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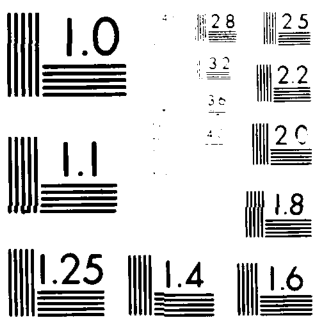

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POINT MASS MODELS AND THE ANOMALOUS GRAVITATIONAL FIELD

HANS SÜNKEL

DEPARTMENT OF GEODETIC SCIENCE AND SURVEYING
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210

DECEMBER 1981

SCIENTIFIC REPORT NO. 12

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFGL-TR-82-0084	2. GOVT ACCESSION NO. AD-A115 216	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) POINT MASS MODELS AND THE ANOMALOUS GRAVITATIONAL FIELD		5. TYPE OF REPORT & PERIOD COVERED Scientific Report No. 12 6. PERFORMING ORG. REPORT NUMBER Report No. 328
7. AUTHOR(s) Hans Sünkel		8. CONTRACT OR GRANT NUMBER(s) F19628-79-C-0075
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Geodetic Science and Surveying The Ohio State University Columbus, Ohio 43210		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62101F 760003AL
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Geophysics Laboratory Hanscom AFG, Massachusetts 01731 Contract Monitor - George Hadgigeorge/LWG		12. REPORT DATE December 1981 13. NUMBER OF PAGES 40 pages
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION DOWNGRADING SCHEDULE Unclassified -

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

point mass model
anomalous potential
gravity anomalies
integral kernel
eigenvalue
degree variances
covariance function
fast Fourier transform

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Single- and multi-layer continuous and discrete anomalous mass distributions and its relation to the spectrum of the anomalous gravitational field are investigated. A fast procedure for the calculation of gravity anomaly degree variances from regularly distributed point masses is outlined. Emphasis is put on the central role of Fast Fourier Transform methods on the sphere for all numerical determinations of point mass models from harmonic coefficients and/or surface gravity anomaly data.

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FOREWORD

This report was prepared by Dr. Hans Sünkel, Technical University at Graz, Austria, under Air Force Contract No. F19628-79-C-0075, The Ohio State University Research Foundation, Project No. 711715, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory (AFGL), Hanscom Air Force Base, Massachusetts, with George Hadgigeorge/LWG Contract Monitor.

CONTENTS

1. INTRODUCTION

Various mathematical models exist for the purpose of representing the anomalous gravity field of the Earth: integral formulas, series of harmonic functions, etc. Due to the harmonicity of the underlying anomalous potential outside the Earth's surface, the integral kernels and the solid spherical harmonics are different from zero almost everywhere; spherical harmonic representations have, in addition, the following disadvantages: the solution reacts globally to local changes; in order to represent the short-periodic part of the spectrum, the series development would require a very high degree; although spherical harmonics are evaluated by the use of recursion formulas, a series development of that kind remains an expensive task. Low degree spherical harmonic coefficients are known to permit a simple physical interpretation in terms of mass, coordinates of the center of gravity, moments of inertia, etc.; for higher degrees no simple physical interpretation is known. Its determination is possible without knowing anything about the mass distribution inside the Earth. The price that has to be paid for this ignorance is the global behavior. Neither the numerical evaluation of integral formulas nor the evaluation of a long harmonic series are adequate means for fast (real-time) and flexible (local) representations of the gravity field. The cause of the gravity field, the mass distribution inside the Earth, is not transparent either in such kind of representations.

Recently two virtually quite different gravity field representation techniques are being developed, which will possibly replace (but for sure supplement) existing techniques:

a) The finite element representation of the gravity field outside the Earth. This method is extremely powerful as far as the fast prediction of gravity field quantities is concerned; it is based on functions with local support and therefore, the represented anomalous potential is not strictly harmonic; a combination with spherical harmonic solutions seems to be very difficult; there is practically no relation to the mass distribution inside the Earth.

b) The mass model representation of the anomalous gravity field which, at the present state of the art, is restricted to its simplest form, the point mass models. This approach towards the cause of the gravity field is at the same time an approach of geodesy and geophysics. The relation of the model to the physical reality is transparent, geophysical evidencies in terms of geological structures can be easily implemented into the model. The harmonicity of the represented anomalous potential is guaranteed. The reciprocal distance and its derivatives provide the simplest possible relation between the data (point masses) and any derived gravity field quantity. Due to the global support of the reciprocal distance, all point masses contribute to the prediction of a single quantity; however, the remote zone effect can be used advantageously to reduce the number of actually used point masses. The relation between point masses and harmonic coefficients of the anomalous gravitational potential is straightforward and simple.

There is of course an essential drawback in the application of this technique: theoretically there exists an infinite number of mass distributions which are compatible with the observed gravity field. From geophysical evidences we know the main features of the density distribution. We can make a virtue of necessity and select a solution which is both simple and geophysically relevant. As far as the distribution of point masses is concerned, we strongly emphasize the importance of regular patterns. A proper design of the data pattern can reduce the calculation efforts dramatically if the algorithm is sophisticated enough to realize the data geometry.

Point mass models will be determined primarily on the basis of harmonic coefficients and/or surface mean gravity anomalies. In order to get an idea about the relation between these quantities, we have to start at the very beginning and investigate the response of essential statistical quantities of the anomalous gravitational field to changes in the spatial point mass model arrangement. Statistical models of continuous mass distributions provide important insight from a theoretical point of view, an investigation of models of discrete distributions in terms of point masses is indispensable from a practical point of view. In the latter case the enormously powerful tool of frequency domain methods on the sphere is a prerequisite for serious model calculations. The variation of the variance and correlation length of a homogeneous and isotropic gravity anomaly covariance function, generated by mass distributions in various depths, is discussed in detail. For discrete distributions the relation to harmonic coefficients of the anomalous gravitational potential is studied; guidelines for actual computations, using the concept of Fast Fourier Transform on the sphere, are given. Since a mass model should reproduce as close as possible the power per degree of the anomalous gravitational field, emphasis is put on the numerical estimation of degree variances. The actual calculation of a point mass model from real world data will be the subject of a forthcoming report.

2. DEPTHS OF ANOMALOUS MASS DISTRIBUTIONS AND THE GRAVITY ANOMALY COVARIANCE FUNCTION

The essential statistical properties of the gravity anomaly field can be described by a few parameters, the variance C_0 , the correlation length ξ , and the variance of the horizontal gravity gradient , G_0 . The latter shows a strong response to topographical and local density anomalies; the correlation length and the variance are much less affected by local anomalies and should therefore respond to deeper density anomalies. It is of primary concern to know the dependence of these two quantities on the depth of the mass point level D .

In order to investigate this problem, let us start with a highly unlikely but nevertheless very instructive case. we assume a continuous mass distribution at the depth D below the surface of the mean terrestrial sphere, with zero average (positive and negative density anomalies), and a white noise covariance function; the center of gravity is supposed to coincide with the origin of the coordinate system. Needless to say, this model is far from reality; it is a very pessimistic model insofar as the anomalous masses are assumed to be uncorrelated. In the following we shall investigate, how the gravity anomaly covariance function at mean sea level responds to that white noise mass distribution at depth D.

The disturbing potential T is given by (Heiskanen and Moritz, 1967, p. 5)

$$T(P) = G \iint_{\sigma} \frac{\mu(Q)}{\ell(P.Q)} \, d\sigma(Q) \tag{2.1}$$

with G .... gravitational constant,

ℓ .... distance (P,Q)

$d\sigma$ .... element of solid angle

$\mu d\sigma$ .... element of anomalous mass.

With

$$\ell^2 = r^2 + (R-D)^2 - 2r(R-D)\cos\psi$$

and

$$\frac{\partial \ell}{\partial r} = r - (R-D)\cos\psi \ ,$$

the radial derivate of T at $r = R$ is obtained by

$$\left.\frac{\partial T(P)}{\partial r}\right|_{r=R} = -G \iint_{\sigma} \frac{[R-(R-D)\cos\psi_{PQ}]\,\mu(Q)}{\ell^3(P.Q)} \, d\sigma(Q) \tag{2.2}$$

with

R ... mean earth radius,

D ... depth of mass anomaly layer,

ψ ... spherical distance

Both (2.1) and (2.2) describe a linear system with input μ and output T and $\partial T/\partial r$, respectively; the integral kernel $G\ell^{-1}$ and $-G[R-(R-D)\cos\psi]\,\ell^{-3}$, resp., is the corresponding system's impulse response. Due to $\Delta g = -\partial T/\partial r - 2T/r$, the impulse response of the linear system with input μ and output Δg is equal to $G\ell^{-3}[R - (R-D)\cos\psi - 2\ell^2/R]$.

In order to derive the gravity anomaly covariance function C_{gg} from the (white noise) mass anomaly covariance function $C_{\mu\mu}$, we have to know the system "function" which is nothing else than the infinite vector of the integral kernel's eigenvalues. The eigenvalues λ_n of the isotropic integral kernel $G\ell^{-3}[R - (R-D)\cos\psi - 2\ell^2/R]$ equal its projection onto the set of Legendre polynomials, multiplied by 2π,

$$\lambda_n = 2\pi \int_{-1}^{1} G\ell^{-3}[R - (R-D)t - 2\ell^2/R]P_n(t)\,dt \qquad (2.3)$$

with $t = \cos\psi$ and P_n denoting the Legendre polynomial of degree n (Müller, 1966, p. 20). The integral can be solved for arbitrary n by representing ℓ^{-1} and ℓ^{-3} in terms of a series of Legendre polynomials, and observing the well-known recurrence and orthogonality relations. A much simpler way is to consider the linear system with input μ and output Δg as a linear system in cascade, consisting of two linear subsystems; the output of the first system is input to the second one. Here the first system is represented by equation (2.1), the second system is simply the boundary condition

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2T}{r} . \qquad (2.4)$$

The cascade system function equals the product of the individual system functions (Papoulis, 1968, pp. 50, 51). System 1 (eq. 2.1) has eigenvalues

$$\lambda_n^{(1)} = 2\pi G \int_{-1}^{1} \ell^{-1} P_n(t)\,dt \quad ; \tag{2.5}$$

expressing ℓ^{-1} in terms of the series

$$\ell^{-1} = \frac{1}{R}\sum_{n=0}^{\infty} \alpha^n P_n$$

with $\alpha := 1 - \frac{D}{R}$, and observing the orthogonality relations of Legendre polynomials $\langle P_n, P_m\rangle = \delta_{n,m} 2/(2n+1)$, we obtain

$$\lambda_n^{(1)} = \frac{4\pi G\,\alpha^n}{R(2n+1)} \quad . \tag{2.5'}$$

System 2 (eq. 2.4) has eigenvalues

$$\lambda_n^{(2)} = \frac{n-1}{R} \tag{2.6}$$

(Heiskanen and Moritz, 1967, p. 97); therefore the cascade eigenvalues (Cascade system "function") are given by $\lambda_n = \lambda_n^{(1)} \cdot \lambda_n^{(2)}$

$$\lambda_n = \frac{4\pi G}{R^2} \cdot \frac{n-1}{2n+1}\,\alpha^n \quad . \tag{2.7}$$

We have assumed that the mass anomalies are uncorrelated (white noise); the corresponding covariance function is

$$C_{\mu\mu}(t) = M_0\,\delta(t-1) \tag{2.8}$$

with the variance M_0 and the Dirac distribution $\delta(t)$. Expressing $C_{\mu\mu}$ in terms of a series of Legendre polynomials,

$$C_{\mu\mu}(t) = \sum_{2}^{\infty} \mu_n P_n(t)$$

we obtain with (2-8) the degree variances μ_n ,

$$M_0 \int_{-1}^{1} \delta(t-1) P_n(t) dt = \frac{2\mu_n}{2n+1} \quad ,$$

which reduces, due to the reproducing property of $\delta(t)$, to

$$\mu_n = \frac{M_0}{2} (2n+1) \quad . \tag{2.9}$$

$\{\mu_n\}$ is the power spectrum of the white noise anomalous mass distribution. The power spectrum $\{g_n\}$ of the gravity anomaly field is then given by

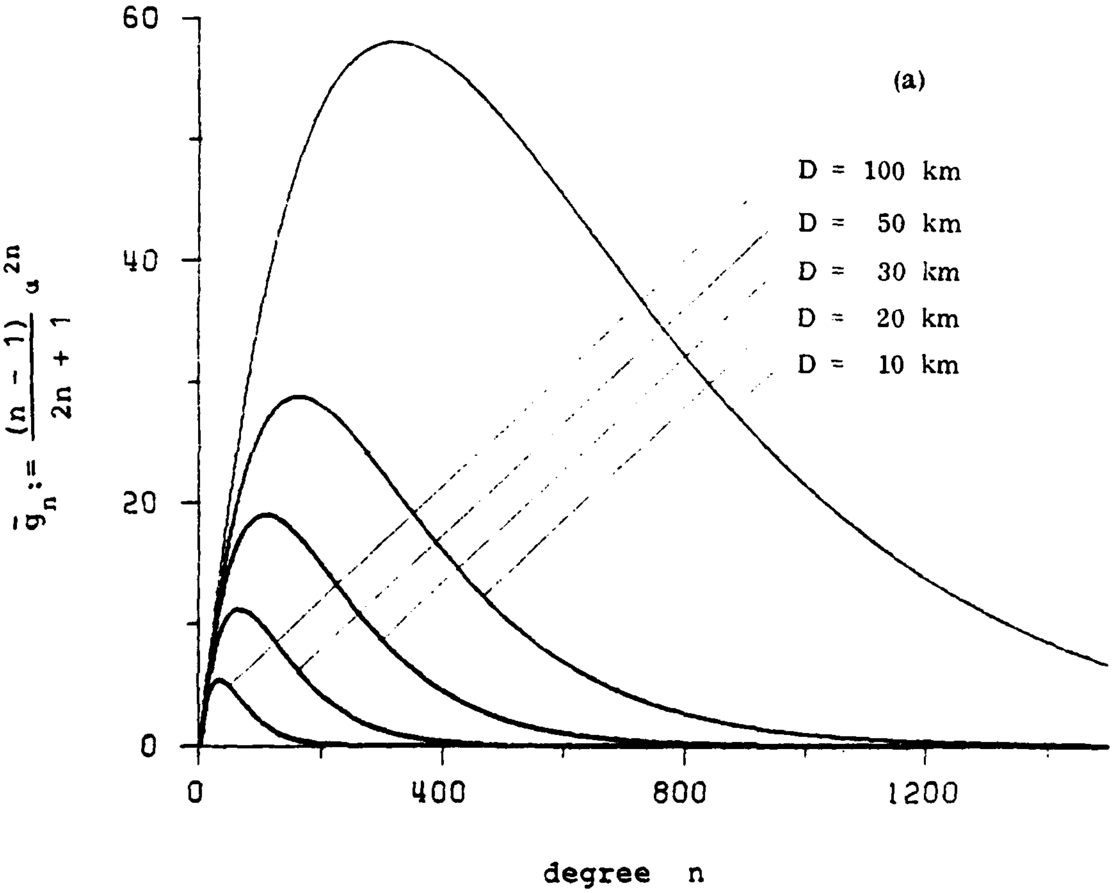
$$g_n = \lambda_n^2 \mu_n \quad ,$$

$$g_n = \left(\frac{4\pi G}{R^2}\right)^2 \frac{M_0}{2} \frac{(n-1)^2}{2n+1} \alpha^{2n} \quad , \tag{2.10}$$

and the gravity anomaly covariance function by

$$C_{gg}(t) = \sum_{2}^{\infty} g_n P_n(t) \quad . \tag{2.11}$$

The behavior of $\alpha^{2n}(n-1)^2/(2n+1)$ is graphically represented in Fig. 2.1a,b. The area below the curves is very closely related to the total variance $C_{gg}(0)$; it is quite obvious that the variance drops dramatically with increasing depth D . (Note that for all curves a common factor $(4\pi G/R^2)^2 \cdot M_0/2 = 1$ has been used.) The maximum contribution to the variance decreases towards the low degree with increasing depth.



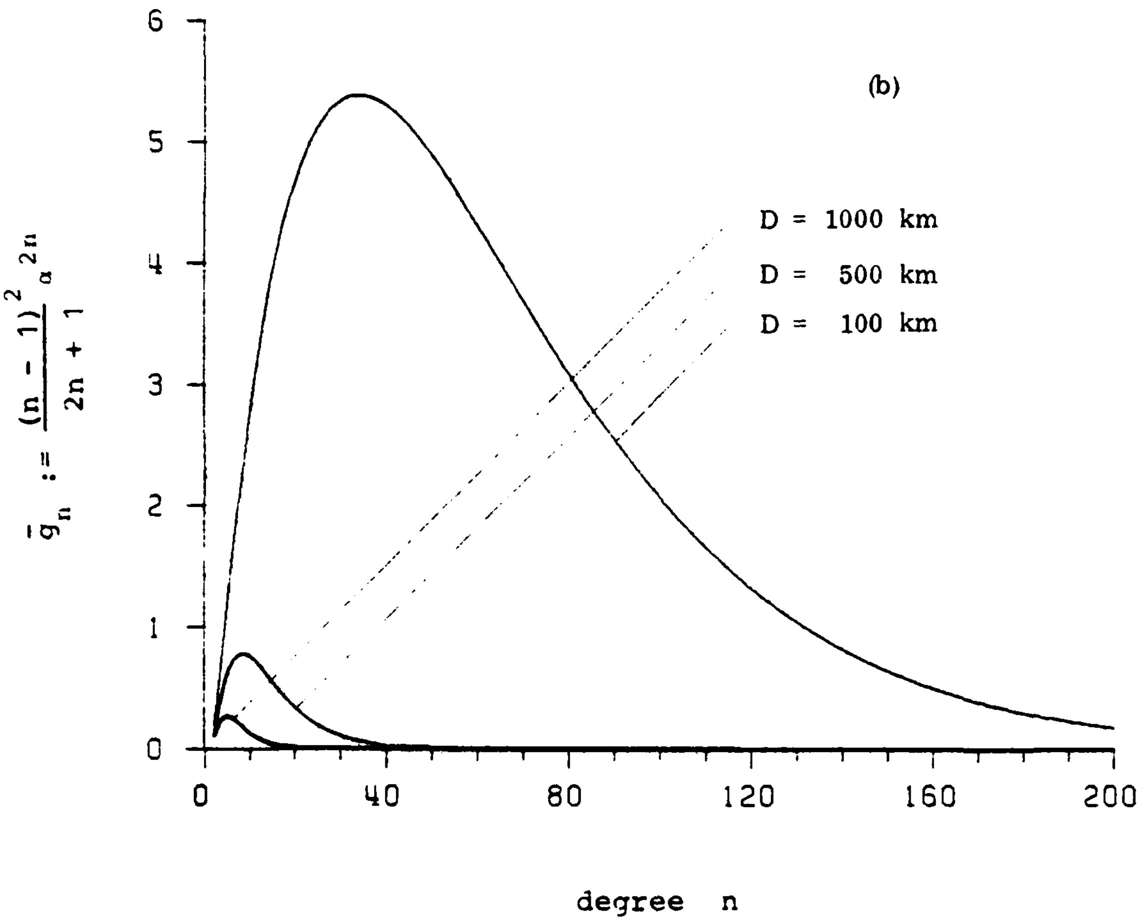


Fig. 2.1a,b Gravity anomaly degree variances due to a white noise anomalous mass distribution at various depths D ; $(4\pi G/R^2)^2 \cdot M_0/2 = 1$ has been assumed.

The simple form of the gravity anomaly degree variances (2.10) allows the variance to be represented by a closed expression.

Observing

$$\frac{(n-1)^2}{2n+1} = \frac{n}{2} - \frac{5}{4} + \frac{9}{4}\,\frac{1}{2n+1} \quad ,$$

$$\sum_{n=0}^{\infty} \alpha^{2n} = \frac{1}{1-\alpha^2} \quad , \qquad |\alpha| < 1$$

(Ryshik and Gradstein, 1963, No. 1.231, p. 24) and noting that

$$\sum_{n=0}^{\infty} n\alpha^{2n} = \frac{\alpha}{2}\frac{d}{d\alpha}\left(\sum_{n=0}^{\infty}\alpha^{2n}\right) = \frac{\alpha^2}{(1-\alpha^2)^2} \quad ,$$

$$\sum_{n=0}^{\infty} \frac{\alpha^2}{2n+1} = \frac{1}{\alpha}\int \sum_{n=0}^{\infty} \alpha^{2n} d\alpha = \frac{1}{2\alpha}\ln\frac{1+\alpha}{1-\alpha} \quad ,$$

we obtain the closed expression

$$C_{gg}(0) = M_0\left(\frac{\pi G}{R^2}\right)^2\left[\frac{9}{\alpha}\ln\frac{1+\alpha}{1-\alpha} + \frac{14\alpha^2-10}{(1-\alpha^2)^2} - 8\right] . \qquad (2.12)$$

For moderate values of D $(D \lesssim 100$ km$)$, the variance $C_{gg}(0)$ can easily be shown to be approximately proportional to D^{-2},

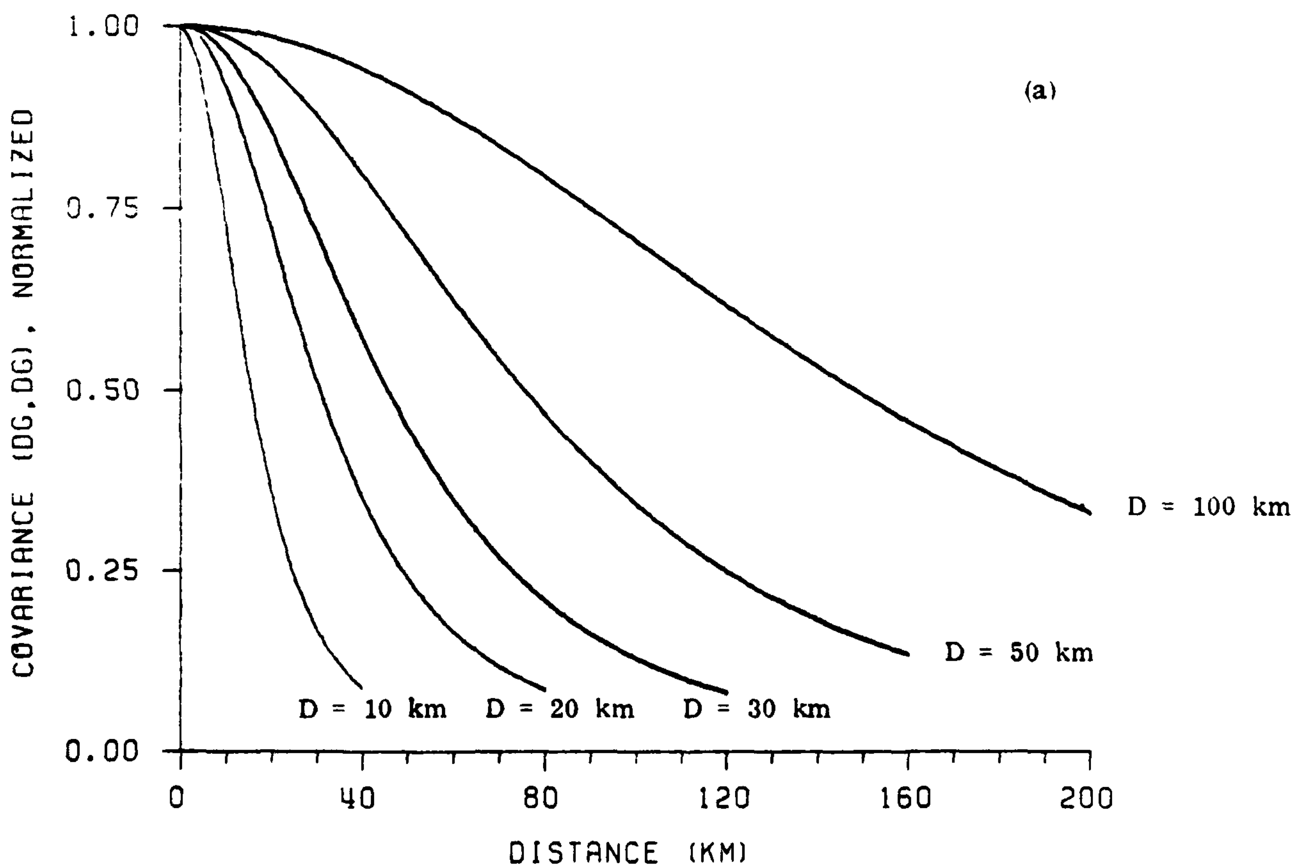
$$C_{gg}(0) \approx \frac{\text{const.}}{D^2} . \qquad (2.13)$$

With other words, the gravity anomaly variance due to a white noise anomalous mass distribution at depth D decreases with the square of the depth - a very remarkable result. Consequently, approximately the same gravity anomaly variance at zero level is generated by two anomalous mass distributions at level D_1, and D_2, if the corresponding mass anomaly variances M_1 and M_2 benave like

$$\frac{M_1}{M_2} = \frac{D_1^2}{D_2^2} \quad .$$

Therefore, we conclude that the ratios M_i/D_i^2 must be selected properly, if the gravity anomaly field's power is to be reproduced by mass anomaly distributions at various depths.

Fig. 2.2a,b show the actual gravity anomaly (normalized) covariance functions, produced by white noise anomalous mass distributions. It is obvious that the correlation length ξ increases



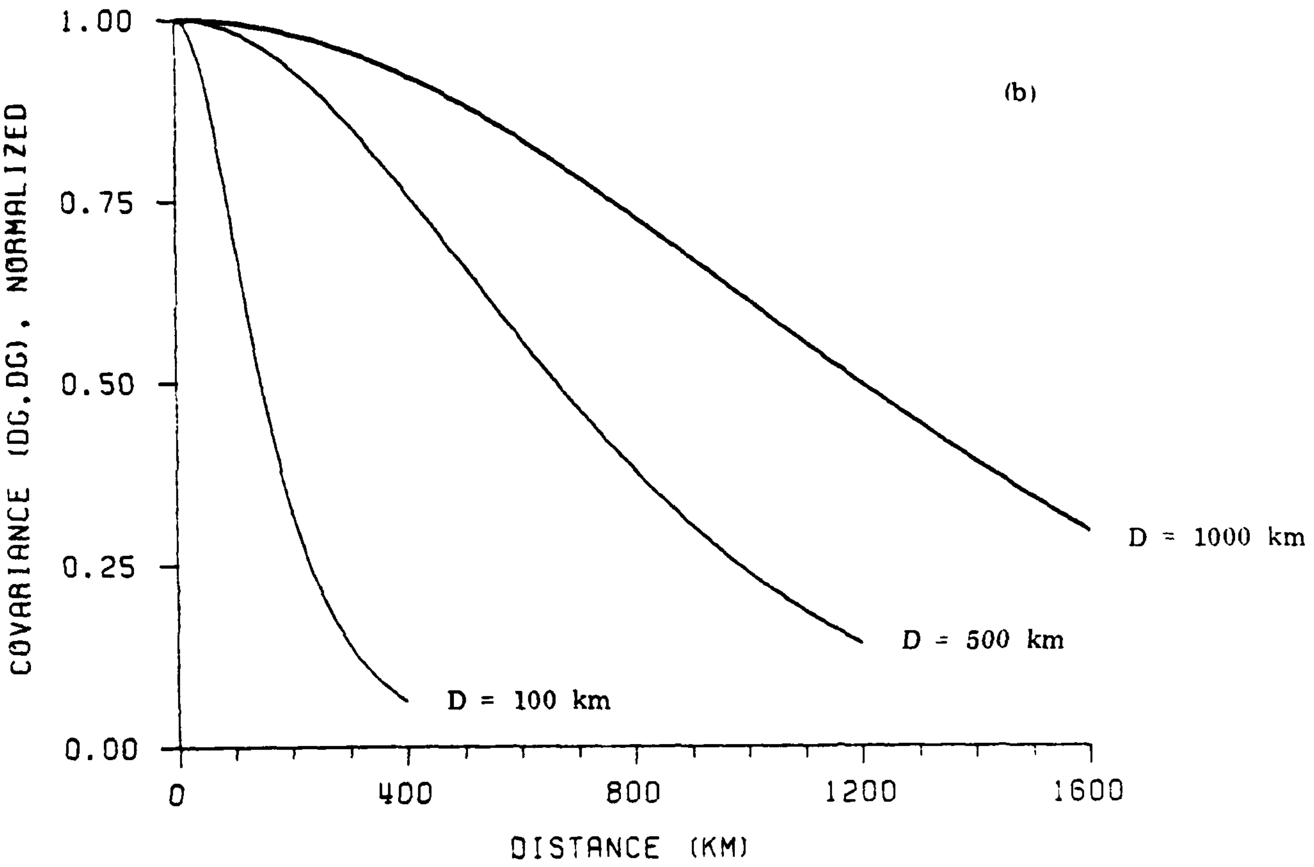


Fig. 2.2a,b Gravity anomaly covariance functions at sea level due to a white noise anomalous mass distribution at various depths D ; the covariances are normalized to $C_{gg}(0) = 1$.

with increasing depth D ; it is less obvious that ξ depends almost linearly on D (at least for moderate values of D) with a proportionality factor close to 3/2 ,

$$\xi = \frac{3}{2} D \quad . \tag{2.14}$$

From these figures we conclude that the global gravity anomaly covariance function with a correlation length of about 45 km cannot be generated by mass anomalies, located below 30 km , alone; shallow mass anomalies must considerably contribute to the observed 45 km correlation length. The topographic masses and near surface mass anomalies alone, vice versa, can hardly account for a correlation length of 45 km. The problem of relating the known gravity anomaly field to unknown mass anomalies is a (difficult) matter of tuning our sensors to proper frequencies. This delicate situation has been described best by Alfred Wegener (1929):

"We are like a judge confronted by a defendent who declines to answer, and we must determine the truth from the circumstancial evidence".

3. DISCRETE POINT MASSES AND CORRESPONDING COVARIANCE FUNCTIONS

An anomalous mass distribution represented by a white noise covariance function as investigated in the foregoing chapter, provides insight into the relation between anomalous mass and anomalous gravity. That highly artificial case is quite unlikely to be met in real world environments. The concept of discrete point masses is not very realistic either as far as geology is concerned; however, the geodetic goal is not to model geological features adequately and accurately, but to find an easy, fast and simple way to model the external anomalous gravity field; point masses present themselves as one of its simplest representations. Moreover, using point masses at discrete points does not necessarily mean that masses are concentrated at points, since the gravity field of a point mass is not distinguishable from that of a spherically symmetric mass distribution. Therefore, we could as well think in terms of a continuous anomalous mass distribution which is such that it can be uniquely described by a point mass model. In this chapter we are primarily concerned with the gravity anomaly covariance function which is generated by a discrete distribution of point masses.

Let us assume that I point masses $\{\mu_i\}$, $i = 1, \ldots, I$ are given at the points $\{Q_i\}$, $i = 1, \ldots, I$ which are located on a concentric sphere with radius $R - D$; we assume furthermore that the gravitational constant G is already contained in μ_i. The potential generated by these point masses is trivially given by

$$T(P) = \sum_{i=1}^{I} \frac{\mu_i}{\ell(P,Q_i)} \tag{3.1}$$

where $\ell(P,Q_i)$ denotes the spatial distance between the calculation point and the mass point. On the sphere $r = R$, T can be represented by a Fourier series

$$T(P) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{a}_{nm} \bar{\Phi}_{nm}(P) \tag{3.2}$$

where $\{\bar{\Phi}_{nm}\}$ denotes the orthonormal set of spherical harmonics and $\bar{a}_{nm}$ the corresponding Fourier coefficients given by

$$\bar{a}_{nm} = \langle T, \bar{\Phi}_{nm} \rangle = \frac{1}{4\pi} \int_{\sigma} T(P) \bar{\Phi}_{nm}(P) d\sigma(P) \quad . \tag{3.3}$$

Using the representation (3.1), we obtain for the coefficients $\bar{a}_{nm}$

$$\bar{a}_{nm} = \frac{1}{4\pi} \sum_{i=1}^{I} \mu_i \int_{\sigma} \frac{\bar{\Phi}_{nm}(P)}{\ell(P,Q_i)} d\sigma(P) \quad . \qquad (3.3)'$$

Taking into account the representation of the harmonic function ℓ^{-1} in terms of a Fourier series (Heiskanen and Moritz, 1967, p. 33)

$$\ell^{-1}(P,Q_i) = \frac{1}{R}\sum_{r=0}^{\infty}\frac{\alpha^r}{2r+1}\sum_{s=-r}^{r}\bar{\phi}_{rs}(P)\bar{\phi}_{rs}(Q_i) \tag{3.4}$$

with $\alpha = 1 - \frac{D}{R}$, and using the orthonormality relations of the fully normalized spherical harmonics

$$<\bar{\phi}_{nm},\bar{\phi}_{rs}> = \frac{1}{4\pi}\int_{\sigma}\bar{\phi}_{nm}(P)\bar{\phi}_{rs}(P)d\sigma(P) = \delta_{nr}\,\delta_{ms}$$

(δ_{nr} ... Kronecker-symbol), the integral in equation (3.3)' reduces to

$$\frac{1}{4\pi}\int_{\sigma}\frac{\bar{\phi}_{nm}(P)}{\ell(P,Q_i)}d\sigma(P) = \frac{1}{R}\,\frac{\alpha^n}{2n+1}\,\bar{\phi}_{nm}(Q_i) \tag{3.5}$$

and the Fourier coefficients assume the simple form

$$\bar{a}_{nm} = \frac{1}{R}\,\frac{\alpha^n}{2n+1}\sum_{i=1}^{I}\mu_i\bar{\phi}_{nm}(Q_i) \quad ;$$

let us repeat: $\bar{a}_{nm}$ is the Fourier coefficient of the potential, generated by point masses $\{\mu_i\} = \{\mu(Q_i)$, $i = 1, \ldots, I$. Denoting by

$$\bar{\mu}_i := \frac{\mu_i}{R} \tag{3.6a}$$

the individual potentials, the coefficients $\bar{a}_{nm}$ are given by

$$\bar{a}_{nm} = \frac{\alpha^n}{2n+1} \sum_{i=1}^{I} \bar{\mu}_i \bar{\phi}_{nm}(Q_i) \quad , \tag{3.3)"}$$

and the point mass generated potential by

$$T(P) = \sum_{i=1}^{I} \bar{\mu}_i \sum_{n=0}^{\infty} \frac{\alpha^n}{2n+1} \sum_{m=-n}^{n} \bar{\phi}_{nm}(P) \bar{\phi}_{nm}(Q_i) \quad . \tag{3.1)'}$$

At this point it is very instructive to consider the special case of D approaching R ; in other words, we investigate equation (3.1)' for the extreme case of all point masses concentrated at the origin of the coordinate system. As a matter of fact, the corresponding potential should reduce to that of a single centered mass point

$$\frac{GM}{R} = \sum_{i=1}^{I} \bar{\mu}_i \quad .$$

It can be easily shown that

$$\lim_{D \to R} \alpha^n = \lim_{D \to R} \left(1 - \frac{D}{R}\right)^n = \delta_{n,0} ;$$

all powers of α vanish apart from power zero; therefore, only $\bar{a}_{00}$ is different from zero and

$$T(P) = \bar{a}_{00} = \sum_{i=1}^{I} \bar{\mu}_i = \frac{GM}{R}$$

is the corresponding potential as anticipated.

Transition to gravity anomalies.

The gravity anomaly field at sea level is related to the anomalous potential through a convolution (Stokes and inverse Stokes formula); the corresponding relation in the frequency domain is given by the well-known product (Heiskanen and Moritz, 1967, p. 97)

$$\Delta g_n = \frac{n-1}{R} T_n \; . \tag{3.7}$$

Therefore, the Fourier coefficients $\bar{b}_{nm}$ of the gravity anomaly Fourier series

$$\Delta g(P) = \sum_n \sum_m \bar{b}_{nm} \bar{\phi}_{nm}(P) \tag{3.8}$$

are related to the Fourier coefficients of the anomalous potential through

$$\bar{b}_{nm} = \frac{n-1}{R} \bar{a}_{nm} \tag{3.9}$$

and with

$$\bar{\bar{\mu}}_i := \frac{\bar{\mu}_i}{R} = \frac{\mu_i}{R^2} \quad , \tag{3.6b}$$

the coefficients are obtained by

$$\bar{b}_{nm} = \alpha^n \frac{n-1}{2n+1} \sum_{i=1}^{I} \bar{\bar{\mu}}_i \bar{\Phi}_{nm}(Q_i) \quad . \tag{3.10}$$

Consequently, the gravity anomaly at $r = R$ is represented by

$$\Delta g(P) = \sum_{i=1}^{I} \bar{\bar{\mu}}_i \sum_{n=0}^{\infty} \alpha^n \frac{n-1}{2n+1} \sum_{m=-n}^{n} \bar{\Phi}_{nm}(P) \bar{\Phi}_{nm}(Q_i) \quad ,$$

or, considering the decomposition formula (Heiskanen and Moritz, 1967, p. 33)

$$P_n(\cos\psi_{PQ}) = \frac{1}{2n+1} \sum_{m=-n}^{n} \bar{\Phi}_{nm}(P) \bar{\Phi}_{nm}(Q) \quad , \tag{3.11}$$

even simpler by

$$\Delta g(P) = \sum_{i=1}^{I} \bar{\bar{\mu}}_i \sum_{n=0}^{\infty} \alpha^n (n-1) P_n(\cos\psi_{P,Q_i}) \quad . \tag{3.8'}$$

The gravity anomaly degree variances.

In order to study the statistical characteristics of the gravity anomaly field, generated by a finite number of point masses, we need to know the power of the field distributed over all frequencies (degrees) n ; for a specific n this power is denoted degree variance of degree n and given by (Heiskanen and Moritz, 1967, p. 259)

$$c_n = \sum_{m=-n}^{n} \bar{b}^2_{nm} \quad . \tag{3.12}$$

With $\bar{b}_{nm}$ given by (3.10), the degree variances can be written explicitly

$$c_n = \left(\alpha^n \frac{n-1}{2n+1}\right)^2 \sum_{m=-n}^{n} \left[\sum_{i=1}^{I} \bar{\bar{\mu}}_i \bar{\Phi}_{nm}(Q_i)\right]^2 \quad ,$$

or, interchanging the sequence of summation,

$$c_n = \left(\alpha^n \frac{n-1}{2n+1}\right)^2 \sum_{i=1}^{I} \sum_{j=1}^{I} \bar{\bar{\mu}}_i \bar{\bar{\mu}}_j \sum_{m=-n}^{n} \bar{\Phi}_{nm}(Q_i) \bar{\Phi}_{nm}(Q_j) \quad .$$

The last sum represents, apart from the factor $(2n+1)$, the decomposition of the Legendre polynomial P_n into fully normalized spherical harmonics (eq. (3.11)); therefore, c_n reduces to

$$c_n = \frac{\alpha^{2n}(n-1)^2}{2n+1} \sum_{i=1}^{I} \sum_{j=1}^{I} \bar{\bar{\mu}}_i \bar{\bar{\mu}}_j P_n(\cos\psi_{Q_i,Q_j}) \quad . \tag{3.13}$$

The double sum equals the quadratic form

$$\sum_{i=1}^{I} \sum_{j=1}^{I} \bar{\bar{\mu}}_i \bar{\bar{\mu}}_j P_n(\cos\psi_{i,j}) = \bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} \tag{3.14}$$

with

$$\bar{\bar{\mu}}^T := (\bar{\bar{\mu}}_1, \bar{\bar{\mu}}_2, \ldots, \bar{\bar{\mu}}_I) \tag{3.15a}$$

and the symmetric matrix

$$A_n := \begin{bmatrix} P_n(\cos\psi_{1,1}), & P_n(\cos\psi_{1,2}), & \ldots, & P_n(\cos\psi_{1,I}) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ P_n(\cos\psi_{I,1}), & P_n(\cos\psi_{I,2}), & \ldots, & P_n(\cos\psi_{I,I}) \end{bmatrix} \tag{3.15b}$$

The matrix A_n can be considered a covariance matrix derived from the covariance function $C(\psi) = P_n(\cos\psi)$, with elements of A_n depending on the mutual spherical distance between the individual point masses. As a consequence, A_n has only positive and zero eigenvalues for $n > 0$ and as a matter of fact, $\bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} \geq 0$. This property guarantees the non-negativity of the

$\{c_n\}$ which, in turn, represents a homogeneous and isotropic gravity anomaly covariance function, derived from a finite set of point masses, in the frequency domain,

$$c_n = \frac{\alpha^{2n}(n-1)^2}{2n+1} \; \bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} \quad . \qquad (3.13)'$$

The corresponding covariance function of gravity anomalies, at zero level, is obtained through (Heiskanen and Moritz, 1967, p. 256)

$$C(\psi) = \sum_{n=2}^{\infty} \frac{\alpha^{2n}(n-1)^2}{2n+1} \bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} P_n(\cos\psi) \; ; \qquad (3.16)$$

its spatial extension can be easily derived by covariance propagation in the frequency domain, using the upward continuation operator applied to gravity anomalies (Heiskanen and Moritz, 1967, pp. 88-89) ,

$$C(P,Q) = \sum_{n=2}^{\infty} \frac{\alpha^{2n}(n-1)^2}{2n+1} \left(\frac{R^2}{r_P r_Q}\right)^{n+2} \bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} P_n(\cos\psi_{PQ}) \quad . \qquad (3.17)$$

The product αR equals the radius of the point mass sphere $R_D = R - D$,

$$\alpha R = \left(1 - \frac{D}{R}\right) R = R - D = R_D$$

and (3.17) can be simplified to become

$$C(P,Q) = \frac{1}{\alpha^4} \sum_{n=2}^{\infty} \frac{(n-1)^2}{2n+1} \left(\frac{R_D^2}{r_P r_Q}\right)^{n+2} \bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} P_n(\cos\psi_{PQ}) \ . \qquad (3.17)'$$

The corresponding spatial covariance function of the anomalous potential is given by

$$K(P,Q) = \frac{1}{\alpha^2} \sum_{n=2}^{\infty} \frac{1}{2n+1} \left(\frac{R_D^2}{r_P r_Q}\right)^{n+1} \bar{\mu}^T A_n \bar{\mu} P_n(\cos\psi_{PQ}) \ . \qquad (3.18)$$

It is both interesting and instructive to compare the gravity anomaly degree variances (2.10) with (3.13)' : equation (2.10) has been derived from a white noise anomalous mass model at depth D , eq. (3.13)' from an (arbitrary) discrete distribution of a finite number of point masses located at the same depth D . The *common* properties are a) the transition from mass to gravity, and b) the upward continuation of gravity; the combination of both is represented by the common degree-dependent factor $\alpha^{2n}(n-1)^2/(2n+1)$; the most striking difference is the degree-independent multiplication factor

$$\left(\frac{4\pi G}{R^2}\right)^2 \frac{M_O}{2}$$

in (2.10) which is due to the introduced white noise model of the anomalous mass distribution, and the degree-dependent multiplication factor $\bar{\bar{\mu}}^T A_n \bar{\bar{\mu}}$ in (3.13)', which is determined, for each n , by the actual point masses. It can be shown that (3.13)' converges to (2.10) if the number of point masses becomes infinite with no correlation between neighboring point masses. Consequently, (2.10) is a special limit case of (3.13)' and this is why we investigated it before. (3.13)' is general and can be used with real point mass data.

4. GRAVITY ANOMALY DEGREE VARIANCES DUE TO REGULARLY DISTRIBUTED POINT MASSES

Gravity anomaly degree variances represent the power of the gravity anomaly field broken up by degree. A distribution of point masses (at a certain depth D) yields a gravity anomaly field at zero level which can be statistically described by its degree variances. For low to moderately high degree variances are known from observations. Any physically meaningful point mass model has therefore to meet an essential requirement: it should generate gravity anomaly degree variances which are close to the "observed" degree variance model. Equation (3.13)' relates discrete point masses to corresponding gravity anomaly degree variances.

Given a discrete point mass distribution, the very problem of calculating degree variances consists obviously in the numerical calculations of the quadratic forms.

$$\bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} \ .$$

This harmless-looking expression poses severe problems if the point mass distribution is irregular and if $\bar{\bar{\mu}}$ is large: the elements of each matrix A_n are Legendre polynomials of degree n , evaluated for the mutual spherical distance between the data (point masses),

$$a_{ij}^{(n)} = P_n(\cos\psi_{ij}) \ , .$$

$$\left\{a_{ij}^{(n)}\right\} = A_n \ ;$$

therefore, the calculation of each element a_{ij} requires one spherical distance calculation. The recurrence relation of Legendre polynomials (Heiskanen and Moritz, 1967, p.23)

$$P_n(t) = \frac{2n-1}{n}\, t\, P_{n-1}(t) - \frac{n-1}{n} P_{n-2}(t)$$

carries over to the matrices A_n ,

$$A_n = \frac{2n-1}{n} A_1 \boxdot A_{n-1} - \frac{n-1}{n} A_{n-2} \;, \qquad (4.1)$$

where " $\boxdot$ " denotes the Hadamard product of matrices,

$$C = A \boxdot B \;:\; c_{ij} = a_{ij} \cdot b_{ij} \;.$$

Despite of the pleasant feature of A_n (note that the spherical distances have to be calculated only once -- in order to set up the matrix A_1), a calculation of the quadratic form $\bar{\bar{\mu}}^T A_n \bar{\bar{\mu}}$ becomes prohibitive for irregularly distributed point masses, even if its number I is as low as a few hundred.

Is our problem surmountable? Yes, it is. The picture changes dramatically if we assume a regular distribution of the point masses at the grid points of a geographical grid. In this case we can take advantage of the structure of A_n , transform the quadratic form into the spectral domain using fast Fourier transform methods, and apply Parseval's theorem. This method is not new to geodesy; the interested reader will find a detailed treatment of such kind of problems in (Heller et al., 1977) and (Colombo, 1979). Therefore, we restrict ourselves to a comprehensive and somewhat simplified presentation of the procedure.

Assume J point masses to be distributed with constant spacing along a single parallel. The corresponding discrete Fourier transform matrix F is given by (Heller et al., 1977, p.23)

$$F = \left[F_{jm}\right] = \frac{1}{\sqrt{J}} e^{-\frac{i2\pi}{J} jm} , \quad o \leq j,m \geq J - 1 \tag{4.2}$$

with the imaginary unit $i = \sqrt{-1}$. The discrete Fourier transform vector of the vector of point masses will be denoted by X and is obtained through the linear transformation

$$X = F^{*}\bar{\bar{\mu}} \quad , \tag{4.3}$$

where " * " denotes the complex conjugate transpose. As a matter of fact X is in general a complex vector; it consists of a real (X^o) and imaginary (X^1) part

$$X = X^o + iX^1$$

with elements

$$\begin{Bmatrix} X_m^o \\ X_m^1 \end{Bmatrix} = \frac{1}{\sqrt{J}} \sum_{j=o}^{J-1} \bar{\bar{\mu}}_j \begin{Bmatrix} \cos\frac{2\pi}{J} jm \\ \sin\frac{2\pi}{J} jm \end{Bmatrix}, \; m = 0,1,\ldots,J-1 \; . \tag{4.4}$$

If A_n would be the unit matrix I , the norm $\bar{\bar{\mu}}^T\bar{\bar{\mu}}$ can easily be shown to be equal to $X^{*}X$, which is the corresponding norm in the frequency domain,

$$\bar{\bar{\mu}}^T \bar{\bar{\mu}} = X^* X \; ; \tag{4.5}$$

this is the discrete form of Parseval's theorem (Cooley et al., 1967). The proof is straightforward: we introduce (4.4) in (4.5) and obtain

$$X^* X = \sum_{m=0}^{J-1} |X_m|^2 = \frac{1}{J} \sum_{m=0}^{J-1} \sum_{j=0}^{J-1} \bar{\bar{\mu}}_j e^{\frac{i2\pi}{J} jm} \sum_{j'=0}^{J-1} \bar{\bar{\mu}}_{j'} e^{-\frac{i2\pi}{J} j'm}$$

$$= \sum_{j=0}^{J-1} \bar{\bar{\mu}}_j \sum_{j'=0}^{J-1} \bar{\bar{\mu}}_{j'} \left\{ \frac{1}{J} \sum_{m=0}^{J-1} e^{\frac{i2\pi}{J} jm} e^{-\frac{i2\pi}{J} j'm} \right\} .$$

The expression between the parentheses equals $\delta_{jj'}$ (δ ... Kronecker symbol) according to the orthogonality relation of exponentials (Brigham, 1974, p.99),

$$\frac{1}{J} \sum_{m=0}^{J-1} e^{\frac{i2\pi}{J} jm} e^{-\frac{i2\pi}{J} j'm} = \delta_{jj'} \quad , \tag{4.6}$$

and the identity

$$X^* X = \sum_{j=0}^{J-1} \bar{\bar{\mu}}_j^2 = \bar{\bar{\mu}}^T \bar{\bar{\mu}}$$

follows immediately.

Our problem is a bit more complicated: the matrix A_n is not a unit matrix, but a full one, whose elements depend only on the spherical distance of the point masses (and on the degree n). Due to the geometry of the data pattern, which is fully represented in A_n, the matrix has a very special structure: A_n is a Toeplitz circulant matrix; (row k equals row $k-1$ shifted one element to the right, with the last element of row $k-1$ wrapped around to the first place in row k.) Therefore, A_n has only J distinct elements; (in our case even only $\mathrm{int}[(J+1)/2]$ due to the dependence of the spherical distance.) It is very essential for the following that circulant matrices are diagonalized under the discrete Fourier transformation. Denoting the discrete Fourier transform of A_n by the diagonal matrix Λ_n,

$$\Lambda_n = FA_nF^* \quad , \tag{4.7}$$

the diagonal terms are simply given by

$$\lambda_m^{(n)} = \sum_{j=0}^{J-1} a_{0j}^{(n)} \cos\frac{2\pi}{J} jm \quad , \qquad m = 0,1,\ldots,J-1 \quad , \tag{4.8}$$

where $\left\{a_{0j}^{(n)}\right\}$, $j=0,1,\ldots,J-1$ denote the elements of the first row of A_n. The proof of equation (4.8) is simple; it relies on the circulant property of A_n and on the orthogonality relation (4.6). The discrete form of Parseval's theorem with A_n as metric is given by

$$X^*\Lambda_n X = \bar{\bar{\mu}}^T A_n \bar{\bar{\mu}} \quad . \tag{4.9}$$

(Note that Λ_n is the diagonal matrix of eigenvalues, F the orthogonal matrix of eigenvectors of A_n ; X is the image of $\bar{\bar{\mu}}$ in the eigenvector system.) The discrete Fourier transformations (4.3) and (4.7) are accomplished by the enormously powerful tool of Fast Fourier Transform (FFT) (Brigham, 1974; Singleton, R.C., 1968). (The CPU-time increases with $J \cdot \ln J$; the constant multiplication factor is about $2.7 \cdot 10^{-5}$ sec for the Singleton-algorithm on a UNIVAC 1100/81.) In view of this speed, the transformation into the frequency domain does not pose any problems. The matrix A_n can be computed recursively using the relation (4.1); (actually only half of the elements of the first row have to be calculated because of the circulant character of A_n .) Parseval's theorem (4.9) gives immediately the desired degree variances (apart from the factor $\alpha^{2n}(n-1)^2(2n+1)^{-1}$).

Sofar we have considered a regular data distribution restricted to a single parallel. Let us now investigate the case of a regular distribution on K parallels. Now the data vector $\bar{\bar{\mu}}$ consists of K subvectors $\bar{\bar{\mu}}^{(k)}$ with J elements each. Analogously, the symmetric matrix A_n can be subdivided into K^2 submatrices of dimension J^2 each. Each submatrix $A_n^{(k,k')}$, which corresponds to row k combined with k' , is Toeplitz circulant due to the geometry of the data pattern. Therefore, A_n is a symmetric block matrix with Toeplitz circulant blocks. If the data pattern is symmetric with respect to the equator, A_n is even persymmetric. Transforming each subvector $\bar{\bar{\mu}}^{(k)}$ and each submatrix $A_n^{(k,k')}$ into the frequency domain according to equ.(4.4) and (4.8) leads to Parseval's identity with a block-diagonal eigenvalue matrix Λ_n , corresponding to the block-circulant matrix A_n . As a matter of fact, a persymmetry of A_n is also reflected by Λ_n . More details on that particular procedure are contained in (Colombo, 1979).

5. MULTI-LAYER MASS DISTRIBUTIONS

We have seen in chapter 2 that a single-layer anomalous mass distribution is neither able of modelling the observed Earth's gravity field adequately, nor is it very realistic from a geophysical point of view. Therefore, multi-layer models have been proposed by numerous authors (Schwarz, 1977, 1981; Jordan, 1978; Heikkinen, 1981; Davenport, 1978). In the sequel we investigate the multi-layer mass distribution problem with an emphasis on the statistics of the resulting anomalous gravity field. In the same order as before, we consider first white noise distributions and second discrete point mass models with regular distribution.

Denoting the depths of the mass anomaly layers by D_l ($l = 1,\ldots,L$), the eigenvalues of the integral kernel, which is responsible for the transition from mass to gravity, are obtained from (2.7),

$$\lambda_n^{(l)} = \frac{4\pi G}{R^2} \frac{n-1}{2n+1} \alpha_l^n \quad , \qquad l = 1,\ldots,L \tag{5.1}$$

with $\alpha_l = 1 - \frac{D_l}{R}$. The gravity anomaly power spectrum corresponding to the white noise anomalous mass distribution at depth D_l is obtained by (2.10)

$$g_n^{(l)} = \left(\frac{4\pi G}{R^2}\right)^2 \frac{M_o^{(l)}}{2} \frac{(n-1)^2}{2n+1} \alpha_l^{2n} \tag{5.2}$$

and the total power spectrum is obviously the sum of the individual ones,

$$g_n = \sum_{l=1}^{L} g_n^{(l)} \tag{5.3}$$

provided two mutually different mass layers are uncorrelated. The total gravity anomaly covariance function is formally equal to (2.11) with g_n defined by (5.3),

$$C_{gg}(t) = \sum_{n=2}^{\infty} g_n P_n(t) \ .$$

If the anomalous mass distribution is discrete and regular, such that the data pattern is common to all levels, the disturbing potential harmonic coefficients (3.3)" are represented in terms of

$$\bar{a}_{nm} = \frac{1}{2n+1} \sum_{i=1}^{I} \bar{\phi}_{nm}(Q_i) \sum_{l=1}^{L} \bar{\mu}_{il} \alpha_l^n \ , \tag{5.4}$$

where $\bar{\mu}_{il}$ denotes the point mass (multiplied by the gravitational constant G and divided by the mean Earth radius R), which has Q_i as horizontal and $R - D_l$ as vertical position. (Note that $\bar{\mu}_{il}$ can be considered as the potential, which is generated by that particular point mass on a ball with radius R and centered at the point mass.) Analogously, the gravity anomaly coefficients (3.10) are given by

$$\bar{b}_{nm} = \frac{n-1}{2n+1} \sum_{i=1}^{I} \bar{\phi}_{nm}(Q_i) \sum_{l=1}^{L} \bar{\bar{\mu}}_{il} \alpha_l^n \ . \tag{5.5}$$

For a specific degree n the last sum represents (apart from the constant G/R^2) the weighted sum of mass points, which are located on the same radius vector; the weights are given by the individual depths. Denoting this sum by $\bar{\bar{\mu}}_i^{(n)}$,

$$\bar{\bar{\mu}}_i^{(n)} := \sum_{l=1}^{L} \bar{\bar{\mu}}_{il} \alpha_l^n \quad , \tag{5.6}$$

the gravity anomaly degree variances c_n can be represented in a similar form as in equation (3.13)',

$$c_n = \frac{(n-1)^2}{2n+1} \bar{\bar{\mu}}^{(n)T} A_n \bar{\bar{\mu}}^{(n)} \quad . \tag{5.7}$$

All degree variances (for arbitrarily high n) are theoretically affected by all point masses. However, due to the weight factors α_l^n, deep point masses have practically no significant contribution to high degree variances. Needless to say, the gravity anomaly variance equals the sum of all degree variances.

ACKNOWLEDGEMENTS

The author would like to thank the colleagues at the Department of Geodetic Science and Surveying of the Ohio State University in Columbus, Ohio, and to Dr. K.-P. Schwarz, Division of Surveying Engineering, University of Calgary, for clarifying discussions and instructive comments. Computer time has been made available by the Instruction and Research Computer Center of the Ohio State University and the EDV-Zentrum Graz.

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