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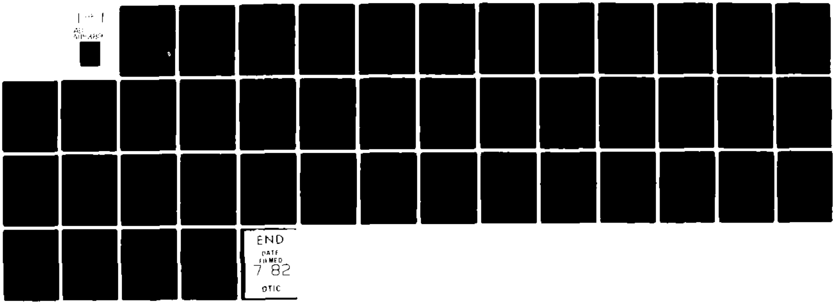

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AN OVERVIEW OF LANCHASTER-TYPE COMBAT MODELS FOR MODERN WARFARE SCENARIOS

by

L.K. Ekchian

ABSTRACT

This paper examines the suitability of Lanchaster-type combat models for analysis of modern warfare. Initially a brief synopsis of basic Lanchaster theory is provided, followed by a discussion of the shortcomings of the original Lanchaster models vis-à-vis real-world warfare setting, e.g. assumption of homogeneity of forces, separation distance between opposing forces, disregard of C^3 effectiveness, lack of reinforcements. Recent papers propound models that account for some of these shortcomings. <u>In particular this paper examines in detail the optimal target selection problem</u>. It encompasses all the aforementioned issues.

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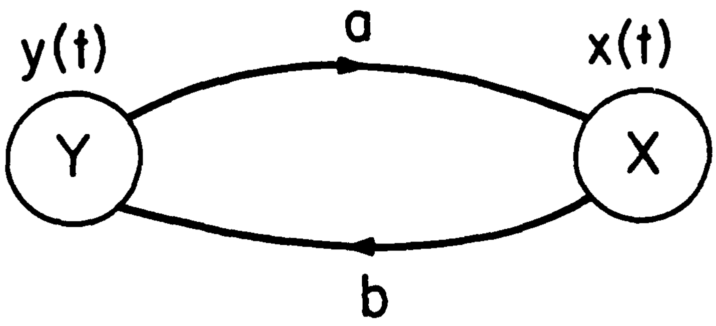
1. Introduction

In 1914 F.W. Lanchaster, in an effort to provide insight into the dynamics of combat under "modern warfare" conditions and to justify the principle of concentration of force, formulated a set of mathematical equations that model combat between two opposing homogeneous forces. "Modern warfare" conditions entail combat with sophisticated weapons capable of accurately delivering payloads to distant targets. Hence, geographically dispersed forces can focus their fire power on a single target. This is in contrast to "ancient warfare" conditions that primarily involves hand-to-hand combat.

Lanchaster proposed two distinct models. The first is entitled the "square"-law attrition process expressed mathematically as: (For definition of notation see Figure 1.1)

$$\frac{dx}{dt} = -ay$$
$$\frac{dy}{dt} = -bx \qquad (1.1)$$

The parameters "a" and "b" denote the attrition-rate coefficients. The structure of "a" and "b" depends on the particular combat scenario being modelled. For instance, in Appendix A an attrition model is developed for the "salvo fire" engagement problem [6]. It accounts for the kill probabilities of both sides (P_x and P_y) and the time interval between the respective salvos (δt_x and δt_y). The reader is referred to Appendix A for the details of the derivation of the following set of differential equations



x(t) = number of units in X at t

y(t) = number of units in Y at t

a = attrition-rate on X

b = attrition-rate on Y

Fig. 1.1 Lanchaster Combat

$$\frac{dx}{dt} = -\frac{x}{\delta t_y}\left[1 - (1 - \frac{P_y}{x})^y\right]$$
$$\frac{dy}{dt} = -\frac{y}{\delta t_x}\left[1 - (1 - \frac{P_x}{y})^x\right] \tag{1.2}$$

When $P_x << 1$ and $P_y << 1$ (i.e., the single shot probabilities of kill are small), expansion of (1.2) into a Taylor series yields

$$\frac{dx(t)}{dt} = -\frac{P_y}{\delta t_y}\, y(t)$$
$$\frac{dy(t)}{dt} = -\frac{P_x}{\delta t_x}\, x(t) \tag{1.3}$$

Comparing (1.1) to (1.3) one notes that the attrition coefficients $a = \frac{P_y}{\delta t_y}$ and $b = \frac{P_x}{\delta t_x}$. It is important to reassert at this point that the format of the attribtion coefficient is highly scenario dependent. Furthermore, this yields a deterministic approximation to what is really a stochastic process. (Chapter 2 of this report contains further discussion of this crucial subject.)

Integration of (1.1) reveals the reason why this attrition process was coined "square" attrition:

$$b(x_o^2 - x^2) = a(y_0^2 - y^2)$$
or
$$bx^2 - ay^2 = \text{constant} \tag{1.4}$$

A key assumption in the "square" law is that all the units of the defender must be within weapon range of all the units of the attacker, and vice versa. An interesting question that immediately arises is how much force should each side dedicate to the conflict. Consider the simple example in Appendix B. Clearly there is a significant advantage (fewer casualties) for Y to concentrate all its forces. Based on equations (1.1) and (1.4) one can assert that effective tactical use of concentration of forces may adequately counter balance superiority in the opponents weapon system.

Furthermore, note from equation (1.1) that

$$\frac{-\left[\frac{dx}{dt}\right]}{y} = a \qquad (1.5)$$

i.e., the rate of destruction of enemy forces produced by each element of Y is constant. This phenomenon is prevalent in situations where each unit is aware of the exact location of remaining enemy units, so that when a target is destroyed, fire may immediately be shifted to a new target (i.e., excellent information provided by C^3 system)*. This type of fire will hereinafter be referred to as "aimed" fire. The issue of C^3 support is critical and will be focused upon in Chapter 2, in particular how well Lanchaster-type models account for the degree of C^3 effectiveness.

The other model for target attrition formulated by Lanchaster is referred to as the "linear" attrition law expressed mathematically as: (For definition of notation see Figure 1.1)

$$\frac{dx}{dt} = -\ axy$$
$$\frac{dy}{dt} = -\ bxy \qquad (1.6)$$

*The reader is referred to Wohl [15] for a discussion on tactical C^3 (i.e., Command, Control and Communication.)

where integration yields Lanchaster's "linear" law:

$$b(x_0 - x) = a(y_0 - y) \tag{1.7}$$

Again considering the simple example presented in Appendix B we see that in the "linear" case there is no advantage to concentration of forces. Taylor [7] points out that such an attrition process arises under two general circumstances (1) fire is uniformly distributed over a constant target area or (2) the meantime of target acquisition is much larger than target destruction time and is inversely proportional to target density. It's assumed that units are only aware of the general position of enemy forces and are not informed of the consequences of their fire (i.e., supported by a highly ineffective C^3 system). Hereinafter, this type of fire will be referred to as "area" fire. An example, that exhibits such behavior is saturation bombing conducted by high-altitude bomber sorties. This is in marked contrast to an air-launched missile attack from low-flying aircraft against unconcealed surface targets. (The latter falls into the class of "aimed" fire combat.)

Moreover, from (1.6) we know that

$$\frac{-\left[\frac{dx}{dt}\right]}{y} = -ax \tag{1.8}$$

i.e., the rate of destruction of enemy forces produced by each unit of Y is dependent linearly on x(t) (and hence decreases monotonically with time.) In our "area" attrition fire example, one would expect that the incremental damage inflicted by successive high-altitude bombing raids would gradually decrease.

In this chapter a synopsis of the original Lanchester combat theory has been presented. In Chapter 2 shortcomings of the simple "square" and "linear" attrition models are delineated, in particular focusing on the "structure" of the attrition-rate coefficients and their intrinsic relationship to C^3 effectiveness. Wohl [15] notes that "a command and control system is a technological, procedural, and organizational extension of the sensing, processing, and communication capabilities of the military commanders whose decisions it supports". Hence, accounting for this relationship is crucial in building realistic models of warfare. Chapter 3 contains an overview and critique of a number of recent papers that use Lanchester's models as a rudimentary base for building more complex models that overcome some of the shortcomings. Particular attention is given to the optimal target selection problem discussed in [5], [7] and [14]. Lastly in Chapter 4 a summary and suggestions for further research are presented.

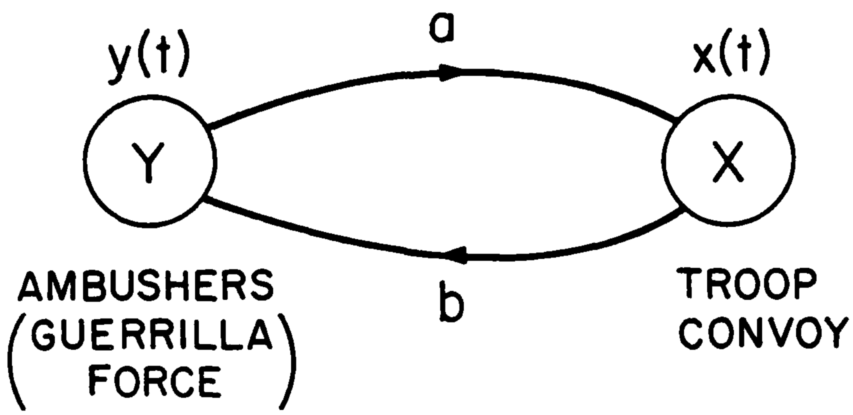
2. Shortcomings of Original Lanchaster Models

The original Lanchaster models provide a very simplified view of modern warfare-neglecting to account for many critical facets. In this chapter we will outline several shortcomings and set the stage for the last two chapters which discuss attempts by various researchers to bolster these simple models and present our suggestions for future research.

Symmetric Attrition Fire

In Chapter 1 a brief review of Lanchaster's "square" and "linear" attrition combat model was presented. However, most combat scenarios do not fall neatly into the "square" or "linear" framework. For instance, the guerilla ambush example depicted in Figure 2.1 (and discussed by Deitchman [3]) exhibits both kinds of attrition. For the sake of argument consider a guerilla force (heavily camouflage) ambushing a convoy of troops and support material. The defender would clearly suffer a "square" attrition loss from the "aimed" fire of the guerilla forces, who have clear sight of the convoy. Conversely, the guerilla casualties would best be modelled by a "linear" attrition process since they have the advantage of being heavily camouflaged).

The model in Figure 2.1, however, fails to account for the following crucial element: Over the course of the battle the troops will gradually ascertain the exact positions of the guerilla forces. Thus the retaliatory fire of the troops against the guerillas will become more accurate, i.e, better presented by "aimed" Lanchaster fire models. Although simple the following model captures the essence of the guerilla ambush problem:



$$\frac{dx}{dt} = -ay \quad \text{"Aimed" fire}$$

$$\frac{dy}{dt} = -bxy \quad \text{"Area" fire}$$

Fig. 2.1 Guerrilla Ambush of Troop Convoy

$$\frac{dx}{dt} = -\, ay \quad \Big\} \text{"Aimed" Fire}$$

$$\frac{dy}{dt} = \underbrace{-\, b_1 xy e^{-\alpha t}}_{\text{"Area" Fire}} \; \underbrace{-\, b_2 x(1-e^{-\alpha t})}_{\text{"Aimed" Fire}} \tag{2.1}$$

Note that initially (t=0) x's (troop's) retaliatory fire upon Y is "area" and as $t \to \infty$ it is "aimed". α is a parameter that reflects how fast the troops ascertain the guerilla locations and is thus tied in with the troop C^3 effectiveness.

Often circumstances that are favorable for the ambusher (who are fewer in number) at the outset can turn unfavorable if they engage in a prolonged conflict. Hence, they must determine the "best" time to retreat after having inflicted considerable casualties.

Disregard of C^3 Effectiveness

Throughout this report we have made references to the issue of C^3 effectiveness. In Chapter 1 implications were made that the attrition coefficients characterized the effectiveness of the weapon system, which is a function of not only fire power but also accuracy of delivery (the latter depending on accurate information on enemy positions).

The reader is reminded that the enemy units can have high maneuverability and acceleration. Moreover, surveillance tasks can be further complicated by enemy counter - C^3 activities, like jamming "radars, launching decoys, etc.

From equation (1.4) one can note that in the "aimed" fire case, the attrition inflicted is proportional to the square of the force levels but only linear with the parameter that characterizes the weapon system effectiveness. A question that arises is whether the "square" attrition model is realistic given that force is much more heavily emphasized than weapon effectiveness. However, closer examination reveals that the casualities inflicted upon the opponents is really related to the square of the effectivness. Consider the following argument. From (1.3) we know that

$$\frac{dx}{dt} = - \frac{P_y}{\delta t_y} \; y \tag{2.2}$$

for the salvo fire scenario described in Chapter 1 where the two sides are comprised of artillery units pounding the opponent with shells. Assume the following simple structure for P_y. Let r_y denote the kill radius, i.e., any target within the circle of radius r_y and centered at the point of impact of the particular Y shell is destroyed with probability one. Let R_y denote a measure of the uncertainty in the exact location of the target the fire is directed at , i.e., <u>measurement error</u>. Hence the probability of kill P_y (assuming $r_y < R_y$) can be expressed as

$$P_y = \frac{\pi \, r_y^2}{\pi \, R_y^2} = \frac{r_y^2}{R_y^2} \tag{2.3}$$

Substituting (2.3) into (1.4) yields ($a = \frac{P_y}{\delta t_y}$, $b = \frac{P_x}{\delta t_x}$)

$$\left[\frac{r_x^2/R_x^2}{\delta t_x}\right] x^2 - \left[\frac{r_y^2/R_y^2}{\delta t_y}\right] y^2 = \text{constant} \tag{2.4}$$

Note that the measurement error, which is strongly coupled with weapon effectiveness, is squared in equation (2.4). Moreover, measurement error is a direct measure of the performance of the C^3 system (in particular surveillance system).

This example has served to illustrate the significance of effective surveillance (tracking capability). Another interesting facet is the significance of target identification capability. Consider the following scenario: X has excellent C^3 support and is never fooled by enemy counter - C^3 activities (like the use of decoys). Hence, when X fires it is guaranteed that it is firing at an actual Y target. On the other hand assume Y has poor C^3 support and can only distinguish between real X targets and decoys (or clutter) 50% percent of the time. Then a reasonable model would be the following modified version of (1.3).

$$\frac{dx}{dt} = -\frac{P_y}{\delta t_y} y \; (.5)$$

$$\frac{dy}{dt} = -\frac{P_x}{\delta t_x} x \qquad (2.5)$$

and the corresponding casualty expression.

$$2\left[\frac{P_x}{\delta t_x}\right] x^2 - \left[\frac{P_y}{\delta t_y}\right] y^2 = \text{constant} \qquad (2.6)$$

Thus we can conclude that there is a strong relationship between C^3 and Lanchaster's models. The C^3 effectiveness in the particular combat environment should enter in determining the attrition-rate coefficients.

Homogeneity of Forces

A major assumption of Lanchaster's laws are that the two opposing forces are comprised of "identical" units (although the rates of attrition may be different). However, in most realistic war scenarios forces are generally multi-component, e.g., comprised of infantry, armour, artillery, air power, etc. This fact immediately complicates the problem since side x must now decide how to allocate his fire and vice versa. Factors that would enter the allocation (target selection) procedure are: attrition fire rate from each type of enemy unit, number of each type of enemy unit (target), and significance associated with each type of enemy target that survive the battle. For example, consider the scenario where X represents the defenders of a strategic town (that eventually lose) and Y represents the attackers. Assume Y is comprised of infantry and armour. During the course of the combat armour poses a greater threat than infantry but at the end of combat infantry is more critical since the enemy needs soldiers to occupy the town and not tanks. Thus, intuitively one would surmise that <u>initially</u> more fire should be directed at tanks than troops and gradually more and more fire power would be diverted to the enemy infantry. We will see in the next chapter (in the discussion of Taylor [7]), that the optimal fire allocation policy for this problem is "bang-bang" [1], i.e. initally <u>all</u> the fire is directed at the armour and then abruptly shifted to the infantry (at a switching time determined by the optimal control procedure. The reader is referred to Chapter 3 of this report for an in depth discussion of this issue.)

<u>Disregard of Separation between Forces</u>

In most cases weapon lose their effectiveness with increasing distance from the target. Since most battle scenarios include moving targets, moving attacking units, or motion of FEBA (Forward Edge of Battle Area), we feel it is imperative to account for force separation. We shall see in the next chapter that this shortcoming is not only restricted to the original Lanchaster models, but is present in very recent papers like Kawara [5] and Wohl [14] that assume forward motion of attacking forces. An exception is Taylor and Brown [11], which considers a constant speed assault on a static defensive position. The authors account for the gradual shrinking of the separation by making the attrition-rate coefficients a function of this distance. Further discussion is provided in Chapter 3 in the discussion of equation (3.21) which represents mathematically the attrition process of [11].

Disregard of Replacements or Withdrawls

Original Lanchaster theory assumes that all the units on both sides are all simultaneously committed to battle. In real battle situations commanders often choose to only allocate part of their forces and bring up replacements whenever necessary. Moreover, often troops faced by a hopeless situation may choose to withdraw to avert certain annihilation.

Battle Termination Conditions not Defined

Rarely do battles develop into a fight-to-the-end conflict. Generally either one side surrenders, withdraws or attains a well-defined objective, e.g. occupying a strategic position or cutting the opponents supply lines. The battle duration issue is critical and will be carefully examined in the next chapter.

Disregard Multiple Killing Hits

In Chapter 1 (and Appendix A) it was noted that if P_x and P_y (single shot probabilities) are small then the differential equations in (1.2) reduce to the Lanchaster "square" law attrition models (1.3). But what if the probabilities of kill are not small? Then one must account for the possibility that a target may be delivered "killing" hits emanating from more than one source. This is the case in some of today's battles with sophisticated weapon systems.

For example consider the surface-to-air war that occurs when a formation of enemy aircraft tries to attack positions defended by batteries of SAM (Surface to Air Missile) sites. Because of communication delays and bandwidth limitations, often more than one missile may be launched by different SAM sites at the same enemy aircraft.

Maugulis [6] presents modifications to classical Lanchaster models to account for the possibility of multiple killing bits.

Stochastic Effects are not Modelled

The original Lanchaster models assume a deterministic world which is obviously unrealistic. But they are popular because of simplicity and computational tractability. However, recently researchers have been devoting increasing effort to constructing stochastic Lanchaster models of warfare. (Taylor [10] provides a list of references). Wohl and Gootkind [14] construct models of Lanchaster combat to account for multi-component forces, second echelon reinforcement activity, close air support and to permit the inclusion of uncertainty.

In this report we shall not address stochastic versions of Lanchester-type combat and instead concentrate on discussing various salient facets of the deterministic models.

3. More Sophisticated Lanchaster-Type Models

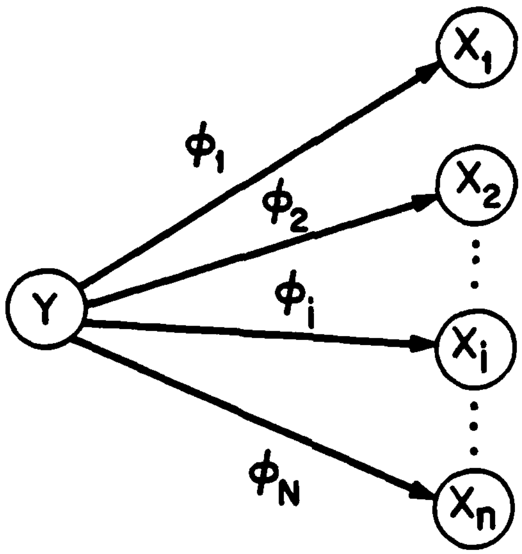
In this chapter a number of recent papers are examined. They propose models that account for a number of the shortcomings of the original Lanchaster models that were outlined in Chapter 2. In particular the optimal target selection problem addressed in [5], [7], and [14] provides the focus of the discussion of this chapter.

First let us consider Taylor [7] which presents a study of optimal distribution of fire over different enemy target types (i.e., the homogeneity assumption is relaxed.) A sequence of simple examples are investigated which examine the structure of the optimal fire allocation policy for the following factors: objectives of the combatants, termination conditions of the conflict, number of target types, special cases of time dependent attrition-rate, and types of ("linear" or "square") attrition processes.

In particular he studies four different scenarios. The first is depicted in Figure 3.1 (along with the definition of the notation that will be used in this section). Note that the attrition processes are "square" and the battle duration is prescribed (i.e., T is fixed a priori). The two main results are:

(1) All the fire is always concentrated on one target type (i.e. bang-bang). This is intuitively reassuring given the argument in Chapter 1 calling for force concentration in the case of "square" attrition fire.

(2) The fire allocation (and target prioritization) is not dependent upon the force levels but on the following parameters.



Problem: $\max_{\phi_i(t)} \{\nu y(T) - \sum_{i=1}^{n} \omega_i x_i(T)\}$

s.t. $\frac{dx_i}{dt} = -\phi_i a_i y$

$\frac{dy}{dt} = - \sum_{i=1}^{n} b_i x_i$

$x_i, y \geq 0, \phi_i \geq 0 \quad \forall i \in \{1,2,...,n\}$ <u>and</u> $\sum_{i=1}^{n} \phi_i = 1$

Fig. 3.1 Prescribed Duration Battle with Several Target Types (Taylor [7])

ν - weight on surviving Y units

ω_i - weights on surviving X_i units in the terminal cost function.

b_i - attrition-rate coefficient of fire from X_i to Y

a_i - attrition-rate coefficient of fire from Y to X_i

As one would expect, "close" to the terminal time the fire will be directed at the target which will provide Y with the greatest effect on the outcome per unit time, i.e., largest $a_i\omega_i$. (In the armour-infantry assault example sketched in Chapter 2 we argued intuitively that the defenders of the town would be firing mainly at the enemy infantry).

However, if the combat lasts sufficiently long, target prioritization can change. In fact working backwards in time from the terminal time T, Taylor [7] demonstrates that the first switch occurs to that target type that satisfies the following expression

$$R_k = \min_{\substack{R_i > 0 \\ a_i b_i > a_m b_m}} (R_1, R_2, \ldots, R_n) \qquad i \in \{1,2,\ldots,n\} \text{ but } i \neq m$$

where

$$R_i = \frac{a_i(b_i\omega_m - b_m\omega_i)}{(a_i b_i - a_m b_m)} \tag{3.1}$$

$$a_m\omega_m = \max(a_1\omega_1, a_2\omega_2, \ldots, a_n\omega_n)$$

The target "k" that satisfies equation (3.1) satisfies simultaneously the two conditions

(1) $a_k b_k > a_m b_m$ (3.2)

(i.e. directing fire at target k destroys more retaliatory fire from X to Y than firing at target m).

(2) $\frac{b_k}{\omega_k} > \frac{b_m}{\omega_m}$ (3.3)

(i.e. "k" has a higher kill rate per unit of x_k per value per unit of x_k survivors than "m" does.)

Basically this implies that Y cannot start by firing at the target with the highest $a_i \omega_i$ and ignore (during combat) the remaining X's that are firing incessantly at Y. The retaliatory attrition-fire coefficients (b_i's) must enter the prioritization so as to minimize the Y casualties. The reader is reminded that Y seeks through his choice of control $\phi_i(t)$ to not only minimize the weighted sum of $x_i(T)$'s but to also maximize $y(T)$ - (see Figure 3.1). (In the armour-infantry example sketched in Chapter 2, we argued that initially the defenders will mainly fire upon the armour units, since they have higher retaliatory attrition capability than infantry).

Taylor [7] further noted that that the target selection greatly simplifies when

$$\omega_i = kb_i \tag{3.4}$$

i.e., the utility assigned per unit of surviving x_i is linear with the attrition coefficient b_i of fire from x_i to Y. <u>In this case target priorities do not change</u>. This can readily be seen by substituting (3.4) into (3.1). Fire is directed at the highest priority target (i.e. the one with the largest $a_i b_i$) till it is annihilated and only then shifted to the second highest priority target type and so forth.

At this point let us recapitulate the two key results of this example and assess their validity in real world settings. First the optimal fire distribution policy is bang-bang and does not depend on the initial force levels. Clearly the latter result is highly questionable. Intuitively one would expect force levels to enter significantly in the optimal allocation policy. In fact this result stems from the assumption made that the combat duration is fixed (which is in itself a questionable assumption. Rarely in combat is one endowed with the luxury of knowing when it will end, e.g. the "weaker" side after suffering significant casualties may elect to withdraw to avoid annihilation).

In the second scenarios studied by Taylor [7] the combat duration is not predetermined. Hence, the problem falls into the class of free terminal time dynamic optimization problems. (This introduces the extra boundary condition that the Hamiltonian at T is zero). Moreover, now the combat is a fight-to-the-end engagement, thus introducing target sets as boundary conditions.

In contrast to the fixed terminal time problem the allocation policy is <u>no longer independent</u> of the force levels. It may depend on the initial force levels and thus is more realistic. The bang-bang structure of the control policy, however, still prevails.

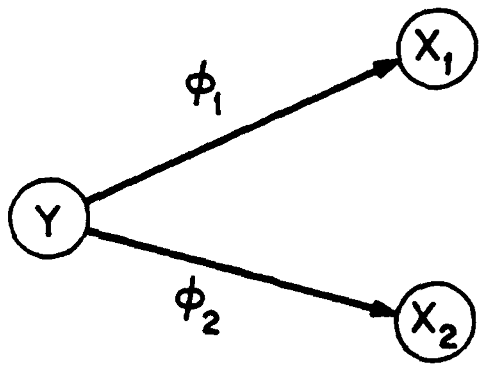
In Scenario three a special case of the time dependent attrition-rate coefficient case is addressed. Unfortunately the results are obvious since it is a trivial extension of scenario one. In particular the coefficients are permitted to be time varying as long as their ratio is constant. Clearly the more interesting issues arise when the restrictive conditions on the attrition rates are relaxed.

Of great interest are the results from Scenario four. (See Figure 3.2 for definition of problem and related notation). Note that the attrition processes are "linear", i.e., "area" fire. We argued with the help of a simple example in Chapter 1 (Appendix B) that force concentration was not always advantageous in the "linear" attrition case. In fact in this instance the optimal allocation policy is not restricted to the extremes of the space of admissable controls. Singular subarcs may be present in the battle trajectory. In particular the singular control is

$$\phi^* = \frac{a_2}{a_1+a_2} \tag{3.5}$$

(In Appendix C a derivation is provided to allow us to refer to it later in the discussion. Taylor presents the details of the derivation in a related paper [8].)

Once again as in Scenario one the structure of the optimal allocation policies depends on how Y weights surviving X forces relative to their retaliatory attrition-fire rates against Y. The reason for including the derivation in Appendix C is to enable us to make the following point. Consider the two target type version of Scenario one, except now one of the attrition processes is "linear" while the other remains "square". From equation (C.18) it is clear that there cannot be a singular control.



Problem: $\max\limits_{\phi(t)} \{ry(T) - px_1(T) - qx_2(T)\}$; T specified

s.t. $\frac{dx_1}{dt} = -\phi a_1 x_1 y$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2$$

$$x_1, x_2, y \geq 0, \quad 0 \leq \phi \leq 1, \quad T \leq T_1$$

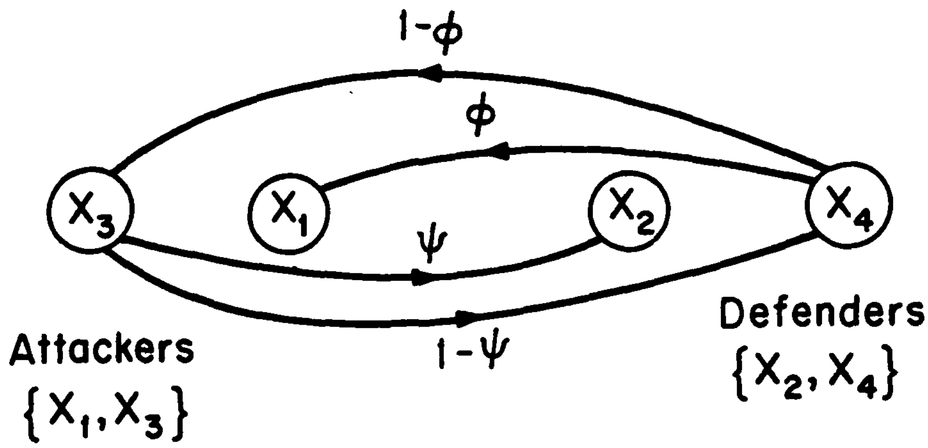
Fig. 3.2 Fire Distribution over Targets Undergoing Linear Attrition

Although Taylor did not make this point it has been included in this paper because it nicely bridges the gap between Taylor [7] and Kawara [5]. The latter deals with the problem of support fire distribution either as "linear" attrition on enemy ground forces or "square" attrition on enemy support units.

The situation examined by Kawara is an extension of the one studied by Weiss [13]. Weiss claims that one of the major problems of weapon system analysis is that of modelling the intricate relationships present between the different types of weapon systems (that may simultaneously be engaged in a single conflict). In particular Weiss' problem may be stated as: "how should the first support system of two heterogeneous forces (each comprised of a combination of ground forces and supplemental fire support) be allocated in real-time against enemy ground forces and their support units." The objective of each side is to maximize the number of one's own ground forces and minimize those of the adversary. (Ground forces are not endowed with fire power.)

Taylor in a related paper [9] applies differential game theory to solve the above problem. (Weiss' solution was heuristic). Kawara [5] goes one step further and introduces "linear" attrition processes (See Figure 3.3). His solution techniques are based on Isaacs' [4] differential game approach. The major result is that the allocation policy is bang-bang on both sides. If the combat duration exceeds a limited time, the optimal strategy requires concentrating all the fire on enemy support units and then switching fire onto enemy ground forces.

However, as we shall see in the ensuing discussion, Kawara's paper suffers from several deficiences: insufficient motivation of the model



$$\text{Problem:} \quad \min_{\phi} \max_{x} \frac{x_1(T)}{x_2(T)}$$

$$\text{s.t.} \quad \dot{x}_1 = -k_2 x_1 \phi x_4$$

$$\dot{x}_2 = -k_1 x_2 \psi x_3$$

$$\dot{x}_3 = -K_2 (1-\phi) x_4$$

$$\dot{x}_4 = -K_1 (1-\psi) x_3$$

$$\dot{T} = -1$$

Fig. 3.3 Dual Support Fire Allocation Problem (Kawara [5]) employed and the omission of discussion of key issues (e.g., singular controls, existence of saddle points.)

Thus far in this chapter we have outlined the contributions of Taylor [7] and Kawara [5]. Before proceeding with a discussion of those made in Wohl and Gootkind [14], Taylor and Brown [11] and Mangulis [6] we will make the discussion of [7] and [5] complete by providing some observations on (1) the validity of the mathematical models proposed by the respective authors and (2) on various technical issues.

Earlier in this chapter it was pointed out that Kawara extended the results of support-fire allocation obtained in [13] and [9] by introducing linear attrition. In particular the support unit direct "area" fire at the opponents ground units.

Although he suggests that the reader picture the primary forces as infantry and the support units as artillery, he makes no use of this representation to motivate his choice of "linear" attrition. (An example, would be support artillery units pounding distant enemy infantry units.)

The next issue raised by Kawara that we question is the statement: "The areas held by both attacker (x_1) and defender (x_2) are constant during combat." But he also assumed that the attacking ground forces (x_1) advance forward during the combat. Generally one would expect areas held by advancing armies to increase while those held by retreating defenders to decrease. Kawara's motive for enforcing such a restriction must have been to permit the use of constant "area" attrition fire rates. Otherwise, the attrition coefficients k_1, k_2 would have to vary to reflect the changing areas. However, there is still the problem of decreasing separation between advancing ground forces x_1 and stationary defensive support units x_4. Based on the discussion in Chapter 2 we can assert that the attrition coefficient k_2 should be a function of the separation. By drawing closer, x_1 units should sustain more and more casualties from improving accuracy of x_4 fire.

Earlier in this chapter we questioned Taylor's assumption (in Scenario one) that the combat duration is fixed (a priori). It was shown that a free terminal time setting (Scenario two) was more realistic and provided a more reasonable fire allocation policy. Similarly we feel that Kawara should use a free terminal time battle model.

Lastly we would like to examine Kawara's choice of the cost function. He states that "in our model the objective of fire support is to assist the primary units to gain superiority over the enemy primary units at the time of contact. It is assumed reasonable to use the following ratio of numbers of the two sides' primary units as a measure of superiority," i.e.

$$J = \frac{x_1(T)}{x_2(T)} \tag{3.6}$$

Consider the following simple example. Let $x_1(0) = 1000$, $x_2(0) = 1000$ (denoting the initial ground force strengths). With the aforementioned ratio as the cost, the optimization procedure for the attacker $\{x_1, x_3\}$ (which seeks to maximize J) would prefer the outcome

$$\frac{x_1(T)}{x_2(T)} = \frac{2}{1} \quad \text{to} \quad \frac{x_1(T)}{x_2(T)} = \frac{650}{326} < 2.$$

Would a "rational" commander, though, sacrifice almost all his troops so as to gain a 2 to 1 edge instead of a 650 to 326? Although he would have twice as many units as the defender it would really only constitute a one unit advantage. Hence the force ratio cost function has two inherent weaknesses:

(1) casualties suffered in not stressed

(2) final difference in strengths is not accounted for

(Kawara makes no attempt to support his choice of the cost function). Asides from the aforementioned issues the force ratio cost function introduces co-states that are sensitive to the duration of combat, in particular

$$p_1(T) = \frac{1}{x_1(T)}$$
$$p_2(T) = -\frac{y_1(T)}{x_1^2(T)} \tag{3.7}$$

are the boundary conditions at t=T (See Athans and Falb [1]).

On the other hand, Taylor [7] avoids all the aforementioned problems associated with the force ratio cost function by selecting one that is the weighted difference of the primary (ground) forces, e.g. the simple scenario he considers on page 82 of [7] (special case of Figure 3.1 where Y must distribute its fire over two different types of X targets, i.e.

$$\nu y(T) - \omega_1 x_1(T) - \omega_2 x_2(T) \tag{3.8}$$

Note that in this case the corresponding co-states are insensitive to T (e.g., the costate corresponding to y is equal to ν). More importantly, since we can rewrite equation (2.8) as

$$\nu(y(0) - y(T)) - \omega_1(x_1(0) - x_1(T)) - \omega_2(x_2(0) - x_2(T)) \quad (3.9)$$

without affecting the optimization, it is clear that the casualties suffered is accounted for in the cost function and the final numerical strength difference is stressed.

However, it is only fair to point out that the coin is a two-sided one. Now the relative (ratio) superiority is not stressed. Again for illustrative purposes consider the following simple example. Let $y(0) = y_0$, $x_1(0) = x_1^0$, and $x_2(0) = x_2^0$ and $\nu = \omega_1 = \omega_2 = 1$. With (3.8) as the cost function, the optimization procedure of Y (which seeks to maximize (3.8) would prefer the outcome $\{y = 511, x_1 = 500, x_2 = 0\}$ over $\{y = 11, x_1 = 1, x_2 = 0\}$. Although the latter outcome entails more Y casualties, the relative numerical superiority is awesome and in some cases may be preferred by commanders.

A reasonable compromise is to strike a balance and use a cost function that captures both measures of superiority, e.g.,

$$J = \alpha \frac{x_1(T)}{x_2(T)} + \beta[\mu_1 x_1(T) - \mu_2 x_2(T)] \quad (3.10)$$

where α, β, μ_1, μ_2 are constants that must be selected by the designers relying on "good" engineering judgement.

Thus far in the discussion of Taylor [7] and Kawara [5] we have examined their contributions and validity of mathematical models. Lastly we would like to address various key technical issues that arise in the solution techniques employed by the authors.

Kawara [5] uses Isaac's [4] approach for solving differential games. In his pioneering work Isaacs assumed that the Hamiltonian for a differential game has a saddle point in pure strategies. He further conjectured that the minmax optimization guarantees that the associated differential game has a value.

However, saddle points in pure strategies are not guaranteed [2]. Hence, Kawara neglects to consider a critical issue when he makes the following statement on page 944: "According to Isaacs, if the value of the game exists, the following relation holds between ---" No discussion of the necessary conditions that guarantee existence is provided. Unlike when Isaacs investigated related problems in the 1950's, by 1973 when Kawara examined the support allocation problem, some conditions for the existence of saddle points and the value of the differential game had been established in the literature.

Consider the following summary of the discussion of this issue provided by Bryson and Ho [2]: Let the dynamic system be given by

$$x = f(x,u,v,t); \quad x(0) = x_0 \quad \text{(free terminal time)} \qquad x(T) = x_T \tag{3.11}$$

and $J = K(x(T)) + \int L(x,u,v,t)\,dt$.

The problem is to find (u^*,v^*) that satisfies the following saddle point relationship:

$$J(u^*,v) \leq J(u^*,v^*) \leq J(u,v^*) \tag{3.12}$$

The necessary conditions for the minmax problem posed above (u seeks to min J while v seeks to max J) can be obtained as follows

$$H = L + p^T f \qquad \text{(Hamiltonian)}$$

$$\dot{x}^* = \left.\frac{\partial H}{\partial p}\right|_* \qquad \dot{p}^* = -\left.\frac{\partial H}{\partial x}\right|_* \qquad (3.13)$$

$$\left.\frac{\partial H}{\partial u}\right|_* = 0 \qquad \left.\frac{\partial H}{\partial v}\right|_* = 0$$

i.e. $H^* = \max_v \min_u H$

The existence of H* is only guaranteed if H is separable. However, separability of H does not imply separability of J which is necessary to guarantee the existence of the (u*, v*) that satisfies the saddle point expression (3.12).

Now consider the problem in Kawara [5]. From Figure (3.3) we know that the Hamiltonian is given by

$$H = -p_1k_2x_1\phi x_4 - p_2k_1x_2\psi x_3 - p_3K_2(1-\phi)x_4 - p_4K_1(1-\psi)x_3 - p_5 \qquad (3.14)$$

Clearly the Hamiltonian is separable in ϕ and ψ But $J = \frac{x_1(T)}{x_2(T)}$ is not separable since it contains terms which are ratios of ϕ and ψ. Hence, we can conclude that with the force ratio cost function one is not guaranteed a saddle point solution.

Aside from existence of saddle points, another critical issue (overlooked in Kawara [5]) is the possible existence of singular controls. We saw earlier in this chapter, that in Taylor [7,8] the use of "linear" attrition-rates introduces singular controls. Moreover, we showed that if one of them is "linear" while the other is "area" then there are no singular controls. Kawara's [5] problem falls in the latter class and thus there are no singular controls. However, had the attrition-fire on the primary units also been "area", then there could be singular subarcs in the solution.

Thus far in this chapter an overview of [5] and [7] has been presented. In the remainder of this chapter a brief discussion is presented of [14], [11] and [6] focusing on their contributions to overcoming several of the shortcomings of the original Lanchaster models (see Chapter 2).

Wohl and Gootkind [14] extend the original Lanchaster models to account for multi-component forces, second echelon reinforcement activity, close air support and interdiction effects. Reinforcements are incorporated into the original Lanchaster models (1.1) by adding a constant term which denotes the number of units per day. Assuming that only X has second echelon reinforcements, they provide Y with an air force to attack X's second echelon. However, now the size of the underlying state-space has to be increased to include the number of planes Y has available, i.e. the Lanchaster model is as follows

$$\frac{dx}{dt} = - ay + V[1 - \frac{z}{A}] \tag{3.15}$$

$$\frac{dy}{dt} = - bx \tag{3.16}$$

$$\frac{dp}{dt} = - cz \tag{3.17}$$

$$0 \leq z \leq kp \tag{3.18}$$

where V denotes the rate of reinforcement.

P denotes the number of plane in Y's air force

Z denotes the sortie rate

A denotes the maximum sortie rate

K denotes the sorties/day/available plane.

The cost function proposed in [14] is

$$J = (y(0)-y) + \Omega(p(0)-p) \tag{3.19}$$

i.e. y seeks to minimize is troop and plane losses. However, note that x does not enter the cost function. Clearly Y seeks to maximize X-casualties. Thus a more reasonable cost would be

$$J = \alpha(y(0)-y) + \beta(p(0)-p) - \gamma(x(0)-x) \tag{3.20}$$

Note that this problem falls into the class of free terminal time optimal control problems (with target set $Y = Kx$ where $K = \frac{\ell y}{\ell x}$, ℓ_y-dimension of Y force area, ℓ_x-dimension of x force area.) The control variable is "z"-sortie rate. The optimal policy is bang-bang (see [14] page 19).

Note the similarity of the structure of the scenario in [14] and the one in (5). Both contain primary units where one is the advancing attacker. Both have support units: In [5] they are artillery batteries while in [14] they are reinforcements and an interdiction air force. One major difference is that Kawara assumes that the primary units do not fire upon each other while in [14] they direct "square" attrition fire at each other.

In Chapter 2 we cited that a major shortcoming prevalent in the models proposed by the literature is disregard of the separation of forces, e.g. Kawara [5]. Wohl and Gootkind [14] similarly assume advancing primary units and fail to take it into account in the attrition coefficients (they assume them to be constant).

One of the few papers that models force separation is Taylor and Brown [11]. In particular they consider a scenario consisting of a constant-speed attack of a mobile force against a static defense; with the following combat dynamics

$$\frac{dx}{dt} = -a(r)y = -[a_0(1-r/R_a)^m]y \tag{3.21}$$

$$\frac{dy}{dt} = -b(r)x = -[b_0(1-r/R_b)^n]x \tag{3.22}$$

where R_a and R_b denote the maximum effective ranges of the Y and X weapon systems. Range is related to time by $r(t) = R_0 - vt$ where R_0 denotes the opening range of battle and v is the constant forward speed of the attacking units.

Lastly consider Mangulis [6] which accounts for the possibility of multiple killing hits. In salvo fire several weapons may be launched at the same target because of a lack of communication or intentionally to enhance the probability of kill. He also considers the case of random fire. In this case multiple hits can occur since confirmation of kill takes some time (communication delays), or the weapon may already by launched when confirmation arrives, tc. This ties in with the C^3 effectiveness issue raised earlier. Generally the more effective the C^3 system the less chance of wasting fire power due to multiple hits.

4. Summary and Suggestions for Further Research

This paper has presented an overview of Lanchaster-type combat models for modern warfare scenarios. Chapter 2 provided a discussion of a number of shortcomings of basic Lanchaster models. One of the critical issues that arose in the need to incorporate various features of the particular battle scenario is the attrition-coefficients and not merely treat them as constant parameters to be chosen in some ad hoc fashion. The optimal target selection problem (and the underlying optimal control problem formulation) served as a backdrop for most of our discussion. In all three papers [5], [7] and [14] we observed a bang-bang fire allocation policy. Interesting issue that needs investigation is whether continuous controls would be more appropriate than bang-bang. Wold [14] comments that an integrated cost (e.g., a quadratic loss function in the control variable) would introduce continuous controls. The next issue would be to ascertain if LQ theory can be brought to bear on this problem in any reasonable way.

The more interesting (but much more complex) issues lie in accounting for the stochastic elements of warfare. Wohl [14] addresses some uncertainty issues but much more carefully formulated models are necessary in order to gain deeper insight into the problem. On the other hand one should bear in mind that modelling a problem in greater and greater detail can render the model too complex to analyze with current theory!

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APPENDIX A

In this appendix the derivation of the set of equations in (1.3) is presented, i.e. the attrition models of two sides exchanging salvo fire (all X and Y weapons are fired simultaneously every δt_x and δt_y respectively) [6].

Let

- P_x denote the single shot probability of kill of a unit of Y by a unit of X
- P_y denote the single shot probability of kill of a unit of X by a unit of Y
- δt_x denote the time interval between X salvos
- δt_y denote the time interval between Y salvos
- x denote number of units in X
- y denote number of units in Y

Assume X fires at random at any Y and vice versa.

The probability that a particular x^0 is firing upon a particular y^0 is $\frac{1}{y}$. Hence the probability that this x^0 kills the particular y^0 is $\frac{P_x}{y}$.

The probability that this y^0 is not killed during the salvo exchange by any of the x-units is

$$\left(1 - \frac{P_x}{y}\right)^x \qquad \text{(A.1)}$$

The Y casualtities during δt_x can be expressed as

$$\Delta y = y\left[1 - \left(1 - \frac{P_x}{y}\right)^x\right]$$

Dividing by δt_x and taking limits yield the differential equation

$$\frac{dy}{dt} = \frac{y}{\delta t_x} \left[1 - \left(1 - \frac{P_x}{y}\right)^x\right] \tag{A.2}$$

If the single shot probabilities of kill are small then expansion of (A.2) into a Taylor series yields

$$\frac{dy}{dt} = - \frac{P_x}{\delta t_x} x \tag{A.3}$$

and similarly

$$\frac{dx}{dt} = - \frac{P_y}{\delta t_y} y \tag{A.4}$$

APPENDIX B

The purpose of this appendix is to demonstrate the advantage of concentration of forces in "Square" attrition. Consider the following simple example [10]: For definition of notation see Figure (1-1)).

(1) Square Attrition: $b(x^2(0)-x^2(T)) = a(y^2(0)-y^2(T))$

Set $b/a = 1$ $x(0) = 100;$ $x(T) = 0$

Then			
$y(0)$	100	150	300
$y(t)$	0	112	283
Y casualties	100	38	17

Thus there is an advantage in concentrating forces

(2) Linear Attrition: $b(x(0) - x(T)) = a(y(0) - y(T)))$

Set $b/a = 1$ $x(0) = 100$; $x(T) = 0$

Then			
$y(0)$	100	150	300
$y(T)$	0	50	200
Y casualties	100	100	100

Hence no apparent advantage in concentrating forces in the "linear" case.

APPENDIX C

In this appendix the derivation of the singular control for Taylor [7] "linear" attrition case is presented. (See [7] page 99 for problem definition).

Problem: $$\max_{\phi(t)} \underbrace{\{ry(T) - px_1(t) - qx_2(T)\}}_{K},\ T_1 \text{ specified}$$

$$\text{s.t. } \frac{dx_1}{dt} = -\phi a_1 x_1 y \tag{C.1}$$

$$\frac{dx_2}{dt} = -(1-\phi) a_2 x_2 y \tag{C.2}$$

$$\frac{dy}{dt} = -b_1 x_1 - b_2 x_2 \tag{C.3}$$

$x_1, x_2, y \geq$ $\qquad 0 \leq \phi \leq 1$ and $T \leq T_1$

Necessary Conditions

I. State and a State Equations

$$\frac{dx_1^*}{dt} = -\phi^* a_1 x_1^* y^* \tag{C.4}$$

$$\frac{dx_2^*}{dt} = -(1-\phi^*) a_2 x_2^* y^* \tag{C.5}$$

$$\frac{dy^*}{dt} = -b_1 x_1^* - b_2 x_2^* \tag{C.6}$$

$$\dot{p}_1^* = -\frac{\partial H}{\partial x_1} = p_1^* \phi^* a_1 y^* + p_3^* b_1 \tag{C.7}$$

$$\dot{p}_2^* = -\frac{\partial H}{\partial x_2} = p_2^*(1-\phi^*) a_2 y^* + p_3^* b_2 \tag{C.8}$$

$$\dot{p}_3^* = -\frac{\partial H}{\partial y} = \phi^* a_1 p_1^* x_1^* + (1-\phi^*) a_2 p_2^* x_2^* \quad \text{(C.9)}$$

II. Boundary Conditions

$$x_1^*(0) = x_1^0 \quad \text{(C.10)}$$

$$x_2^*(0) = x_2^0 \quad \text{(C.11)}$$

$$y^*(0) = y^0 \quad \text{(C.12)}$$

$$p_1(T) = \frac{\partial K}{\partial x_1(T)} = p \quad \text{(C.13)}$$

$$p_2(T) = \frac{\partial K}{\partial x_2(T)} = q \quad \text{(C.14)}$$

$$p_3(T) = -r \quad \text{(C.15)}$$

III. Hamiltonian Minimization

$$H(\underline{x}^*(t), \underline{p}^*(t), \phi^*(t), t) \leq H(\underline{x}^*(t), \underline{p}^*(t), \underline{\omega}, t) \quad \text{(C.16)}$$

$$\Rightarrow \phi = \begin{cases} 0 & [-a_1 x y p_1 + p_2 a_2 x_2 y] > 0 \\ 1 & [-a_1 x_1 y p_1 + p_2 a_2 x_2 y] < 0 \end{cases} \quad \text{(C.17)}$$

Singular controls can exist if $[-a_1 x y p_1 + p_2 a_2 x_2 y] = 0$ over a finite interval.

Let $M = -a_1 x_1 p_1 + p_2 a_2 x_2 = 0$

$\Rightarrow \dot{M} = 0$

$$-a_1 \dot{x}_1 p_1 - a_1 x_1 \dot{p}_1 + \dot{p}_2 a_2 x_2 + p_2 a_2 \dot{x}_2 = 0$$

$$= a_1^2 \phi x_1 y p_1 - a_1^2 x_1 p_1 \phi y - a_1 p_3 b_1 x_1$$

$$+ p_2 (1-\phi) a_2^2 y x_2 + p_3 b_2 a_2 x_2 - p_2 a_2^2 (1-\phi) x_2 y = 0$$

hence

$$x_2 a_2 b_2 = x_1 b_1 a_1 \tag{C.18}$$

Differentiating yields

$$\dot{x}_2 a_2 b_2 = \dot{x}_1 b_1 a_1 \tag{C.19}$$

$$\Rightarrow \quad -(1-\phi) a_2^2 x_2 y b_2 = -\phi a_1^2 x_1 y b_1$$

$$\phi [a_1^2 x_1 b_1 + a_2^2 x_2 b_2] = a_2^2 x_2 b_2$$

$$\phi^* = \frac{a_2^2 x_2 b_2}{a_1^2 x_1 b_1 + a_2^2 x_2 b_2} \tag{C.20}$$

$$= \frac{a_2}{a_1 + a_2} \tag{C.21}$$

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