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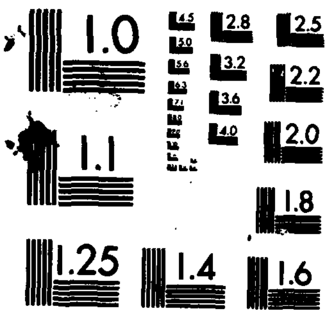

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END STRESS CALCULATIONS
ON
ELASTIC CYLINDERS

by

P.J.D. Mayes and D.A. Spence

March 1982

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For a semi-infinite circular elastic cylinder $z \geq 0$, $r \leq 1$ deformed solely by a distribution of stress and displacements on its flat end $z = 0$, the Love stress function can be expanded in a series of eigenfunctions of known form. For problems in which mixed stress and displacements boundary conditions are prescribed on $z = 0$ the coefficients appearing in the expansion can be determined in an explicit form via sets of biorthogonal functions. When normal and shear stresses are prescribed on $z = 0$ no such closed expressions for the coefficients exist and approximate methods usually

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20 - Abstract (continued)

lead to infinite systems of linear equations which are solved by truncation. Stability of solution as the order of truncation is increased can only be guaranteed theoretically when the infinite matrix is diagonally dominated, and this is not the case for existing methods. A Galerkin method has been developed using weighting functions chosen so as to optimise the diagonal dominance of the infinite matrix, and numerical results show that although the resulting matrix is not completely diagonally dominated, the resulting coefficients show an improvement in stability, and accurate solutions can be obtained using smaller matrices thus producing a much more efficient method of solution. Calculations are presented numerically and graphically for representative distributions for three classes of data:-

(i) Smooth continuous data

(ii) Smooth data violating compatibility at $r = 1$

(iii) Data containing discontinuities.

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END STRESS CALCULATIONS ON ELASTIC CYLINDERS

Final Technical Report

by

P.J.D. Mayes and D.A. Spence

March 1982

United States Army

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End Stress Calculations on Elastic Cylinders

by P.J.D. Mayes and D.A. Spence

ABSTRACT

For a semi-infinite circular elastic cylinder $z>0$, $r<1$ deformed solely by a distribution of stress and displacements on its flat end $z=0$, the Love stress function can be expanded in a series of eigenfunctions of known form. For problems in which mixed stress and displacements boundary conditions are prescribed on $z=0$ the coefficients appearing in the expansion can be determined in an explicit form via sets of biorthogonal functions. When normal and shear stresses are prescribed on $z=0$ no such closed expressions for the coefficients exist and approximate methods usually lead to infinite systems of linear equations which are solved by truncation. Stability of solution as the order of truncation is increased can only be guaranteed theoretically when the infinite matrix is diagonally dominated, and this is not the case for existing methods. A Galerkin method has been developed using weighting functions chosen so as to optimise the diagonal dominance of the infinite matrix, and numerical results show that although the resulting matrix is not completely diagonally dominated, the resulting coefficients show an improvement in stability, and accurate solutions can be obtained using smaller matrices thus producing a much more efficient method of solution. Calculations are presented numerically and graphically for representitive distributions for three classes of data:-

(i) Smooth continuous data

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Table of Contents

End Stress Calculations on Elastic Cylinders

by P.J.D. Mayes and D.A. Spence

1. Introduction

The Love stress function $\Phi(r,z)$ in an elastic cylinder $z>0$, $r<1$ subjected to homogeneous boundary conditions on the curved boundary $r=1$ can be expressed as an eigenfunction expansion of the form

$$\sum c_n e^{-\lambda_n z} \phi(r;\lambda_n) \tag{1.1}$$

where λ_n is an eigenvalue determined from the conditions on $r=1$. For the case of a traction-free curved face, λ_n is a root of

$$\lambda^2\left\{J_0^2(\lambda) + J_1^2(\lambda)\right\} = 2(1-\nu)J_1^2(\lambda) \tag{1.2}$$

Little and Childs [1967] have given a construction for determining the coefficients c_n in the expansion (1.1) for cases in which the data on the flat end $z=0$ takes the form of prescribed values of either of the pairs

$$\begin{array}{ll} & \sigma_{zz} \text{ and } u_r \\ \text{or} & \\ & \sigma_{rz} \text{ and } u_z \end{array} \tag{1.3}$$

For these "canonical" problems the $\{c_m\}$ are found explicitly as quadratures of the data with appropriate biorthogonal functions derived from the $\phi(r;\lambda_n)$.

In the present report we consider the problem of determining the coefficients when σ_{zz} and σ_{rz} are prescribed. It is known that no explicit solution exists for this case, and the $\{c_n\}$ must be found by approximate methods leading in general to infinite matrices which can only be inverted in truncated form.

This problem has already been studied extensively for the elastic strip, $x>0$, $|y|<1$. Spence [1978] introduced a set of weighting functions derived from members of the family of biorthogonal functions, which in the case of the traction problem for the strip, namely

$$\sigma_{xx},\ \sigma_{xy} \text{ defined on } x=0$$

lead to a diagonally dominated system of equations

$$\sum_n A_{mn} c_n = d_m \tag{1.4}$$

where $\underset{\sim}{A} = \underset{\sim}{I} - \underset{\sim}{G}$, with the row sum norm $\|\underset{\sim}{G}\| < 1$. For such a system, the solution $\underset{\sim}{c}^{(N)}$ say, of the truncated system

$$\sum_n A_{mn}^{(N)} c_n^{(N)} = d_m^{(N)} \tag{1.5}$$

is known to converge to the solution of the full system as $N \to \infty$, and this was borne out for the cases tested, in which it was found that changing the order of truncation N did not lead to changes in the coefficients. This was not found to be the case with other published methods that were tested.

2. The New Formulation

The construction given by Little and Childs [1967] for obtaining biorthogonal functions for the two canonical end problems for the elastic cylinder, thus enabling them to obtain the coefficients appearing in (1.1) explicitly, has not proved to be the most suitable for the present studies. The main disadvatage is that for the stress problem it is not possible to "optimise" the weighting functions, thus improving the diagonal dominance of the infinite matrix arising in this problem. Consequently we choose a different but equivalent set of four stress- and displacement-related variables which will be prescribed on $z=0$.

In terms of the biharmonic "Love" stress function (Love [1927], Art.188) the stresses and displacements are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left\{\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right\} \qquad \sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\} \tag{2.1,2}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\} \qquad \sigma_{\theta\theta} = \frac{\partial}{\partial z}\left\{\nu\nabla^2\Phi - \frac{1\partial\Phi}{r\partial r}\right\} \tag{2.3,4}$$

$$2\mu u_r = -\frac{\partial^2\Phi}{\partial r\partial z} \qquad 2\mu u_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{2.5,6}$$

where ν is Poisson's ratio and

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \equiv B^2 + \frac{\partial^2}{\partial z^2} . \tag{2.7}$$

If the cylinder is subjected to stress-free side conditions on $r=1$ and a self-equilibrating distribution of stresses and displacements on $z=0$ then Φ may be expanded as an eigenfunction expansion

$$\Phi(r,z) = \sum_m c_m \phi(r;\lambda_m) e^{-\lambda_m z} \tag{2.8}$$

where λ_m is a root of

$$\lambda^2\left\{J_0^2(\lambda) + J_1^2(\lambda)\right\} - 2(1-\nu)J_0^2(\lambda) = 0, \tag{2.9}$$

$$\phi(r;\lambda) = \left[2(1-\nu)J_1(\lambda)+\lambda J_0(\lambda)\right]J_0(\lambda r) + \lambda J_1(\lambda) r J_1(\lambda r) \tag{2.10}$$

and the correct interpretation of the summation (2.8) is obtained by numbering the roots of (2.9) in the right half-plane so that $\lambda_{-n} = \overline{\lambda_n}$ [see figure 1] and writing the expansion more precisely as

$$\Phi(r,z) = \sum_{m=-\infty}^{m=+\infty}{}' c_m \phi(r;\lambda_m) e^{-\lambda_m z} \tag{2.11}$$

where the prime means that the term with m=0 does not appear in the summation. This implies that the normal stress distribution is equilibrated. i.e. $\int_0^1 r\sigma_{zz}(r,0)\,dr = 0$.

The present choice of prescribed functions together with their expansions in terms of the "derived" functions $\phi_m^{(\alpha)}(r)$ are given by

$$\begin{bmatrix} f^{(1)}(r) \\ f^{(2)}(r) \\ f^{(3)}(r) \\ f^{(4)}(r) \end{bmatrix} = \begin{bmatrix} \partial\sigma_{zz}/\partial r \\ \sigma_{rz} \\ -(1-2\nu)\frac{\partial}{\partial r}B^2\Phi_z + 2\nu\Phi_{zzzr} \\ (1+\nu)\frac{\partial}{\partial r}\nabla^2\Phi \end{bmatrix}_{z=0} = \sum c_m \begin{bmatrix} \phi_m^{(1)}(r) \\ \phi_m^{(2)}(r) \\ \phi_m^{(3)}(r) \\ \phi_m^{(4)}(r) \end{bmatrix} \tag{2.12}$$

This can be seen to be equivalent to prescribing the unmodified stresses and displacements as in Little and Childs - for example, if σ_{zz} and u_r are known on z=0, then so are $f^{(1)}$ and $f^{(3)}$ as defined above.

In terms of $\phi(r;\lambda)$ the derived functions $\phi_m^{(\alpha)}$ are given by

$$\phi_m^{(1)}(r) = -\lambda_m\left\{(2-\nu)\frac{d}{dr}B^2\phi + (1-\nu)\lambda_m^2\cdot\frac{d\phi}{dr}\right\} \tag{2.13}$$

$$\phi_m^{(2)}(r) = (1-\nu)\frac{d}{dr}B^2\phi + -\nu\lambda_m^2\cdot\frac{d\phi}{dr} \tag{2.14}$$

$$\phi_m^{(3)}(r) = -\lambda_m\left\{-(1-2\nu)\frac{d}{dr}B^2\phi + 2\nu\lambda_m^2\cdot\frac{d\phi}{dr}\right\} \tag{2.15}$$

$$\phi_m^{(4)}(r) = (1+\nu)\left\{\frac{d}{dr}B^2\phi + \lambda_m^2\cdot\frac{d\phi}{dr}\right\} \tag{2.16}$$

and explicit expressions for these functions in terms of Bessel functions are

$$\phi_m^{(1)}(r) = \lambda_m^4\{\lambda_m J_1(\lambda_m) r J_0(\lambda_m r) + [2J_1(\lambda_m) - \lambda_m J_0(\lambda_m)]J_1(\lambda_m r)\} \tag{2.17}$$

$$\phi_m^{(2)}(r) = \lambda_m^4\{-J_1(\lambda_m) r J_0(\lambda_m r) + J_0(\lambda_m) J_1(\lambda_m r)\} \tag{2.18}$$

$$\phi_m^{(3)}(r) = \lambda_m^4\{-\lambda_m J_1(\lambda_m) r J_0(\lambda_m r) + [2\nu J_1(\lambda_m) + \lambda_m J_0(\lambda_m)]J_1(\lambda_m r)\} \tag{2.19}$$

$$\phi_m^{(4)}(r) = -2(1+\nu)\lambda_m^3 J_1(\lambda_m) J_1(\lambda_m r) \tag{2.20}$$

3. Derivation of Biorthogonal Functions

$\phi(r;\lambda)$ is a solution of the reduced biharmonic equation

$$\left[\frac{1}{r}\,\frac{d}{dr}r\frac{d}{dr} + \lambda^2\right]^2 \phi = 0, \tag{3.1}$$

and as in Spence [1978] this equation may be expressed as a matrix differential equation in either $\phi_m^{(1)}$ and $\phi_m^{(3)}$ or $\phi_m^{(2)}$ and $\phi_m^{(4)}$ (†) For the (1,3)-canonical problem the matrix equation

$$\begin{bmatrix} -\nu\left[B^2-\frac{1}{r^2}\right] & \left[B^2-\frac{1}{r^2}\right] \\ -\left[B^2-\frac{1}{r^2}\right] & -(2+\nu)\left[B^2-\frac{1}{r^2}\right] \end{bmatrix} \begin{bmatrix} \phi_m^{(1)} \\ \phi_m^{(3)} \end{bmatrix} = (1+\nu)\lambda_m^2 \begin{bmatrix} \phi_m^{(1)} \\ \phi_m^{(3)} \end{bmatrix} \tag{3.2}$$

can readily be shown using (2.13,15) to reduce to

$$\frac{d}{dr}\left[B^2 + \lambda^2\right]^2 \phi = 0 \tag{3.3}$$

The condition $\sigma_{rz} = 0$ on $r=1$ may be written in terms of $\phi_m^{(1)}$ and $\phi_m^{(3)}$ as

$$\nu\phi_m^{(1)}(1) - \phi_m^{(3)}(1) = 0 \tag{3.4}$$

The corresponding boundary condition for σ_{rr} is

$$(1-\nu)D\phi_m^{(1)}(1) + 2D\phi_m^{(3)}(1) + (1+\nu)\phi_m^{(3)}(1) = 0 \tag{3.5}$$

(†) This is another advantage of the present formulation. The Little and Childs derived functions do not appear to be the solutions of any underlying matrix differential equation.

where $D = d/dr$. The derivatives of $\phi_m^{(1)}$ and $\phi_m^{(3)}$ contain the fourth derivatives of ϕ, and in obtaining (3.5) it has been necessary to use the reduced biharmonic equation (3.1) evaluated at r=1 to express the σ_{rr} condition in the required form.

As in Spence [1978] the function $\psi_n^{(1)}$ and $\psi_n^{(3)}$ which are biorthogonal to $\phi_m^{(1)}$ and $\phi_m^{(3)}$ are obtained as the eigenfunctions of the differential operator adjoint to (3.2) which are constructed[†] as follows:-

Using the differential equation (3.2) we may write

$$(1+\nu)\lambda_m^2\left\langle \phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_m^{(3)} \right\rangle \qquad \left[\ \langle * \rangle = \int_\epsilon^1 *.rdr\ \right]$$

$$= \left\langle \left\{(1+\nu)\lambda_m^2\phi_m^{(1)}\right\}\psi_n^{(1)} + \left\{(1+\nu)\lambda_m^2\phi_m^{(3)}\right\}\psi_n^{(3)} \right\rangle$$

$$= \left\langle \left\{-\nu\left[B^2-\frac{1}{R^2}\right]\phi_m^{(1)} + \left[B^2-\frac{1}{r^2}\right]\phi_m^{(3)}\right\}\psi_n^{(1)} \right.$$

$$\left. + \left\{-\left[B^2-\frac{1}{R^2}\right]\phi_m^{(1)} - (2+\nu)\left[B^2-\frac{1}{r^2}\right]\phi_m^{(3)}\right\}\psi_n^{(3)} \right\rangle = (*)$$

We may now integrate twice by parts, transferring the B^2 derivatives onto the $\psi_n^{(\alpha)}$ and introducing boundary conditions at r=1:-

---

(†) The construction of biorthogonal functions for the (2,4)-problem is a modification of the work of Klemm [1970], who treated the full non-axisymmetric end loading problem. Putting $\theta=0$, $\partial/\partial\theta=0$ in his construction gives the biorthogonality given here. However, his construction for the (1,3)-problem does not lead to a pure biorthogonality from which the coefficients can be determined explicitly, and the construction described below is new.

$$(*) = -\nu\left\langle \phi_m^{(1)}\left[B^2-\tfrac{1}{r^2}\right]\psi_n^{(1)}\right\rangle + \left\langle \phi_m^{(3)}\left[B^2-\tfrac{1}{r^2}\right]\psi_n^{(1)}\right\rangle$$

$$- \left\langle \phi_m^{(1)}\left[B^2-\tfrac{1}{r^2}\right]\psi_n^{(3)}\right\rangle - (2+\nu)\left\langle \phi_m^{(3)}\left[B^2-\tfrac{1}{r^2}\right]\psi_n^{(3)}\right\rangle$$

$$-\nu\left[\psi_n^{(1)}D\phi_m^{(1)} - \phi_m^{(1)}D\psi_n^{(1)}\right]_{r=1} + \left[\psi_n^{(1)}D\phi_m^{(3)} - \phi_m^{(3)}D\psi_n^{(1)}\right]_{r=1}$$

$$- \left[\psi_n^{(3)}D\phi_m^{(1)} - \phi_m^{(1)}D\psi_n^{(3)}\right]_{r=1} - (2+\nu)\left[\psi_n^{(3)}D\phi_m^{(3)} - \phi_m^{(3)}D\psi_n^{(3)}\right]_{r=1}$$

If $\underset{\sim}{\psi}_n^{(1,3)} = (\psi_n^{(1)},\psi_n^{(3)})^T$ is an eigenfunction of the adjoint differential equation

$$\begin{bmatrix} -\nu\left[B^2-\frac{1}{r^2}\right] & -\left[B^2-\frac{1}{r^2}\right] \\ \left[B^2-\frac{1}{r^2}\right] & -(2+\nu)\left[B^2-\frac{1}{r^2}\right] \end{bmatrix}\begin{bmatrix}\psi_n^{(1)} \\ \psi_n^{(3)}\end{bmatrix} = (1+\nu)\lambda_n^2\begin{bmatrix}\psi_n^{(1)} \\ \psi_n^{(3)}\end{bmatrix} \qquad (3.6)$$

then using the boundary conditions (3.4,5) we may write

$$(1+\nu)(\lambda_m^2-\lambda_n^2)\left\langle \phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_n^{(3)}\right\rangle =$$

$$+ \tfrac{1}{2}(1+\nu)\phi_m^{(1)}(1)\left\{-\nu\psi_n^{(1)}(1) + 2(1+\nu)D\psi_n^{(3)}(1) + \nu(2+\nu)\psi_n^{(3)}(1)\right\}$$

$$+ \tfrac{1}{2}(1+\nu)D\phi_m^{(3)}(1)\left\{-\psi_n^{(1)}(1) - \nu\psi_n^{(3)}(1)\right\} \qquad (3.7)$$

Thus if $\underset{\sim}{\psi}_n^{(1,3)}$ satisfies the adjoint boundary conditions

$$\psi_n^{(1)}(1) + \nu\psi_n^{(3)}(1) = 0 \qquad (3.8)$$

$$-\nu\psi_n^{(1)}(1) + 2(1+\nu)D\psi_n^{(3)}(1) + \nu(2+\nu)\psi_n^{(3)}(1) = 0, \qquad (3.9)$$

or more compactly

$$\psi_n^{(1)}(1) + \nu\psi_n^{(3)}(1) = 0 \tag{3.10}$$

$$D\psi_n^{(3)}(1) - \psi_n^{(1)}(1) = 0, \tag{3.11}$$

we find

$$(1+\nu)(\lambda_m^2-\lambda_n^2)\left\langle\phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_n^{(3)}\right\rangle = 0 \tag{3.12}$$

and hence

$$\left\langle\phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_n^{(3)}\right\rangle = 0 \quad \text{for} \quad m\neq n. \tag{3.13}$$

Exactly the same construction may be used for the (2,4)-canonical problem. This time the required matrix differential equation is

$$\begin{bmatrix} -(1+\nu)\left[B^2-\frac{1}{r^2}\right] & \left[B^2-\frac{1}{r^2}\right] \\ 0 & -(1+\nu)\left[B^2-\frac{1}{r^2}\right] \end{bmatrix}\begin{bmatrix}\phi_m^{(2)} \\ \phi_m^{(4)}\end{bmatrix} = (1+\nu)\lambda_m^2\begin{bmatrix}\phi_m^{(2)} \\ \phi_m^{(4)}\end{bmatrix} \tag{3.14}$$

with corresponding boundary conditions

$$\phi_m^{(2)}(1) = 0 \tag{3.15}$$

$$(1+\nu)D\phi_m^{(2)}(1) = D\phi_m^{(4)}(1) + \nu\phi_m^{(4)}(1) \tag{3.16}$$

and the adjoint equation and boundary conditions are

$$\begin{bmatrix} -(1+\nu)\left[B^2-\frac{1}{r^2}\right] & 0 \\ \left[B^2-\frac{1}{r^2}\right] & -(1+\nu)\left[B^2-\frac{1}{r^2}\right] \end{bmatrix}\begin{bmatrix}\psi_n^{(2)} \\ \psi_n^{(4)}\end{bmatrix} = (1+\nu)\lambda_n^2\begin{bmatrix}\psi_n^{(2)} \\ \psi_n^{(4)}\end{bmatrix} \tag{3.17}$$

$$(1+\nu)D\psi_n^{(4)}(1) = D\psi_n^{(2)}(1) + \nu\psi_n^{(2)}(1) \qquad (3.18)$$

$$\psi_n^{(4)}(1) = 0 \qquad (3.19)$$

resulting in the biorthogonality

$$\left\langle \phi_m^{(2)}\psi_n^{(2)} + \phi_m^{(4)}\psi_n^{(4)} \right\rangle = 0 \quad \text{for} \quad m \neq n. \qquad (3.20)$$

In terms of the Bessel functions the two biorthogonal vectors are given by

$$\begin{bmatrix} \psi_n^{(1)}(r) \\ \psi_n^{(3)}(r) \end{bmatrix} = A_n \begin{bmatrix} -\lambda_n J_1(\lambda_n) r J_0(\lambda_n r) + [-2\nu J_1(\lambda_n) + \lambda_n J_0(\lambda_n)] J_1(\lambda_n r) \\ -\lambda_n J_1(\lambda_n) r J_0(\lambda_n r) + [2J_1(\lambda_n) + \lambda_n J_0(\lambda_n)] J_1(\lambda_n r) \end{bmatrix} \qquad (3.21)$$

$$\begin{bmatrix} \psi_n^{(2)}(r) \\ \psi_n^{(4)}(r) \end{bmatrix} = B_n \begin{bmatrix} 2(1+\nu) J_1(\lambda_n) J_1(\lambda_n r) \\ \lambda_n J_1(\lambda_n) r J_0(\lambda_n r) - \lambda_n J_0(\lambda_n) J_1(\lambda_n r) \end{bmatrix} \qquad (3.22)$$

where

$$A_n = \frac{1}{2(1+\nu)\lambda_n^{\,2} J_1^{\,2}(\lambda_n) P(\lambda_n)} \qquad (3.23)$$

$$B_n = \frac{1}{2(1+\nu)\lambda_n J_1^{\,2}(\lambda_n) P(\lambda_n)} \qquad (3.24)$$

$$P(\lambda_n) = -\lambda_n^2 J_0^2(\lambda_n) + 2(1-\nu)\lambda_n J_0(\lambda_n) J_1(\lambda_n) - 2(1-\nu) J_1^2(\lambda_n) \qquad (3.25)$$

and the normalising factor $P(\lambda_n)$ has been introduced so that

$$\left\langle \phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_n^{(3)} \right\rangle = \delta_{mn} \qquad (3.26)$$

$$\left\langle \phi_m^{(2)} \psi_n^{(2)} + \phi_m^{(4)} \psi_n^{(4)} \right\rangle = \delta_{mn}. \tag{3.27}$$

It is interesting to note that as in Spence [1978] this formulation exhibits what might be called a "self-biorthogonality" where

$$\begin{bmatrix} \psi_n^{(1)} \\ \psi_n^{(3)} \end{bmatrix} = \frac{A_n}{(1+\nu)\lambda_n^4} \begin{bmatrix} -2\nu\phi_n^{(1)} + (1-\nu)\phi_n^{(3)} \\ (1-\nu)\phi_n^{(1)} + 2\phi_n^{(3)} \end{bmatrix} \tag{3.28}$$

and

$$\begin{bmatrix} \psi_n^{(2)} \\ \psi_n^{(4)} \end{bmatrix} = \frac{-B_n}{2(1+\nu)\lambda_m^3} \begin{bmatrix} \phi_n^{(4)} \\ 2(1+\nu)\phi_n^{(2)} \end{bmatrix} \tag{3.29}$$

By contrast, in the formulation of Little and Childs, the (1,3)-biorthogonal functions are given in terms of the (2,4)-derived *functions* and vice versa.

4. Optimal Weighting Functions

In this section we consider the stress problem in which

$$\frac{\partial}{\partial r}(\sigma_{zz})_{z=0} = f^{(1)}(r)$$

and (4.1)

$$(\sigma_{rz})_{z=0} = f^{(2)}(r)$$

are prescribed functions of r. This does not fall into the class of canonical end problems categorised in Section 1. As was done for the strip problem, we now seek weighting functions of the form

$$\chi_m^{(1)} = A\phi_m^{(1)} + B\phi_m^{(3)} \tag{4.2}$$

$$\chi_m^{(2)} = C\lambda_m^2\phi_m^{(2)} + D\lambda_m^2\phi_m^{(4)} \tag{4.3}$$

where A, B, C and D are constants to be determined. [The choice $A = -2\nu$, $B = (1-\nu)$, $C = 0$, $D = -(1+\nu)$ would produce the biorthogonal functions $\psi_m^{(1)}$, $\psi_m^{(2)}$ defined in Section 3, but as will be seen these are not optimal for the non-canonical problem].

An infinite set of linear equations for the coefficients c_n in the derived expansions

$$f^{(1)} = \sum c_n\phi_n^{(1)} \tag{4.4}$$

$$f^{(2)} = \sum c_n\phi_n^{(2)} \tag{4.5}$$

is obtained by combining the scalar products of (4.4) with $\chi_m^{(1)}$ and (4.5) with $\chi_m^{(2)}$ for each n. This yields the set

$$\sum_n A_{mn} c_n = d_m \tag{4.6}$$

where

$$A_{mn} = \left\langle \chi_m^{(1)} \phi_n^{(1)} + \chi_m^{(1)} \phi_n^{(2)} \right\rangle \tag{4.7}$$

and

$$d_m = \left\langle \chi_m^{(1)} f^{(1)} + \chi_m^{(2)} f^{(2)} \right\rangle \tag{4.8}$$

We now choose the constants A, B, C and D so as to make the off-diagonal elements of the matrix $\underset{\sim}{A}$ as small as possible in absolute value compared with the diagonal elements. For this purpose the scalar products

$$\left\langle \phi_n^{(1)} \phi_m^{(1)} \right\rangle, \quad \left\langle \phi_n^{(1)} \phi_m^{(3)} \right\rangle, \quad \left\langle \phi_n^{(2)} \phi_m^{(2)} \right\rangle \text{ and } \left\langle \phi_n^{(2)} \phi_m^{(4)} \right\rangle \tag{4.9}$$

have been calculated and are listed in Appendix A. The expressions are cumbersome, but the salient feature is that the first three contain the factor $(\lambda_m^2 - \lambda_n^2)^{-3}$. As was noted by Spence for the strip problem, the presence of any negative power of $(\lambda_m - \lambda_n)$ in the matrix A_{mn} leads to divergent row sum norms. The four constants A, B, C and D provide just sufficient freedom to suppress all such factors in the denominator.

The procedure for determining the optimal choice for the constants A,B,C and D given the choice of weighting functions (4.2,3) involves taking the matrix elements (4.7) with $\chi_m^{(1)}$ and $\chi_m^{(2)}$ given by (4.2,3), and dividing out the unwanted factors $(\lambda_m^2 - \lambda_n^2)^{-1}$ giving three equations for the four constants.

Using the quadratures given in Appendix B the general matrix element A_{mn} is

$$A_{mn} = A\left\langle \phi_n^{(1)}\phi_m^{(1)}\right\rangle + B\left\langle \phi_n^{(1)}\phi_m^{(3)}\right\rangle + C\lambda_m^2\left\langle \phi_n^{(2)}\phi_m^{(2)}\right\rangle + D\lambda_m^2\left\langle \phi_n^{(2)}\phi_m^{(4)}\right\rangle$$

$$4\lambda_m^3\lambda_n^3 J_1(\lambda_m)J_1(\lambda_n)\times$$

$$\left\{\frac{1}{\lambda_m{}^2-\lambda_n{}^2}\left[-(A+B\nu)\lambda_m\lambda_n(\lambda_m J_0(\lambda_m)J_1(\lambda_n)-\lambda_n J_1(\lambda_m)J_0(\lambda_n))\right]\right.$$

$$+\frac{1}{(\lambda_m{}^2-\lambda_n{}^2)^2}\left[-(A+B\nu)\lambda_m^2\lambda_n^2(\lambda_m J_1(\lambda_m)J_0(\lambda_n)+\lambda_n J_0(\lambda_m)J_1(\lambda_n))\right.$$

$$+2B(1+\nu)\lambda_m^3\lambda_n^2 J_1(\lambda_m)J_0(\lambda_n) - C\lambda_m^3\lambda_n(\lambda_m J_1(\lambda_m)J_0(\lambda_n)+\lambda_n J_0(\lambda_m)J_1(\lambda_n))$$

$$\left.+D(1+\nu)\lambda_m^3\lambda_n(\lambda_m J_1(\lambda_m)J_0(\lambda_n)-\lambda_n J_0(\lambda_m)J_1(\lambda_n))\right]$$

$$+\frac{1}{(\lambda_m{}^2-\lambda_n{}^2)^3}\left[2(A-B)\lambda_m^3\lambda_n^3(\lambda_m J_0(\lambda_m)J_1(\lambda_n)-\lambda_n J_1(\lambda_m)J_0(\lambda_n))\right.$$

$$\left.\left.+2C\lambda_m^4\lambda_n^2(\lambda_m J_0(\lambda_m)J_1(\lambda_n)-\lambda_n J_1(\lambda_m)J_0(\lambda_n))\right]\right\}$$

$$+4(1-\nu)\lambda_m^3\lambda_n^3 J_1^2(\lambda_m)J_1^2(\lambda_n)\left\{\frac{1}{\lambda_m{}^2-\lambda_n{}^2}\left[-B(1+\nu)\lambda_m\lambda_n-D(1+\nu)\lambda_m^2\right]\right.$$

$$\left.+\frac{1}{(\lambda_m{}^2-\lambda_n{}^2)^2}\left[A\lambda_m\lambda_n(\lambda_m^2+\lambda_n^2)-B\lambda_m\lambda_n(\lambda_m^2+\lambda_n^2)+C(\lambda_m^2+\lambda_n^2)\lambda_m^2\right]\right.$$

We shall try to eliminate the $(\lambda_m^2-\lambda_n^2)^{-1}$ from the dominant term, which over a common denominator can be written

$$\frac{4\lambda_m^4\lambda_n^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3(\lambda_m-\lambda_n)^3}\left\{\left[-(C-D(1+\nu))\lambda_m^5+2B(1+\nu)\lambda_m^4\lambda_n\right.\right.$$

$$\left.-(C+D(1+\nu))\lambda_m^3\lambda_n^2-(3A+2B\nu)\lambda_m^2\lambda_n^3+(A+B\nu)\lambda_n^5\right]J_1(\lambda_m)J_0(\lambda_n)$$

$$+\left[-(A+B\nu)\lambda_m^5+(C-D(1+\nu))\lambda_m^4\lambda_n\right.$$

$$\left.\left.+(3A-2B+B\nu)\lambda_m^3\lambda_n^2+(C+D(1+\nu))\lambda_m^2\lambda_n^3\right]J_0(\lambda_m)J_1(\lambda_n)\right\}$$

The condition that both factors multiplying the Bessel functions have a factor $\lambda_m-\lambda_n$ is the same, namely

$$A - B + C = 0, \tag{4.10}$$

and if this condition is satisfied the dominant term becomes

$$\frac{4\lambda_m^4\lambda_n^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3(\lambda_m-\lambda_n)^2}\Big\{J_1(\lambda_m)J_0(\lambda_n)\left[(C-D(1+\nu))\lambda_m^4-(2A+2B\nu+C+D(1+\nu))\lambda_m^3\lambda_n\right.$$

$$\left.-(2A+2B\nu)\lambda_m^2\lambda_n^2+(A+B\nu)\lambda_m\lambda_n^3+(A+B\nu)\lambda_n^4\right]$$

$$\left.+\lambda_m^2 J_0(\lambda_m)J_1(\lambda_n)\left[(A+B\nu)\lambda_m^2+(3A-2B+B\nu+C+D(1+\nu))\lambda_m\lambda_n+(C+D(1+\nu))\lambda_n^2\right]\right\}$$

Again the condition that both terms inside the square brackets have the factor $\lambda_m-\lambda_n$ is the same:

$$A + B\nu + D(1+\nu) = 0 \tag{4.11}$$

giving a dominant term

$$\frac{-4\lambda_m^4\lambda_n^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3(\lambda_m-\lambda_n)}\Big\{J_1(\lambda_m)J_0(\lambda_n)\left[(2A+2B\nu+C+D(1+\nu))\lambda_m^3\right.$$

$$\left.-(2A+2B\nu)\lambda_m\lambda_n^2-(A+B\nu)\lambda_n^3\right]$$

$$+\lambda_m^2 J_0(\lambda_m) J_1(\lambda_n)\left[(A+B\nu)\lambda_m-(C+D(1+\nu))\lambda_n\right]\Big\}$$

The last relation suppressing all factors $(\lambda_m-\lambda_n)^{-1}$ in the denominator is

$$A + B\nu - C - D(1+\nu) = 0 \qquad (4.12)$$

giving as the dominant term

$$\frac{4(A+B\nu)\lambda_m^4\lambda_n^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3}\left\{(3\lambda_m^2+3\lambda_m\lambda_n+\lambda_n^2)J_1(\lambda_m)J_0(\lambda_n)+\lambda_m^2 J_0(\lambda_m)J_1(\lambda_n)\right\}$$

The three equations (4.10,11,12) lead to the values

$$A = -(1-2\nu); \quad B = -3; \quad C = -2(1+\nu); \quad D = 1. \qquad (4.13)$$

The resulting weighting functions are thus

$$\chi_m^{(1)} = 2(1+\nu)\lambda_m^4\left[\lambda_m J_1(\lambda_m) r J_0(\lambda_m r) - \{J_1(\lambda_m)+\lambda_m J_0(\lambda_m)\}J_1(\lambda_m)\right] \qquad (4.14)$$

$$\chi_m^{(2)} = 2(1+\nu)\lambda_m^5\left[\lambda_m J_1(\lambda_m) r J_0(\lambda_m r) - \{J_1(\lambda_m)+\lambda_m J_0(\lambda_m)\}J_1(\lambda_m)\right] \qquad (4.15)$$

and the matrix elements are

$$A_{mn} = \frac{4(1+\nu)\lambda_m^4\lambda_n^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3}\left\{(3\lambda_m^2+3\lambda_m\lambda_n+\lambda_n^2)J_1(\lambda_m)J_0(\lambda_n)+\lambda_m^2 J_0(\lambda_m)J_1(\lambda_n)\right\}$$

$$- \frac{4(1-\nu^2)\lambda_m^4\lambda_n^3 J_1^2(\lambda_m) J_1^2(\lambda_n)(3\lambda_m+\lambda_n)}{(\lambda_m+\lambda_n)^2} \tag{4.16}$$

$$A_{mm} = 2(1+\nu)\lambda_m^7 J_0(\lambda_m) J_1^2(\lambda_m)\left\{2\nu J_1(\lambda_m) + \lambda_m J_0(\lambda_m)\right\} \tag{4.17}$$

In order to see why this choice of coefficients should give rise to a more stable matrix, it is of interest to determine the asymptotic form of the matrix elements. These are determined by using the asymptotic form for the eigenvalues

$$\lambda_m \sim m\pi + \frac{1}{2} i \log(4m\pi) \tag{4.18}$$

so that we simply replace λ_m by $m\pi$ to first order in the matrix elements (4.16,17). Using the asymptotic forms of the Bessel functions given, for example, in Abramovitch and Stegun, it can be shown that the Bessel functions of the eigenvalues, namely $J_0(\lambda_m)$ and $J_1(\lambda_m)$, have the asymptotic form

$$J_0(\lambda_m) \sim (-1)^m \frac{(1+i)}{\sqrt{\pi}} \qquad J_1(\lambda_m) \sim (-1)^{m+1} \frac{(1-i)}{\sqrt{\pi}} \tag{4.9}$$

Thus the matrix elements have the asymptotic form

$$A_{mn} \sim 16(1+\nu)\pi^5 i \frac{m^4 n^4}{(m+n)^3}\left\{4m^2+3mn+n^2\right\}$$

for λ_m in the first quadrant, and

$$A_{mn} \sim -16(1+\nu)\pi^5 i \frac{m^4 n^4}{(m+n)^2}\left\{2m+n\right\},$$

for λ_m in the fourth quadrant, with

$$\Lambda_{mm} \sim 8(1+\nu)\pi^6 m^8$$

If the factors $(\lambda_m - \lambda_n)^{-1}$ had not been eliminated, then the row sums would grow with m, for exactly the same reasons as given in Spence [1978] for the strip problem.

5. Details of the Numerical Results

In order to test the optimal weighting functions derived in section 4 and compare them with unmodified biorthogonal weighting functions the following sample stress distributions were considered

Case 1 $\sigma_{zz} = 1 - 2r^2$

$\sigma_{rz} = 0$

Smooth continuous data (Cases 1 and 2)

Case 2 $\sigma_{zz} = 0$

$\sigma_{rz} = r - r^3$

Case 3

$$\sigma_{zz} = \begin{cases} 1 - 32r^2/7 & (0<r<\tfrac{1}{2}) \\ -1/7 & (\tfrac{1}{2}<r<1) \end{cases}$$

$\sigma_{rz} = 0$

Case 4 $\sigma_{zz} = 0$

$$\sigma_{rz} = \begin{cases} -\tfrac{3}{4}r & (0<r<\tfrac{1}{2}) \\ r - r^3 & (\tfrac{1}{2}<r<1) \end{cases}$$

Data containing discontinuities (Cases 3 to 5)

Case 5

$$\sigma_{zz} = \begin{cases} 3 & (0<r<\tfrac{1}{2}) \\ -1 & (\tfrac{1}{2}<r<1) \end{cases}$$

$\sigma_{rz} = 0$

Case 6 $\sigma_{zz} = 0$

$\sigma_{rz} = r$

Incompatible with edge conditions

In order that the stresses should decay as $z \to \infty$ the normal stress must be self-equilibrated. All the distributions tested satisfy this condition. For non-self-equilibrated distributions a simple polynomial term can be added to a stress function representing an equilibrated distribution. In addition to this condition on the normal stress, the shear stress, as well as vanishing at the origin, must also be zero at r=1 if the end distribution is to be compatible with the zero stress condition on r=1.

The first two cases satisfy the conditions of equilibration and compatibility and are continuous. Cases 3 and 4 have simple jump discontinuities in the first derivative of the prescribed stresses, Case 5 is equilibrated but discontinuous, and case 6 is incompatible with the side conditions on the shear stress.

It is only possible to find closed forms for integrals of the form

$$\int_0^1 t^k J_0(\lambda_n t)\,dt \qquad \int_0^1 t^k J_1(\lambda_n t)\,dt$$

when k is even for the J_0 integrals and k is odd for the J_1 integrals. Therefore the prescribed normal stress distribution may only contain even powers of r and the shear stress distribution odd powers, if the right-hand sides of the truncated systems are to be evaluated in closed form. Using integration by parts the other integrals may be reduced to

$$\int_0^1 J_0(\lambda_n t)\,dt$$

which could be evaluated numerically. However the real and imaginary parts of $J_s(\lambda_n t)$ become more oscillatory as n increases, which presents problems for library integration subroutines. Although a general program for solving the end stress problem would need to include the possibility of general polynomial stress distributions, for the purposes of this report it was decided that sufficient test could be devised with the above restrictions.

Three salient features of the numerical results presented in appendices C and D are worthy of note, showing the advantages offered by Optimal Weighting functions. These are

(i) The improvement in diagonal dominance of the truncated matrices.

(ii) The increase in stability of the earlier coefficients as the order of truncation is increased.

(iii) Improved convergence to the data for various orders of truncation.

The improvement in diagonal dominance of the truncated matrices can be seen in appendix C. Not only are the row sum norms less for Optimal Weighting Functions than for Unmodified Biorthogonal Weighting functions, but they are decreasing with the row index, and they are less subject to the effects of truncation.

As an example of the increased stability in the early coefficients, the first two coefficients for all the orders of truncation shown in Appendix D for $\sigma_{zz} = 1-2r^2$, $\sigma_{rz} = 0$ are

	c_1		c_2	
N=5	-0.11902E-1	0.11986E-1	0.34817E-3	-0.16901E-3
N=10	-0.16985E-1	0.14775E-1	0.47215E-4	-0.25240E-3
N=20	-0.16622E-1	0.14578E-1	0.66895E-4	-0.24615E-3
N=50	-0.16582E-1	0.14556E-1	0.69037E-4	-0.24546E-3
N=100	-0.16566E-1	0.14547E-1	0.69904E-4	-0.24518E-3

for Unmodified Biorthogonal Weighting Functions, and

	c_1		c_2	
N=5	-0.16470E-1	0.14509E-1	0.75161E-3	-0.24251E-3
N=10	-0.16588E-1	0.14566E-1	0.68057E-4	-0.24554E-3
N=20	-0.16572E-1	0.14549E-1	0.69270E-4	-0.24531E-3
N=50	-0.16558E-1	0.14543E-1	0.70286E-4	-0.24505E-3
N=100	-0.16557E-1	0.14542E-1	0.70409E-4	-0.24502E-3

for Optimal Weighting functions. The corresponding coefficients for the incompatible distribution $\sigma_{zz} = 0$, $\sigma_{rz} = r$, which presents a much more severe test of convergence and stability, are

	c_1		c_2	
N=5	-0.10644E+0	0.67713E-1	-0.68485E-2	-0.14244E-2
N=10	-0.31706E-1	0.26662E-1	-0.23732E-2	-0.20218E-3
N=20	-0.11428E-1	0.15645E-1	-0.12832E-2	0.14838E-3
N=50	0.83730E-2	0.48957E-2	-0.22527E-3	0.49061E-3
N=100	0.54865E-1	-0.20350E-1	0.22660E-2	0.12874E-2

for Unmodified biorthogonal weighting functions and

	c_1		c_2	
N=5	0.27844E-1	-0.64030E-2	0.63268E-3	0.82867E-3
N=10	0.29602E-1	-0.72516E-2	0.77081E-3	0.85079E-3
N=20	0.31084E-1	-0.79308E-2	0.88376E-3	0.87495E-3
N=50	0.32631E-1	-0.86280E-2	0.99925E-3	0.90250E-3
N=100	0.33516E-1	-0.90251E-2	0.10648E-2	0.91860E-3

for Optimal weighting functions. The increase in stability for the smooth first distribution is marked, and for the incompatible case O.W.F. coefficients are still reasonably stable, whereas the U.B.W.F. coefficients lose all stability.

The third advantage can be seen in the improvement in accuracy of the summed expansions tested against the prescribed stresses on z=0. Although the difference is only slight for the well-behaved distribution $1-2r^2$, U.B.W.F. completely fail to converge to the incompatible shear stress, whereas the O.W.F. produce reasonably good results when the Cesaro sums are calculated rather than partial sums, as shown by the graphs in appendix F.

Appendix A

Eigenfunction Quadratures

In this appendix we give explicit expressions for the eigenfunction quadratures of the form $\left\langle \phi_n^{(\alpha)} \phi_m^{(\beta)} \right\rangle$ required in the construction of the matrix discussed in section 4 of this report.

$$\left\langle \phi_n^{(1)} \phi_m^{(1)} \right\rangle \quad (m \neq n)$$

$$4\lambda_m^3\lambda_n^3 J_1(\lambda_m) J_1(\lambda_n) \left\{ \frac{1}{\lambda_m{}^2-\lambda_n{}^2} \left[-\lambda_m\lambda_n(\lambda_m J_0(\lambda_m) J_1(\lambda_n) - \lambda_n J_1(\lambda_m) J_0(\lambda_n)) \right] \right.$$

$$+ \frac{1}{(\lambda_m{}^2-\lambda_n{}^2)^2} \left[(1-\nu)\lambda_m\lambda_n(\lambda_m^2+\lambda_n^2) J_1(\lambda_m) J_1(\lambda_n) \right.$$

$$\left. - \lambda_m^2\lambda_n^2(\lambda_m J_1(\lambda_m) J_0(\lambda_n) + \lambda_n J_0(\lambda_m) J_1(\lambda_n)) \right]$$

$$\left. + \frac{1}{(\lambda_m{}^2-\lambda_n{}^2)^3} \left[2\lambda_m^3\lambda_n^3(\lambda_m J_0(\lambda_m) J_1(\lambda_n) - \lambda_n J_1(\lambda_m) J_0(\lambda_n)) \right] \right\}$$

$$\left\langle \phi_n^{(1)} \phi_m^{(3)} \right\rangle \quad (m \neq n)$$

$$4\lambda_m^3\lambda_n^3 J_1(\lambda_m) J_1(\lambda_n) \left\{ \frac{1}{\lambda_m^2-\lambda_n^2}\left[-(1-\nu^2)\lambda_m\lambda_n J_1(\lambda_m) J_1(\lambda_n) \right.\right.$$

$$\left. -\nu\lambda_m\lambda_n(\lambda_m J_0(\lambda_m) J_1(\lambda_n) - \lambda_n J_1(\lambda_m) J_0(\lambda_n)) \right]$$

$$+ \frac{1}{(\lambda_m^2-\lambda_n^2)^2}\left[2(1+\nu)\lambda_m^3\lambda_n^2 J_1(\lambda_m) J_0(\lambda_n) - (1-\nu)\lambda_m\lambda_n(\lambda_m^2+\lambda_n^2) J_1(\lambda_m) J_1(\lambda_n) \right.$$

$$\left. -\ \nu\lambda_m^2\lambda_n^2(\lambda_m J_1(\lambda_m) J_0(\lambda_n) + \lambda_n J_0(\lambda_m) J_1(\lambda_n)) \right]$$

$$\left. + \frac{1}{(\lambda_m^2-\lambda_n^2)^3}\left[-2\lambda_m^3\lambda_n^3(\lambda_m J_0(\lambda_m) J_1(\lambda_n) - \lambda_n J_1(\lambda_m) J_0(\lambda_n)) \right] \right\}$$

$$\left\langle \phi_n^{(2)} \phi_m^{(2)} \right\rangle \quad (m \neq n)$$

$$4\lambda_m^3\lambda_n^3 J_1(\lambda_m) J_1(\lambda_n) \left\{ \frac{1}{(\lambda_m^2-\lambda_n^2)^2}\left[(1-\nu)(\lambda_m^2+\lambda_n^2) J_1(\lambda_m) J_1(\lambda_n) \right.\right.$$

$$\left. -\lambda_m\lambda_n(\lambda_m J_1(\lambda_m) J_0(\lambda_n) + \lambda_n J_0(\lambda_m) J_1(\lambda_n)) \right]$$

$$\left. + \frac{1}{(\lambda_m^2-\lambda_n^2)^3}\left[2\lambda_m^2\lambda_n^2(\lambda_m J_0(\lambda_m) J_1(\lambda_n) - \lambda_n J_1(\lambda_m) J_0(\lambda_n)) \right] \right\}$$

$$\left\langle \phi_n^{(2)} \phi_m^{(4)} \right\rangle \quad (m \neq n)$$

$$4\lambda_m^3\lambda_n^3 J_1(\lambda_m) J_1(\lambda_n) \left\{ \frac{1}{\lambda_m^2-\lambda_n^2}\left[-(1-\nu^2) J_1(\lambda_m) J_1(\lambda_n) \right] \right.$$

$$\left. + \frac{1}{(\lambda_m^2-\lambda_n^2)^2}\left[(1+\nu)\lambda_m\lambda_n(\lambda_m J_1(\lambda_m) J_0(\lambda_n) - \lambda_n J_0(\lambda_m) J_1(\lambda_n)) \right] \right\}$$

$$\left\langle \phi_m^{(1)} \phi_m^{(1)} \right\rangle$$

$$\lambda_m^7\Big\{\frac{2}{3}\lambda_m^3 J_0^2(\lambda_m) J_1^2(\lambda_m) + \frac{1}{6}\lambda_m^3 J_1^4(\lambda_m) + \frac{1}{2}\lambda_m^3 J_0^4(\lambda_m) - \frac{8}{3}\lambda_m^2 J_0(\lambda_m) J_1^3(\lambda_m)$$

$$+ \frac{11}{3}\lambda_m J_1^4(\lambda_m) + 6\lambda_m J_0^2(\lambda_m) J_1^2(\lambda_m) - 3\lambda_m^2 J_0^3(\lambda_m) J_1(\lambda_m) - 4J_0(\lambda_m) J_1^3(\lambda_m)\Big\}$$

$$\left\langle \phi_m^{(1)} \phi_m^{(3)} \right\rangle$$

$$\lambda_m^7\Big\{-\frac{2}{3}\lambda_m^3 J_0^2(\lambda_m) J_1^2(\lambda_m) - \frac{1}{6}\lambda_m^3 J_1^4(\lambda_m) - \frac{1}{2}\lambda_m^3 J_0^4(\lambda_m) + (\frac{5}{3}-\nu)\lambda_m^2 J_0(\lambda_m) J_1^3(\lambda_m)$$

$$+ (-\frac{2}{3}+3\nu)\lambda_m J_1^4(\lambda_m) + (-2+4\nu)\lambda_m J_0^2(\lambda_m) J_1^2(\lambda_m) + (2-\nu)\lambda_m^2 J_0^3(\lambda_m) J_1(\lambda_m)$$

$$- 4\nu J_0(\lambda_m) J_1^3(\lambda_m)\Big\}$$

$$\left\langle \phi_m^{(2)} \phi_m^{(2)} \right\rangle$$

$$\lambda_m^6\Big\{\frac{2}{3}\lambda_m^2 J_0^2(\lambda_m) J_1^2(\lambda_m) + \frac{1}{6}\lambda_m^2 J_1^4(\lambda_m) - \frac{2}{3}\lambda_m J_0(\lambda_m) J_1^3(\lambda_m) - \frac{1}{3}J_1^4(\lambda_m)$$

$$+ \frac{1}{2}\lambda_m^2 J_0^4(\lambda_m) - \lambda_m J_0^3(\lambda_m) J_1(\lambda_m)\Big\}$$

$$\left\langle \phi_m^{(2)} \phi_m^{(4)} \right\rangle$$

$$-2(1+\nu)\lambda_m^6\Big\{-\frac{1}{2}J_1^4(\lambda_m) + \frac{1}{2}\lambda_m J_0^3(\lambda_m) J_1(\lambda_m) + \frac{1}{2}\lambda_m J_0(\lambda_m) J_1^3(\lambda_m)$$

$$- J_0^2(\lambda_m) J_1^2(\lambda_m)\Big\}$$

Appendix B

Right-hand sides for Infinite Systems

This appendix lists explicit expressions for the right-hand sides d_m corresponding to the six special cases of section 5, obtained from optimal weighting functions.

Case 1

$$d_m = -8(1+\nu)\lambda_m^2 J_1(\lambda_m)\left\{\lambda_m J_0(\lambda_m)-2(2+\nu)J_1(\lambda_m)\right\}$$

Case 2

$$d_m = 4(1+\nu)\lambda_m J_1(\lambda_m)\left\{3\lambda_m^2 J_1(\lambda_m)+12\lambda_m J_0(\lambda_m)-8(4+\nu)J_1(\lambda_m)\right\}$$

Case 3

$$d_m = -\frac{16(1+\nu)}{7}\lambda_m^2\{\lambda_m^2 J_1(\lambda_m)J_1(\tfrac{1}{2}\lambda_m)+2\lambda_m^2 J_0(\lambda_m)J_0(\tfrac{1}{2}\lambda_m)$$

$$+6\lambda_m J_1(\lambda_m)J_0(\tfrac{1}{2}\lambda_m)-8\lambda_m J_0(\lambda_m)J_1(\tfrac{1}{2}\lambda_m)-24J_1(\lambda_m)J_1(\tfrac{1}{2}\lambda_m)\}$$

Case 4

$$d_m = \frac{1}{2}(1+\nu)\lambda_m\Big\{\frac{1}{2}\lambda_m^3 J_0(\tfrac{1}{2}\lambda_m)J_1(\lambda_m)-\lambda_m^3 J_1(\tfrac{1}{2}\lambda_m)J_0(\lambda_m)-7\lambda_m^2 J_1(\tfrac{1}{2}\lambda_m)J_1(\lambda_m)$$

$$-8\lambda_m^2 J_0(\tfrac{1}{2}\lambda_m)J_0(\lambda_m)+56\lambda_m^2 J_1^2(\lambda_m)+32\lambda_m^2 J_0^2(\lambda_m)-40\lambda_m J_0(\tfrac{1}{2}\lambda_m)J_1(\lambda_m)$$

$$+32\lambda_m J_1(\tfrac{1}{2}\lambda_m)J_0(\lambda_m)+96\lambda_m J_0(\lambda_m)J_1(\lambda_m)+160J_1(\tfrac{1}{2}\lambda_m)J_1(\lambda_m)-320J_1^2(\lambda_m)\Big\}$$

Case 5

$$d_m = 2(1+\nu)\lambda_m^4\Big\{2\lambda_m J_1(\tfrac{1}{2}\lambda_m)J_0(\lambda_m)-\lambda_m J_0(\tfrac{1}{2}\lambda_m)J_1(\lambda_m)+2J_1(\tfrac{1}{2}\lambda_m)J_1(\lambda_m)\Big\}$$

Case 6

$$d_m = 2(1+\nu)\lambda_m^3 J_1(\lambda_m)\Big\{\lambda_m J_0(\lambda_m)-2(2+\nu)J_1(\lambda_m)\Big\}$$

Appendix C

Tables of Row Sum Norms

This appendix lists the row sum norms

$$\sum_n |A_{mn}| \;/\; |A_{mm}|$$

for various orders of truncation (by the order of truncation we mean the number of pairs of eigenvalues used in truncating the matrix. Thus N=10 means that a 20×20 matrix has been inverted). It should be noted that although the norms for the Optimal weighting functions are not less than one, they decrease with n, (ignoring the effects of truncation for n at either end of the range) unlike those for the unmodified biorthogonal weighting functions, and the improvement this affords is demonstrated by the results in the next appendix.

	Biorthogonal Weighting Functions	Optimal Weighting Functions
n	N = 5	N = 5
1	3.60230	1.71249
2	3.36248	2.11606
3	3.51319	1.99478
4	3.29536	1.83499
5	2.16175	1.68600
	N = 10	N = 10
1	5.61611	2.34593
2	4.11965	2.88797
3	4.22286	2.77801
4	4.46599	2.61097
5	4.68817	2.44794
6	4.86089	2.29953
7	4.95964	2.16658
8	4.90947	2.04769
9	4.50296	1.94108
10	3.12498	1.84507

	N = 20	N = 20
1	9.59345	3.00200
2	5.47667	3.69238
3	4.91307	3.60346
4	4.90790	3.44068
5	5.07706	3.27596
6	5.30875	3.12280
7	5.56248	2.98317
8	5.81666	2.85629
9	6.06628	2.74077
10	6.30916	2.63518
11	6.54364	2.53827
12	6.76732	2.44895
13	6.97645	2.36631
14	7.16500	2.28958
15	7.32200	2.21808
16	7.42656	2.15128
17	7.43424	2.08868
18	7.24075	2.02987
19	6.57820	1.97450
20	4.77745	1.92225

	N = 50	N = 50
2	9.55000	4.77807
4	6.15013	4.56876
6	5.87586	4.25654
8	6.15164	3.98821
10	6.58065	3.76178
12	7.05966	3.56802
14	7.55387	3.39950
16	8.04895	3.25085
18	8.53647	3.11817
20	9.01408	2.99860
22	9.48194	2.88994
24	9.94036	2.79052
26	10.38957	2.69903
28	10.82965	2.61440
30	11.26041	2.53578
32	11.68129	2.46245
34	12.09109	2.39383
36	12.48757	2.32943
38	12.86643	2.26881
40	13.21917	2.21161
42	13.52738	2.15753
44	13.74547	2.10628
46	13.73585	2.05763
48	12.95004	2.01136
50	8.77780	1.96730

	N = 100	N = 100
4	8.23533	5.43116
8	6.70562	4.85933
12	7.30349	4.43930
16	8.18078	4.11988
20	9.10390	3.86388
24	10.02212	3.65091
28	10.92245	3.46894
32	11.80173	3.31037
36	12.65852	3.17007
40	13.49438	3.04447
44	14.31146	2.93093
48	15.11152	2.82748
52	15.89599	2.73260
56	16.66597	2.64509
60	17.42227	2.56399
64	18.16532	2.48851
68	18.89513	2.41800
72	19.61101	2.35193
76	20.31100	2.28983
80	20.99060	2.23130
84	21.63895	2.17603
88	22.22654	2.12370
92	22.64989	2.07408
96	22.32804	2.02692
100	14.21699	1.98204

Appendix D

Coefficients and Summed Expansions

Convergence to $\sigma_{zz} = 1-2r^2$, $\sigma_{rz} = 0$

n	Biorthogonal Weighting Functions N = 5		Optimal Weighting Functions N = 5	
1	-0.11902E-1	0.11986E-1	-0.16468E-1	0.14509E-1
2	0.34817E-3	-0.16901E-3	0.75162E-4	-0.24251E-3
3	0.88271E-4	0.22602E-4	0.32191E-4	-0.10014E-4
4	0.22726E-4	0.23230E-4	0.76576E-5	0.13609E-5
5	-0.10792E-4	0.39198E-4	0.21059E-5	0.11227E-5

r	Normal Stress σ_{zz}	Normal Stress UBWF	Normal Stress OWF	Shear Stress σ_{rz}	Shear Stress UBWF	Shear Stress OWF
0.0	1.00	0.8283	1.0243	0.00	0.0000	0.0000
0.1	0.98	0.8834	0.9830	0.00	-0.1076	0.0183
0.2	0.92	0.8936	0.9070	0.00	-0.0177	-0.0036
0.3	0.82	0.7422	0.8277	0.00	0.0817	-0.0121
0.4	0.68	0.5895	0.6882	0.00	0.0376	0.0130
0.5	0.50	0.4723	0.4841	0.00	0.0017	0.0087
0.6	0.28	0.2426	0.2781	0.00	0.0414	-0.0180
0.7	0.02	-0.0451	0.0386	0.00	0.0621	0.0007
0.8	-0.28	-0.2866	-0.2892	0.00	0.0896	0.0228
0.9	-0.62	-0.5559	-0.6262	0.00	0.0965	-0.0051
1.0	-1.00	-0.7252	-1.0034	0.00	0.0000	0.0000

NOTE

UBWF - Unmodified Biorthogonal Weighting Functions

OWF - Optimal Weighting Functions

	Biorthogonal Weighting Functions, N = 10		Optimal Weighting Functions, N = 10	
1	-0.16985E-1	0.14775E-1	-0.16588E-1	0.14556E-1
2	0.47215E-4	-0.25240E-3	0.68058E-4	-0.24554E-3
3	0.27679E-4	-0.13361E-4	0.30973E-4	-0.11049E-4
4	0.65262E-5	0.67046E-7	0.73391E-5	0.95283E-6
5	0.17761E-5	0.52755E-6	0.20022E-5	0.93628E-6
6	0.59203E-6	0.27933E-6	0.62190E-6	0.49396E-6
7	0.27527E-6	0.13832E-6	0.21096E-6	0.24802E-6
8	0.19793E-6	0.11598E-6	0.74706E-7	0.12711E-6
9	0.12656E-6	0.19600E-6	0.26008E-7	0.67366E-7
10	-0.96880E-7	0.13266E-7	0.78339E-8	0.36903E-7

r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	1.00	1.0173	0.9794	0.00	0.0000	0.0000
0.1	0.98	0.9825	0.9880	0.00	0.0032	-0.0038
0.2	0.92	0.9298	0.9135	0.00	-0.0032	0.0012
0.3	0.82	0.8237	0.8264	0.00	-0.0013	0.0002
0.4	0.68	0.6874	0.6744	0.00	-0.0027	-0.0019
0.5	0.50	0.5047	0.5056	0.00	-0.0038	0.0029
0.6	0.28	0.2839	0.2754	0.00	-0.0043	-0.0050
0.7	0.02	0.0241	0.0239	0.00	-0.0045	0.0058
0.8	-0.28	-0.2805	-0.2817	0.00	-0.0083	-0.0081
0.9	-0.62	-0.6254	-0.6213	0.00	-0.0049	0.0072
1.0	-1.00	-1.0257	-0.9990	0.00	0.0000	0.0000

	Biorthogonal Weighting Functions		Optimal Weighting Functions	
	N = 20		N = 20	
1	-0.16622E-1	0.14578E-1	-0.16572E-1	0.14549E-1
2	0.66895E-4	-0.24615E-3	0.69270E-4	-0.24531E-3
3	0.30920E-4	-0.11202E-4	0.31251E-4	-0.10945E-4
4	0.73657E-5	0.91154E-6	0.74367E-5	0.99956E-6
5	0.20277E-5	0.92480E-6	0.20454E-5	0.96002E-6
6	0.64032E-6	0.49145E-6	0.64404E-6	0.50723E-6
7	0.22392E-6	0.24839E-6	0.22349E-6	0.25600E-6
8	0.84011E-7	0.12839E-6	0.82323E-7	0.13220E-6
9	0.32906E-7	0.68881E-7	0.30901E-7	0.70754E-7
10	0.13125E-7	0.38454E-7	0.11116E-7	0.39245E-7
11	0.52022E-8	0.22364E-7	0.32949E-8	0.22481E-7
12	0.19950E-8	0.13609E-7	0.23007E-9	0.13241E-7
13	0.69736E-9	0.87645E-8	-0.88613E-9	0.79837E-8
14	0.12389E-9	0.60958E-8	-0.11986E-8	0.49067E-8
15	-0.30167E-9	0.46801E-8	-0.11890E-8	0.30610E-8
16	-0.97063E-9	0.39408E-8	-0.10627E-8	0.19300E-8
17	-0.22738E-8	0.32481E-8	-0.90673E-9	0.12244E-8
18	-0.43301E-8	0.12425E-8	-0.75622E-9	0.77744E-9
19	-0.43209E-8	-0.42482E-8	-0.62369E-9	0.49075E-9
20	0.19237E-8	-0.76306E-9	-0.51189E-9	0.30513E-9

r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	1.00	0.9985	0.9937	0.00	0.0000	0.0000
0.1	0.98	0.9803	0.9785	0.00	0.0006	0.0012
0.2	0.92	0.9204	0.9188	0.00	0.0001	0.0008
0.3	0.82	0.8206	0.8188	0.00	-0.0003	0.0006
0.4	0.68	0.6807	0.6788	0.00	-0.0005	0.0005
0.5	0.50	0.5008	0.4986	0.00	-0.0006	0.0003
0.6	0.28	0.2809	0.2784	0.00	-0.0006	0.0001
0.7	0.02	0.0208	0.0182	0.00	-0.0006	-0.0002
0.8	-0.28	-0.2798	-0.2820	0.00	-0.0006	-0.0007
0.9	-0.62	-0.6210	-0.6218	0.00	-0.0010	-0.0014
1.0	-1.00	-1.0039	-0.9995	0.00	0.0000	0.0000

	Biorthogonal Weighting Functions		Optimal Weighting Functions	
	N = 50		N = 50	
2	0.69037E-4	-0.24546E-3	0.70286E-4	-0.24505E-3
4	0.74558E-5	0.99827E-6	0.75074E-5	0.10471E-5
6	0.65137E-6	0.51019E-6	0.65842E-6	0.52040E-6
8	0.85310E-7	0.13409E-6	0.86878E-7	0.13720E-6
10	0.12510E-7	0.40352E-7	0.12961E-7	0.41542E-7
12	0.95515E-9	0.13909E-7	0.11042E-8	0.14440E-7
14	-0.78839E-9	0.53280E-8	-0.73710E-9	0.55911E-8
16	-0.81470E-9	0.22066E-8	-0.79879E-9	0.23482E-8
18	-0.59796E-9	0.96547E-9	-0.59576E-9	0.10466E-8
20	-0.40622E-9	0.43691E-9	-0.40949E-9	0.48577E-9
22	-0.27090E-9	0.19987E-9	-0.27630E-9	0.23048E-9
24	-0.18075E-9	0.89619E-10	-0.18689E-9	0.10939E-9
26	-0.12143E-9	0.37238E-10	-0.12773E-9	0.50267E-9
28	-0.82238E-10	0.12297E-10	-0.88466E-10	0.20936E-10
30	-0.56083E-10	0.74857E-12	-0.62144E-10	0.63731E-11
32	-0.38427E-10	-0.41172E-11	-0.44268E-10	-0.70643E-12
34	-0.26417E-10	-0.55884E-11	-0.31959E-10	-0.39460E-11
36	-0.18298E-10	-0.53050E-11	-0.23364E-10	-0.52122E-11
38	-0.13067E-10	-0.41206E-11	-0.17283E-10	-0.54785E-11
40	-0.10321E-10	-0.26105E-11	-0.12956E-10	-0.52578E-11
42	-0.10188E-10	-0.16237E-11	-0.97646E-11	-0.48209E-11
44	-0.12881E-10	-0.34865E-11	-0.74462E-11	-0.43096E-11
46	-0.14402E-10	-0.14170E-10	-0.57276E-11	-0.37959E-11
48	0.13179E-10	-0.24198E-10	-0.44411E-11	-0.33145E-11
50	-0.58778E-11	0.53884E-11	-0.34692E-11	-0.28799E-11

r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	1.00	1.0006	1.0000	0.00	0.0000	0.0000
0.1	0.98	0.9803	0.9800	0.00	0.0000	-0.0001
0.2	0.92	0.9204	0.9201	0.00	-0.0001	0.0001
0.3	0.82	0.8204	0.8200	0.00	-0.0001	-0.0002
0.4	0.68	0.6803	0.6800	0.00	-0.0002	0.0002
0.5	0.50	0.5003	0.5000	0.00	-0.0002	-0.0003
0.6	0.28	0.2802	0.2799	0.00	-0.0003	0.0004
0.7	0.02	0.0202	0.0202	0.00	-0.0003	-0.0005
0.8	-0.28	-0.2800	-0.2803	0.00	-0.0004	0.0005
0.9	-0.62	-0.6203	-0.6195	0.00	-0.0005	-0.0006
1.0	-1.00	-1.0015	-0.9999	0.00	0.0000	0.0000

	Biorthogonal Weighting Functions		Optimal Weighting Functions	
	N = 100		N = 100	
4	0.74939E-5	0.10328E-5	0.75161E-5	0.10529E-5
8	0.86652E-7	0.13645E-6	0.87440E-7	0.13782E-6
12	0.11147E-8	0.14343E-7	0.12125E-8	0.14593E-7
16	-0.78652E-9	0.23318E-8	-0.76611E-9	0.24035E-8
20	-0.50250E-9	0.48385E-9	-0.39691E-9	0.51042E-9
24	-0.18301E-9	0.11034E-9	-0.18125E-9	0.12201E-9
28	-0.86235E-10	0.22270E-10	-0.85654E-10	0.28053E-10
32	-0.42922E-10	0.45789E-12	-0.42755E-10	0.36001E-11
36	-0.22513E-10	-0.43001E-11	-0.22503E-10	-0.24601E-11
40	-0.12360E-10	-0.45703E-11	-0.12412E-10	-0.34212E-11
44	-0.70510E-11	-0.38008E-11	-0.71294E-11	-0.30400E-11
48	-0.41486E-11	-0.29431E-11	-0.42400E-11	-0.24108E-11
52	-0.24975E-11	-0.22295E-11	-0.25980E-11	-0.18365E-11
56	-0.15233E-11	-0.16856E-11	-0.16333E-11	-0.13798E-11
60	-0.92723E-12	-0.12842E-11	-0.10496E-11	-0.10343E-11
64	-0.54758E-12	-0.99108E-12	-0.68737E-12	-0.77781E-12
68	-0.29289E-12	-0.77628E-12	-0.45744E-12	-0.58833E-12
72	-0.10869E-12	-0.61553E-12	-0.30857E-12	-0.44819E-12
76	0.39932E-13	-0.48769E-12	-0.21049E-12	-0.34406E-12
80	0.17718E-12	-0.36936E-12	-0.14488E-12	-0.26617E-12
84	0.31717E-12	-0.22381E-12	-0.10039E-12	-0.20750E-12
88	0.43820E-12	0.22179E-12	-0.69865E-13	-0.16297E-12
92	0.32371E-12	0.47258E-12	-0.48708E-13	-0.12890E-12
96	-0.79584E-12	0.27566E-12	-0.33916E-13	-0.10265E-12
100	-0.77300E-13	0.15096E-12	-0.23500E-13	-0.82266E-13

r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	1.00	1.0027	0.9997	0.00	0.0000	0.0000
0.1	0.98	0.9804	0.9800	0.00	-0.0003	0.0000
0.2	0.92	0.9204	0.9200	0.00	-0.0002	0.0000
0.3	0.82	0.8204	0.8199	0.00	-0.0002	0.0000
0.4	0.68	0.6803	0.6799	0.00	-0.0002	0.0000
0.5	0.50	0.5003	0.4999	0.00	-0.0002	0.0000
0.6	0.28	0.2803	0.2799	0.00	-0.0002	0.0001
0.7	0.02	0.0203	0.0199	0.00	-0.0003	0.0001
0.8	-0.28	-0.2798	-0.2801	0.00	-0.0003	0.0002
0.9	-0.62	-0.6200	-0.6201	0.00	-0.0002	0.0002
1.0	-1.00	-1.0006	-1.0000	0.00	0.0000	0.0000

Convergence to $\sigma_{zz} = 0$; $\sigma_{rz} = r$

NOTE

The partial sums for this stress distribution are not convergent, and the sums shown below are Cesaro sums, i.e.

If S_n is the nth partial sum, then $C_n = \frac{1}{n}\sum_{i=1}^{n} S_i$

n	Biorthogonal Weighting Functions N = 5		Optimal Weighting Functions N = 5	
1	-0.10644E-0	0.67713E-1	0.27844E-1	-0.64030E-2
2	-0.68485E-2	-0.14244E-2	0.63268E-3	0.82867E-3
3	-0.13217E-2	-0.71058E-3	0.51659E-4	0.16962E-3
4	-0.34379E-3	-0.44117E-3	0.39435E-5	0.52381E-4
5	0.20885E-3	-0.51413E-5	-0.18870E-5	0.20690E-4

r	Normal Stress σ_{zz}	UBWF	OWF	Shear Stress σ_{rz}	UBWF	OWF
0.0	0.00	3.8987	0.2517	0.00	0.0000	0.0000
0.1	0.00	2.9468	0.1779	0.10	0.6512	0.0356
0.2	0.00	1.5725	0.0983	0.20	0.2314	0.1574
0.3	0.00	1.5576	0.1361	0.30	-0.7526	0.3124
0.4	0.00	2.1319	0.1394	0.40	-1.0633	0.3889
0.5	0.00	1.9386	0.0377	0.50	-0.6903	0.4027
0.6	0.00	1.3380	0.0267	0.60	-0.7798	0.4898
0.7	0.00	1.1321	0.0766	0.70	-1.5204	0.6656
0.8	0.00	0.7844	-0.1215	0.80	-1.5147	0.7029
0.9	0.00	-1.8012	-0.2834	0.90	-0.2523	0.4080
1.0	0.00	-8.6863	0.4987	1.00	0.0000	0.0000

	Biorthogonal Weighting Functions		Optimal Weighting Functions	
	N = 10		N = 10	
1	-0.31706E-1	0.26662E-1	0.29602E-1	-0.72516E-2
2	-0.23732E-2	-0.20218E-3	0.77081E-3	0.85079E-3
3	-0.40086E-3	-0.16414E-3	0.82572E-4	0.18016E-3
4	-0.10276E-3	-0.68799E-4	0.14477E-4	0.57098E-4
5	-0.31990E-4	-0.31147E-4	0.26161E-5	0.23042E-4
6	-0.99216E-5	-0.15341E-4	0.25682E-7	0.10861E-4
7	-0.15152E-5	-0.76746E-5	-0.52883E-6	0.57046E-5
8	0.21502E-5	-0.28967E-5	-0.57641E-6	0.32458E-5
9	0.27344E-5	0.11829E-5	-0.50190E-6	0.19643E-5
10	-0.51185E-6	0.19050E-6	-0.41006E-6	0.12487E-5

r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	0.00	1.1891	0.0428	0.00	0.0000	0.0000
0.1	0.00	1.0300	0.1108	0.10	0.0311	0.1071
0.2	0.00	1.0541	0.0698	0.20	-0.0559	0.1745
0.3	0.00	0.9870	0.0946	0.30	-0.0084	0.2875
0.4	0.00	0.9331	0.0539	0.40	-0.0862	0.3565
0.5	0.00	0.8375	0.0659	0.50	-0.0698	0.4673
0.6	0.00	0.6878	0.0282	0.60	-0.1509	0.5248
0.7	0.00	0.4734	0.0087	0.70	-0.1756	0.6414
0.8	0.00	0.1128	0.0003	0.80	-0.2722	0.6588
0.9	0.00	-0.7781	-0.1916	0.90	-0.0649	0.7860
1.0	0.00	-3.8650	0.5033	1.00	0.0000	0.0000

	Biorthogonal Weighting Functions		Optimal Weighting Functions	
	N = 20		N = 20	
1	-0.11428E-1	0.15645E-1	0.31084E-1	-0.79308E-2
2	-0.12832E-2	0.14838E-3	0.88376E-3	0.87495E-3
3	-0.22399E-3	-0.44775E-4	0.10765E-3	0.19064E-3
4	-0.57686E-4	-0.23268E-4	0.23082E-4	0.61745E-4
5	-0.18352E-4	-0.10654E-4	0.63527E-5	0.25371E-4
6	-0.64103E-5	-0.50486E-5	0.19086E-5	0.12148E-4
7	-0.21509E-5	-0.24762E-5	0.52231E-6	0.64715E-5
8	-0.46193E-6	-0.12170E-5	0.55949E-7	0.37300E-5
9	0.25341E-6	-0.55559E-6	-0.99179E-7	0.22845E-5
10	0.56504E-6	-0.17913E-6	-0.14183E-6	0.14685E-5
11	0.69655E-6	0.58845E-6	-0.14318E-6	0.98186E-6
12	0.74017E-6	0.23229E-6	-0.13029E-6	0.67836E-6
13	0.73143E-6	0.38204E-6	-0.11368E-6	0.48185E-6
14	0.67518E-6	0.53240E-6	-0.97389E-7	0.35048E-6
15	0.55031E-6	0.69484E-6	-0.82821E-7	0.26023E-6
16	0.30160E-6	0.85477E-6	-0.70304E-7	0.19673E-6
17	-0.16928E-6	0.91311E-6	-0.59745E-7	0.15111E-6
18	-0.90938E-6	0.51023E-6	-0.50911E-7	0.11772E-6
19	-0.11247E-5	-0.10165E-5	-0.43540E-7	0.92877E-7
20	0.48445E-6	-0.18472E-6	-0.37388E-7	0.74119E-7

r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	0.00	-0.0399	0.0326	0.00	0.0000	0.0000
0.1	0.00	0.6073	0.0500	0.10	0.2237	0.0883
0.2	0.00	0.6601	0.0487	0.20	0.2075	0.1830
0.3	0.00	0.6704	0.0453	0.30	0.2054	0.2760
0.4	0.00	0.6528	0.0403	0.40	0.2007	0.3677
0.5	0.00	0.6036	0.0340	0.50	0.1868	0.4577
0.6	0.00	0.5086	0.0262	0.60	0.1604	0.5452
0.7	0.00	0.3382	0.0170	0.70	0.1226	0.6291
0.8	0.00	0.0342	0.0053	0.80	0.0893	0.7049
0.9	0.00	-0.5280	-0.0089	0.90	0.1204	0.7490
1.0	0.00	-2.4519	0.5068	1.00	0.0000	0.0000

	Biorthogonal Weighting Functions		Optimal Weighting Functions	
	N = 50		N = 50	
2	-0.22527E-3	0.49061E-3	0.99925E-3	0.90250E-3
4	-0.13663E-4	0.19719E-4	0.31656E-4	0.66934E-4
6	-0.12284E-5	0.42449E-5	0.37754E-5	0.13609E-4
8	0.16323E-7	0.16176E-5	0.68948E-6	0.42925E-5
10	0.15705E-6	0.78827E-6	0.13239E-6	0.17304E-5
12	0.14700E-6	0.44161E-6	0.80056E-8	0.81676E-6
14	0.11961E-6	0.27186E-6	-0.19961E-7	0.43045E-6
16	0.95898E-7	0.17971E-6	-0.23538E-7	0.24612E-6
18	0.77936E-7	0.12595E-6	-0.20995E-7	0.14984E-6
20	0.64636E-7	0.92907E-7	-0.17368E-7	0.95888E-7
22	0.54708E-7	0.71878E-7	-0.14029E-7	0.63900E-7
24	0.47131E-7	0.58240E-7	-0.11274E-7	0.44040E-7
26	0.41151E-7	0.49425E-7	-0.90829E-8	0.31226E-7
28	0.36190E-7	0.43949E-7	-0.73614E-8	0.22684E-7
30	0.31753E-7	0.40948E-7	-0.60103E-8	0.16828E-7
32	0.27338E-7	0.39922E-7	-0.49457E-8	0.12715E-7
34	0.22319E-7	0.40575E-7	-0.41015E-8	0.97631E-8
36	0.15778E-7	0.42660E-7	-0.34273E-8	0.76050E-8
38	0.62169E-8	0.45703E-7	-0.28846E-8	0.60004E-8
40	-0.88936E-8	0.48255E-7	-0.24444E-8	0.47893E-8
42	-0.33468E-7	0.45685E-7	-0.20846E-8	0.38628E-8
44	-0.70257E-7	0.23722E-7	-0.17885E-8	0.31452E-8
46	-0.95703E-7	-0.52967E-7	-0.15431E-8	0.25832E-8
48	0.50518E-7	-0.13265E-6	-0.13384E-8	0.21385E-8
50	-0.27265E-7	0.25884E-7	-0.11665E-8	0.17834E-8

r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	0.00	2.0988	0.0225	0.00	0.0000	0.0000
0.1	0.00	0.2693	0.0344	0.10	0.3377	0.0965
0.2	0.00	0.4966	0.0300	0.20	-0.1055	0.1888
0.3	0.00	0.4078	0.0321	0.30	0.3475	0.2857
0.4	0.00	0.3462	0.0267	0.40	0.0844	0.3774
0.5	0.00	0.4080	0.0274	0.50	0.3589	0.4738
0.6	0.00	0.2145	0.0208	0.60	0.2812	0.5625
0.7	0.00	0.2317	0.0165	0.70	0.3425	0.6595
0.8	0.00	0.0154	0.0089	0.80	0.4142	0.7390
0.9	0.00	-0.2509	-0.0397	0.90	0.4407	0.8420
1.0	0.00	-0.8829	0.5091	1.00	0.0000	0.0000

	Biorthogonal Weighting Functions		Optimal Weighting Functions	
	N = 100		N = 100	
4	0.95239E-4	0.11908E-3	0.36441E-4	0.69952E-4
8	0.37014E-5	0.83793E-5	0.10350E-5	0.46246E-5
12	0.52896E-6	0.16646E-5	0.83162E-7	0.90043E-6
16	0.13384E-6	0.52082E-6	0.21734E-8	0.27684E-6
20	0.47464E-7	0.21090E-6	-0.61255E-8	0.10983E-6
24	0.21179E-7	0.10102E-6	-0.55460E-8	0.51300E-7
28	0.11166E-7	0.54545E-7	-0.41231E-8	0.26839E-7
32	0.66574E-8	0.32276E-7	-0.29718E-8	0.15265E-7
36	0.43386E-8	0.20572E-7	-0.21538E-8	0.92562E-8
40	0.30025E-8	0.13973E-7	-0.15854E-8	0.59049E-8
44	0.21476E-8	0.10047E-7	-0.11880E-8	0.39253E-8
48	0.15405E-8	0.76165E-8	-0.90606E-9	0.26999E-8
52	0.10587E-8	0.60709E-8	-0.70246E-9	0.19111E-8
56	0.62831E-9	0.50753E-8	-0.55285E-9	0.13863E-8
60	0.19478E-9	0.44357E-8	-0.44107E-9	0.10272E-8
64	-0.29348E-9	0.40310E-8	-0.35626E-9	0.77525E-9
68	-0.89764E-9	0.37738E-8	-0.29097E-9	0.59472E-9
72	-0.17021E-8	0.35772E-8	-0.24007E-9	0.46287E-9
76	-0.28297E-8	0.33077E-8	-0.19989E-9	0.36493E-9
80	-0.44480E-8	0.26844E-8	-0.16784E-9	0.29107E-9
84	-0.66801E-8	0.10234E-8	-0.14201E-9	0.23462E-9
88	-0.89139E-8	-0.33647E-8	-0.12100E-9	0.19092E-9
92	-0.56161E-8	-0.13141E-7	-0.10377E-9	0.15672E-9
96	0.23451E-7	-0.53547E-8	-0.89527E-10	0.12968E-9
100	0.24496E-8	-0.46558E-8	-0.77664E-10	0.10810E-9

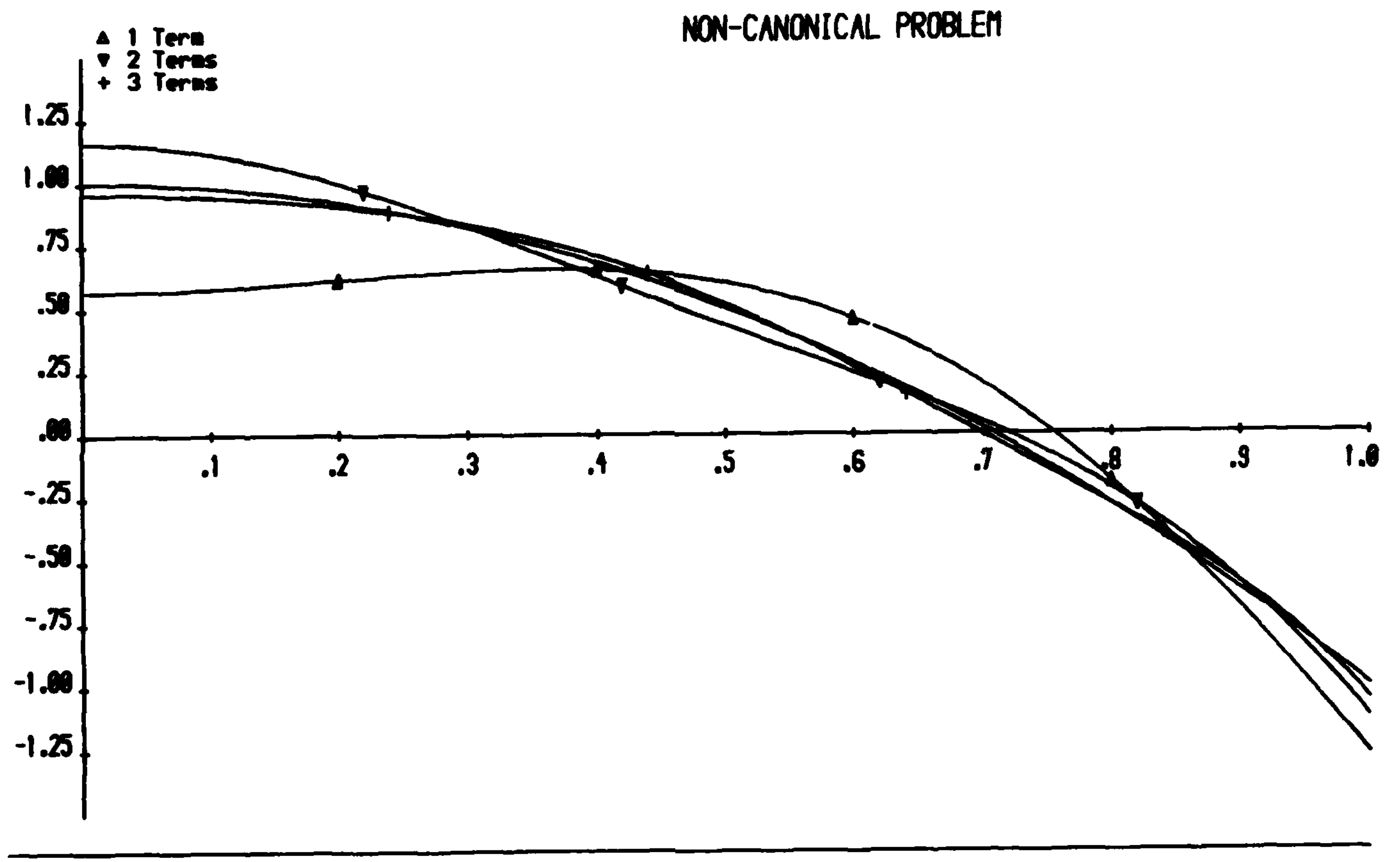
r	Normal Stress			Shear Stress		
	σ_{zz}	UBWF	OWF	σ_{rz}	UBWF	OWF
0.0	0.00	-3.2079	0.0273	0.00	0.0000	0.0000
0.1	0.00	-0.4719	0.0226	0.10	0.4175	0.0967
0.2	0.00	-0.3235	0.0222	0.20	0.4858	0.1926
0.3	0.00	-0.2290	0.0216	0.30	0.5616	0.2884
0.4	0.00	-0.1588	0.0208	0.40	0.6277	0.3839
0.5	0.00	-0.1109	0.0198	0.50	0.6841	0.4790
0.6	0.00	-0.0881	0.0184	0.60	0.7432	0.5734
0.7	0.00	-0.0808	0.0160	0.70	0.8352	0.6667
0.8	0.00	-0.0272	0.0113	0.80	1.0034	0.7579
0.9	0.00	0.2338	-0.0020	0.90	1.1914	0.8412
1.0	0.00	2.7601	0.5116	1.00	0.0000	0.0000

APPENDIX E

Graphical Results

The following are a selection of results obtained for the six special cases discussed in section 5, all obtained by using Optimal Weighting Functions and truncating the infinite matrix using 100 eigenvalues.

Perhaps the most striking feature is the improvement in convergence when Cesaro sums are used rather than partial sums in those cases where the coefficients decay too slowly for the expansions to convergence normally. As with ordinary Fourier series these discontinuous cases show Gibbs phenomena in the neighbourhood of the jumps, as pointed out by Joseph and Sturges [1978] for the semi-infinite strip.

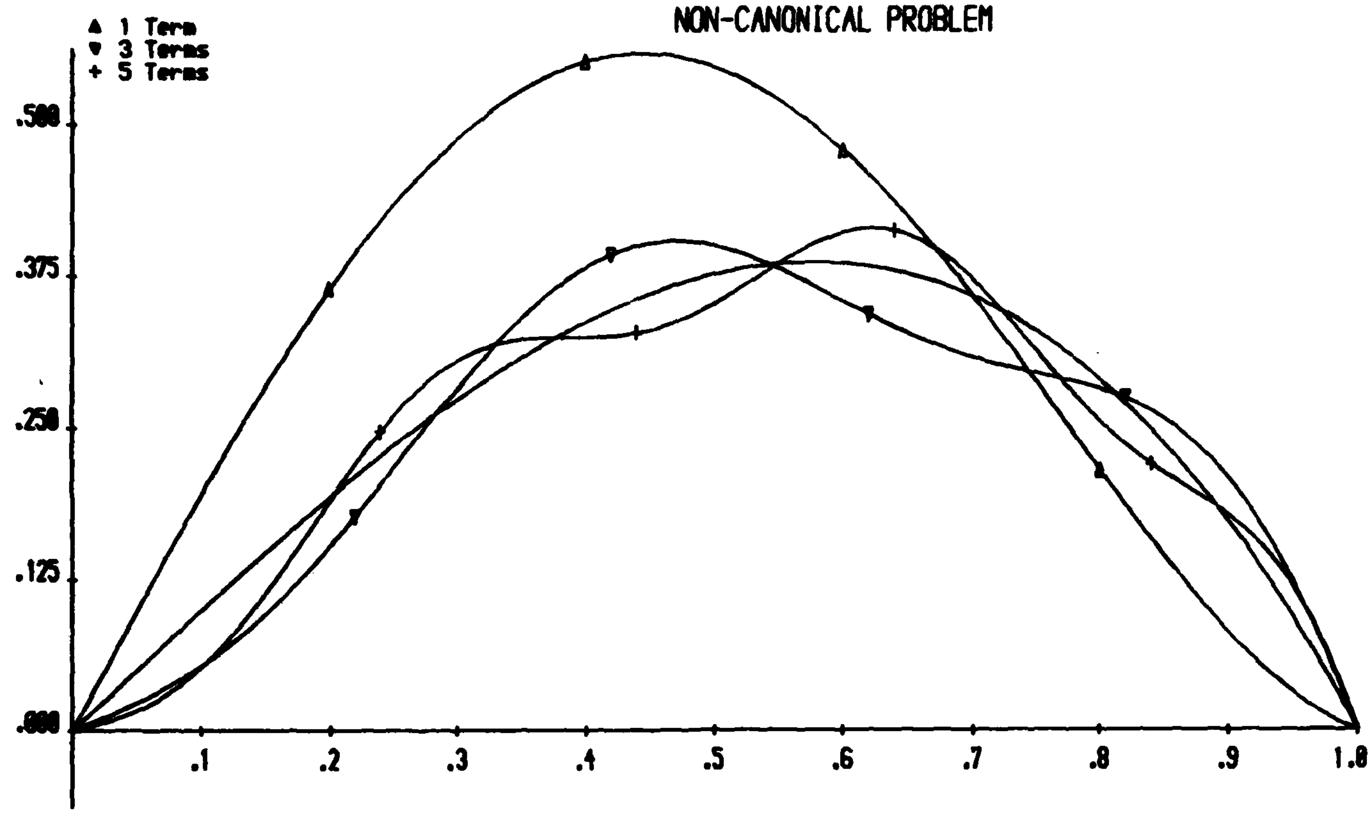


Smooth Normal Stress Distribution

Convergence to the data for z=0

Partial Sums

1, 2, 3 Terms



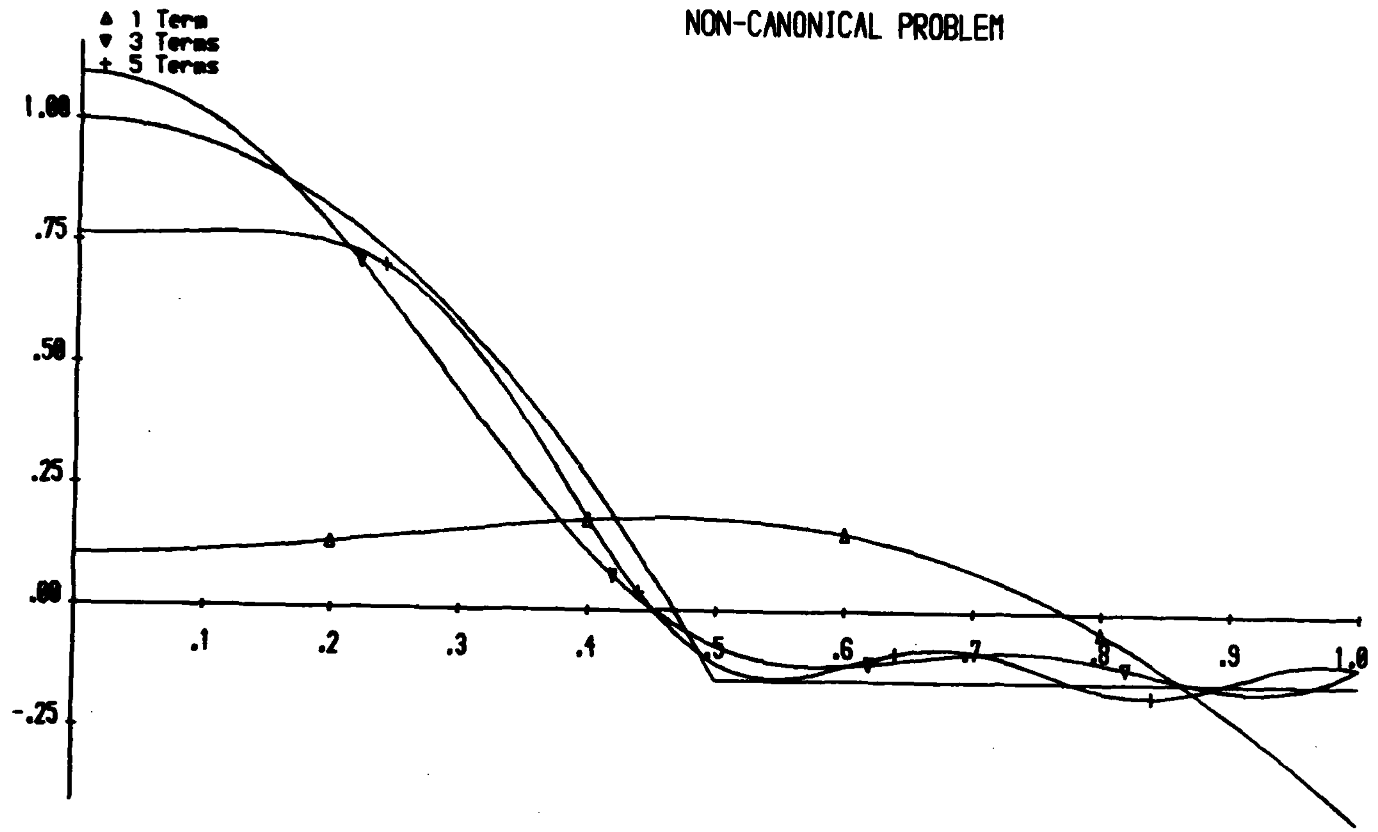
Smooth Shear Stress Distribution

Convergence to the data for z=0

Partial Sums

1, 3, 5 Terms

NON-CANONICAL PROBLEM



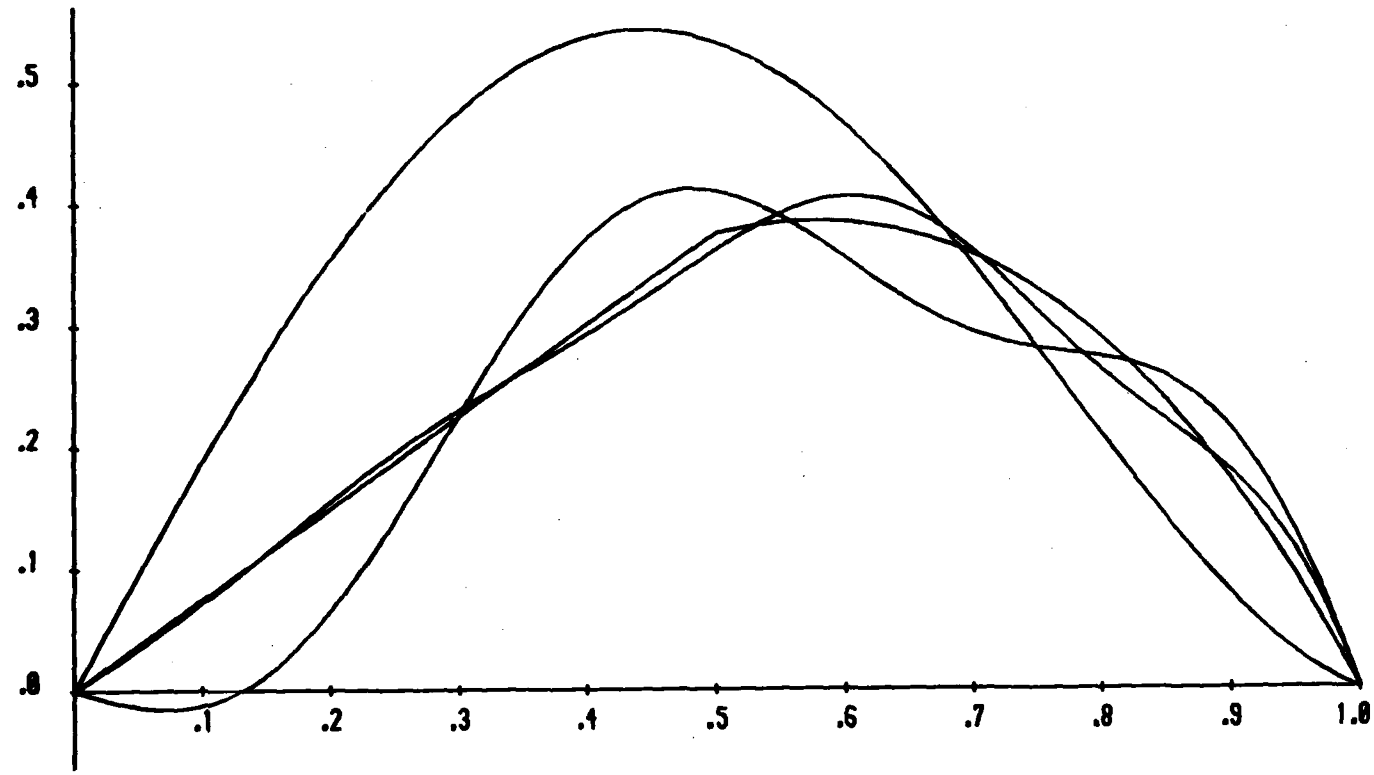
Unsmooth Normal Stress Distribution

Partial Sums

Convergence to the data for z=0

1, 3, 5 Terms

NON-CANONICAL PROBLEM



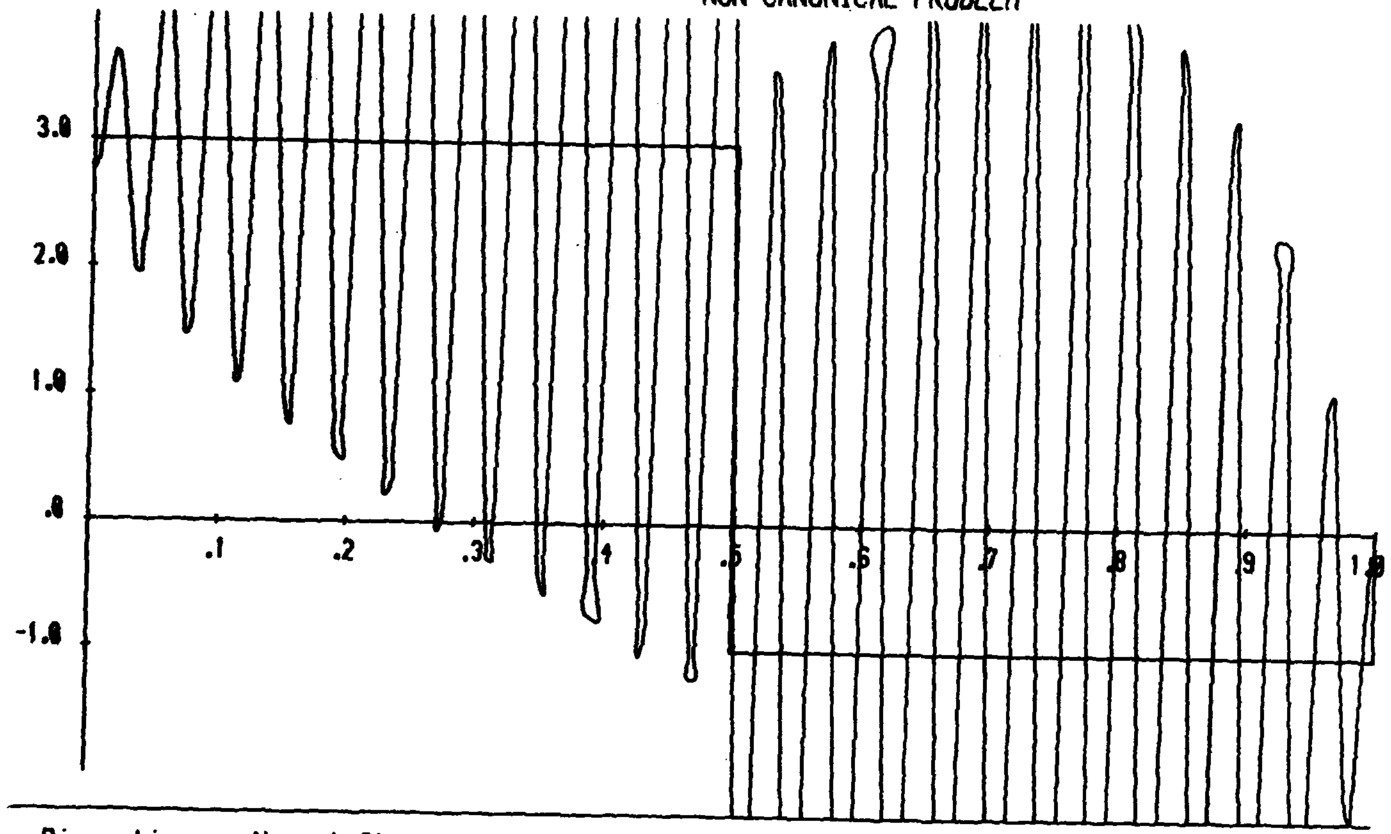
Unsmooth Shear Stress Distribution

Convergence to the data for z=0

Partial Sums

1, 3, 5 Terms

NON-CANONICAL PROBLEM



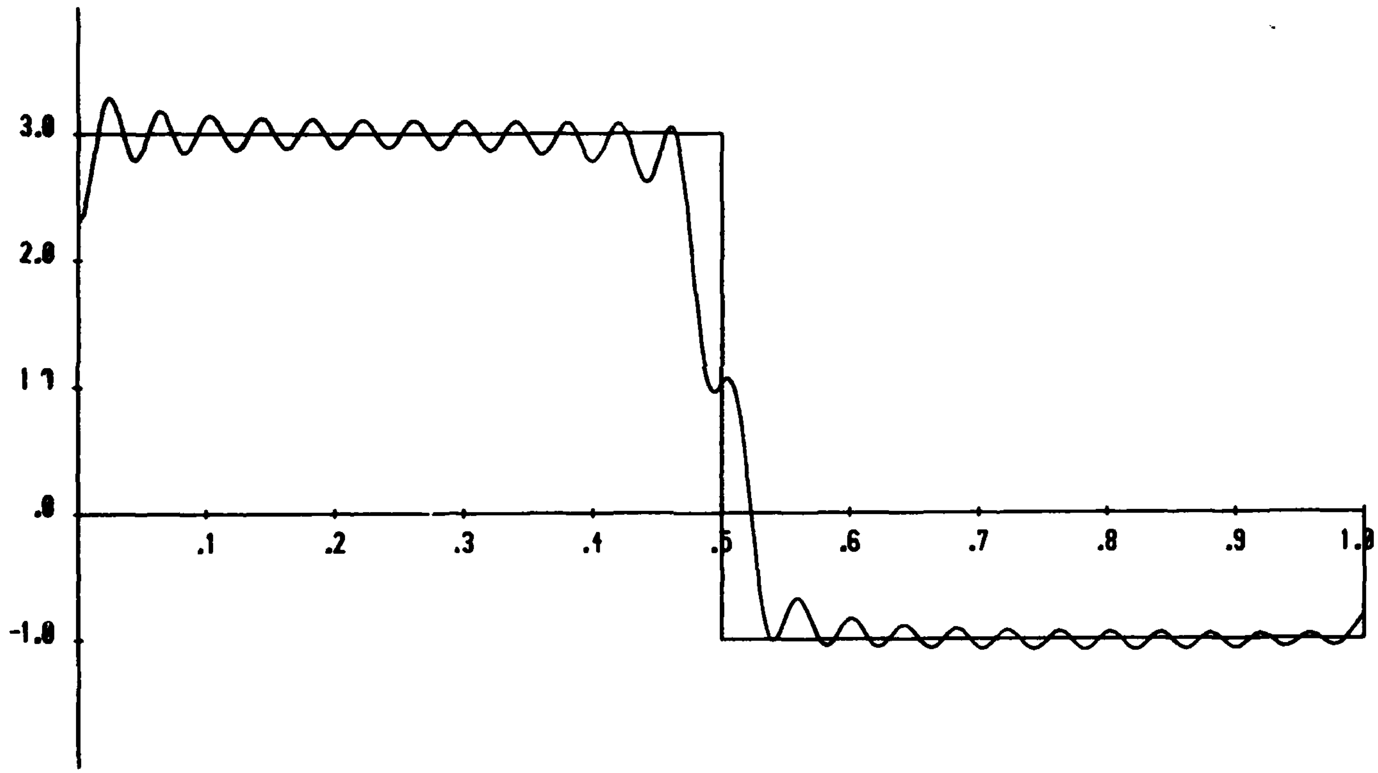
Discontinuous Normal Stress Distribution

Convergence to the data for z=0

Partial Sums

50 Terms

NON-CANONICAL PROBLEM



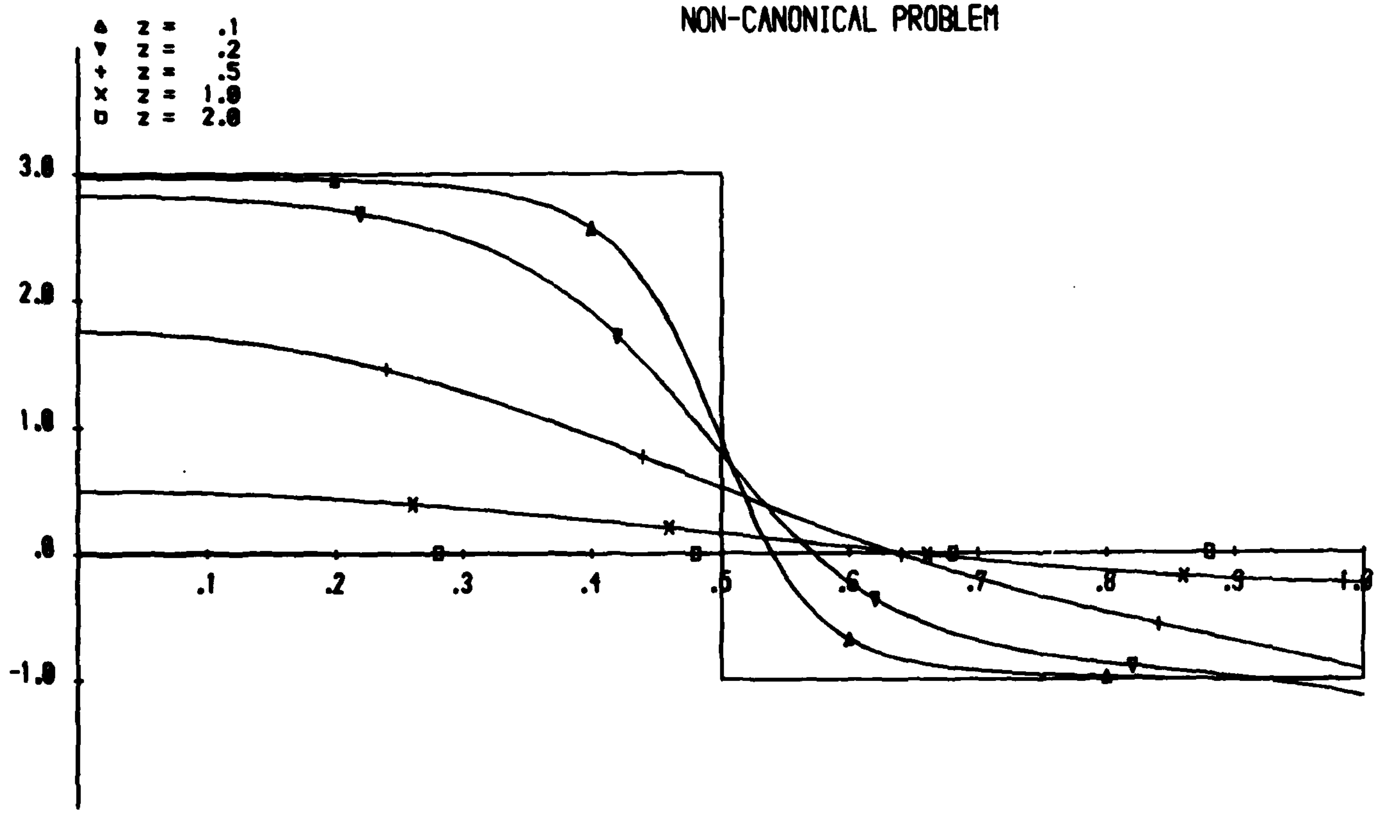
Discontinuous Normal Stress Distribution

Convergence to the data for z=0

Cesaro Sums

50 Terms

NON-CANONICAL PROBLEM



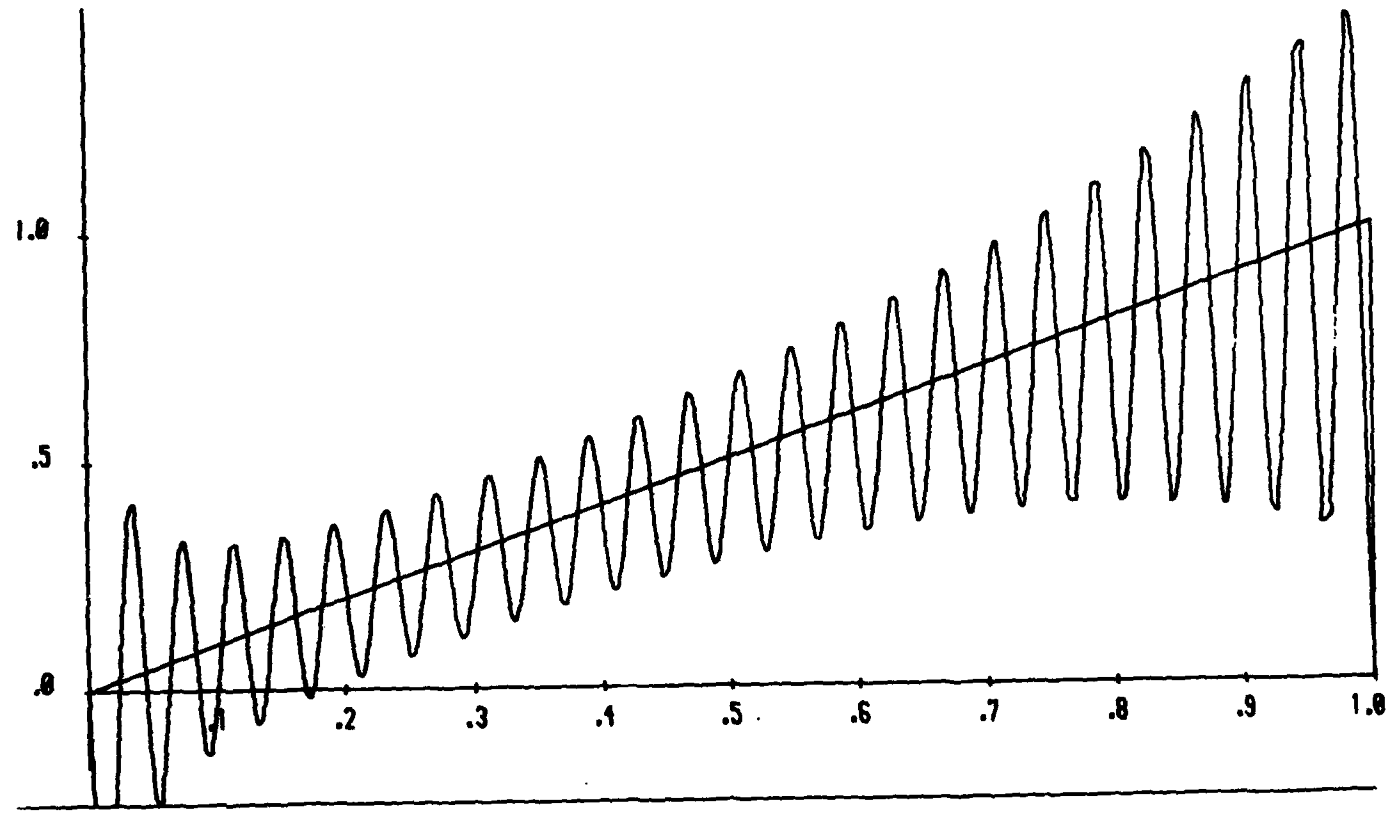
Discontinuous Normal Stress Distribution

Decay of stresses/displacements for z>0

Partial Sums, 100 Terms

Z= .1, .2, .5, 1.0, 2.0

NON-CANONICAL PROBLEM



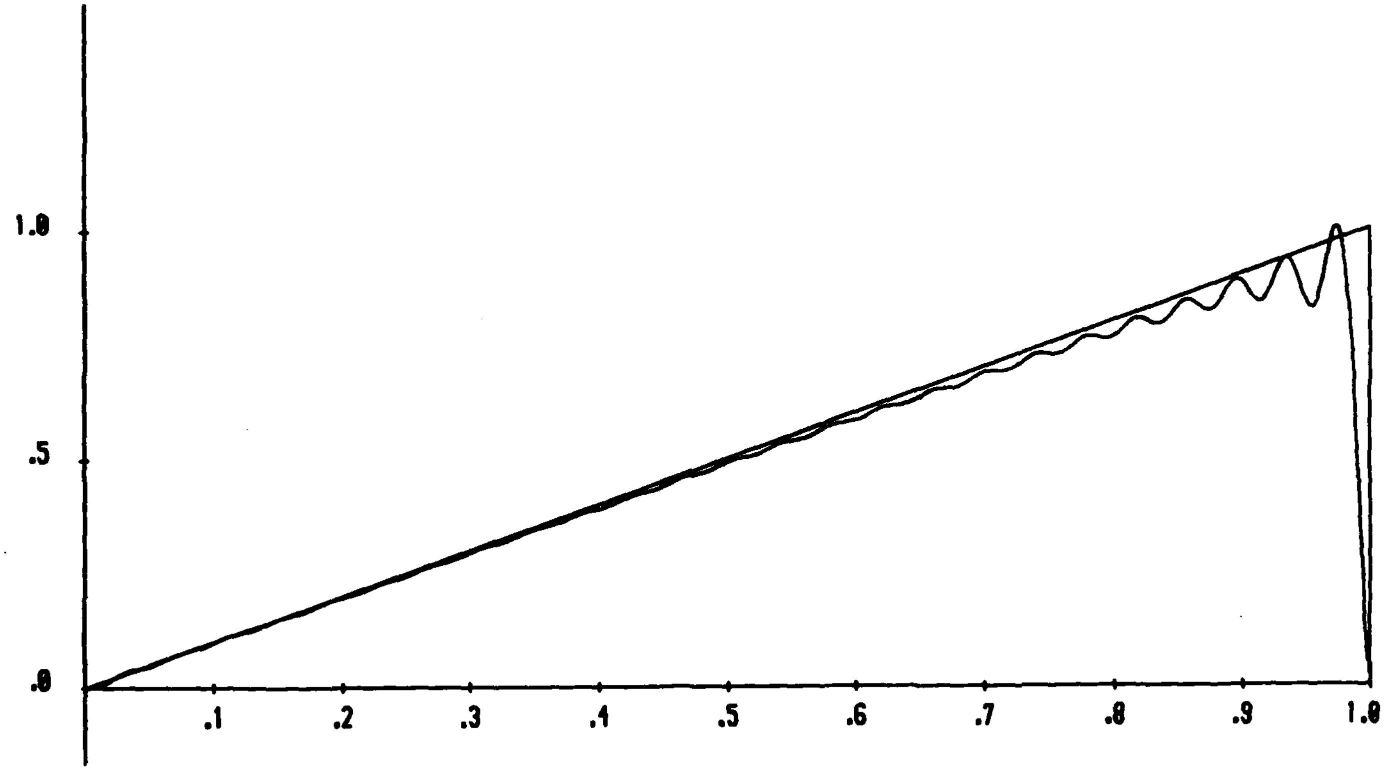
Discontinuous Shear Stress Distribution

Convergence to the data for z=0

Partial Sums

50 Terms

NON-CANONICAL PROBLEM



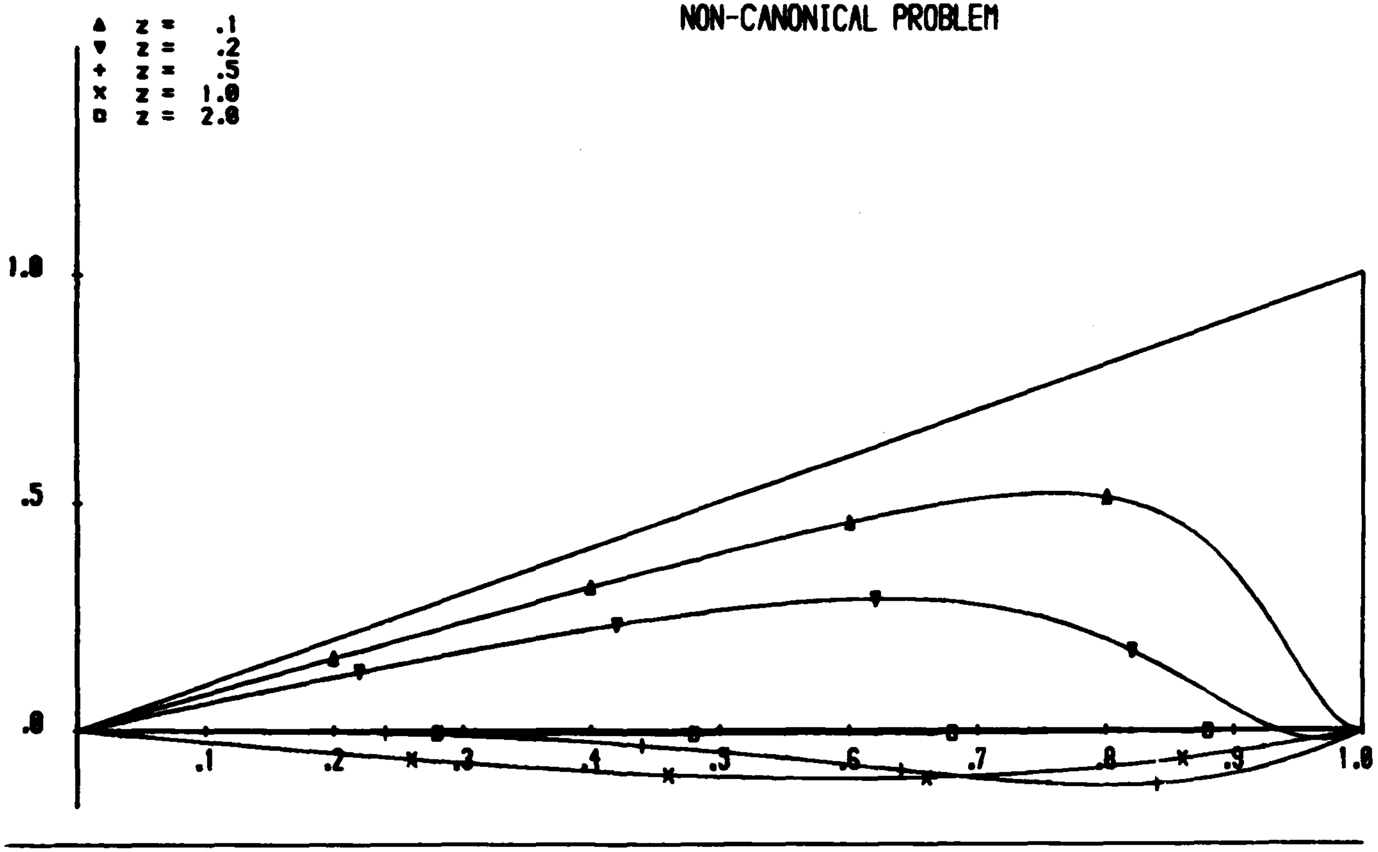
Discontinuous Shear Stress Distribution

Convergence to the data for z=0

Cesaro Sums

50 Terms

NON-CANONICAL PROBLEM



Discontinuous Shear Stress Distribution

Decay of stresses/displacements for z>0

Partial Sums, 100 Terms

Z= .1, .2, .5, 1.0, 2.0

APPENDIX F

Notes on the Computations

All the computations discussed in this report were carried out on the Oxford University Engineering Science department's VAX 11/780 machine.

The programmes to calculate the eigenvalues and all the Bessel functions required for summing expansions were calculated using a slightly modified version of the BRL Bessel function subroutine. The programme was rewritten in DOUBLE COMPLEX (COMPLEX*16) arithmetic, and was simplified slightly so that only the Bessel functions $J_0(z)$ and $J_1(z)$ would be calculated for each call to the subroutine. The eigenvalues could be calculated for any value of Poisson's ratio ν, but all the results in this report have used the value $\nu = 0.3$. The eigenvalues, their Bessel functions $J_0(\lambda_m)$ and $J_1(\lambda_m)$, and Bessel functions of the form $J_0(\lambda_m r)$ for various intermediate values of $r \in [0,1]$, were calculated in advance and stored on disk.

The cylinder eigenvalues were calculated using a simple Newton iteration technique which was found to produce satisfactory convergence to values which agreed to virtually full double precision with those calculated at BRL. The programme for calculating the coefficients for the non-canonical stress problem was built around the NAG library routine F04ADF which solves complex systems of linear equations (with multiple right-hand sides if required) using the Crout factorisation method. The subroutines to set up the infinite matrix and the right-hand sides were coded to test both unmodified biorthogonal weighting functions and optimal weighting functions. The matrix was checked for diagonal dominance and the equations were inverted. Having obtained the coefficients, the run could be terminated if desired. Otherwise the eigenfunction expansions could be summed either for a few points in the range (0,1) to test the convergence to the prescribed data or over a large number of points for various numbers of terms and for increasing values of z for use in a graphics program.

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