q+1} & & & \\ \underline{R}_{-q+p-1} & & \cdots & \underline{R}_{-q} \end{bmatrix}$$

$$\underline{R}_{-q-1,-q-p} = [\underline{R}_{-q-1} \underline{R}_{-q-2} \cdots \underline{R}_{-q-p}]$$

$$\underline{A}^p = (\underline{A}_1^p \ldots \underline{A}_p^p) \; .$$

These equations may be alternately written

$$\underline{\breve{A}}_p \underline{H}_p = -\underline{R}_{-q-1,-q-p} \tag{1b}$$

where

$$\underline{\breve{A}}_p = (\underline{A}_p \ldots \underline{A}_1) = \underline{A}_p \underline{J} \; , \quad \underline{H}_p = \underline{J}\, \underline{R}_p$$

and

$$\underline{J} = \begin{bmatrix} \underline{0} & \cdot & \cdot & \cdot & \cdot & \underline{0} & \underline{I} \\ & & & \cdot & & \underline{I} & \underline{0} \\ & & \cdot & & & & \\ \underline{I} & \vdots & \cdot & \cdot & \cdot & & \underline{0} \end{bmatrix} .$$

Now $\underline{H}_p$ is a block Hankel matrix. If $n_H = \text{rank}(\underline{H}_p) = \text{rank}(\underline{R}_p) = pr$ then the ARMA model is called block identifiable (Akaike [1]) and the p - AR matrices may be obtained by solving the pr^2 equations (1). For the present discussion the assumption of block identifiability will not be restrictive. Thus it is henceforth assumed that $n_H = pr$.

B. Derivation

To derive a lattice algorithm begin by introducing the forwards prediction error

$$\underline{e}_m(n) = \underline{y}_n + \Sigma_1^m \underline{A}_i^m \underline{y}_{n-i} \tag{2a}$$

where $\underline{A}^m = (\underline{A}_1^m \ldots \underline{A}_m^m)$ is defined by an orthogonality condition that ensures (when $m = p$) equations (1) hold: namely

$$E(\underline{e}_m(n)(\underline{y}'_{n-q-1} \ldots \underline{y}'_{n-q-m})) = \underline{0} \tag{2b}$$

i.e.

$$\underline{A}^m \underline{R}_m = -\underline{R}_{q-1,-q-m} \quad .$$

It will be convenient to introduce the IV sequence $\underline{z}_n = \underline{y}_{n-q}$ so that (2b) can be written

$$E(\underline{e}_m(n)(\underline{z}'_{n-1} \ldots \underline{z}'_{n-m})) = \underline{0} \quad . \tag{2c}$$

It will be seen in what follows that we can, to a large extent, forget that $\underline{z}_n = \underline{y}_{n-q}$. It will also prove convenient to denote the fact that $\underline{e}_m(n)$ is a linear combination of $\underline{y}_n \ldots \underline{y}_{n-m}$ with the coefficient on $\underline{y}_n$ being $\underline{I}$, by writing

$$\underline{e}_m(n) - \underline{y}_n \in S_{n+1,n+m} = \text{span}(\underline{y}_{n-1} \ldots \underline{y}_{n-m}) \quad . \tag{3}$$

Then clearly $\underline{e}_m(n)$ is determined uniquely by the two requirements (2c), (3).

The next step in a lattice derivation is to find an order update equation for $\underline{e}_m(n)$. For this, introduce the backward prediction error

$$\underline{r}_m(n-1) = \underline{y}_{n-m-1} + \Sigma_1^m \underline{B}_{m+1-i}^m \underline{y}_{n-i} \tag{4a}$$

where $\underline{B}^m = (\underline{B}_1^m \ldots \underline{B}_m^m)$ is determined by

$$E(\underline{r}_m(n-1)(\underline{z}'_{n-1} \;\ldots\; \underline{z}'_{n-m})) = \underline{0} \tag{4b}$$

that is $\underline{B}^m$ obeys

$$\tilde{\underline{B}}^m \underline{R}_m = (\underline{B}^m \; \underline{J})\underline{R}_m = -\underline{R}_{-q+m,-q+1} \tag{4c}$$

or

$$\underline{B}^m \underline{R}_m = -\underline{R}_{-q+1,-q+m}$$

(in the scalar case cf. expression (f) of Carayannis et. al. [2]).

Note again the $\underline{r}_m(n)$ is uniquely determined by two conditions (4a), (4d)

$$\underline{r}_m(n) - \underline{y}_{n-m} \in S_{n,n+m-1} \quad . \tag{4d}$$

Now recall that, by definition

$$E(\underline{e}_{m+1}(n)(\underline{z}'_{n-1} \;\ldots\; \underline{z}'_{n-m} \; \underline{z}'_{n-m-1})) = \underline{0} \quad . \tag{5}$$

Next observe that for any $\underline{K}$, $\underline{e}_m(n) + \underline{K}\,\underline{r}_m(n-1)-\underline{y}_n \in S_{n+1,n+m+1}$, so if we choose $\underline{K}$ to satisfy

$$E(\underline{e}_m(n) + \underline{K}\,\underline{r}_m(n-1))\underline{z}'_{n-m-1} = \underline{0}$$

i.e.

$$\underline{K} = \underline{K}^r_{m+1} = -\underline{G}^r_m \, \underline{\Sigma}^{-r}_m \tag{6a}$$

where

$$\underline{G}^r_m = E(e_m(n-1)\underline{z}'_{n-m-1}) \tag{6b}$$

$$\underline{\Sigma}^r_m = E(\underline{r}_m(n-1)\underline{z}'_{n-m-1}) \tag{6c}$$

then we must have (via (2c), (3)) that

$$\underline{e}_{m+1}(n) = \underline{e}_m(n) + \underline{K}^r_{m+1}\,\underline{r}_m(n-1) \tag{7a}$$

This is the desired order update for $\underline{e}_m(n)$. (Note the notation $\underline{\Sigma}^{-r}_m = (\underline{\Sigma}^r_m)^{-1}$.)

Note from (2a), (6b) that

$$\underline{G}^r_m = \underline{R}_{-q-m-1} + \underline{A}^m \underline{J}\,\underline{R}_{-q-1,-q-m}$$

(in the scalar case cf. expression (9) of Crayannis et. al. [2]).

On the other hand comparing coefficients of lagged $\underline{y}$'s in (7) yields

$$\underline{A}^{m+1}_{m+1} = \underline{K}^r_{m+1} \tag{7b}$$

$$\underline{A}_j^{m+1} = \underline{A}_j^m + \underline{K}_{m+1} \underline{\tilde{B}}_j^m \qquad 1 < j < m \quad . \tag{7c}$$

Next an order updating formula for $\underline{r}_m(n)$ is given. Again beginning with

$$E(\underline{r}_{m+1}(n)(\underline{z}_n' \ \cdots \ \underline{z}_{n-m-1}')) = \underline{0} \tag{8}$$

we see that for any $\underline{K}$

$$\underline{r}_m(n-1) + \underline{K}\ \underline{e}_m(n) - \underline{y}_{n-m} \in S_{n,n+m}$$

so if we choose $\underline{K}$ to obey

$$E(\underline{r}_m(n-1) + \underline{K}\ \underline{e}_m(n))\underline{z}_n' = \underline{0}$$

i.e.
$$\underline{K} = \underline{K}_{m+1}^e = -\underline{G}_m^e \underline{\Sigma}_m^{-e} \tag{9a}$$

where
$$\underline{G}_m^e = E(\underline{r}_m(n-1)\underline{z}_n') \tag{9b}$$

$$\underline{\Sigma}_m^e = E(\underline{e}_m(n)\underline{z}_n') \tag{9c}$$

we must have, as required

$$\underline{r}_m(n) = \underline{r}_m(n-1) + \underline{K}_{m+1}^e \ \underline{e}_m(n) \quad . \tag{10a}$$

Again observe that from (9a), (9b)

$$\underline{G}_m^e = \underline{R}_{-q+m+1} + \underline{\tilde{B}}^m \underline{R}_{-q+1,-q+m}$$

(in the scalar case cf. expression (11) of Carayannis et. al. [2]). On the other hand comparing once more coefficients of lagged $\underline{y}$'s in (10a) gives

$$\underline{B}_{m+1}^{m+1} = \underline{K}_{m+1}^e \tag{10b}$$

$$\underline{B}_j^{m+1} = \underline{B}_j^m + \underline{K}_{m+1}^e \ \underline{\tilde{A}}_j^m \quad . \tag{10c}$$

An order update can now be obtained for $\underline{\Sigma}_m^e$, $\underline{\Sigma}_m^r$. From (7a), (9c) calculate

$$\underline{\Sigma}_{m+1}^e = E(\underline{e}_{m+1}(n)\underline{z}_n')$$

$$= E(\underline{e}_m(n)\underline{z}_n') + \underline{K}_{m+1}^r \ E(\underline{r}_m(n-1)\underline{z}_n')$$

$$= \underline{\Sigma}_m^e + \underline{K}_{m+1}^r \ \underline{G}_m^e$$

$$\underline{\Sigma}_{m+1}^e = (\underline{I} - \underline{K}_{m+1}^r \ \underline{K}_{m+1}^e)\underline{\Sigma}_m^e \quad \text{by (9a)} \quad . \tag{11}$$

Similarly

$$\underline{\Sigma}_{m+1}^r = (\underline{I} - \underline{K}_{m+1}^e \ \underline{K}_{m+1}^r)\underline{\Sigma}_m^r \quad . \tag{12}$$

Initial conditions are

$$\underline{e}_0(n) = \underline{y}_n = \underline{r}_0(n)$$

$$\Sigma_0^e = E(\underline{y}_{n-1}\,\underline{y}'_{n-q-1}) = E(\underline{y}_n\,\underline{y}'_{n-q}) = \underline{\Sigma}_0^r = \underline{R}_{-q} \quad .$$

Note from equations (11), (12) that in the scalar case $\Sigma_m^e = \Sigma_m^r$.

C. Connections with Triangular Decompositions.

Recalling equation (4c) consider the following indentity

$$\underline{L}_{m+1}\,\underline{R}_{m+1}\,\underline{V}_{m+1} = \begin{pmatrix} \underline{L}_m & \underline{0} \\ \underline{\tilde{B}}^m & \underline{I} \end{pmatrix} \begin{pmatrix} \underline{R}_m & \underline{R}'_{q+m,q+1} \\ \underline{R}_{-q+m,-q+1} & \underline{R}_{-q} \end{pmatrix} \begin{pmatrix} \underline{U}_m & \underline{N}^m \\ \underline{0} & \underline{I} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{L}_m\,\underline{R}_m\,\underline{U}_m & \underline{0} \\ \underline{0}' & \underline{D}_{m+1} \end{pmatrix}$$

where

$$\underline{D}_{m+1} = \underline{R}_{-q} + \underline{\tilde{B}}^m\,\underline{R}'_{q+m,q+1}$$

$$\underline{R}_m\,\underline{N}^m = -\underline{R}'_{q+m,q+1} \quad .$$

Clearly recursive application of this identity beginning with $\underline{L}_1 = \underline{I} = \underline{U}_1$; $\underline{D}_1 = \underline{R}_1$ will produce a triangular decomposition of $\underline{R}_m$ for each m. Just such a scheme was given by Risannen [7] and equations (7), (10), (12) form one half of that scheme. The other half consists of a triple set to produce $\underline{N}^m$. These equations are obtained by replacing $\underline{R}_i$ in (7), (10), (12) by $\underline{R}'_{-i}$ (see Risannen [7]).

To show the equivalence of (7), (10), (12) to Risannen's equations make the following notation changes:

$$\underline{\hat{A}}_i^m = \underline{A}_{m-i}^m,\ \underline{\hat{B}}_i^m = \underline{\tilde{B}}_{i+1}^m \qquad 0 \le i \le m-1$$

then (7b), (7c) read (take $\underline{\hat{A}}_s^m = 0 \quad s < 0$)

$$\underline{\hat{A}}_i^{m+1} = A_{m+1-i}^{m+1} = \underline{A}_{m+1-i}^m + \underline{K}_{m+1}^r\,\underline{\tilde{B}}_{m+1-i}^m$$

$$= \underline{\hat{A}}_{i-1}^m + \underline{K}_{m+1}^r\,\underline{\hat{B}}_{m-i}^m \qquad 0 \le i \le m \tag{13a}$$

similarly (1oc) is (with $\hat{\underline{B}}^m_s = 0 \quad s < 0$)

$$\hat{\underline{B}}_i^{m+1} = \hat{\underline{B}}^m_{i-1} + \underline{K}^e_{m+1} \hat{\underline{A}}^m_{m-i} \quad 0 \leq i \leq m \quad . \tag{13b}$$

These are just equations (3.8), (3.9) of Risannen.

II. ARMA model: finite data IVLA

To use the scheme of Section I the $\underline{R}_i$ must be replaced by estimates $\hat{\underline{R}}_i$. Then however the resulting estimates no longer obey infinite data orthogonality conditions such as (2b), (4b). Now an algorithm is given which does obey finite data versions of these orthogonality conditions. It will be called a finite data LA.

The argument is simplified by using a setting such as that of Shensa [9]. Here though a vector version of that discussion is needed. If $\underline{y}_t$ is an r-dimensional discrete time series with $\underline{y}_t < \underline{0}$ $t < 0$, introduce the infinite data matrix

$$\underline{\mathcal{Y}} = (\underline{y}_0, \underline{y}_1, \underline{y}_2 \cdots \quad .$$

Also introduce the shift operator (i.e. infinite matrix) ζ,

$$\zeta^{-1}\,\underline{\mathcal{Y}} = (\underline{0}, \underline{y}_0, \underline{y}_1, \cdots$$

so

$$\zeta\,\underline{\mathcal{Y}} = (\underline{y}_1, \underline{y}_2, \underline{y}_3, \cdots \quad .$$

Note that $\zeta\,\zeta^{-1} = I$ but $\zeta^{-1}\,\zeta \neq I$. Define now a "matrix inner-product" between two data matrices

$$\langle \underline{\mathcal{X}}, \underline{\mathcal{Y}} \rangle_n = \Sigma_0^n\, \underline{x}_i\, \underline{y}_i' \tag{14}$$

and observe the fundamental relation

$$\begin{aligned} \langle \zeta^{-1}\underline{\mathcal{X}}, \underline{\mathcal{Y}} \rangle_n &= \Sigma_0^n\, \underline{x}_{i-1}\, \underline{y}_i' = \Sigma_1^n\, \underline{x}_{i-1}\, \underline{y}_i' = \Sigma_0^{n-1}\, \underline{x}_j\, \underline{y}_{j+1}' \\ &= \langle \underline{\mathcal{X}}, \zeta\, \underline{\mathcal{Y}} \rangle_{n-1} \quad . \end{aligned} \tag{15a}$$

Notice that $\underline{x}_i$, $\underline{y}_i$ need not be of the same dimension for the definition (14) to make sense. In this respect note for example that if $\underline{\alpha}$ is an arbitrary vector (of dimension that of $\underline{x}_i$) then

$$\underline{\alpha}'\langle \underline{\mathcal{X}}, \underline{\mathcal{Y}} \rangle_n = \Sigma_0^n\, \underline{\alpha}'\, \underline{x}_i\, \underline{y}_i = \langle \underline{\alpha}'\underline{\mathcal{X}}, \underline{\mathcal{Y}} \rangle_n \quad . \tag{15b}$$

Also if w is an infinite data vector $\underline{\alpha w} = (\underline{\alpha w}, \underline{\alpha w}_2\ \underline{\alpha w}_3 \cdots$. (15c)

It will be convenient to denote $\zeta^{-i}\underline{\mathcal{Y}}$ as $\underline{\mathcal{Y}}_i$; so $\underline{\mathcal{Y}} = \underline{\mathcal{Y}}_0$. Thus we will use the IV matrix $\underline{z} = \zeta^{-q}\underline{y} = \underline{y}_{-q}$. Finally the following time update formula will be useful

$$\langle \underline{x}, \underline{y} \rangle_n = \langle \underline{x}, \underline{y} \rangle_{n-1} + \underline{x}_n \underline{y}_n' \quad . \tag{15d}$$

A. Lattice Equations: Statement

Proceding by analogy with the infinite data scheme introduce the forward prediction error

$$\underline{e}_m(n) = \underline{y}_0 - \Sigma_1^m \underline{A}_i^n(n)\underline{y}_{-i} \tag{16}$$

the kth component for $\underline{e}_m(n)$ is $\underline{e}_m(k,n) = \underline{y}_k - \Sigma_1^m \underline{A}_i^m(n)\underline{y}_{k-i}$. Note that the $\underline{A}_i^m(n)$ are $r \times r$ matrices. These coefficients are determined by the orthogonality conditions

$$\langle \underline{e}_m(n), \underline{z}_{-i} \rangle_n = 0 \qquad 1 \leq i \leq m \tag{17a}$$

equivalently, putting $\underline{A}^m(n) = (\underline{A}_1^m(n) \ \ldots \ \underline{A}_m^m(n))$

$$\underline{A}^m(n)\underline{R}_m(n) = \underline{R}_{-q-1,-q-m}(n) \tag{17b}$$

where

$$\underline{R}_m(n) = \Sigma_0^n \begin{pmatrix} \underline{y}_{k-1} \\ \\ \underline{y}_{k-m} \end{pmatrix} (\underline{z}_{k-1} \ \cdots \ \underline{z}_{k-m})'$$

$$\underline{R}_{-q-1,-q-m}(n) = \Sigma_0^n \underline{y}_k (\underline{z}_{k-1} \ \cdots \ \underline{z}_{k-m})' \quad .$$

Thus equations (17) are simply the equations satisfied by the mth order, finite data, IV estimate of the AR parameters. It is assumed that $\underline{R}_m(n)$ is of rank rm for all n: If $E(\underline{R}_m(n))$ is of rank rm then this event occurs with probability one. When $q = 0$ so that $\underline{z} \equiv \underline{y}$ equations (17) are the finite data least-squares equations.

Let us observe that $\underline{e}_m(n)$ is determined uniquely by two conditions: (17a) and (17c)

$$\underline{e}_m(n) - \underline{y}_0 \in S_{1,m} = \mathrm{span}(\underline{y}_{-1}, \ \cdots \ \underline{y}_{-m}) \quad . \tag{17c}$$

This is so since equation (17a) (or (17b)) specifies the coefficients of $\underline{y}_{-1} \cdots \underline{y}_{-m}$ uniquely.

Now introduce a backwards prediction error

$$\underline{r}_m(n) = \underline{y}_{-m} - \Sigma_1^m \underline{B}^m_{m+1-i}(n)\underline{y}_{-i+1}$$

where $\underline{r}_m(n)$ is determined uniquely by

$$\underline{r}_m(n) - \underline{y}_{-m} \in S_{0,m-1} \tag{18a}$$

$$\langle \underline{r}_m(n), \underline{z}_{-i}\rangle_n = 0 \quad 0 \leq i \leq m-1 \ . \tag{18b}$$

Now the lattice equations can be stated as follows: A derivation is given in B below. Note that the equations are easily modified if the inner product is changed to

$$\langle \underline{x}, \underline{y}\rangle_n = \Sigma_0^n w^{n-i} \underline{x}_i \underline{y}_i'$$

where $0 < w < 1$. This is a suitable form for time varying equations. Finally it is clear by inspection that the algorithm requires $0(m)$ operations.

ORDER UPDATES

$$\underline{e}_{m+1}(n) = \underline{e}_n(n) + \underline{K}^r_m(n)\zeta^{-1}\underline{r}_m(n-1) \tag{19a}$$

$$\underline{r}_{m+1}(n) = \zeta^{-1}\underline{r}_m(n-1) + \underline{K}^e_m(n)\underline{e}_m(n) \tag{19b}$$

$$\underline{K}^r_m(n) = -\underline{G}^r_m(n)\underline{\Sigma}^{-r}_n(n-1)$$

$$\underline{K}^e_m(n) = -\underline{G}^e_m(n)\underline{\Sigma}^{-e}_m(n)$$

$$\underline{\Sigma}^e_m(n) = (\underline{I} - \underline{K}^r_m(n)\underline{K}^e_m(n))\underline{\Sigma}^e_{m-1}(n) \tag{19c}$$

$$\underline{\Sigma}^r_m(n) = (\underline{I} - \underline{K}^e_m(n)\underline{K}^r_m(n))\underline{\Sigma}^r_{m-1}(n) \ . \tag{19d}$$

Initial conditions for these are

$$\underline{e}_0(n) = \underline{y}_0 = \underline{r}_0(n) \tag{19e}$$

$$\underline{\Sigma}^r_0(n) = \underline{\Sigma}^e_0(n) = \langle \underline{y}_0, \underline{z}_0\rangle_n = \underline{\Sigma}^e_0(n-1) + \underline{y}_n \underline{z}_n' \ . \tag{19f}$$

TIME UPDATES: $m > 1$

The first wo calculations must be performed before the next two:

$$\underline{G}_m^r(n) = \underline{G}_m^r(n-1) + \underline{e}_m(n,n)\underline{\delta}_{m-1}^{r'}(n-1)(1 - \xi_{m-1}(n-1,n-1))^{-1} \tag{20a}$$

$$\underline{G}_m^e(n) = \underline{G}_m^e(n-1) + \underline{r}_m(n-1,n-1)\underline{\delta}_{n-1}^{e'}(n)(1 - \xi_{m-1}(n-1,n-1))^{-1} \tag{20b}$$

$$\underline{\delta}_m^{e'}(n) = \underline{\delta}_{m-1}^{e'}(n) - \underline{\delta}_{m-1}^{r'}(n-1)\underline{\Sigma}_m^{-r}(n-1)\underline{G}_m^e(n) \tag{20c}$$

$$\underline{\delta}_m^{r'}(n) = \underline{\delta}_{m-1}^{r'}(n-1) - \underline{\delta}_{m-1}^{e'}(n)\underline{\Sigma}_m^{-e}(n)\underline{G}_m^r(n) \quad . \tag{20d}$$

The initial conditions are:

$$\underline{G}_0^e(n) = \underline{G}_0^e(n-1) + \underline{y}_{n-1}\,\underline{z}_n' \tag{20e}$$

$$\underline{G}_0^r(n) = \underline{G}_0^r(n-1) + \underline{y}_n\,\underline{z}_{n-1}' \tag{20f}$$

$$\underline{\delta}_0^{r'}(n) = \underline{z}_{n-1}' - \underline{z}_n'\,\underline{\Sigma}_0^{-e}(n)\underline{G}_0^r(n) \tag{20g}$$

$$\underline{\delta}_0^{e'}(n) = \underline{z}_n' - \underline{z}_{n-1}'\,\underline{\Sigma}_0^{-e}(n-1)\underline{G}_0^e(n) \quad . \tag{20h}$$

Initializing values for the whole recursion are

$$\underline{e}_0(0) = \underline{y}_0 = \underline{r}_0(0)$$

$$\underline{\Sigma}_0^r(0) = \underline{\Sigma}_0^e(0) = \underline{y}_0\,\underline{z}_0'$$

$$\underline{G}_0^e(0) = \underline{0} = \underline{G}_0^r(0)$$

$$\underline{\delta}_0^r(0) = \underline{0} \ : \ \underline{\delta}_0^e(0) = \underline{y}_0 \quad .$$

Finally updates for the scalar $\xi_m(n,n)$ are

$$\xi_{m+1}(n,n) = \xi_m(n,n) + \underline{\delta}_m^{r'}(n)\underline{\Sigma}_{m+1}^{-r}(n)\underline{r}_{m+1}(n,n) \tag{21a}$$

$$\xi_0(n,n) = \underline{z}_n'\,\underline{\Sigma}_0^{-e}(n)\underline{y}_n \quad . \tag{21b}$$

B. Derivation

(1) Order updates for $\underline{e}_m(n)$, $\underline{r}_m(n)$

To find an order update equation for $\underline{e}_m(n)$, begin by observing

$$\begin{aligned} \langle \zeta^{-1}\underline{r}_m(n-1), \underline{z}_{-i}\rangle_n &= \langle \underline{r}_m(n-1), \zeta\, \underline{z}_{-i}\rangle_{n-1} \\ &= \langle \underline{r}_m(n-1), \underline{z}_{-i+1}\rangle_{n-1} \\ &= 0 \quad 1 \le i \le m \ \text{ by (18b)} \ . \end{aligned} \tag{22}$$

In view of (17a), (22) consider that for any $\underline{K}$

$$\underline{e}_m(n) + \underline{K}\,\zeta^{-1}\underline{r}_m(n-1) - \underline{y}_0 \in S_{1,m+1}$$

$$\langle \underline{e}_m(n) + \underline{K}\,\zeta^{-1}\underline{r}_m(n-1), \underline{z}_{-i}\rangle_n = \underline{0} \quad 1 \le i \le m \ .$$

Thus if we choose $\underline{K}$ so that

$$\langle \underline{e}_m(n) + \underline{K}\,\zeta^{-1}\underline{r}_m(n-1), \underline{z}_{-m-1}\rangle_n = \underline{0}$$

i.e.

$$\underline{K} = -\underline{G}^r_m(n)\underline{\Sigma}^{-r}_m(n-1) \tag{23a}$$

where

$$\underline{G}^r_m(n) = \langle \underline{e}_m(n), \underline{z}_{-m-1}\rangle_n \tag{23b}$$

$$\underline{\Sigma}^r_m(n-1) = \langle \zeta^{-1}\underline{r}_m(n-1), \underline{z}_{-m-1}\rangle_n = \langle \underline{r}_m(n-1), \underline{z}_{-m}\rangle_{n-1} \tag{23c}$$

then we must have (19a). Note that when using this update only the nth component need be calculated. Similarly we can find equation (19b) with

$$\underline{K}^e_m(n) = -\underline{G}^e_m(n)\underline{\Sigma}^{-e}_m(n) \tag{24a}$$

$$\underline{G}^e_m(n) = \langle \zeta^{-1}\underline{r}_m(n-1), \underline{z}_0\rangle_n \tag{24b}$$

$$\underline{\Sigma}^e_m(n) = \langle \underline{e}_m(n), \underline{z}_0\rangle_n \ . \tag{24c}$$

(2) Order Updates for $\underline{\Sigma}^e_m(n)$, $\underline{\Sigma}^r_m(n)$

First apply (19a) to

$$\underline{\Sigma}^e_m(n) = \langle \underline{e}_n(n), \underline{z}_0 \rangle_n$$

$$= \langle \underline{e}_{m-1}(n), \underline{z}_0 \rangle_n + \underline{K}^r_{m-1}(n) \langle \zeta^{-1} \underline{r}_{m-1}(n-1), \underline{z}_0 \rangle_n$$

$$= \underline{\Sigma}^e_{m-1}(n) + \underline{K}^r_{m-1}(n)\underline{G}^e_{m-1}(n)$$

$$= (\underline{I} - \underline{K}^r_{m-1}(n)\underline{K}^e_{m-1}(n))\underline{\Sigma}^e_{m-1}(n)$$

which is equation (19c). Similarly (19d) follows.

(3) Time Updates for $\underline{e}_m(n)$, $\underline{r}_m(n)$

To obtain time updates for $\underline{G}^e_m(n)$, $\underline{G}^r_m(n)$ (the usual next step) it will be necessary to find time updates for $\underline{e}_m(n)$, $\underline{r}_m(n)$. It should be noted that in the least squares case $(q = 0)$, $\underline{z} = \underline{y}$ so that $\underline{G}^r_m(n) = \underline{G}^{e'}_m(n)$. This is easily seen by observing that $\underline{z}_{-m-1} = \underline{y}_{-m-1}$ in (23b) can be replaced by $\zeta^{-1}\underline{r}_m(n-1)$ since $\zeta^{-1}\underline{r}_m(n-1) - \zeta^{-1}\underline{y}_m \in S_{1,m}$. Thus $\underline{G}^r_m(n) = \langle \underline{e}_m(n), \zeta^{-1}\underline{r}_m(n-1)\rangle_n$. Similarly $\underline{G}^e_m(n) = \langle \zeta^{-1}\underline{r}_m(n-1), \underline{e}_m(n)\rangle_n$.

To find time updates for $\underline{e}_m(n)$, $\underline{r}_m(n)$ we are guided by conditions (17a), (17c). Consider then, using (15d), that

$$\langle \underline{e}_m(n-1), \underline{z}_{-i} \rangle_{n-1} = \langle \underline{e}_m(n-1), \underline{z}_{-i} \rangle_n - \underline{e}_m(n,n-1)\underline{z}'_{n-i} \quad . \tag{25}$$

This leads us to introduce the infinite data vector

$$\xi_{m-1}(n) = \Sigma_0^{m-1} \underline{c}_i^{(m-1)'}(n)\underline{y}_{-i} \in S_{0,m-1} \tag{26a}$$

with $\underline{c}^{m-1}(n) = (\underline{c}_0^{m-1}(n) \ldots \underline{c}_{m-1}^{m-1}(n))$ (and hence $\xi_{m-1}(n)$) determined by

$$\langle \xi_{m-1}(n), \underline{z}_{-i} \rangle_n = \underline{z}'_{n-i} \qquad 0 \le i \le m-1 \quad . \tag{26b}$$

Note there are mr unknowns in $\underline{c}^{m-1}(n)$ determined by mr equations (26b). Now observe that

$$\langle \zeta^{-1}\xi_{m-1}(n-1), \underline{z}_{-i} \rangle_n = \langle \xi_{m-1}(n-1), \underline{z}_{-i+1} \rangle_{n-1}$$

$$= \underline{z}'_{n-1-(i-1)} = \underline{z}'_{n-i} \qquad 1 \le i \le m-1 \quad .$$

Then consider that (recall (15c))

$$\underline{e}_m(n-1) - \underline{e}_m(n,n-1)\zeta^{-1}\xi_{m-1}(n-1) - \underline{\psi}_0 \in S_{1,m}$$

while from (25) and the definition of $\underline{e}_m(n-1)$

$$\langle \underline{e}_m(n-1) - \underline{e}_m(n,n-1)\zeta^{-1}\xi_{m-1}(n-1), \underline{z}_{-i}\rangle_n$$

$$= \langle \underline{e}_m(n-1), \underline{z}_{-i}\rangle_{n-1} = \underline{0} \quad 1 \leq i \leq m \ .$$

Thus we must have from (17a), (17c)

$$\underline{e}_m(n) = \underline{e}_m(n-1) - \underline{e}_m(n,n-1)\zeta^{-1}\xi_{m-1}(n-1) \ . \tag{27a}$$

Similarly

$$\underline{r}_m(n) = \underline{r}_m(n-1) - \underline{r}_m(n,n-1)\xi_{m-1}(n) \ . \tag{27b}$$

To proceed further it is necessary to provide both order and time updates for $\xi_m(n)$. In the least squares case $(q = 0)$ only time updates were needed.

(4). Time and Order Updates for $\xi_m(n)$

As usual we are guided by the conditions (26). The order update is considered first. Begin by recalling three facts

$$\langle \xi_{m+1}(n), \underline{z}_{-i}\rangle_n = \underline{z}'_{n-i} \quad 0 \leq i \leq m+1$$

$$\langle \xi_m(n), \underline{z}_{-i}\rangle_n = \underline{z}'_{n-i} \quad 0 \leq i \leq m$$

$$\langle \underline{r}_{m+1}(n), \underline{z}_{-i}\rangle_n = \underline{0} \quad 0 \leq i \leq m \ .$$

Further, for any r-vector $\underline{\eta}$

$$\xi_m(n) + \underline{\eta}' \, \underline{r}_{m+1}(n) \in S_{0,m} \ .$$

Thus if we choose $\underline{\eta}$ to satisfy the r equations

$$\langle \xi_m(n) + \underline{\eta}' \, \underline{r}_{m+1}(n), \underline{z}_{-m-1}\rangle_n = \underline{z}'_{n-m-1}$$

i.e. $$\underline{\eta}' = \underline{\delta}^{r'}_m(n)\underline{\Sigma}^{-r}_{m+1}(n)$$

where $$\underline{\delta}^r_m(n) = \underline{z}_{n-m-1} - \langle \underline{z}_{-m-1}, \xi_m(n)\rangle_n \tag{28a}$$

then we must have the desired order update

$$\xi_{m+1}(n) = \xi_m(n) + \underline{\delta}^{r'}_m(n)\underline{\Sigma}^{-r}_{m+1}(n)\underline{r}_{m+1}(n) \ . \tag{29}$$

The nth component of this equation is (21a).

To find (21b) take $m = 1$ in (26b) so

$$<\xi_0(n), \underline{z}>_n = \underline{z}'_n$$

or
$$\underline{c}_0^{0'}(n) <\underline{y}, \underline{z}>_n = \underline{c}_0^{0'}(n)<\underline{y}, \underline{z}>_n = \underline{z}'_n$$

so
$$\underline{c}_0^{0'}(n) = \underline{z}'_n \underline{\Sigma}_0^{-e}(n) \quad \text{whereupon}$$

$$\xi_0(n) = \underline{z}'_n \underline{\Sigma}_0^{-e}(n)\underline{y}_0 \tag{30}$$

the first component of this is (21b).

We turn now to the time update. This time consider the three facts

$$<\xi_m(n), \underline{z}_{-i}>_n = \underline{z}'_{n-i} \quad 0 \leq i \leq m$$

$$<\zeta^{-1}\xi_{m-1}(n-1), \underline{z}_{-i}>_n = <\xi_{m-1}(n-1), \underline{z}_{-i+1}>_{n-1}$$

$$= \underline{z}'_{n-i} \quad 1 \leq i \leq m$$

$$<\underline{e}_m(n), \underline{z}_{-i}>_n = \underline{0} \quad 1 \leq i \leq m \quad .$$

Further, for any r-vector $\underline{\eta}$

$$\zeta^{-1}\xi_{m-1}(n-1) + \underline{\eta}'\underline{e}_m(n) \in S_{0,m}$$

thus if we choose $\underline{\eta}$ to satisfy the r-equations

$$<\zeta^{-1}\xi_{m-1}(n-1) + \underline{\eta}'\underline{e}_m(n), \underline{z}_0>_n = \underline{z}'_n$$

i.e.
$$\underline{\eta}' = \underline{\delta}_{n-1}^{e'}(n)\underline{\Sigma}_m^{-e}(n)$$

where
$$\underline{\delta}_{m-1}^{e}(n) = \underline{z}_n - <\underline{z}_0, \zeta^{-1}\xi_{m-1}(n-1)>_n \tag{28b}$$

then we must have from (26) the desired time update

$$\xi_m(n) = \zeta^{-1}\xi_{m-1}(n-1) + \underline{\delta}_{m-1}^{e'}(n)\underline{\Sigma}_m^{-e}(n)\underline{e}_m(n) \quad . \tag{31}$$

Actually there is also an order change in this equation but that will not cause a problem. Now it is possible to derive an interconnected set of time and order updates for the $\underline{G}_m^e(n)$, $\underline{G}_m^r(n)$, $\underline{\delta}_m^e(n)$, $\underline{\delta}_m^r(n)$.

(5). <u>Time and Order Updates for</u> $\underline{G}_m^e(n)$, $\underline{G}_m^r(n)$, $\underline{\delta}_m^e(n)$, $\underline{\delta}_m^r(n)$

First time updates are given for $\underline{G}_m^r(n)$, $\underline{G}_m^e(n)$ and then time/order updates are given for $\underline{\delta}_m^r(n)$, $\underline{\delta}_m^e(n)$. Begin by applying (27a) to $\underline{G}_m^r(n)$.

$$\underline{G}^{r}_{m}(n) = \langle \underline{e}_m(n),\ \underline{z}_{-m-1}\rangle_n$$

$$= \langle \underline{e}_m(n-1),\ \underline{z}_{-m-1}\rangle_n - \underline{e}_m(n,n-1)\ \langle \zeta^{-1}\xi_{m-1}(n-1),\ \underline{z}_{-m-1}\rangle_n$$

$$= \langle \underline{e}_m(n-1),\ \underline{z}_{-m-1}\rangle_{n-1}$$

$$+\ \underline{e}_m(n,n-1)(\underline{z}'_{n-m-1} - \langle \xi_{m-1}(n-1),\ \underline{z}_{-m}\rangle_{n-1})$$

$$= \underline{G}^{r}_{m}(n-1) + \underline{e}_m(n,n-1)\underline{\delta}^{r'}_{m-1}(n-1) \quad \text{by (28a).} \qquad (32)$$

To reproduce equation (20a) take the nth component of (27a) to see

$$\underline{e}_m(n,n) = \underline{e}_m(n,n-1)(1 - \xi_{m-1}(n-1,n-1))$$

and substitute this into (32): equation (20a) results.

Next apply (27b) to $\underline{G}^{e}_{m}(n)$

$$\underline{G}^{e}_{m}(n) = \langle \zeta^{-1}\underline{r}_m(n-1),\ \underline{z}_0\rangle_n \qquad \text{by (24b)}$$

$$= \langle \zeta^{-1}\underline{r}_m(n-2),\ \underline{z}_0\rangle_n - \underline{r}_m(n-1,n-2)\ \langle \zeta^{-1}\xi_{m-1}(n-1),\ \underline{z}_0\rangle_n$$

$$= \langle \zeta^{-1}\underline{r}_m(n-2),\ \underline{z}_0\rangle_{n-1}$$

$$+\ \underline{r}_m(n-1,n-2)(\underline{z}'_n - \langle \zeta^{-1}\xi_{m-1}(n-1),\ \underline{z}_0\rangle_n)$$

$$= \underline{G}^{e}_{m}(n-1) + \underline{r}_m(n-1,n-2)\underline{\delta}^{e'}_{m-1}(n) \quad \text{by (28b)} \ . \qquad (33)$$

Again (20d) is found from the nth component of (27b) namely

$$\underline{r}_m(n,n) = \underline{r}_m(n,n-1)(1 - \xi_{m-1}(n,n))$$

and (putting n to n - 1) substituting this in (33).

Now equations (20a), (20b) must be completed by time/order updates for $\underline{\delta}^{e}_{m}(n)$, $\underline{\delta}^{r}_{m}(n)$. Consider then applying (31) to (28a)

$$\underline{\delta}^{r'}_{m}(n) = \underline{z}'_{n-m-1} - \langle \xi_m(n),\ \underline{z}_{-m-1}\rangle_n$$

$$= \underline{z}'_{n-m-1} - \langle \zeta^{-1}\xi_{m-1}(n-1),\ \underline{z}_{-m-1}\rangle_n$$

$$-\ \underline{\delta}^{e'}_{m-1}(n)\underline{\Sigma}^{-e}_{m}(n)\langle \underline{e}_m(n),\ \underline{z}_{-m-1}\rangle_n$$

$$= \underline{z}'_{n-m-1} - \langle \xi_{m-1}(n-1), \underline{z}_{-m} \rangle_{n-1} - \underline{\delta}^{e'}_{m-1} \underline{\Sigma}^{-e}_{m}(n)\underline{G}^{r}_{m}(n)$$

$$= \underline{\delta}^{r'}_{m-1}(n-1) - \underline{\delta}^{e'}_{m-1}(n)\underline{\Sigma}^{-e}_{m}(n)\underline{G}^{r}_{m}(n)$$

which is equation (20d). For $\underline{\delta}^{e}_{m}(n)$ using (29) in (28b) gives

$$\underline{\delta}^{e'}_{m}(n) = \underline{z}'_{n} - \langle \zeta^{-1}\xi_{m}(n-1), \underline{z}_{0} \rangle_{n}$$

$$= \underline{z}'_{n} - \langle \zeta^{-1}\xi_{m-1}(n-1), \underline{z}_{0} \rangle_{n}$$

$$- \underline{\delta}^{r'}_{m-1}(n-1)\underline{\Sigma}^{-r}_{m}(n-1)\langle \zeta^{-1}\underline{r}_{m}(n-1), \underline{z}_{0} \rangle_{n}$$

$$= \underline{\delta}^{e'}_{m-1}(n) - \underline{\delta}^{r'}_{m-1}(n-1)\underline{\Sigma}^{-r}_{m}(n-1)\underline{G}^{e}_{m}(n)$$

which is equation (20c).

Finally we can calculate initial conditions. Now

$$\underline{\delta}^{r'}_{0}(n) = \underline{z}'_{n-1} - \langle \xi_{0}(n), \underline{z}_{-1} \rangle_{n}$$

$$\underline{\delta}^{e'}_{0}(n) = \underline{z}'_{n} - \langle \zeta^{-1}\xi_{0}(n-1), \underline{z}_{0} \rangle_{n}$$

however, from equation (30)

$$\underline{\delta}^{r'}_{0}(n) = \underline{z}'_{n-1} - \underline{z}'_{n} \underline{\Sigma}^{-e}_{0}(n) \langle \underline{y}_{0}, \underline{z}_{-1} \rangle_{n}$$

$$= \underline{z}'_{n-1} - \underline{z}'_{n} \underline{\Sigma}^{-e}_{0}(n)\underline{G}^{r}_{0}(n)$$

which is (20g). Similarly (20h) follows.

III. General IV finite data LA

A. Statement

Let $\underline{x}_n$, y_n, $\underline{n}_n$ be r-dimensional discrete Time Series and consider the estimation of the parameters $\underline{A}_x^p = (\underline{A}_1 \ldots \underline{A}_p)$ in the lagged regression model (or finite impulse response model)

$$\underline{x}_n = \Sigma_1^p \underline{A}_i \underline{y}_{n-i} + \underline{n}_n \quad n \geq 0$$

by means of an IV sequence $\underline{z}_n$ which is correlated with the $\underline{y}_i$ but uncorrelated with the coloured noise $\underline{n}_i$. The finite data IV estimator $\underline{A}_x^p(n)$ is obtained by solving the normal equations

$$\Sigma_0^n \underline{x}_k (\underline{z}'_{k-1} \ldots \underline{z}'_{k-p}) = \underline{A}_x^p(n) \Sigma_0^n \begin{pmatrix} \underline{y}_{k-1} \\ \\ \underline{y}_{k-p} \end{pmatrix} (\underline{z}'_{k-1} \ldots \underline{z}'_{k-p}) \quad .$$

Note $\underline{x}_i = \underline{0}$, $\underline{y}_i = \underline{0}$, $\underline{z}_i = 0$ $i < 0$. Now a lattice scheme for solving these equations will be developed.

Introduce, as expected, the forward prediction error

$$\underline{\nu}_m(n) = \underline{x}_0 - \Sigma_i^m \underline{A}_{x_i}^m(n) \underline{y}_{-i} \quad . \tag{34}$$

Now the mr^2 parameters $\underline{A}_x^m(n) = (\underline{A}_{x1}^m(n) \ldots \underline{A}_{xm}^m(n))$ are uniquely determined by the mr^2 <u>orthogonality conditions</u>

$$\langle \underline{\nu}_m(n), \underline{z}_{-i} \rangle_n = \underline{0} \qquad 1 \leq i \leq m \quad . \tag{35a}$$

Thus the $\underline{\nu}_m(n)$ are uniquely determined by the conditions (35a), (35b)

$$\underline{\nu}_m(n) - \underline{x}_0 \in S_{1,m} \quad . \tag{35b}$$

Now in searching for an order update equation for $\underline{\nu}_m(n)$ it becomes immediately clear that we must also use the backwards prediction error

$$\underline{r}_m(n) = \underline{y}_{-m-1} - \Sigma_1^m \underline{B}_i^m(n) \underline{y}_{-i}$$

which is determined as in Section II together with

$$\underline{e}_m(n) = \underline{y}_0 - \Sigma_1^m \underline{A}_i^m(n) \underline{y}_{-i} \quad .$$

Now the finite data lattice IV recursion can be stated.

<u>Order update</u>: Equations (19), (36)

$$\underline{\nu}_{m+1}(n) = \underline{\nu}_m(n) + \underline{K}_{xm}(n)\zeta^{-1}\underline{r}_m(n-1) \tag{36a}$$

$$\underline{K}_{xm}(n) = -\underline{G}_{xm}(n)\underline{\Sigma}_m^{-r}(n-1) \quad . \tag{36b}$$

<u>Time update</u> $(m > 1)$: Equations (20), (37)

$$\underline{K}_{xm}(n) = \underline{K}_{xm}(n-1) + \underline{\nu}_m(n,n)\underline{\delta}^{r'}_{m-1}(n-1)(1 - \xi_{m-1}(n-1,n-1))^{-1} \quad . \tag{37a}$$

The initial conditions are

$$\underline{K}_{x0}(n) = \underline{K}_{x0}(n-1) + \underline{x}_n \, \underline{z}'_{n-1} \quad . \tag{37b}$$

B. <u>Derivation</u>

To find the order update for $\underline{\nu}_m(n)$ it is only necessary to repeat the type of argument used in Section II to find equations (36) where

$$\underline{G}_{xm}(n) = \langle \underline{\nu}_m(n), \underline{z}_{-m-1} \rangle_n \quad . \tag{38}$$

To complete the algorithm it is sufficient to add a time update for $\underline{G}_{xm}(n)$. As in Section II this requires a time update for $\underline{\nu}_m(n)$ which is easily seen to be (cf. derivation of (27a))

$$\underline{\nu}_m(n) = \underline{\nu}_m(n-1) + \underline{\nu}_m(n,n-1)\zeta^{-1}\xi_{m-1}(n-1) \quad . \tag{39}$$

Now we find much as before (cf. (32))

$$\underline{G}_{xm}(n) = \underline{G}_{xm}(n-1) + \underline{\nu}_m(n,n-1)\underline{\delta}^{r'}_{m-1}(n-1) \quad . \tag{40}$$

However the nth component of (39) gives

$$\underline{\nu}_m(n,n) = \underline{\nu}_m(n,n-1)(1 - \xi_{m-1}(n-1,n-1))$$

and substituting this in (40) gives (37a). Equation (37b) follows from (38) and (15d).

IV. Application to IV estimation of scalar ARMAX Time Series Models

Consider the estimation of the parameters $\underline{a}, \underline{b}$ in the ARMAX Time Series model

$$(1 + a(L))x_n = b(L)u_n + (1 + c(L))\varepsilon_n \quad .$$

This model is basic in Identification, Econometrics and Time Series (L is the backshift operator; $a(L) = \Sigma_1^p a_i L^i$; $\underline{a} = (a_1 \ldots a_p)$ etc.). If we write this model in regression form as

$$x_n = \Sigma_1^p \underline{\alpha}'_{xi} \underline{y}_{n-i} + \eta_n$$

where

$$\underline{y}_n = (x_n u_n)'; \; \alpha_{xi} = (a_i b_i)'; \; \eta_n = (1 + c(L))\varepsilon_n$$

then the simplest IV scheme uses the IV sequence

$$\underline{z}_n = \underline{y}_{n-p-1} \quad .$$

To find a finite data lattice recursion we begin as earlier by introducing the forward prediction error

$$e_{xm}(n) = \chi_0 - \Sigma_1^m \underline{\alpha}_{xi}^{m'}(n) \underline{\mathcal{Y}}_{-i} \tag{41}$$

where $\chi, \mathcal{Y}$ are the infinite data quantities

$$\chi = (x_0, x_1, \ldots$$

$$\mathcal{Y} = (\underline{y}_0, \underline{y}_1, \ldots \quad .$$

The parameters $\underline{\alpha}_x^m(n) = (\underline{\alpha}_{x1}^m(n) \ldots \underline{\alpha}_{xm}^m(n))$ are determined by the equations

$$\langle e_{xm}(n), \underline{Z}_{-i} \rangle_n = 0 \qquad 1 \leq i \leq m \tag{42}$$

where $\underline{Z}$ is the infinite data matrix

$$\underline{Z} = (\underline{z}_0, \underline{z}_1 \ldots \quad .$$

As before an order update for $e_{xm}(n)$ necessitates the introduction of the backward prediction order

$$\underline{r}_m(n) = \underline{\mathcal{Y}}_{-m} - \Sigma_1^m \underline{B}_{m+1-i}^m(n) \underline{\mathcal{Y}}_{-i} \quad . \tag{43}$$

Now it is clear from Section II that calculation of this requires parallel calculation (via equations (19), (20)) of the forward prediction error

$$\underline{e}_m(n) = \underline{y}_0 - \Sigma_1^m \underline{A}_i^m(n)\underline{y}_{-i} \quad . \tag{44}$$

Now recall that $\underline{e}_m(n)$ is defined by the equations

$$\langle \underline{e}_m(n), \underline{z}_{-i} \rangle_n = \underline{0} \qquad 1 \leq i \leq m \quad . \tag{17a}$$

However since the first row of $\underline{y}_0$ is x_0 the first row of equation (17a) is just equation (42). To put it another way, the first row of $\underline{e}_m(n)$ is $e_{xm}(n)$. Thus the solution of the present problem is necessarily obtained by the algorithm of Section II. This means in effect that to estimate $\underline{\alpha}_x^p$ by a lattice scheme we must fit the model

$$\underline{y}_n = \Sigma_1^p \underline{A}_i^p \underline{y}_{n-i} + \underline{\eta}_n$$

i.e. the scalar ARMAX problem is naturally imbedded in an ARMA problem (of dimension 2). In the least squares case $(\underline{z}_i = \underline{y}_i)$ this idea has been used by Lee et. al. [4] though they did not point out its natural appearance. Thus it is not necessary as suggested in [4] that u_n obey an ARMA model for the lattice algorithm to work (in the IV case or AR in the least squares case). All that is needed is that the algorithm produce a solution to equation (42); which it does. The fact that it does so by fitting an ARMA model to u_n (or AR model in the least squares case) is (in this respect) incidental.

Finally it is worth observing that the discussion of this section will carry over to the case of a multivariate ARMAX model

$$(I + \underline{A}(L))\underline{x}_n = \underline{B}(L)\underline{u}_n + (1 + \underline{c}(L))\underline{\epsilon}_n$$

(here $\underline{A}(L) = \Sigma_1^p \underline{A}_i L^i$ etc.) in which the block Hankel matrix based on the 2r-vector sequence $\underline{y}_n = (\underline{x}_n' \ \underline{u}_n')'$ has rank 2pr.

Conclusion

This article has provided a finite data lattice algorithm for the fast solution of equations arising for instrumental variable estimation of parameters in Time Series models. Two basic finite data algorithms have been presented. One is for the fast solution of the equations satisfied by the AR parameters in a block identifiable multivariate ARMA model. The other algorithm is for the instrumental variable estimation of parameters in a multivariate logged regression model (or finite impulse response model) with coloured disturbance noise. It turns out that the instrumental variable estimation of the transfer function parameters in scalar ARMAX Time Series model can be naturally embedded into the lagged regression model so the second finite data lattice algorithm can be applied.

A natural problem for future investigation is the derivation of a finite data lattice algorithm for solving equations from multivariate ARMA models that are not block identifiable. This would include the general ARMAX case too.

REFERENCES

[1] H. Akaike, Markovian Representation of Stochastic Processes and its Application to the analysis of Autoregressive Moving Average Processes, Ann. Inst. Stat. Math., 20, p.363, 1974.

[2] G. Carayannis, N. Kalouptsidis, D. Manolakis, Efficient Algorithm for Determination of A.R. part of A.R.M.A. model, Elect. Lett., 656-657, 1980.

[3] J. Makhoul, Stable and Efficient Lattice Methods for Linear Prediction, I.E.E.E. Trans. Acoust., Speech, Signal Processing, vol. ASSP-25, 423-428, 1977.

[4] D. T. Lee, B. Friedlander and M. Morf, Recursive Ladder Algorithms for ARMA modelling, Proc. 19th IEEE Conf. Decision and Control, 1225-1231, Dec. 1980.

[5] D. T. Lee, M. Morf, B. Friedlander, Recursive Least Squares Ladder Estimation Algorithms, IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-29, 627-641, 1981.

[6] L. Ljing, M. Morf, D. Falconer, Fast Calculation of Gain Matrices for Recursive Estimation Schemes, Int. J. Control, vol. 27, 1-19, 1978.

[7] J. Risannen, Algorithms for Triangular Decomposition of Block Hankel and Toeplitz Matrices with Application to Factoring Positive Matrix Polynomials, Math. Comp., vol. 17, 147-154, 1973.

[8] E. H. Sartorius, M. J. Shensa, Recursive Lattice Filters - A Brief Overview, Proc. 19th IEEE Conf. Decision and Control, 955-959, Dec. 1980.

[9] M. J. Shensa, Recursive Least Squares Lattice Algorithms - A Geometrical Approach, IEEE Trans. Autom. Control, vol. AC-26, 695-702, 1981.

VS/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2347 / 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Finite Data Lattice Algorithms for Instrumental Variable Recursions	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Victor Solo	8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 4 - Statistics & Probability
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709	12. REPORT DATE March 1982
	13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Time series, Signal processing, Instrumental Variables, Hankel matrix, Recursive least squares, ARMA model, Fast Algorithm, Toeplitz matrix

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The last few years have seen a rapid development of the so-called Lattice Algorithms for the fast solution of finite data least squares problems. While a fast algorithm has been given for finite data Instrumental Variable Recursions as yet no finite data lattice schemes have been given. In this work a lattice algorithm is derived for a finite data Instrumental Variable Recursion and its use in both ARMA and ARMAX Time Series models is indicated.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

FILMED

6-8