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RELATIVE NUMERICAL DISPERSION OF SHALLOW WATER GRAVITY WAVES CA--ETC(U)
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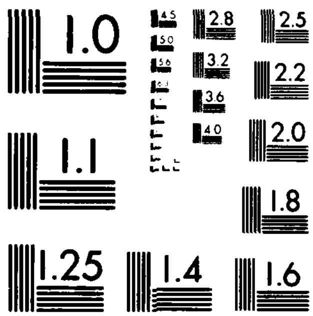

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Naval Ocean Research and Development Activity
NSTL Station, Mississippi 39529

[illegible] Numerical Dispersion of Shallow Water Gravity Waves Caused by Three Time Differencing Techniques

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ABSTRACT

Using analytical and numerical techniques we compare the effect of three different time-differencing methods on the propagation of shallow-water gravity waves. We compare the numerical dispersion of both phase and group velocities caused by a centered-in-time explicit formulation, a centered-in-time implicit method, and a trapezoidal-in-time implicit treatment. All three techniques slow the waves relative to their theoretical phase and group velocities. For each method, the slowing increases with decreasing spatial resolution and with decreasing temporal resolution. For $\Delta t \sqrt{gH}/d \leq 1$, the slowing increases as we switch from explicit, to trapezoidal implicit, to centered implicit methods. In this formula, $\sqrt{gH}$ = theoretical shallow water gravity wave speed, Δt = time increment, and d = smallest space increment between like variable grid points. For $\Delta t \sqrt{gH}/d > 1$, the trapezoidal implicit scheme again outperforms the centered implicit scheme.

ACKNOWLEDGEMENTS

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I. Introduction

Implicit time differencing in a finite difference approach to an initial value problem indicates that the solution for a given variable at a given time level requires knowledge of variables at the same time level in the forcing functions. In an explicit technique, only knowledge of variables at previous time levels is required. This study discusses a semi-implicit scheme for the equations of motion that treats gravity waves implicitly, but terms for other physical processes explicitly. This technique numerically retards the phase speed of the shallow-water gravity waves. This slowing then allows the use of large time-steps, which would violate the Courant limitation affecting explicit schemes (Mesinger and Arakawa, 1976).

Kwizak and Robert (1971) pioneered the use of the semi-implicit technique in an atmospheric grid point model. O'Brien and Hurlburt (1972) and Hamilton and Rattray (1978), among others, applied this method in coastal upwelling problems. Hamilton (1976) and Wang and Kravitz (1980) used this form of time differencing in applications to estuarine circulation. All these applications include possible effects of gravity waves, but only Wang and Kravitz (1980) mentioned any effect of time-differencing induced numerical dispersion on their results. They note the retarding effect of the semi-implicit scheme by stating that it is "apparently the result of damping of surface gravity waves by the semi-implicit method." They further discuss the apparent limitations of the method in transient conditions when seiching becomes considerable. One can generalize the limitations of the semi-implicit approach to any situation where shallow water gravity waves are significant, including ocean forecasting in coastal areas where transient forcing due to winds or tsunamis may arise.

Following the methods of Mesinger and Arakawa (1976) and Grotjahn and O'Brien (1976), we demonstrate the actual effect of the semi-implicit technique on the phase and group velocities of shallow-water gravity waves. We compare the analytic results with a centered space and time explicit technique, the centered space and time implicit method of Kwizak and Robert (1971) and the centered in space trapezoidal in time method used by Wang and Kravitz (1980). We include the additional slowing effects of the space differencing to make our results as comparable as possible to numerical simulation applications. For clarity in the analysis, we shall ignore all terms but those generating shallow-water gravity waves.

2. Linearized Shallow Water Gravity Wave Equations

The one-dimensional, linearized, shallow-water gravity wave equations may be written

$$u_t = -g\eta_x \tag{2.1}$$

$$\eta_t = -Hu_x \tag{2.2}$$

where u = horizontal velocity, g = gravitational acceleration, η = surface height deviation, and H = total depth; subscripts indicate differentiation by the subscripted variable.

We assume wave form solutions

$$u(x,t) = \mathrm{Re}[\hat{u}e^{i(kx-\omega t)}] \tag{2.3}$$

$$\eta(x,t) = \mathrm{Re}[\hat{\eta}e^{i(kx-\omega t)}] \tag{2.4}$$

with k = radian wave number and ω = radian wave frequency. We then substitute (2.3) and (2.4) into (2.1) and (2.2) treating H as a constant. These solutions satisfy (2.1) and (2.2) for the frequency

$$\omega = \pm k\sqrt{gH}\ . \tag{2.5}$$

We calculate phase velocity, C, from

$$C = \frac{\omega}{k} = \pm\sqrt{gH} \tag{2.6}$$

and group velocity, C_g, from

$$C_g = \frac{d\omega}{dk} = \pm \sqrt{gH} \ . \tag{2.7}$$

These results tell us that the solutions to (2.1) and (2.2) are nondispersive waves with equal phase and group speeds that depend only on gravitational acceleration and water depth. Recall that the group speed represents the energy propagation speed.

We will demonstrate that solutions to (2.1) and (2.2) by finite difference techniques introduce numerical dispersion into the solutions and that this dispersion is highly technique dependent.

3. Explicit Treatment, Centered in Space and Time

We write the explicit, centered in space and time, finite difference form of (2.1) and (2.2) as

$$\frac{u_{jd^*}^{(n+1)\Delta t} - u_{jd^*}^{(n-1)\Delta t}}{2\Delta t} = -g\left[\frac{\eta_{(j+1)d^*}^{n\Delta t} - \eta_{(j-1)d^*}^{n\Delta t}}{2d^*}\right] \tag{3.1}$$

$$\frac{\eta_{jd^*}^{(n+1)\Delta t} - \eta_{jd^*}^{(n-1)\Delta t}}{2\Delta t} = -H\left[\frac{u_{(j+1)d^*}^{n\Delta t} - u_{(j-1)d^*}^{n\Delta t}}{2d^*}\right] , \tag{3.2}$$

where Δt = finite time increment, d^* = finite space increment between grid points*, superscripts represent position in time, and subscripts indicate position in space.

We now write solutions (2.3) and (2.4) as

$$u(jd^*, n\Delta t) = \mathrm{Re}[\hat{u}e^{i(kjd^* - \omega n\Delta t)}] \tag{3.3}$$

$$\eta(jd^*, n\Delta t) = \mathrm{Re}[\hat{\eta}e^{i(kjd^* - \omega n\Delta t)}] . \tag{3.4}$$

Substituting (3.3), (3.4) into (3.1), (3.2) yields solutions if

$$\omega^* = \pm \frac{1}{\Delta t} \sin^{-1} \left[\frac{\Delta t}{d^*} \sqrt{gH} \ \sin(kd^*)\right] \tag{3.5}$$

*Note $d^* = \frac{1}{2}d$ where d is distance between like labeled grid points on Arakawa grid c. (Mesinger and Arakawa, 1976).

This result then implies a dispersive wave with phase speed, C*, and group speed, C_g^*,

$$C^* = \frac{\omega}{k} = \pm \frac{1}{k\Delta t} \sin^{-1}[\frac{\Delta t}{d^*} \sqrt{gH} \sin(kd^*)] \qquad (3.6)$$

$$C_g^* = \frac{d\omega}{dk} = \pm \sqrt{gH} \cos(kd^*) [1 - (\frac{\Delta t}{d^*} \sqrt{gH})^2 \sin^2(kd^*)]^{-\frac{1}{2}} . \qquad (3.7)$$

We now introduce the effects of a time-centered implicit formulation.

4. Implicit Treatment, Centered in Space and Time

Kwizak and Robert (1971) use the implicit, centered in space and time, finite difference form of (2.1) and (2.2), which we write,

$$\frac{u_{jd^*}^{(n+1)\Delta t} - u_{jd^*}^{(n-1)\Delta t}}{2\Delta t} = \frac{-g}{2} [\frac{\eta_{(j+1)d^*}^{(n-1)\Delta t} - \eta_{(j-1)d^*}^{(n-1)\Delta t}}{2d^*} + \frac{\eta_{(j+1)d^*}^{(n+1)\Delta t} - \eta_{(j-1)d^*}^{(n+1)\Delta t}}{2d^*}] \qquad (4.1)$$

$$\frac{\eta_{jd^*}^{(n+1)\Delta t} - \eta_{jd^*}^{(n-1)\Delta t}}{2\Delta t} = \frac{-H}{2} [\frac{u_{(j+1)d^*}^{(n-1)\Delta t} - u_{(j-1)d^*}^{(n-1)\Delta t}}{2d^*} + \frac{u_{(j+1)d^*}^{(n+1)\Delta t} - u_{(j-1)d^*}^{(n+1)\Delta t}}{2d^*}] \qquad (4.2)$$

Substituting the finite difference solutions (3.3) and (3.4) into (4.1) and (4.2) now yields a different numerically generated dispersion. The radian frequency now involves the inverse tangent;

$$\omega^* = \pm \frac{1}{\Delta t} \tan^{-1} [\frac{\Delta t}{d^*} \sqrt{gH} \sin(kd^*)]. \qquad (4.3)$$

The phase and group speeds then are

$$C^* = \pm \frac{1}{k\Delta t} \tan^{-1} [\frac{\Delta t}{d^*} \sqrt{gH} \sin(kd^*)] \qquad (4.4)$$

and

$$C_g^* = \frac{d\omega}{dk} = \pm \sqrt{gH} \cos(kd^*)[1 + (\frac{\Delta t}{d^*} \sqrt{gH})^2 \sin^2(kd^*)]^{-1}. \qquad (4.5)$$

Note that no information at the nth time level appears above. The significance of this becomes apparent as we look at a trapezoidal implicit scheme similar to that used by Wang and Kravitz (1980).

5. Implicit Treatment, Centered in Space, Trapezoidal in Time

We write the implicit, centered in space, trapezoidal in time, finite difference form of (2.1) and (2.2) as

$$\frac{u_{jd*}^{(n+1)\Delta t} - u_{jd*}^{n\Delta t}}{\Delta t} = \frac{-g}{2}\left[\frac{\eta_{(j+1)d*}^{n\Delta t} - \eta_{(j-1)d*}^{n\Delta t}}{2d*} + \frac{\eta_{(j+1)d*}^{(n+1)\Delta t} - \eta_{(j-1)d*}^{(n+1)\Delta t}}{2d*}\right] \quad (5.1)$$

$$\frac{\eta_{jd*}^{(n+1)\Delta t} - \eta_{jd*}^{n\Delta t}}{\Delta t} = \frac{-H}{2}\left[\frac{u_{(j+1)d*}^{n\Delta t} - u_{(j-1)d*}^{n\Delta t}}{2d*} + \frac{u_{(j+1)d*}^{(n+1)\Delta t} - u_{(j-1)d*}^{(n+1)\Delta t}}{2d*}\right] \quad (5.2)$$

Substituting solutions (3.3) and (3.4) into these equations yields a radian frequency

$$\omega^* = \pm \frac{2}{\Delta t} \tan^{-1} \left[\frac{\Delta t}{2d*} \sqrt{gH} \sin(kd^*)\right] . \quad (5.3)$$

The phase and group speeds are then

$$C^* = \frac{\omega^*}{k^*} = \pm \frac{2}{k\Delta t} \tan^{-1} \left[\frac{\Delta t}{2d*} \sqrt{gH} \sin(kd^*)\right] \quad (5.4)$$

and

$$C_g^* = \frac{d\omega}{dk*} = \pm \sqrt{gH} \cos(kd^*) \left[1 + \left(\frac{\Delta t}{2d*} \sqrt{gH}\right)^2 \sin^2(kd^*)\right]^{-1} . \quad (5.5)$$

6. Model Runs

We performed numerical experiments to test the application of the analyses in sections 3 and 4 to models with additional physical processes covering limited domains. We used a barotropic, free-surface, explicit, primitive equation model (PL1FBE) to compare with the time-centered explicit analysis. We used a barotropic, free-surface, time-centered, implicit, primitive equation model (PL1FBI) to compare with the time-centered implicit analysis. Hurlburt and Thompson (1980) describe this second model more fully.

Both models computed on 1000 x 1000 x 4 km basins, with d = 20 km. For each timestep we investigated the effect of varying the wave resolution by initializing the basin with 0.5, 1, 2.5, 5 and 10 waves in the basin. We ran both models for $\Delta t = 30$ sec and ran PL1FBI additionally at $\Delta t = 178.6$ sec and 505.1 sec.

7. Discussion

The one-dimensional Courant-Friedrich-Lewy (C.F.L.) stability criterion (neglecting advection) requires that the gravity wave speed of the fluid remain less than that which would traverse the grid mesh distance d in timestep Δt, that is $\sqrt{gH}\,\frac{\Delta t}{d} \leq 1$. We define the Courant number for a one-dimensional mesh as

$$C_o(1) = \sqrt{gH}\,\frac{\Delta t}{d} \qquad (7.1)$$

and for two dimensions

$$C_o(2) = \sqrt{2gH}\,\frac{\Delta t}{d} \qquad (7.2)$$

This number gives the amount by which we either exceed or fall within the theoretical C.F.L. criterion. Table 1 summarizes the relevant phase and group speed formula in terms of $C_o(2)$ and kd*.

Figures 1 and 2 stand as checks on the analysis. Figure 1 compares the analysis in section 5 with values taken from a plot resulting from a similar analysis in two dimensions performed by Mesinger and Arakawa (1976). Figure 2 compares the analysis in sections 3 and 4 with results calculated from the explicit and centered in time implicit models discussed in section 6. First, note the agreement between the different results, which indicates the usefulness of the analysis. Second, note the effects of spatial and temporal resolution.

Figure 3 summarizes these effects. As expected for all schemes, gravity waves slow as wave resolution by the grid spacing, d, decreases. For the implicit schemes, gravity waves slow as Courant number increases. At sub-C.F.L. conditions, the explicit scheme reproduces theoretical phase speed better than the trapezoidal implicit scheme which, in turn, reproduces wave speed better than

TABLE 1

Method	$C^*/\sqrt{gH}$	$C_g^*/\sqrt{gH}$
Explicit Centered Space Centered Time	$\frac{\sqrt{2}}{2}[(kd^*)C_o(2)]^{-1}\sin^{-1}[\sqrt{2}\,C_o(2)\sin(kd^*)]$	$\cos(kd^*)\{1-[\sqrt{2}\,C_o(2)\sin(kd^*)]^2\}^{-1/2}$
Implicit Centered Space Centered Time	$\frac{\sqrt{2}}{2}[(kd^*)C_o(2)]^{-1}\tan^{-1}[\sqrt{2}\,C_o(2)\sin(kd^*)]$	$\cos(kd^*)\{1+[\sqrt{2}\,C_o(2)\sin(kd^*)]^2\}^{-1}$
Implicit Centered Space Trapezoidal Time	$\sqrt{2}[(kd^*)C_o(2)]^{-1}\tan^{-1}[\frac{\sqrt{2}}{2}C_o(2)\sin(kd^*)]$	$\cos(kd^*)\{1+[\frac{\sqrt{2}}{2}C_o(2)\sin(kd^*)]^2\}^{-1}$

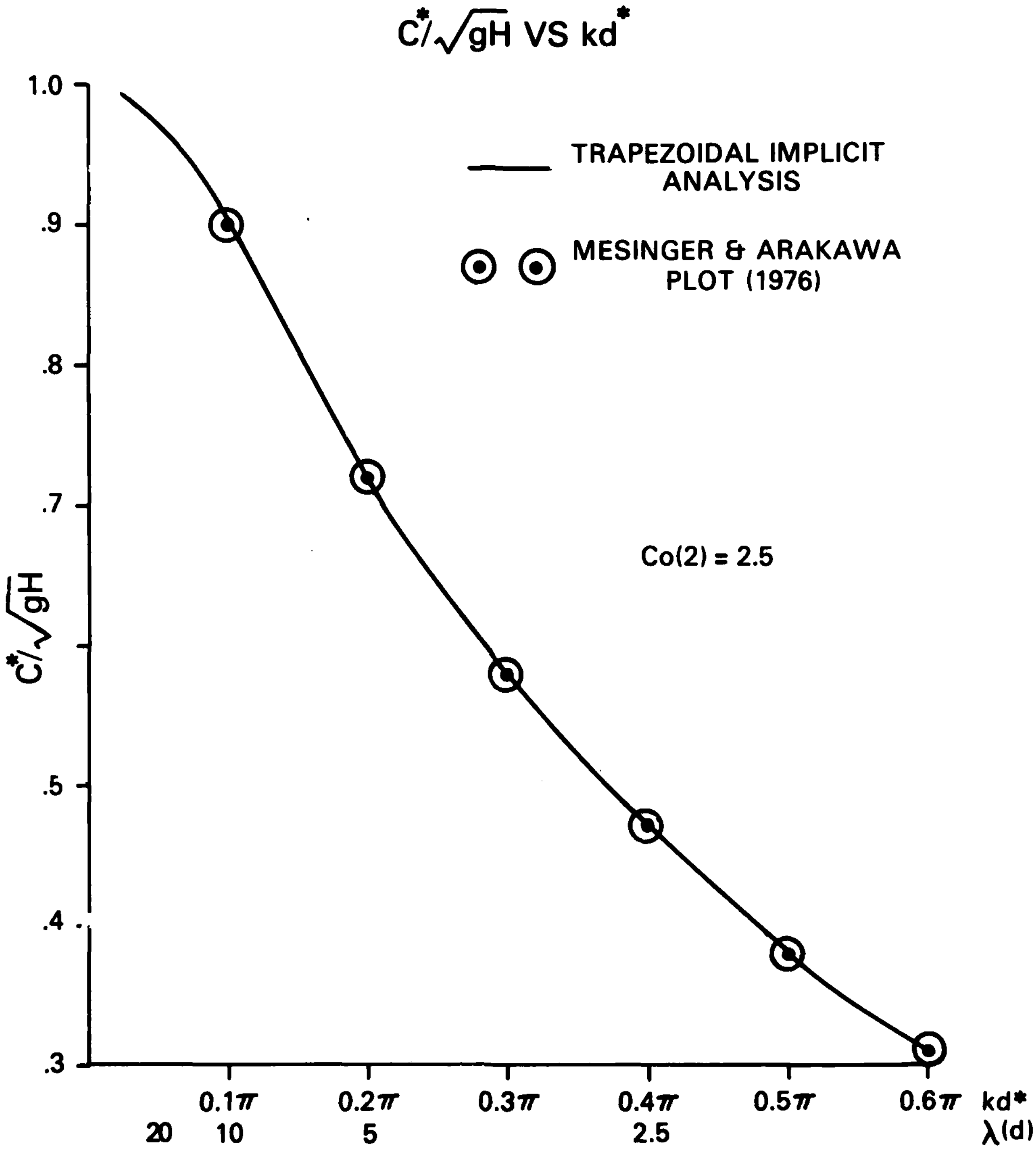

Figure 1. Ratio of numerical to theoretical shallow water gravity wave, phase velocity, $c^*/\sqrt{gH}$, vs. kd^* from trapezoidal implicit analysis and from plot of similar two-dimensional analysis by Mesinger and Arakawa (1976). $\lambda = \frac{2\pi}{k}$ = wavelength

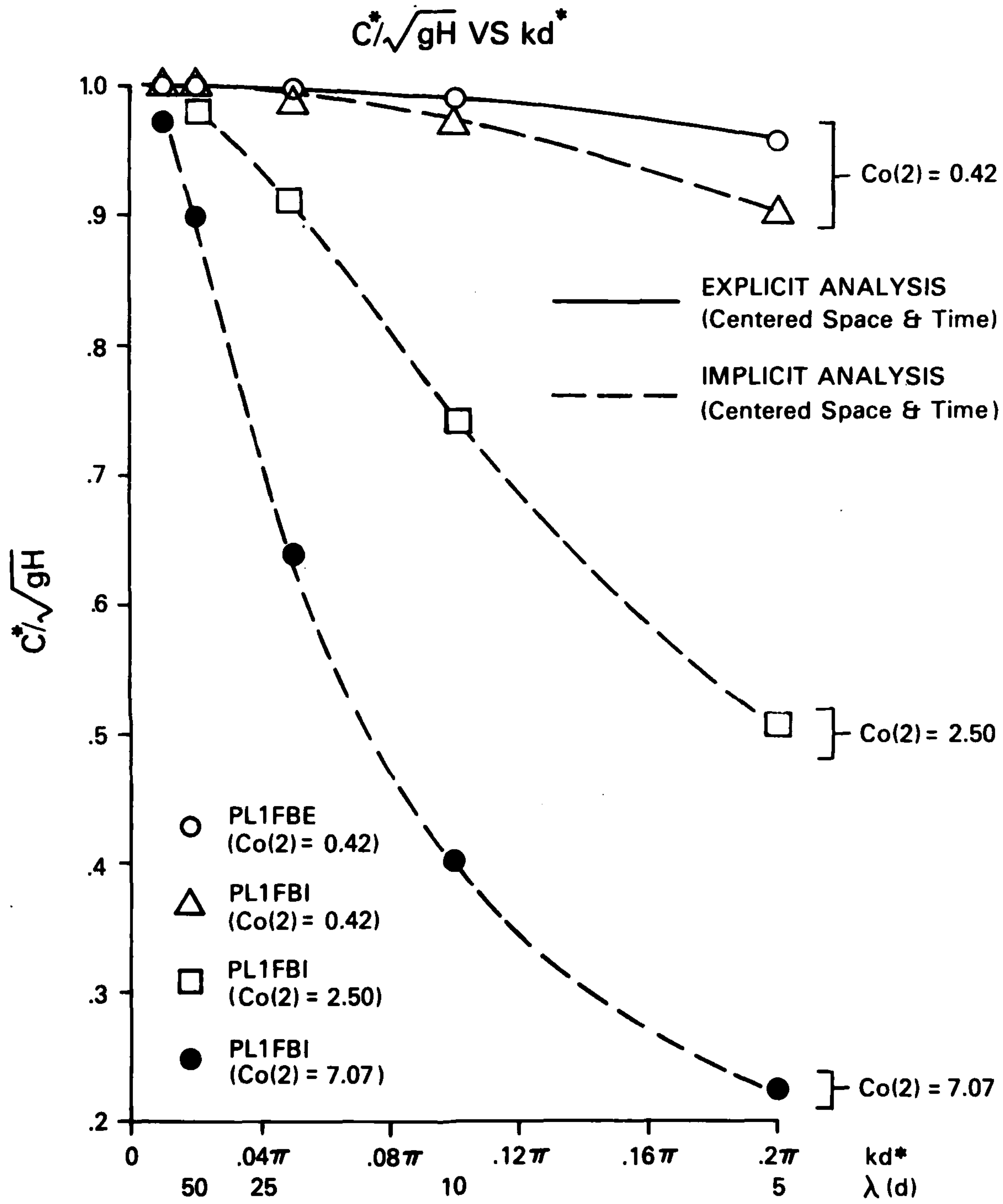

Figure 2. Same as Figure 1 for explicit and centered implicit analyses and for comparable explicit and implicit numerical model runs.

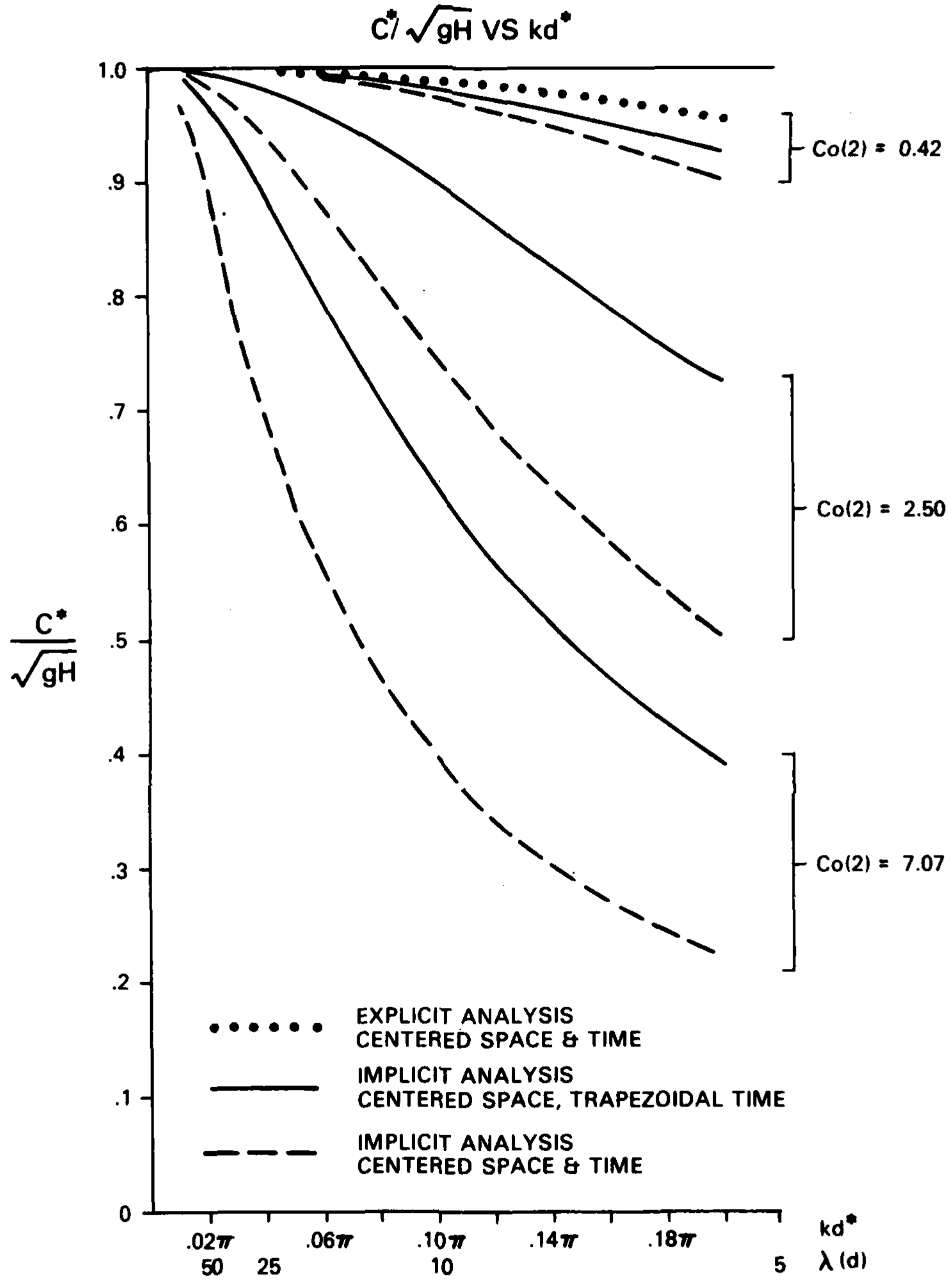

Figure 3. Same as Figure 1 for explicit, trapezoidal implicit, and centered implicit analyses at $C_0(2) = 0.4$, and for the implicit analyses, additionally for $C_0(2) = 7.1$.

the centered implicit method. At super-C.F.L. conditions the trapezoidal scheme continues to outperform the centered implicit scheme.

For a quantitative feel for these comparisons, consider a wave resolved by 10 d (a rule-of-thumb minimum for adequate spatial resolution). At $C_o(2) = 0.42$, the explicit scheme produces 99% of the phase speed, the trapezoidal implicit scheme 98%, and the centered implicit scheme 97.5%. Exceeding C.F.L. results in rapid deterioration of calculated phase speed. At $C_o(2) = 2.5$, the trapezoidal implicit scheme yields 90% of the theoretical speed while the centered scheme yields only 75%. At $C_o(2) = 7.07$, the trapezoidal method gives only 64% while the centered implicit scheme results in only 40% of the actual value.

For the group velocity, the statement given for the numerical effects on phase velocity also apply qualitatively to the group velocity. Quantitatively, the effect of decreased spatial and temporal resolution is more drastic (Fig. 4). For $C_o(2) = 0.42$, and 10d/wave, the explicit, trapezoidal implicit, and centered implicit schemes yield 97, 94, and 92% of theoretical group speed, respectively. For $C_o(2) = 2.5$, the trapezoidal and centered implicit methods give 73 and 43%, respectively. For $C_o(2) = 7.07$, the trapezoidal and centered schemes yield 28 and 9%, respectively.

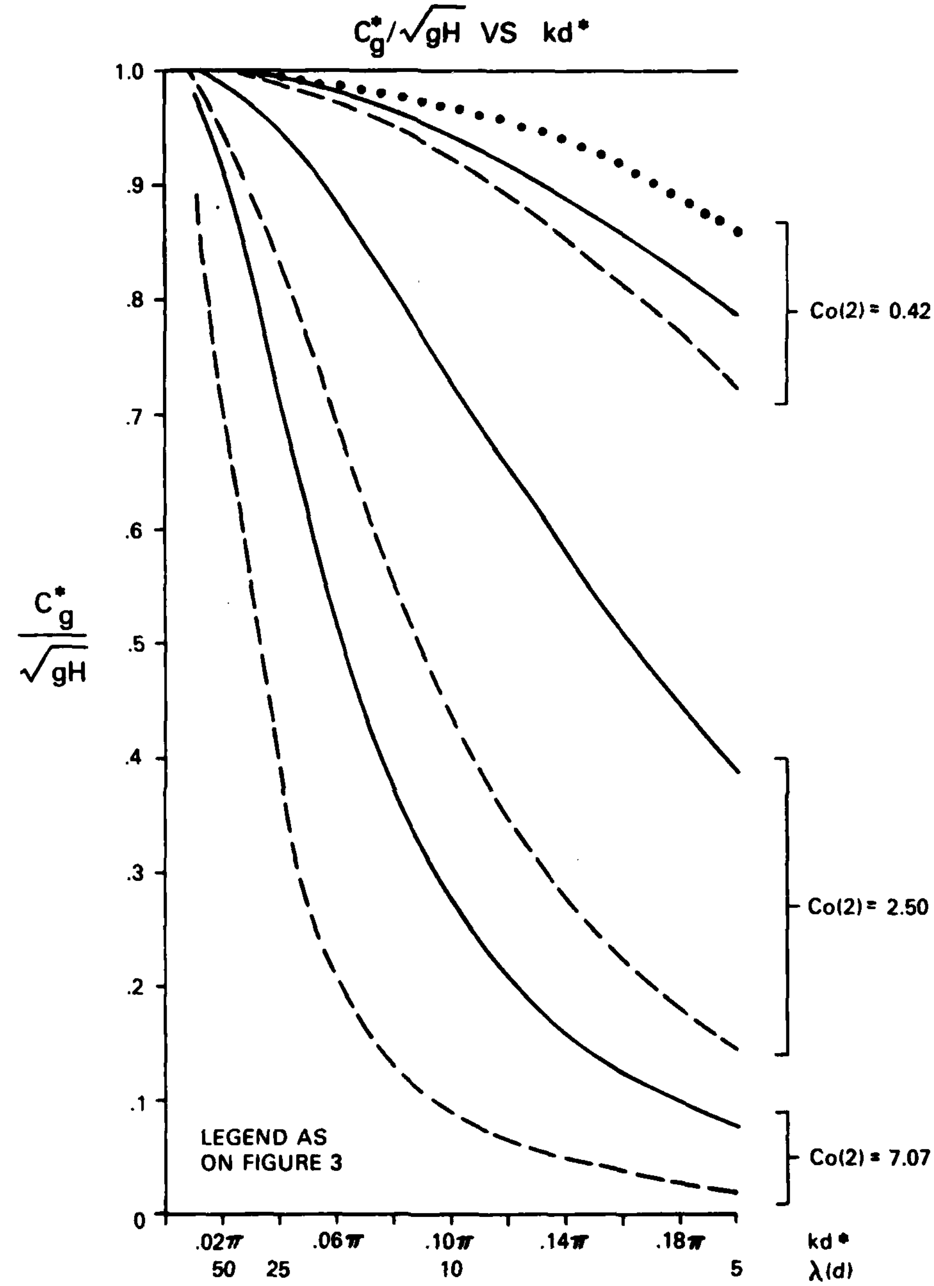

Figure 4. Same as Figure 3. except numerical group velocity, C_g^* is used instead of numerical phase velocity, C*.

8. Conclusions

Consider now the task of choosing a numerical scheme to simulate a situation where gravity waves may be important. This situation might include storm surge and tsunami predictions or just forecasts of transient, wind-driven circulations in a coastal region. How should we choose between the three schemes analyzed? If the horizontal resolution only minimally resolves the wave form and a sub-C.F.L. timestep is affordable, choose the explicit scheme. If we require $C_0(2) > 1$, perhaps for economic reasons, use the trapezoidal implicit scheme, but keep in mind its limitations. If spatial resolution allows over 50d per wave and super-C.F.L. timestep is required, either of the implicit techniques can be used.

In summary, this report demonstrates the importance of and relative ease of analysis in determining the effect of time-differencing schemes on shallow water gravity wave simulations.

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