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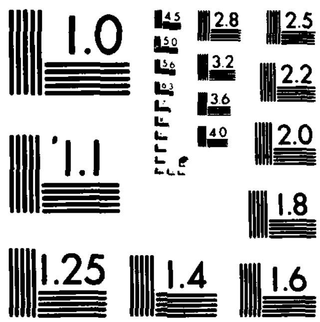

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ON ASYMMETRIC STOCHASTIC BANG-BANG CONTROL

By

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On Asymmetric Stochastic Bang-Bang Control

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I. Introduction

Three stochastic bang-bang control problems are considered, the predicted miss, the linear regulator, and a simple complete observation model ([1],[2],[3]), which have been solved for symmetric constant bound on the control function $u(t)$, $0 \leq t \leq T$. Here asymmetric, finite bounds on $u(t)$ are considered.

To state these problems we use the notation and definitions of ([1],[2],[3]).

II. Predicted Miss

Let $A(t)$, $B(t)$, $C(t)$ be, respectively, dxd, rxd, and dxd matrix-valued continuous functions on $[0,T]$ with

$$\langle \underline{\alpha}, C(t)C'(t)\underline{\alpha} \rangle \geq \beta \langle \underline{\alpha}, \underline{\alpha} \rangle > 0$$

for all $t \in [0,T]$ and $\underline{\alpha} \in R^d$, where $\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^{d} a_i b_i$.

Denote the system equation by

$$dX(t) = A(t)X(t)dt + B(t)u(t)dt + C(t)dW(t)$$

$$X(0) = G \in R^d \tag{1}$$

and $W(t)$ is standard d-dimensional Wiener process. Transpose A is indicated by A'.

Assumption: The admissible control set $\mathcal{C}$ consists of the set of processes

$u(t) \equiv u(t,w)$ such that $-\infty < a < f(t) \leq u(t,w) \leq g(t) < b < \infty$

where $f(t) < g(t)$ are bounded continuous non-random given functions on $[0,T]$, and $f(0) + g(0) = 0$.

The Girsanov Theorem may be used to solve (1) in law (i.e. there is a weak solution. The boundedness of $u(t,w)$ insures that for any given $u \in \mathcal{C}$, there is a unique solution to (1) by ([4]), Theorem 1).

Given a fixed vector $\underline{\gamma}$, the cost corresponding to $u \in \mathcal{C}$ is

$$J(u) = E_u(\ell(<\gamma,X(T)>))$$

where for each u, there is a probability space $(\Omega,\mathcal{F},P_u)$ with $\Omega = C^d[0,T]$, $X(t,w) : \Omega \to R^d$ is the coordinate map $X(t,w) = w(t)$. Then for $u \in \mathcal{C}$, P_u on $(\Omega,\mathcal{F})$, where $\mathcal{F} = \sigma(X(s),\ 0 \leq s \leq T)$ is such that $P_u[X(0)=G] = 1$ and the process $W(t,u)$ defined by

$$W(t,u) = \int_0^t C^{-1}(s)dX(s) - \int_0^t C^{-1}(s)[A(s)X(s)+B(s)U(s)]ds \text{ is}$$

d-dimensional Wiener process. Hence

$$J(u) \equiv \int_\Omega \ell(<\gamma, X(T)>)\, dP_u = E_u\ell(<\gamma, X(T)>).$$

where

$$\ell : R \to R^+ \text{ has these properties:}$$

(i) ℓ is even : $\ell(x) = \ell(-x)$.

(ii) ℓ is continuously differentiable for $x > 0$

and $\ell'(x) \geq 0$ all $x > 0$

(iii) $\ell(x) = O(\exp \delta|x|)$, some $\delta > 0$.

The object is to find optimal $u_0(t)$ such that

$$J(u_0) = \min_{u \in \mathcal{U}} J(u).$$

Continuing the exposition in [2], if $X(t) = x$, and $u(\xi) = 0$, $t \leq \xi \leq T$,

then

$$E[\langle \gamma, X(T)\rangle | X(t) = x] = \langle \gamma, \Phi'(t,T)x\rangle$$

where Φ is the solution operator for (1) when $u = 0$.

Define

$$s(t) = \Phi'(t,T)\gamma.$$

Then $s(t)$ satisfies

$$\frac{ds(t)}{dt} = -A'(t)s(t), \qquad s(T) = \gamma .$$

and

$$E_u[\langle \gamma, X(T)\rangle | X(t)] = \langle s(t), X(t)\rangle \equiv m(t).$$

Then $m(t)$ satisfies

$$dm(t) = \langle B'(t)s(t), u(t)\rangle dt$$

$$+ \langle C'(t)s(t), dW(t,u)\rangle$$

or, equivalently,

$$dm(t) = \langle b(t), u(t)\rangle dt + dv(t)$$

$$m(0) = \langle s(0), G\rangle. \qquad (2).$$

The cost is expressed as

$$J(u) = E_u[\ell(m(T))]$$

Theorem 1 Under conditions above in II, the optimal control $u_0(t)$ is expressible by components, $1 \leq i \leq d$, with $u_0 = (u_{01}, \ldots u_{0d})$, as

$$u_{0i}(t) = \frac{f_i(t) + g_i(t)}{2}$$

$$- \frac{(g_i(t)-f_i(t))}{2} \operatorname{sign}\left([m(t) - \frac{1}{2} \sum_{\ell=1}^{d} \int_t^T b_\ell(s)(f_\ell(s)+g_\ell(s))ds]b_i(t)\right) \quad (3)$$

Proof. First assume $f_i(t) + g_i(t) = 0$, all $1 \leq i \leq d$ and set

$$h_i(t) = \frac{g_i(t) - f_i(t)}{2}$$

In this case, note $0 < h_i(t) \leq |a|+|b|$, $i \leq i \leq d$ and by [4], since $|u_i(t)| \leq h_i(t)$, then (2) has a unique solution with $u = u_0$.

Set

$dm(t) = \langle b,u \rangle dt + dv$ then it follows by the same argument as in [2] that

$$u_{0i}(t) = -(h_i(t) \operatorname{sign}\left(m(t)b_i(t)\right). \quad (4)$$

is optimal for the symmetric control case. In general, by the argument of ([2], pp. 207-208), one may invoke symmetry by utilizing a switching curve $k(t)$ such that if

$$\bar{m}(t) \equiv m(t) - k(t), \text{ then it would follow that}$$

$$d\bar{m}(t) = \langle b, u - \frac{f+g}{2} \rangle dt + dv(t). \tag{5}$$

To accomplish (5) it clearly suffices to set

$$\frac{dk(t)}{dt} = \frac{1}{2} \sum_{\ell=1}^{d} b_\ell(t)(f_\ell(t)+g_\ell(t))$$

and $$k(T) = 0,$$

which allows $$J(\bar{m}(T)) = J(m(T)). \tag{6}$$

Hence (4) - (6) suffice for the proof of (3).

III. Linear Regulator.

A one-dimensional linear regulator problem, following [3] is defined as follows: The one-dimensional process $X(t,w)$ is given by

$$dX(t,w) = (aX(t,w) + u(t,w))dt + dW_1(t,w)$$

with

$$X(0,w) = X_0(w)$$

and observation equation

$$dY(t,w) = cX(t,w)dt + dW_2(t,w) \tag{7}$$

and

$$Y(0,w) = 0$$

for

$$0 \leq t \leq T$$

where

$a > 0$, $c > 0$ are constants, W_1, W_2 are independent one-dimensional Wiener processes. Let $(\Omega,\mathcal{F},\mathcal{P}_{u},)$ be $\Omega = C[0,T]$, $\mathcal{F} = \sigma(X(s),\ 0 \leq s \leq T)$, and $\mathcal{P}$ correspond to W_1,W_2.

Let the performance index, as a function of a given control u be

$$J(u) = \int_0^T E(X^2(s))ds \equiv \int_0^T X^2(s,u)d\mathcal{P} \tag{8}$$

and the set of admissible controls $\mathcal{A}$ is given by

$$\mathcal{A} = \{u \mid |u(t,w)| \leq g(t), \quad 0 \leq t \leq T, \tag{9}$$

with continuous $g(t) > 0$, $0 \leq t \leq T$.

A control $U_0 \in \mathcal{A}$ is optimal if

$$J(u_0) \leq J(u) \text{ all } u \in \mathcal{A}.$$

This problem may be recast as a complete observation control problem, following ([3], eq. (3.19) - (3.21)). The new state variables are $R(t,w)$ and satisfy

$$dR(t,w) = (aR(t,w) + u(t,w))dt + c\ p(t)\ dW_3(t,w)$$

$$R(0,w) = E\ X_0(w)$$

and

$$J(u) = \int_0^T E(R^2(s))ds \tag{10}$$

where W_3 is a Wiener process, and the function $p(t)$ satisfies a Riccati equation

$$\frac{dp}{dt} = 2ap(t) + 1 - c^2p^2(t) \qquad 0 \leq t \leq T$$

$$p(0) = E(X_0^2) - (EX_0)^2 = \mathrm{Var}X_0. \tag{11}$$

The admissible control set $\mathcal{A}$ is unchanged and it is noted that (11) does not depend on $\mathcal{A}$ by ([3], eq. (2.15) - (2.20)).

Also $J(u) = \int_0^T E(R^2(s))ds$

Theorem 2 The equation, for any fixed $u \in \mathcal{A}$,

$$dV(t,w) = (aV(t,w) + u(t,w))dt$$
$$+ cp(t)dW_3(t,w)$$
$$V(0,w) = EX_0(w) \tag{12}$$

has a unique solution.

Proof. This follows from the boundedness of p, u, f, g in [0,T] by ([4], Theorem 1).

Theorem 3 The optimal $u \in \mathcal{A}$ for the system (10), (11) is expressible as

$$u_0(t,w) = -g(t)\text{sign } X(t,w). \tag{13}$$

Proof. This follows since $g(t) > 0$ factors out of both sides of ([3], eq. (2.26)), and Theorem 2.

IV Complete Observation

Consider a one-dimensional complete observation control problem with state $X(t,w)$, control $u(t,w)$ and Wiener process $W(t,w)$ defined by

$$dX(t,w) = u(t,w)dt + dW(t,w) \tag{14}$$

and

$$X(0,w) = x \; .$$

The $(\Omega \mathcal{F} P)$ are as in III.

With admissible control set

$$\mathcal{A} = \{u \mid |u(t,w)| \leq g(t), \quad 0 \leq t \leq T\}$$

with continuous $g(t) > 0$, $0 \leq t \leq T$, as in (9).

The performance index for given $u \in \mathcal{A}$

is

$$J(u) = \int_0^T E|X(t,w)|^{\ell} dt \tag{15}$$

for ℓ a fixed positive integer. The object is to find $u_0 \in \mathcal{A}$ so that

$$J(u_0) \leq J(u), \; u \in \mathcal{A}.$$

For $\ell = 1, 2$, this problem was solved in [1].

<u>Theorem 4</u> Under (9), (14), (15), for all $\ell \geq 1$, the solution is

$$u_0^o(t,w) = -g(t)\text{sign } X(t,w). \tag{16}$$

<u>Proof</u>. The equation, with $X(0,w) = x$

$$X(t,w) = -g(t)\text{sign } X(t,w)dt + dW(t,w)$$

has a unique solution ([4], Theorem 1) by boundedness of g.

The reasoning of ([1], pp. 93, 96, eq. (2.15)) implies that the same optimal u_0 holds for all $\ell \geq 1$ in (15). The argument is as in Theorem 3.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Three stochastic bang-bang control problems, the predicted miss, the linear regulator, and a complete observation model are shown to have formally similar solutions under asymmetric boundedness condition on the control $u(t)$, $0 < t < T$.

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