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**THE PHASE PROBABILITY DENSITY  
FUNCTION OF A RANDOM WALK IN  
TWO DIMENSIONS**

by

B. C. Barber

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Procurement Executive, Ministry of Defence  
Farnborough, Hants

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## 1 INTRODUCTION

The random walk is a classical mathematical problem. The definitive statement of the problem was given, many years ago, by K. Pearson<sup>1</sup>. Pearson required to know the probability of finding a random walker (or - to paraphrase Ref 2 - "a drunken man in open country") within a given radius of his starting point after  $N$  equal steps. The random walker changed his direction in a random manner after each step.

In the classical random walk all angles through which the walker may turn after each step are equally probable. The angle probability density function is uniform. In this paper the non-uniform (or non-isotropic) walk is considered in which the step angle probability density function is not uniform. Probability density functions which are mapped on to the unit circle have been discussed by, for example, Wintner<sup>3</sup>, who gives further references. In this paper the existence of phase probability density functions is taken for granted; they can be easily justified on physical grounds. Consider the example of an electromagnetic wave incident upon a point in space. The wave is scattered. Suppose now that the phase of the scattered wave is measured relative to the unscattered wave at the point of emission. Suppose, further, that random fluctuations in the path length between the scatterer and the measurement point make the phase measurement uncertain, then the phase has a probability density function. If the fluctuations are such that the path length has a Gaussian density then the resulting phase probability density function will be a Gaussian with its 'tails' periodically folded into the interval  $-\pi$  to  $+\pi$ , because measurement of phase is only unambiguous within this interval. This case has been examined by Wintner in Ref 3 (it corresponds to the case of the parameter  $\lambda$  in equation (1) of Ref 3 equal to 2). It is examined further in section 4 of this paper but from a physical viewpoint.

R. von Mises seems to have been among the first\* to consider the idea of a probability density function mapped on to a circle<sup>4</sup> and von Mises density function (Ref 4, equation (12) and Fig 5, page 494) is employed in this work in section 5.3. In connection with Ref 4, see the comments by Greenwood and Durand in Ref 6, section 2. Greenwood and Durand call von Mises density function the 'circular normal distribution' but it would be more apt to call the Gaussian (or 'normal') density mapped onto a unit circle the circular normal density, although in this paper that function will be called the theta density for reasons which will become apparent in section 4.

The non-isotropic two-dimensional random walk appears to have been considered first by Rayleigh<sup>7</sup> who considered the walk in which the angle density functions of each step were uniform over an interval not equal to  $2\pi$ .

Returning now to the example of the fluctuating-point scatterer, consider an experiment in which the phase and amplitude of the scattered wave is measured at a number of intervals of time. It is assumed that sufficient time is allowed to lapse between each measurement so that the samples are not significantly correlated. The measured vectors are added together on the complex plane to form a random walk. It is the

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\* Although see footnote (1) on page 8 of Ref 5 where it appears that P. Lévy had considered probability distributions mapped on to the unit circle several years earlier.

object of the work presented in this paper to investigate the phase probability density function of the resultant of such a random walk. In particular, the case of most interest was that in which the phase density function of each step was almost uniform and it is to this case that most attention is directed. It will be shown that, for a large number of steps, even slight deviations from uniformity in each step phase density function can make the phase density of the resultant sharply peaked. In other words the effect of the random walk is to amplify the non-uniformity in the step phase probability density function.

One avenue of approach to the problem, at least for a large number of steps, is to invoke the central limit theorem (see Ref 8, §5-4). The central limit theorem predicts that for a large number of steps the real and imaginary components of the resultant of a walk on the complex plane will each tend to have a Gaussian distribution. One would then work out the first and second moments, write down the joint density function (Ref 8, §8-2), and then compute the phase density, using the standard methods of transforming probability functions given in, for example, Ref 8, §3-6. This method was not adopted for two main reasons. Firstly it was desired to see how far one could go in constructing an analysis valid for a small number of steps as well as a large number. Secondly an analysis based on the central limit theorem rapidly becomes unwieldy and awkward to handle in the case of the phase density function. The analysis which follows is based largely on the theory of generalised functions (concentrated on smooth hyper-dimensional manifolds) given by Gel'fand and Shilov in Ref 9. No attempt has been made to be consistently rigorous and some of the analysis is based on physical ideas. In the final section the results of this analysis are checked against numerically computed phase probability density functions obtained by a Monte-Carlo method.

Throughout this paper the term 'probability density function' is used in preference to 'probability distribution' in order to avoid confusion with the completely different usage of the term distribution in generalised function theory.

## 2 THE JOINT PROBABILITY DENSITY FUNCTION

### 2.1 Introduction

In this section, the joint probability density function (which henceforth will be called 'the density function') of magnitude and phase of the resultant of a two-dimensional non-uniform walk is derived. It is first introduced in an intuitive way, and then in sections 2.2 to 2.4 it is obtained in a more rigorous manner.

The density function is denoted by  $p(a, \theta)$  and the probability of finding the resultant of a random walk within a distance  $da$  and within an angle  $d\theta$  of the point  $ae^{i\theta}$  on the complex plane is:

$$dp = p(a, \theta) da d\theta$$

Since it is a certainty that the vector has somewhere on the complex plane:

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$$\int_{-\pi}^{\pi} \int_0^{\infty} p(a, \theta) da d\theta = 1 \quad (2)$$

### 2.1.1 A one step 'walk'

Consider a 'walk' consisting of one step of length  $r$  in a direction  $\phi$ ,  $\phi$  being measured anticlockwise from the real line to the vector  $\underline{r}$ , then the 'probability density function' corresponding to this 'walk' of fixed, certain, length  $r$  and fixed, certain, angle  $\phi$  is

$$p(a, \theta) = \delta(a - r)\delta(\theta - \phi) \quad (3)$$

$p(a, \theta)$  is to be regarded as a generalised function defined in terms of the delta functions  $\delta(a - r)$  and  $\delta(\theta - \phi)$ . It is necessary that  $r > 0$ . The angle delta function is periodic, with a period of  $2\pi$ . Thus

$$\int_0^{\infty} \int_{-\pi}^{\pi} p(a, \theta) d\theta da = \int_0^{\infty} \int_{-\pi}^{\pi} \delta(a - r)\delta(\theta - \phi) dr d\phi = 1 \quad (4)$$

for all  $\theta \in [-\pi, \pi]$  and  $r > 0$ .

Suppose now that  $\phi$  has a continuous density function over  $-\pi, \pi$ . Let this function be  $P(\phi)$ . Then

$$p(a, \theta) = P(\phi)\delta(a - r) = \delta(a - r) \int_{-\pi}^{\pi} P(\phi)\delta(\theta - \phi) d\phi, \quad (5)$$

where a well-known convolution property of the delta function has been used, see Ref 9, §5.2, equation (3), page 104. So (5) is obtained by convolving (3) with the angle density function. Note that this is easily extended to the case where  $r$  is also continuously distributed for then:

$$p(a, \theta) = \int_0^{\infty} Q(r)\delta(a - r) dr \int_{-\pi}^{\pi} P(\phi)\delta(\theta - \phi) d\phi \quad (6)$$

for  $r > 0$ ,  $\theta \in [-\pi, \pi]$ .

### 2.1.2 The two step walk

Now suppose that the walk has two steps of fixed and certain length  $r_1$  and  $r_2$  at fixed certain angles of  $\phi_1$  and  $\phi_2$ . Then evidently

$$p(a, \theta) = \delta\left(a - |r_1 e^{i\phi_1} + r_2 e^{i\phi_2}|\right)\delta(\theta - \psi) \quad (7)$$

where the resultant of vectors  $\underline{r}_1$  and  $\underline{r}_2$  is

$$\text{resultant} = |r_1 e^{i\phi_1} + r_2 e^{i\phi_2}| e^{i\psi} \quad (8)$$

If  $\phi_1$  and  $\phi_2$  are allowed to have any value with probability densities  $P_1(\phi_1)$  and  $P_2(\phi_2)$  then convolving (7) with  $P_1(\phi_1)$  and  $P_2(\phi_2)$  gives:

$$p(a, \theta) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_1(\phi_1) P_2(\phi_2) \delta\left(a - |r_1 e^{i\phi_1} + r_2 e^{i\phi_2}|\right) \delta(\theta - \psi) d\phi_1 d\phi_2 \quad (9)$$

and again the result may be extended to the case where  $r_1$  and  $r_2$  have continuous densities  $Q_1(r_1)$  and  $Q_2(r_2)$  by convolving over  $r_1$  and  $r_2$ . Physically, (9) may be regarded as the sum of all combinations of vectors of lengths  $r_1$  and  $r_2$  over all angles weighted by their respective probabilities, the integrations being regarded as limits of sums over the probability densities. It will be seen further that if  $\phi_1$  and  $\phi_2$  are not independent then  $p_1(\phi_1)p_2(\phi_2)$  is replaced by the joint density  $p(\phi_1, \phi_2)$  in equation (9), although in this paper only independent steps are considered.

### 2.1.3 The N step walk

It is straightforward to continue the argument to the random walk of N steps each step having a length  $r_k$ ,  $k \in [1, N]$  and angle  $\phi_k$ ,  $\phi_k \in [-\pi, \pi]$ . Then

$$p(a, \theta) = \delta\left(a - \left| \sum_{k=1}^N r_k e^{i\phi_k} \right|\right) \delta(\theta - \psi) \quad (10)$$

where

$$\text{the resultant} = \left| \sum_{k=1}^N r_k e^{i\phi_k} \right| e^{i\psi} \quad (11)$$

If the vectors have continuous phase angle density functions  $P_k(\phi_k)$ , then

$$p(a, \theta) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_1(\phi_1) \dots P_N(\phi_N) \delta\left(a - \left| \sum_{k=1}^N r_k e^{i\phi_k} \right|\right) \delta(\theta - \psi) d\phi_1 \dots d\phi_N \quad (12)$$

where there are N convolutions over the  $\phi_k$  and the  $\phi_k$  are all independent. Also  $r_k > 0$  and  $\phi_k \in [-\pi, \pi]$ .

(12) is the required expression for the joint  $(a, \theta)$  density function of an N step non-uniform walk. In general two of the integrations may be performed (since there are two delta functions) but the other integrations are very difficult. The density function has been obtained here in an intuitive way but it can be shown to follow from the characteristic function of the walk and this is the subject of sections 2.2 to 2.4.

## 2.2 The characteristic function

The angle density function of each step is periodic over  $-\pi, \pi$  and is defined by a Fourier series:

$$P_k(\phi_k) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} B_n(k) e^{ni\phi_k} \quad (13)$$

where the notation  $B_n(k)$  means the  $n$ th Fourier coefficient of the  $k$ th step,  $B_n(k)$  may be complex and  $B_0(k) = 1$  since  $\int_{-\pi}^{\pi} P_k(\phi_k) d\phi_k = 1$ .

The characteristic function =  $f(\xi, \zeta)$  and is defined as

$$f(\xi, \zeta) = \langle \exp i(\xi x + \zeta y) \rangle, \quad (14)$$

see Ref 8, section 4-3, page 52, also Ref 19, section 3.2, page 41

where

$$x(\phi_1 \dots \phi_N; r_1 \dots r_N) = \sum_{k=1}^N r_k \cos \phi_k \quad (15)$$

and

$$y(\phi_1 \dots \phi_N; r_1 \dots r_N) = \sum_{k=1}^N r_k \sin \phi_k. \quad (16)$$

The average, implied in the notation  $\langle \rangle$ , is taken over the angles  $\phi_k$ . So

$$f(\xi, \zeta) = \langle \exp i \left[ \xi \sum_1^N r_k \cos \phi_k + \zeta \sum_1^N r_k \sin \phi_k \right] \rangle \quad (17)$$

$$= \langle \prod_1^N \exp i(\xi r_k \cos \phi_k + \zeta r_k \sin \phi_k) \rangle \quad (18)$$

$$= \prod_1^N \langle \exp i(\xi r_k \cos \phi_k + \zeta r_k \sin \phi_k) \rangle. \quad (19)$$

The density function of the angles  $\phi_k$  is given by (13). Hence performing the averaging over the  $\phi_k$  gives:

$$f(\xi, \zeta) = \prod_{k=1}^N \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp i(\xi r_k \cos \phi_k + \zeta r_k \sin \phi_k) \sum_{n=-\infty}^{\infty} B_n(k) \exp in\phi_k d\phi_k \right]. \quad (20)$$

Put  $\xi = \alpha \cos \eta$  and  $\zeta = \alpha \sin \eta$ , then integrating the Fourier series term by term gives

$$g(\alpha, \eta) = \prod_{k=1}^N \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} b_n(k) \int_{-\pi}^{\pi} \exp[i\alpha r_k \cos \phi_k + ni(\phi_k + \eta)] d\phi_k \quad (21)$$

$$= \prod_{k=1}^N \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} B_n(k) \exp ni\eta \int_{-\pi}^{\pi} \exp(i\alpha r_k \cos \phi_k + ni\phi_k) d\phi_k \quad (22)$$

Now

$$\int_{-\pi}^{\pi} \exp(i\alpha r_k \cos \phi_k + ni\phi_k) d\phi_k = 2\pi i^n J_n(\alpha r_k) \quad (23)$$

where  $J_n$  is a Bessel function of order  $n$ . See Ref 10, §2.2 (5).

Hence

$$g(\alpha, \eta) = \prod_{k=1}^N \sum_{n=-\infty}^{\infty} B_n(k) \exp in(\eta + \pi/2) J_n(\alpha r_k) \quad (24)$$

The sum over  $n$  in (24) is absolutely convergent. This may easily be seen by putting  $B_n(k) = 1$ . The sum is then

$$\sum_{n=-\infty}^{\infty} \exp in(\eta + \pi/2) J_n(\alpha r_k) = \exp(i\alpha \cos \eta) \quad (25)$$

which is absolutely convergent (Watson Ref 10, §2.1). Since all the  $B_n(k)$  are finite, i.e. less than some constant  $K$  the series

$$\sum_{n=-\infty}^{\infty} B_n(k) \exp in(\eta + \pi/2) J_n(\alpha r_k)$$

is also absolutely convergent.

So the series in (24) may be multiplied together and the finite product may be evaluated. Multiplying out the product in (24) gives

$$\prod_{k=1}^N \sum_{n=-\infty}^{\infty} B_n(k) \exp in(\eta + \pi/2) J_n(\alpha r_k) = \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_{n_1}(1) \dots B_{n_N}(N) \exp im(\eta + \pi/2) \times J_{n_1}(\alpha r_1) \dots J_{n_N}(\alpha r_N) \quad (26)$$

where  $m = n_1 + \dots + n_N$ .

Hence

$$g(\alpha, \eta) = \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_n(1) \dots B_n(N) \exp im(\eta + \pi/2) J_{n_1}(\alpha r_1) \dots J_{n_N}(\alpha r_N) \quad (27)$$

which is the required expression for the characteristic function expressed in radial and angular frequencies.

### 2.3 The density function as a sum of Hankel transforms

The density function is given by the inverse Fourier transform of the characteristic function (Ref 8, §4-3). In polar frequency coordinates this is

$$p(x, y) = \frac{1}{4\pi^2} \int_0^{\infty} \int_{-\pi}^{\pi} \exp -i\alpha(x \cos \eta + y \sin \eta) g(\alpha, \eta) d\eta d\alpha \quad (28)$$

or if  $x = a \cos \theta$  and  $y = a \sin \theta$

$$p(a, \theta) = \frac{1}{4\pi^2} \int_0^{\infty} \int_{-\pi}^{\pi} \exp[-i\alpha a \cos(\eta - \theta)] g(\alpha, \eta) d\eta d\alpha \quad (29)$$

Now, consider

$$I = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i\alpha a \cos(\eta - \theta)] g(\alpha, \eta) d\eta \quad (30)$$

Then

$$I = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i\alpha a \cos(\eta - \theta)] \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_n(1) \dots B_n(N) \exp im(\eta + \pi/2) \times \\ \times J_{n_1}(\alpha r_1) \dots J_{n_N}(\alpha r_N) d\eta \quad (31)$$

$$= \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_n(1) \dots B_n(N) J_{n_1}(\alpha r_1) \dots J_{n_N}(\alpha r_N) \exp im\pi/2 \times \\ \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i\alpha a \cos(\eta - \theta) + im\eta] d\eta \quad (32)$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i\alpha a \cos(\eta - \theta) + im\eta] d\eta = \exp im(\theta - \pi/2) J_m(\alpha a) \quad (33)$$

So

$$I = \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_n(1) \dots B_n(N) J_{n_1}(\alpha r_1) \dots J_{n_N}(\alpha r_N) \exp im\theta J_m(\alpha a) \quad (34)$$

and

$$p(a, \theta) = \frac{1}{2\pi} \int_0^{\infty} I \alpha da \quad . \quad (35)$$

Hence

$$p(a, \theta) = \frac{a}{2\pi} \int_0^{\infty} \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_{n_1}(1) \dots B_{n_N}(N) J_{n_1}(\alpha r_1) \dots J_{n_N}(\alpha r_N) \exp im^{\alpha} J_m(\alpha a) \alpha da \quad , \quad (36)$$

which is a sum of Hankel transforms. This is the required result. Notice that if the angle density functions are uniform then the only non-zero term in the sum is that for which  $n_1 = \dots = n_N = 0$ . In which case,

$$p(a, \theta) = \frac{a}{2\pi} \int_0^{\infty} J_0(\alpha r_1) \dots J_0(\alpha r_N) J_0(\alpha a) \alpha da \quad , \quad (37)$$

ie

$$p(a) = a \int_0^{\infty} J_0(\alpha r_1) \dots J_0(\alpha r_N) J_0(\alpha a) \alpha da \quad , \quad (38)$$

which is Klyver's integral for the uniform walk - Ref 11 and Ref 12, also Ref 19, section 4.4, page 93.

Integrals similar to those in (36) have been studied by several writers. The most recent appears to be Bailey<sup>13</sup> who gives references to earlier work. Watson<sup>10</sup> also gives further references. In particular Bailey shows that (Ref 13, equation (10.3)):

$$\int_0^{\infty} J_{\rho}(bt) J_{\mu}(a_1 t) J_{\nu}(a_2 t) \dots J_k(a_n t) t^{\rho-\mu-\nu-\dots-k+1} dt = 0 \quad , \quad (39)$$

when  $b > a_1 + a_2 + \dots + a_n$  and all  $a_k$  are positive.

In our case  $\rho = \mu + \nu + \dots + k$  and this obviously reduces to the integrals in (36). The result in (39) means that the integral has bounded support - or a finite span - physically the integral has a finite span because the resultant  $a$  cannot be greater than  $r_1 + r_2 + \dots + r_N$ . The probability that  $a > r_1 + \dots + r_N$  is zero and this is the physical interpretation of Bailey's result. Unfortunately Bailey's technique cannot be used to evaluate the integral when  $a \leq r_1 + \dots + r_N$ . Under certain circumstances the resultant  $a$  may be confined to an annulus. Suppose that one step  $r_1$ , say is greater than the sum of all the others, ie  $r_1 > r_2 + \dots + r_N$ . Then the probability that  $a < r_1 - r_2 - \dots - r_N$  is zero. This is also contained in Bailey's result since  $a < r_1 - r_2 - \dots - r_N \Rightarrow r_1 > r_2 + r_3 + \dots + r_N + a$  and this is just a matter of redefining the walk. It will next be shown how (36) may be transformed into (12).

## 2.4 Transformation of the density function

We have

$$p(a, \theta) = \frac{a}{2\pi} \int_0^\infty \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_n(1) \dots B_n(n) J_{n_1}(ar_1) \dots J_{n_N}(ar_N) \exp im\theta J_m(\alpha a) \alpha d\alpha, \quad (36)$$

and the Fourier coefficients  $B_n(k)$  are given by

$$B_n(k) = \int_{-\pi}^{\pi} P(\phi_k) e^{-in_k \phi_k} d\phi_k. \quad (40)$$

Replacing the  $B_n(k)$  in (36) by (40) gives:

$$p(a, \theta) = \frac{a}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} J_{n_1}(ar_1) \dots J_{n_N}(ar_N) \times \exp[im\theta - i(n_1\phi_1 + \dots + n_N\phi_N)] J_m(\alpha a) \alpha d\alpha, \quad (41)$$

where  $\int_{-\pi}^{\pi} P(\phi_k) d\phi_k$  means  $N$  repeated integrals on the same interval,

ie

$$\int_{-\pi}^{\pi} P(\phi_k) d\phi_k \equiv \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} P(\phi_1) \dots P(\phi_N) d\phi_1 \dots d\phi_N. \quad (42)$$

Now the point on the complex plane at which the density function is measured is  $\underline{a} = ae^{i\theta}$ . Subtracting the last step  $r_N$  from  $\underline{a}$  gives:

$$\underline{c} = \underline{a} - r_N \quad (43)$$

so that

$$c^2 = a^2 + r_N^2 - 2ar_N \cos(\theta - \phi_N). \quad (44)$$

See Fig 1.

Let  $m_j = \sum_{k=1}^j n_k$  so  $m \equiv m_N$  and  $m_N = n_N + m_{N-1}$ . Then the sum over  $n_N$  in (41) may be written

$$\begin{aligned} & \sum_{n_N=-\infty}^{\infty} J_{n_N}(ar_N) J_m(\alpha a) \exp i(m\theta - n_N\phi_N) \\ &= \exp im_{N-1}\theta \sum_{n_N=-\infty}^{\infty} J_{(n_N+m_{N-1})}(\alpha a) J_{n_N}(ar_N) \exp in_N(\theta - \phi_N). \quad (45) \end{aligned}$$

The sum on the RHS in (45) may be summed by a Bessel function addition theorem (Graf's formula), see Watson Ref 10, §11.3 (2). Then

$$\sum_{n_N=-\infty}^{\infty} J(n_N + m_{N-1})(\alpha a) J_{n_N}(\alpha r_N) \exp i n_N(\theta - \phi_N) = J_{m_{N-1}}(\alpha c) \exp i m_{N-1}(\eta - \phi) \quad (46)$$

where  $c$  and  $\eta$  are defined in Fig 1.

Then

$$\sum_{n_N=-\infty}^{\infty} J_{n_N}(\alpha r_N) J_{m_{N-1}}(\alpha a) \exp i(m_{N-1}\theta - n_N\phi_N) = J_{m_{N-1}}(\alpha c) \exp i m_{N-1}\eta \quad (47)$$

And so

$$p(a, \theta) = \frac{a}{2\pi} \int_0^{\infty} \alpha d\alpha \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_{N-1}=-\infty}^{\infty} J_{n_1}(\alpha r_1) \dots J_{n_{N-1}}(\alpha r_{N-1}) \times \\ \times \exp[i m_{N-1}\eta - i(n_1\phi_1 + \dots + n_{N-1}\phi_{N-1})] J_{m_{N-1}}(\alpha c) \quad (48)$$

Evidently this process can be continued by subtracting the next step  $r_{N-1}$  from  $\underline{c}$  and so on. When the walk has been reduced by  $N-1$  steps then:

$$p(a, \theta) = \frac{a}{2\pi} \int_0^{\infty} \alpha d\alpha \int_{-\pi}^{\pi} P(\phi_x) d\phi_k \sum_{n_1=-\infty}^{\infty} J_{n_1}(\alpha r_1) J_{n_1}(\alpha v) e^{i n_1(\theta - \phi_1)} \quad (49)$$

where 
$$\underline{v} = |\underline{a} - \underline{r}_N - \dots - \underline{r}_2| e^{i\chi} \quad (50)$$

Applying Graf's formula for the last time gives

$$p(a, \theta) = \frac{a}{2\pi} \int_0^{\infty} \alpha d\alpha \int_{-\pi}^{\pi} P(\phi_k) d\phi_k J_0(\alpha w) \quad (51)$$

where 
$$w = |\underline{a} - \sum_{k=1}^N \underline{r}_k| \quad (52)$$

To proceed further it is necessary to reverse the order of integrations in (51); difficulties then arise, however, because the integral over  $\alpha$  diverges at the upper limit. In fact as  $\alpha \rightarrow \infty$

$$\alpha J_0(\alpha w) \sim \sqrt{2\alpha/\pi w} \cos(\alpha w - \pi/4) \quad (53)$$

It will be shown that (51) can be interpreted as a generalised function, and to this end a function  $q_t(a, \theta)$  is introduced:

$$q_t(a, \theta) = \frac{a}{2\pi} \int_0^\infty \alpha \exp\left(-\frac{4\alpha^2}{t}\right) d\alpha \int_{-\pi}^{\pi} P(\phi_k) d\phi_k J_0(\alpha w) \quad (54)$$

and  $\lim_{t \rightarrow \infty} q_t(a, \theta) = p(a, \theta)$ ,  $t$  is an integer. Hence  $p(a, \theta)$  is the limit function of a sequence of function  $q_t(a, \theta)$  as  $t \rightarrow \infty$ . The convergence is now dominated by the exponential factor and the order of integration is reversed to give:

$$q_t(a, \theta) = \frac{a}{2\pi} \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \int_0^\infty \alpha \exp\left(-\frac{4\alpha^2}{t}\right) J_0(\alpha w) d\alpha, \quad (55)$$

where  $w = \left| \underline{a} - \sum_1^N \underline{r}_k \right|$ .

Now consider a fixed walk consisting of  $N$  steps each of length  $r_1 \dots r_N$  at fixed angles of  $\phi_1 \dots \phi_N$ . Let the resultant of this fixed walk be  $s \exp i\psi$ , i.e.

$$s \exp i\psi = r_1 \exp i\phi_1 + \dots + r_N \exp i\phi_N. \quad (56)$$

Then

$$w = \left| \underline{a} - \underline{s} \right| \quad (57)$$

and

$$q_t(a, \theta) = \frac{a}{2\pi} \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \int_0^\infty \alpha \exp\left(-\frac{4\alpha^2}{t}\right) J_0(\alpha |\underline{a} \exp i\theta - \underline{s} \exp i\psi|) d\alpha. \quad (58)$$

Graf's formula is now applied but in reverse. See Fig 2.

Then

$$J_0(\alpha |\underline{a} - \underline{s}|) = \sum_{n=-\infty}^{\infty} J_n(\alpha a) J_n(\alpha s) \exp in(\theta - \psi). \quad (59)$$

Hence

$$q_t(a, \theta) = \frac{a}{2\pi} \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \int_0^\infty \alpha \exp\left(-\frac{4\alpha^2}{t}\right) d\alpha \sum_{n=-\infty}^{\infty} J_n(\alpha a) J_n(\alpha s) \exp in(\theta - \psi). \quad (60)$$

The Neumann series in (60) is uniformly convergent, so integrating term by term:

$$q_t(a, \theta) = \frac{a}{2\pi} \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \sum_{n=-\infty}^{\infty} \exp in(\theta - \psi) \int_0^\infty J_n(\alpha a) J_n(\alpha s) \exp\left(-\frac{4\alpha^2}{t}\right) \alpha d\alpha. \quad (61)$$

Now

$$\int_0^\infty J_n(\alpha a) J_n(\alpha s) \exp\left(-\frac{4\alpha^2}{t}\right) \alpha d\alpha = 2t^2 \exp[-t^2(a^2 + s^2)] I_n(2t^2 as) \quad (62)$$

where  $I_n$  is the modified Bessel function of order  $n$ , see Watson Ref 10, §13.31 (1).

Also, for large  $t$

$$I_n(2t^2as) \sim \frac{1}{2t\sqrt{\pi as}} \exp(2t^2as) \quad (63)$$

See Watson, Ref 10, §7.23.

Hence

$$\int_0^\infty J_n(ua)J_n(us) \exp\left(-\frac{4a^2}{t^2}\right) u du \sim \frac{t}{\sqrt{\pi}} \frac{1}{\sqrt{as}} \exp - t^2(a-s)^2 \quad (64)$$

and this is a delta convergent sequence. See Gel'fand and Shilov, Ref 9, §2.5, ex 2.

The integral  $\int_0^\infty J_n(ua)J_n(us)u du$  can therefore be interpreted as a delta function and we have

$$\int_0^\infty J_n(ua)J_n(us)u du = \frac{1}{\sqrt{as}} \delta(a-s) \quad , \quad (65)$$

and this is independent of  $n$ .

Hence

$$\lim_{t \rightarrow \infty} q_t(a, \theta) = \frac{a}{2\pi} \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \frac{1}{a} \delta(a-s) \sum_{n=-\infty}^{\infty} \exp in(\theta - \psi) \quad (66)$$

The Fourier series  $\sum_{n=-\infty}^{\infty} \exp in(\theta - \psi)$  defines a 'row of deltas' at  $\theta - \psi + 2n\pi$ , or in the case of this problem the delta function at  $\theta - \psi$  since the domain of  $\theta - \psi$  is the interval  $-\pi, \pi$ .

Hence

$$p(a, \theta) = \int_{-\pi}^{\pi} P(\phi_k) d\phi_k \delta(a-s) \delta(\theta - \psi) \quad , \quad (67)$$

ie

$$p(a, \theta) = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} P_1(\phi_1) \dots P_N(\phi_N) d\phi_1 \dots d\phi_N \delta(a-s) \delta(\theta - \psi) \quad (68)$$

where  $|s| = |r_1 \exp i\phi_1 + \dots + r_N \exp i\phi_N|$  (69)

and

$$\underline{s} = |s| \exp i\psi \quad (70)$$

which is the required result.

It has thus been shown how (68), *ie* (12) which was obtained in an intuitive way in section 2.1.3 may be obtained from the characteristic function of the walk.

Two results which were shown above are repeated, that is

$$\frac{1}{2\pi} \int_0^{\infty} J_0(\alpha z) \alpha \, d\alpha = \frac{1}{\sqrt{xy}} \delta(x-y) \delta(\alpha-\beta) \quad (71)$$

where 
$$z^2 = x^2 + y^2 - 2xy \cos(\alpha - \beta) \quad (72)$$

and 
$$\int_0^{\infty} J_n(\alpha x) J_n(\alpha y) \alpha \, d\alpha = \frac{1}{\sqrt{xy}} \delta(x-y) \quad (73)$$

In passing it is pointed out that by making use of the convolution property of the  $\delta$  function (Gel-fand and Shilov<sup>9</sup> §5.2) it is straightforward to show that (71) leads to Neumann's integral theorem (Watson<sup>10</sup> §14.3 (1)) and (73) leads to Hankel's integral transform theorem (Watson<sup>10</sup> §14.3 (3) and the rest of Chapter 14).

### 3 SOLUTION OF THE JOINT DENSITY FUNCTION

In this section an attempt is made to solve (68) directly. It will become apparent that by and large (68) is intractable but it may be integrated in one or two cases.

#### 3.1 The two step walk

The steps have lengths  $r_1$  and  $r_2$  and angle density functions  $P_1(\phi_1)$  and  $P_2(\phi_2)$ .

Then:

$$p(a, \theta) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_1(\phi_1) P_2(\phi_2) \delta\left(a - |r_1 e^{i\phi_1} + r_2 e^{i\phi_2}|\right) \delta(\theta - \psi) d\phi_1 d\phi_2 \quad (74)$$

Now let

$$c = |r_1 e^{i\phi_1} + r_2 e^{i\phi_2}| e^{i\psi} \quad (75)$$

See Fig 3.

Then

$$c^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos \beta = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\phi_2 - \phi_1) \quad (76)$$

since  $\beta = \pi - (\phi_2 - \phi_1)$ , see Fig 3.

Also

$$r_1^2 = c^2 + r_2^2 - 2cr_2 \cos \alpha = c^2 + r_2^2 - 2cr_2 \cos(\phi_2 - \psi) \quad (77)$$

since  $\alpha = \phi_2 - \psi$ .

Hence

$$\psi = \phi_2 - \cos^{-1} \left[ \frac{r_1^2 - c^2 - r_2^2}{2cr_2} \right] \quad (78)$$

and

$$d\phi_1 = cdc / \left[ r_1 r_2 \sqrt{1 - \left( \frac{c^2 - r_1^2 - r_2^2}{2r_1 r_2} \right)^2} \right] = \frac{cdc}{2\Delta(r_1 r_2 c)} \quad (79)$$

where  $\Delta(r_1 r_2 c)$  is the area of triangle  $r_1 r_2 c$ .

So transforming the integral over  $\phi_1$  in (74) to an integral over  $c$  gives:

$$\begin{aligned} p(a, \theta) &= \int_{-\pi}^{\pi} P_2(\phi_2) d\phi_2 \int_{|r_1 - r_2|}^{r_1 + r_2} P_1 \left[ \phi_2 - \cos^{-1} \left( \frac{c^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right] \delta(a - b) \times \\ &\quad \times \delta \left[ \theta - \phi_2 + \cos^{-1} \left( \frac{r_1^2 - c^2 - r_2^2}{2cr_2} \right) \right] \frac{cdc}{2\Delta(r_1 r_2 c)} \cdot \quad (80) \\ &= \frac{2a}{\Delta(r_1 r_2 a)} \int_{-\pi}^{\pi} P_2(\phi_2) d\phi_2 P_1 \left[ \phi_2 - \cos^{-1} \left( \frac{a^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right] \delta \left[ \theta - \phi_2 + \cos^{-1} \left( \frac{r_1^2 - a^2 - r_2^2}{2ar_2} \right) \right] \\ &\quad \dots (81) \end{aligned}$$

Note that in the integration over  $\phi_2$  there are two delta functions in the interval  $\phi_2 \in [-\pi, \pi]$ .

And so

$$\begin{aligned} p(a, \theta) &= \frac{4a}{\Delta(r_1 r_2 a)} P_2 \left[ \theta + \cos^{-1} \left( \frac{r_1^2 - a^2 - r_2^2}{2ar_2} \right) \right] \times \\ &\quad \times P_1 \left[ \theta + \cos^{-1} \left( \frac{r_1^2 - a^2 - r_2^2}{2ar_2} \right) - \cos^{-1} \left( \frac{a^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right] \quad (82) \end{aligned}$$

which is the required result.

Note that when the angle density function is uniform  $P_1 = P_2 = 1/2\pi$  and then:

$$p(a) = \int_{-\pi}^{\pi} \left( \frac{1}{2\pi} \right)^2 \frac{4a}{\Delta(r_1 r_2 a)} d\theta = \frac{2a}{\pi \Delta(r_1 r_2 a)} \quad (83)$$

which is the result for the uniform walk given by Rayleigh<sup>12</sup> and others.

### 3.2 The three step walk

The steps have lengths  $r_1$ ,  $r_2$  and  $r_3$  and angle density functions  $P_1(\phi_1)$ ,  $P_2(\phi_2)$  and  $P_3(\phi_3)$ . The walk is shown in Fig 4. It will be noted that the walk has been divided up into two two-step walks. If  $a_1$  is the resultant of  $r_1$  and  $r_2$  a new walk may be defined which is the sum of  $a_1$  and  $r_3$ . The density function of  $a_1$  is given by (82):

$$p(a_1, \theta_1) = \frac{2a_1}{\Delta(r_1 r_2 a_1)} P_1 \left[ \theta_1 + \cos^{-1} \left( \frac{r_1^2 - a_1^2 - r_2^2}{2a_1 r_2} \right) - \cos^{-1} \left( \frac{a_1^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right] \times \\ \times P_2 \left[ \theta_1 + \cos^{-1} \left( \frac{r_1^2 - a_1^2 - r_2^2}{2a_1 r_2} \right) \right] . \quad (84)$$

In forming the density function  $p(a, \theta)$  it is now necessary to convolve over  $a_1$  since  $a_1$  is now continuously distributed over the interval  $a_1 \in [ |r_1 - r_2|, r_1 + r_2 ]$ . Hence the density function of the walk formed by  $a_1$  and  $r_3$  is:

$$p(a, \theta) = \int_{|r_1 - r_2|}^{r_1 + r_2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{2a_1}{\Delta(r_1 r_2 a_1)} P_1 \left[ \theta_1 + \cos^{-1} \left( \frac{r_1^2 - a_1^2 - r_2^2}{2a_1 r_2} \right) - \cos^{-1} \left( \frac{a_1^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right] \times \\ \times P_2 \left[ \theta_1 + \cos^{-1} \left( \frac{r_1^2 - a_1^2 - r_2^2}{2a_1 r_2} \right) \right] P_3(\phi_3) \times \\ \times \delta(a - |a_1 e^{i\theta_1} + r_3 e^{i\phi_3}|) \delta(\theta - \psi) d\theta_1 d\phi_3 da_1 . \quad (85)$$

The same transformation as before is now applied:

Let

$$\underline{c} = |a_1 e^{i\theta_1} + r_3 e^{i\phi_3}| e^{i\psi} , \quad (86)$$

so

$$c^2 = a_1^2 + r_3^2 + 2a_1 r_3 \cos(\phi_3 - \theta_1) , \quad (87)$$

also

$$a_1^2 = c^2 + r_3^2 - 2r_3 c \cos(\phi_3 - \psi) \quad (88)$$

$$\psi = \phi_3 - \cos^{-1} \left( \frac{a_1^2 - r_3^2 - c^2}{2cr_3} \right) , \quad (89)$$

and

$$d\theta_1 = 2cdc / \Delta(a_1 r_3 c) . \quad (90)$$

Also

$$\theta_1 = \phi_3 - \cos^{-1} \left( \frac{c^2 - a_1^2 - r_3^2}{2a_1 r_3} \right) . \quad (91)$$

Hence

$$\begin{aligned}
p(a, \theta) = & \int_{|r_1 - r_2|}^{r_1 + r_2} \int_{-\pi}^{\pi} \int_{|a_1 - r_3|}^{a_1 + r_3} \frac{2a_1}{\Delta(r_1 r_2 a_1)} P_1 \left[ \phi_3 - \cos^{-1} \left( \frac{c^2 - a_1^2 - r_3^2}{2a_1 r_3} \right) + \cos^{-1} \left( \frac{r_1^2 - a_1^2 - r_2^2}{2a_1 r_3} \right) \right. \\
& \left. - \cos^{-1} \left( \frac{a_1^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right] P_2 \left[ \phi_3 - \cos^{-1} \left( \frac{c^2 - a_1^2 - r_3^2}{2a_1 r_3} \right) + \cos^{-1} \left( \frac{r_1^2 - a_1^2 - r_2^2}{2a_1 r_2} \right) \right] \times \\
& \times P_3(\phi_3) \delta(a - c) \delta \left[ \theta - \phi_3 + \cos^{-1} \left( \frac{a_1^2 - c^2 - r_3^2}{2cr_3} \right) \right] \frac{2cdc}{\Delta(a_1 r_3 c)} d\phi_3 da_1 \quad (92)
\end{aligned}$$

Performing the integrations over  $c$  and  $\phi_3$  gives:

$$\begin{aligned}
p(a, \theta) = & \int_{|r_1 - r_2|}^{r_1 + r_2} \frac{4a_1}{\Delta(r_1 r_2 a_1)} \frac{4a}{\Delta(a_1 r_3 a)} P_1 \left[ \theta + \cos^{-1} \left( \frac{a_1^2 - a^2 - r_3^2}{2ar_3} \right) \right. \\
& \left. - \cos^{-1} \left( \frac{a^2 - a_1^2 - r_3^2}{2a_1 r_3} \right) + \cos^{-1} \left( \frac{r_1^2 - a_1^2 - r_2^2}{2a_1 r_2} \right) - \cos^{-1} \left( \frac{a_1^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right] \times \\
& \times P_2 \left[ \theta + \cos^{-1} \left( \frac{a_1^2 - a^2 - r_3^2}{2ar_3} \right) - \cos^{-1} \left( \frac{a^2 - a_1^2 - r_3^2}{2a_1 r_3} \right) \right] \\
& \times P_3 \left[ \theta + \cos^{-1} \left( \frac{a_1^2 - a^2 - r_3^2}{2ar_3} \right) \right] da_1 \quad (93)
\end{aligned}$$

And this is as far as it is possible to go unless the angle density functions are uniform in which case the integral can be performed and leads to a complete elliptic integral - see for example Ref 14 and also Ref 15 Appendix 6.3.

### 3.3 The N step walk

In (82) the angle density functions are  $P_2 \left[ \theta + \cos^{-1} \left( \frac{r_1^2 - a^2 - r_2^2}{2ar_2} \right) \right]$  which is  $P_2[\theta + (\phi_2 - \theta)]$  or  $P_2(\phi_2)$  and  $P_1 \left[ \theta + \cos^{-1} \left( \frac{r_1^2 - a^2 - r_2^2}{2ar_2} \right) - \cos^{-1} \left( \frac{a^2 - r_1^2 - r_2^2}{2r_1 r_2} \right) \right]$

and this is  $P_1[\theta + (\phi_2 - \theta) - (\phi_2 - \phi_1)]$  or  $P_1(\phi_1)$ . Likewise in (93) it will be seen that the angle density functions are  $P_1[\theta + (\phi_3 - \theta) - (\phi_3 - \phi_2) + (\phi_1 - \phi_2)]$  or  $P_1(\phi_1)$  and so on.

Hence it will be seen that extending the analysis to four steps from three is a matter of adding another triangle to the walk and another angle density function,  $\phi_4$

$$\begin{aligned}
P(a, \theta) = & \int_{|a_1 - r_3|}^{a_1 + r_3} \int_{|r_1 - r_2|}^{r_1 + r_2} \frac{4a_1}{\Delta(r_1 r_2 a_1)} \frac{4a_2}{\Delta(a_1 r_3 a_2)} \frac{4a}{\Delta(a_2 r_4 a)} P_1(\phi_1) P_2(\phi_2) P_3(\phi_3) P_4(\phi_4) da_1 da_2 \\
& \dots \dots (94)
\end{aligned}$$

$$\text{where } \phi_1 = \theta + (\phi_4 - \theta) - (\phi_4 - \phi_3) + (\phi_2 - \phi_3) - (\phi_2 - \phi_1) \quad , \quad (95)$$

$$\phi_2 = \theta + (\phi_4 - \theta) - (\phi_4 - \phi_3) + (\phi_2 - \phi_3) \quad , \quad (96)$$

$$\phi_3 = \theta + (\phi_4 - \theta) - (\phi_4 - \phi_3) \quad , \quad (97)$$

$$\phi_4 = \theta + (\phi_4 - \theta) \quad , \quad (98)$$

where the angle differences are written in terms of the lengths of each triangle in the walk.

It is thus possible to extend the result to  $N$  steps, although the integrals appear to be totally intractable. It is not possible to find the phase density function by evaluating the joint density function  $p(a, \theta)$  and then integrating over  $a$ . The problem of finding  $p(\theta)$  is pursued further in section 5 of this Report, but in the next section the most interesting of the angle densities - the Gaussian density - is examined.

#### 4 THE GAUSSIAN PHASE DENSITY FUNCTION OR THE THETA DENSITY

Consider the following physical situation: a wave is incident upon a point in space and is scattered. The phase of the scattered wave is measured at some point. The position of the point is uncertain. Suppose that the distance between the point and the phase measuring observer has a probability density function which is Gaussian. The measurement of the phase has an ambiguity of  $2n\pi$ ,  $n$  being an integer. Hence phase angles greater than  $\pi$  are mapped back on to the interval  $-\pi$  to  $\pi$  as are phase angles less than  $-\pi$ . The phase angle density function as measured by the observer therefore consists of a Gaussian density with its 'tails' periodically folded back onto  $[-\pi, \pi]$ . The function may be represented on the interval  $[-\pi, \pi]$  by a Gaussian periodically repeated at intervals of  $2\pi$ . This is illustrated in Fig 5. The Gaussian phase density function is then:

$$P(\theta) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(\theta - 2n\pi)^2}{2\sigma^2}\right], \quad \theta \in [-\pi, \pi] \quad . \quad (99)$$

Evidently

$$\begin{aligned} \int_{-\pi}^{\pi} P(\theta) d\theta &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{(\theta - 2n\pi)^2}{2\sigma^2}\right] d\theta \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{\theta^2}{2\sigma^2}\right] d\theta = 1 \quad . \quad (100) \end{aligned}$$

The function  $P(\theta)$  will be recognised as a theta function with an imaginary period. It is in fact the third of Jacobi's four theta functions and in the notation of Whittaker and Watson (Ref 16, Chapter 21)

$$P(\theta) = \vartheta_3(\theta/2, i\sigma^2/2\pi)/\sigma\sqrt{2\pi} \quad (101)$$

The corresponding Fourier series may be written down immediately (Poisson's summation formula, Ref 16, Chapter 6, Example 18, page 124, also section 21.51, page 474)

$$P(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2\sigma^2}{2}\right) \exp ni\theta \quad (102)$$

$P(\theta)$  will henceforth be referred to as the theta density, this term being considerably shorter than 'Gaussian phase density'. It has been assumed that the theta density is 'centred' on  $\theta = 0$ . Centring the density function on  $\theta_0$  merely adds a phase shift on to each Fourier coefficient of course:

$$P(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2\sigma^2}{2} - in\theta_0\right) \exp ni\theta \quad (103)$$

The theta function is very rapidly convergent. Rewriting (102):

$$P(\theta) = \frac{1}{2\pi} [1 + 2 \exp\left(-\frac{\sigma^2}{2}\right) \cos \theta + 2 \exp(-2\sigma^2) \cos 2\theta + \dots] \quad (104)$$

let

$$\sigma^2/2 = \pi \quad .$$

Then

$$P(\theta) = \frac{1}{2\pi} [1 + 0.08643 \cos \theta + 6.9 \times 10^{-6} \cos 2\theta + \dots] \quad (105)$$

Hence to a very good approximation when  $\sigma^2/2 > \pi$ ,  $P(\theta)$  can be approximated by the mean term and first harmonic only. This is the case which is of interest, as mentioned in the Introduction and this approximation is used later.

In passing it is mentioned that  $P(\theta)$  may be written in a number of different forms. If the cosines of multiple angles in (104) are written as polynomials in  $\cos \theta$  it is possible to prove the following:

$$P(\theta) = \frac{1}{2\pi} [\vartheta_4 + \vartheta_1^i \cos \theta + \frac{1}{2} \vartheta_4^{ii} \cos^2 \theta + \frac{1}{6} (\vartheta_1^{iii} + \vartheta_1^i) \cos^3 \theta + \frac{1}{24} (\vartheta_4^{iv} + 4\vartheta_4^{ii}) \cos^4 \theta + \dots] \quad (106)$$

where Roman numerals denote derivatives with respect to  $z$  and the first and fourth theta functions  $\vartheta_1$  and  $\vartheta_4$  and derivatives are evaluated at  $z = 0$ ,  $q = e^{-2\sigma^2}$ , i.e. for example

$$\vartheta_1^i = \frac{\partial}{\partial z} \vartheta_1(z|q) \Big|_{z=0, q=e^{-2\sigma^2}} \quad (107)$$

The proof is straightforward but somewhat tedious and will not be given here.

5 PHASE DENSITY FUNCTION

Returning to equation (12):

$$p(a, \theta) = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} P_1(\phi_1) \dots P_N(\phi_N) \delta\left(a - \left| \sum_{k=1}^N r_k e^{i\phi_k} \right|\right) \delta(\theta - \psi) d\phi_1 \dots d\phi_N \dots \quad (12)$$

Integrating over  $a$  gives:

$$p(\theta) = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} P_1(\phi_1) \dots P_N(\phi_N) \delta(\theta - \psi) d\phi_1 \dots d\phi_N \quad (108)$$

and this is the required formula for  $p(\theta)$ . The integrations, however, are not at all straightforward. Firstly, attention is directed towards the delta function  $\delta(\theta - \psi)$ .

5.1  $\delta(\theta - \psi)$ 

The angle  $\psi$  is the phase of the resultant of a fixed walk with step lengths  $r_k$  and step phases of  $\phi_k$ . That is:

$$\psi = \arg(r_1 \exp i\phi_1 + \dots + r_N \exp i\phi_N) \quad (109)$$

If  $\psi$  is limited to a half plane (any half plane), then

$$\arg(r_1 \exp i\phi_1 + \dots + r_N \exp i\phi_N) = \tan^{-1} \left[ \frac{r_1 \sin \phi_1 + \dots + r_N \sin \phi_N}{r_1 \cos \phi_1 + \dots + r_N \cos \phi_N} \right] \quad (110)$$

Outside the half plane there is ambiguity because:

$$\tan^{-1} \left[ \frac{r_1 \sin \phi_1 + \dots + r_N \sin \phi_N}{r_1 \cos \phi_1 + \dots + r_N \cos \phi_N} \right] = \tan^{-1} \left[ \frac{r_1 \sin(\phi_1 + n\pi) + \dots + r_N \sin(\phi_N + n\pi)}{r_1 \cos(\phi_1 + n\pi) + \dots + r_N \cos(\phi_1 + n\pi)} \right] \dots \quad (111)$$

for any positive or negative integer  $n$ .

Hence if  $\delta(\theta - \psi)$  is written as

$$\delta(\theta - \psi) = \delta \left[ \theta - \tan^{-1} \left( \frac{r_1 \sin \phi_1 + \dots + r_N \sin \phi_N}{r_1 \cos \phi_1 + \dots + r_N \cos \phi_N} \right) \right] \quad (112)$$

we have the required delta function at  $\psi = \theta$  and, in addition, 'spurious' delta functions at  $\psi = \theta \pm n\pi$ . There are then two delta functions in the interval  $[-\pi, \pi]$ . The effects of the additional delta are considered in section 5.1.2.

5.1.1 Transformation of  $\delta(\theta - \psi)$ 

We have

$$\tan \psi = \frac{r_1 \sin \phi_1 + \dots + r_N \sin \phi_N}{r_1 \cos \phi_1 + \dots + r_N \cos \phi_N} \quad (113)$$

Now change the variables, so that  $u_j = \phi_j - \theta$  etc. This is a simple linear rotation by an angle  $\theta$  with no change of scale and does not scale the  $\delta$  function. Note that in (108) the convolution is performed by rotating the delta function relative to the plane containing the various combinations of vectors which make up the walk. The transformation simply means that the delta function is now held stationary and the planes containing the combinations of vectors are rotated relative to it.

Then

$$\tan \psi = \frac{r_1 \sin(\phi_1 + \theta) + \dots + r_N \sin(\phi_N + \theta)}{r_1 \cos(\phi_1 + \theta) + \dots + r_N \cos(\phi_N + \theta)}, \quad (114)$$

ie

$$\tan(\psi - \theta) = \frac{r_1 \sin u_1 + \dots + r_N \sin u_N}{r_1 \cos u_1 + \dots + r_N \cos u_N}. \quad (115)$$

And so

$$\delta(\theta - \psi) = \delta \left[ \tan^{-1} \left( \frac{r_1 \sin u_1 + \dots + r_N \sin u_N}{r_1 \cos u_1 + \dots + r_N \cos u_N} \right) \right]. \quad (116)$$

Generalised functions are uniquely determined by their local properties. Indeed in showing that a delta function  $\delta[f(u_1 \dots u_N)]$  is equal to another generalised function  $k(u_1 \dots u_N) \delta[g(u_1 \dots u_N)]$ , say, it is only necessary to consider the behaviour of  $f(u_1 \dots u_N)$  and  $g(u_1 \dots u_N)$  in a sufficiently small neighbourhood of the support of the delta function. The support of the delta function is the set  $[a_i]$  for which  $f(u_1 \dots u_N) = 0$ , or  $g(u_1 \dots u_N) = 0$ . In other words the support is the  $N - 1$  dimensional hypersurface (manifold)  $f(u_1 \dots u_N) = 0$ . In connection with those local properties of generalised functions which are assumed here, see Gel'fand and Shilov<sup>9</sup>, §A1.3, pages 144-146. Note that in (116)  $\tan^{-1} \left( \frac{r_1 \sin u_1 + \dots + r_N \sin u_N}{r_1 \cos u_1 + \dots + r_N \cos u_N} \right)$  is smooth and infinitely differentiable in a 'sufficiently small neighbourhood' of the manifold  $r_1 \sin u_1 + \dots + r_N \sin u_N = 0$ . Hence expanding  $\tan^{-1} \left( \frac{r_1 \sin u_1 + \dots + r_N \sin u_N}{r_1 \cos u_1 + \dots + r_N \cos u_N} \right)$  about the manifold  $r_1 \sin u_1 + \dots + r_N \sin u_N = 0$  by Taylor's series gives:

$$\begin{aligned} \tan^{-1}[f(u_1 \dots u_N)] &= \tan^{-1}[f(a_1 \dots a_N)] \\ &+ \sum_{k=1}^N \left[ \frac{1}{1 + [f(u_1 \dots u_N)]^2} \frac{\partial f(u_1 \dots u_N)}{\partial u_k} \right]_{u=a} (u_k - a_k) \\ &+ 0(u - a)^2 \end{aligned} \quad (117)$$

$$= \sum_{k=1}^N \frac{\partial f(a_1 \dots a_N)}{\partial u_k} (u_k - a_k) + 0(u - a)^2 \quad (118)$$

where

$$f(u_1 \dots u_N) \equiv \frac{r_1 \sin u_1 + \dots + r_N \sin u_N}{r_1 \cos u_1 + \dots + r_N \cos u_N}. \quad (119)$$

Expanding  $f(u_1 \dots u_N)$  about the same manifold obviously gives:

$$f(u_1 \dots u_N) = \sum_{k=1}^N \frac{\partial f}{\partial u_k} (a_1 \dots a_N)(u_k - a_k) + O(u - a)^2 \quad (120)$$

Hence in the neighbourhood of the manifold

$$\tan^{-1}[f(u_1 \dots u_N)] = f(u_1 \dots u_N) + O(u - a)^2 \quad (121)$$

Hence everywhere,

$$\delta[\tan^{-1}\{f(u_1 \dots u_N)\}] = \delta[f(u_1 \dots u_N)] \quad (122)$$

Now, if  $Q$  is a non-vanishing function of  $u_1 \dots u_N$  and  $P$  is a function of  $u_1 \dots u_N$  which can be zero, then:

$$\delta(PQ) = Q^{-1}\delta(P) \quad (123)$$

see Gel'fand and Shilov<sup>9</sup>, §17, page 236. This can also be easily proved by a Taylor series expansion about the manifold  $P = 0$ . Note that  $r_1 \sin u_1 + \dots + r_N \sin u_N$  can be zero, but  $(r_1 \cos u_1 + \dots + r_N \cos u_N)^{-1}$  can never be zero, in fact:

$$|r_1 \cos u_1 + \dots + r_N \cos u_N| \leq r_1 + \dots + r_N \quad (124)$$

Hence

$$\delta\left[\frac{r_1 \sin u_1 + \dots + r_N \sin u_N}{r_1 \cos u_1 + \dots + r_N \cos u_N}\right] = (r_1 \cos u_1 + \dots + r_N \cos u_N)\delta(r_1 \sin u_1 + \dots + r_N \sin u_N) \quad (125)$$

And so:

$$\delta\left[\tan^{-1}\left(\frac{r_1 \sin u_1 + \dots + r_N \sin u_N}{r_1 \cos u_1 + \dots + r_N \cos u_N}\right)\right] = (r_1 \cos u_1 + \dots + r_N \cos u_N)\delta(r_1 \sin u_1 + \dots + r_N \sin u_N) \quad (126)$$

and in this form the delta function is amenable to calculation.

### 5.1.2 Some properties of $\delta(\theta - \psi)$

There are two deltas for each set  $\{a_i\}$  satisfying  $r_1 \sin u_1 + \dots + r_N \sin u_N = 0$ . One delta lies on the positive half of the real line and one lies on the negative half. This can be rectified by replacing  $(r_1 \cos u_1 + \dots + r_N \cos u_N)\delta(r_1 \sin u_1 + \dots + r_N \sin u_N)$  by:

$$\delta\left[\arg\left(\sum_{k=1}^N r_k \exp iu_k\right)\right] = \frac{1}{2}\left[|\sum_{k=1}^N r_k \cos u_k| + \sum_{k=1}^N r_k \cos u_k\right]\delta\left[\sum_{k=1}^N r_k \sin u_k\right] \quad (127)$$

and then on the positive half of the real time  $|\sum_{k=1}^N r_k \cos u_k| = \sum_{k=1}^N r_k \cos u_k$ , and on the

negative half  $|\sum_k r_k \cos u_k| = -\sum_k r_k \cos u_k$ . So the deltas on the negative half of the real line subtract leaving only those on the positive half. Formula (127) therefore enables the analysis to be extended over the whole plane, and although the modulus term makes the expression awkward to use, its presence is inevitable.

So equation (108) can be written:

$$p(\theta) = \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} P_1(u_1 + \theta) \dots P_N(u_N + \theta) \frac{1}{2} \left[ \left| \sum_{k=1}^N r_k \cos u_k \right| + \sum_{k=1}^N r_k \cos u_k \right] \delta \left[ \sum_{k=1}^N r_k \sin u_k \right] du_1 \dots du_N, \quad (128)$$

where  $u_1 = \phi_1 - \theta$  etc, and all the  $P_k(u_k + \theta)$  are periodic with period  $2\pi$ .

Now replace the  $P_k(u_k + \theta)$  by their Fourier series representations:

$$P_k(u_k + \theta) = \frac{1}{2\pi} \sum_{n_k=-\infty}^{\infty} B_n(k) \exp i n_k (u_k + \theta). \quad (129)$$

Hence:

$$p(\theta) = \left(\frac{1}{2\pi}\right)^N \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_{n_1}(1) \dots B_{n_N}(N) \exp i m \theta \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \exp i (n_1 u_1 + \dots + n_N u_N) \times \frac{1}{2} \left( \left| \sum_{k=1}^N r_k \cos u_k \right| + \sum_{k=1}^N r_k \cos u_k \right) \delta \left( \sum_{k=1}^N r_k \sin u_k \right) du_1 \dots du_N. \quad (130)$$

where, as usual,  $m = n_1 + \dots + n_N$ . It is evident that if  $m$  is odd adding  $\pi$  on to the  $u_1 \dots u_N$  reverses the sign of  $\exp i (n_1 u_1 + \dots + n_N u_N)$ . Now the delta on the negative half of the real line generated by the term  $\sum_k r_k \cos u_k \delta \left( \sum_k r_k \sin u_k \right)$  is negative

and that on the positive half of the line is positive. Therefore, if  $m$  is odd

$\sum_k r_k \cos u_k \delta \left( \sum_k r_k \sin u_k \right)$  picks out all combinations twice since both deltas cover the same interval, the changes in sign cancelling each other. If  $m$  is even then adding a  $\pi$  on to the  $u_1 \dots u_N$  leaves the sign of  $\exp i (n_1 u_1 + \dots + n_N u_N)$  unchanged. In this case the deltas generated by  $\sum_k r_k \cos u_k \delta \left( \sum_k r_k \sin u_k \right)$  cancel, and the result is zero. Therefore  $\sum_k r_k \cos u_k \delta \left( \sum_k r_k \sin u_k \right)$  pick out the odd harmonics in  $p(\theta)$ . A similar argument for  $\left| \sum_k r_k \cos u_k \right| \delta \left( \sum_k r_k \sin u_k \right)$  can be made, but with a sign reversed.

And so

$$\left( \sum_{k=1}^N r_k \cos u_k \right) \delta \left( \sum_{k=1}^N r_k \sin u_k \right) \quad \text{generates the odd numbered harmonics in } p(\theta), \quad (131)$$

$$\left( \left| \sum_{k=1}^N r_k \cos u_k \right| \right) \delta \left( \sum_{k=1}^N r_k \sin u_k \right) \quad \text{generates the even numbered harmonics in } p(\theta). \quad (132)$$

Unfortunately the modulus in (132) renders its application somewhat awkward. There is, however, one class of problems for which (131) on its own is sufficient and these are problems involving phase density functions which are non-zero only on an acute segment.

### 5.2 The phase density function on an acute segment

If the  $P_k(\phi_k)$  are limited to an acute segment and are zero outside this segment, then the resultant of the walk must also lie within this segment. This is also true if the segment is an open half plane. It is also true if the segment on which the  $P_k(\phi_k)$  are non-zero is a closed half plane, with the exception of one open half plane, for then the resultant still lies in the open half plane.

Suppose that  $P_k(\phi_k) = 0$  for  $|\phi_k| > a$ ,  $a \leq \pi/2$ , then  $p(\theta) = 0$  for  $|\theta| > a$ . Now if  $p(\theta)$  is computed on the basis of formula (131), i.e. if  $p(\theta)$  is written:

$$p(\theta) = \left( \frac{1}{2\pi} \right)^N \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_{n_1}(1) \dots B_{n_N}(N) \exp i n \theta \times \\ \times \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \exp i(n_1 u_1 + \dots + n_N u_N) \left( \sum_{k=1}^N r_k \cos u_k \right) \delta \left( \sum_{k=1}^N r_k \sin u_k \right) \quad (133)$$

Then only odd harmonics are obtained, and a function of the type shown in Fig 6 is obtained. For  $-a \leq \theta \leq a$  the 'true'  $p(\theta)$  is obtained and for  $-\pi \leq \theta \leq a - \pi$  and  $\pi - a \leq \theta \leq \pi$ ,  $-p(\theta)$  is obtained. Of course, since it is necessary that  $p(\theta) \geq 0$  only the result for  $-a \leq \theta \leq a$  has any significance, but within this range it is correct, and  $p(\theta)$  must be identically zero outside it.

On the other hand if  $\left( \left| \sum_{k=1}^N r_k \cos u_k \right| \right) \delta \left( \sum_{k=1}^N r_k \sin u_k \right)$  had been used in (133) then the function shown in Fig 7 would have been obtained, i.e.  $p(\theta)$  for  $-a \leq \theta \leq a$  and  $p(\theta)$  again for  $-\pi \leq \theta \leq a - \pi$  and  $\pi - a \leq \theta \leq \pi$ .

It is straightforward to show that a function such as is shown in Fig 6 where for  $-a \leq \theta \leq a$  the function equals  $p(\theta)$  and for  $n\pi - a \leq \theta \leq n\pi + a$ ,  $n$  odd,  $-p(\theta)$  has a Fourier series which must have only odd harmonics, i.e. if

$$p(\theta) = \frac{1}{2\pi} \sum_n c_n \exp i n \theta \quad (134)$$

Then

$$c_n = \left[ 1 + (-1)^{n+1} \right] \int_{-a}^a f(\theta) \exp - in\theta d\theta, \quad (135)$$

and for a function of the type shown in Fig 7 there are only even harmonics:

$$c_n = \left[ 1 + (-1)^n \right] \int_{-a}^a f(\theta) \exp - in\theta d\theta. \quad (136)$$

One of the many sequences which define a delta function is

$$\delta(x) = \lim_{v \rightarrow \infty} \frac{1}{2v} \int_{-v}^v \exp i\xi x d\xi, \quad (137)$$

see Gel'fand and Shilov<sup>9</sup>, §2.5, page 38 (5).

And so:

$$\delta(r_1 \sin u_1 + \dots + r_N \sin u_N) = \lim_{v \rightarrow \infty} \frac{1}{2v} \int_{-v}^v \exp i\xi(r_1 \sin u_1 + \dots + r_N \sin u_N) d\xi. \quad (138)$$

Applying this expression to (133) gives:

$$\begin{aligned} p(\theta) &= \left( \frac{1}{2\pi} \right)^N \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_{n_1}(1) \dots B_{n_N}(N) \exp im\theta \times \\ &\times \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \exp i(n_1 u_1 + \dots + n_N u_N) [r_1 \cos u_1 + \dots + r_N \cos u_N] \times \\ &\times \lim \frac{1}{2v} \int_{-v}^v \exp i\xi(r_1 \sin u_1 + \dots + r_N \sin u_N) du_1 \dots du_N, \quad (139) \end{aligned}$$

or, rearranging:

$$\begin{aligned} p(\theta) &= \left( \frac{1}{2\pi} \right)^{N+1} \lim \int_{-v}^v d\xi \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_{n_1}(1) \dots B_{n_N}(N) \exp im\theta \times \\ &\times \sum_{k=1}^N \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} r_k \cos u_k \exp(in_1 u_1 + ir_1 \xi \sin u_1) \dots \\ &\exp(in_N u_N + ir_N \xi \sin u_N) du_1 \dots du_N. \quad (140) \end{aligned}$$

Now consider the integral:

$$I = \int_{-\pi}^{\pi} \cos u_j \exp(in_k u_k + ir_k \xi \sin u_k) du_k \quad (141)$$

There are two cases  $j \neq k$  and  $j = k$ . Both are covered by:

$$\begin{aligned} I_i &= \int_{-\pi}^{\pi} \cos u \exp(ir_k \xi \sin u + in_k u) du \\ &= 2(-)^{n_k+1} \frac{1}{\xi} [J_{n_k+1}(r_k \xi) + J_{n_k-1}(r_k \xi)] \quad (142) \end{aligned}$$

Hence for  $\alpha = 0$  and  $\alpha = 1$ :

$$\int_{-\pi}^{\pi} \exp(in_k u + ir_k \xi \sin u) du = (-)^{n_k} 2\pi J_{n_k}(r_k \xi) \quad (143)$$

$$\int_{-\pi}^{\pi} \cos u \exp(in_k u + ir_k \xi \sin u) du = (-)^{n_k+1} 2\pi \frac{n_k}{r_k \xi} J_{n_k}(r_k \xi) \quad (144)$$

where a standard Bessel function recurrence formula has been used (Watson<sup>10</sup>, §2.12, page 17 (1)).

Therefore:

$$\begin{aligned} &\int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} r_k \cos u_k \exp(in_1 u_1 + ir_1 \xi \sin u_1) \dots \exp(in_N u_N + ir_N \xi \sin u_N) du_1 \dots du_N \\ &= (2\pi)^N (-)^{m+1} \frac{n_k}{\xi} J_{n_1}(r_1 \xi) \dots J_{n_N}(r_N \xi) \quad (145) \end{aligned}$$

where, as usual,  $m = n_1 + \dots + n_N$ .

Hence

$$\begin{aligned} p^{(N)} &= \frac{1}{2\pi} \lim_{\xi \rightarrow 0} \int_{-\pi}^{\pi} d\xi \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_n(1) \dots B_n(N) \exp im\theta \\ &= (-)^{m+1} \frac{1}{\xi} J_{n_1}(r_1 \xi) \dots J_{n_N}(r_N \xi) \sum_{k=1}^N n_k \quad (146) \end{aligned}$$

ie

$$p(\theta) = \frac{1}{2\pi} \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} B_n(1) \dots B_n(N) (-)^{m+1} m \exp im\theta \lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} J_{n_1}(r_1\xi) \dots J_{n_N}(r_N\xi) \frac{d\xi}{\xi} \dots (147)$$

If  $m$  is even then  $J_{n_1}(r_1\xi) \dots J_{n_N}(r_N\xi)$  is an even function and  $J_{n_1}(r_1\xi) \dots J_{n_N}(r_N\xi)/\xi$  is an odd function. The integral is then zero. Hence there are only odd numbered harmonics in (147).

Equation (147) can alternatively be written:

$$p(\theta) = \frac{1}{2\pi} \lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} \frac{3\xi}{\xi} \left[ i \frac{3}{3\xi} \prod_{k=1}^N \left\{ \sum_{n_k=-\infty}^{\infty} (-)^{n_k} B_n(k) \exp in_k \xi J_{n_k}(r_k \xi) \right\} \right] \dots (148)$$

and this is a useful form.

The series  $\sum_{n_k=-\infty}^{\infty} (-)^{n_k} B_n(k) \exp in_k \xi J_{n_k}(r_k \xi)$  is a Neumann series and the coefficients  $B_n(k)$  are the coefficients in the Fourier expansion of the angle density function of the  $k$ th step. For example, if each step is certain to be on the real line at  $\theta = 0$  then the density is a delta function and then  $B_n(k) = 1$ .

Hence

$$\sum_{n_k=-\infty}^{\infty} (-)^{n_k} B_n(k) \exp in_k \xi J_{n_k}(r_k \xi) = \sum_{n_k=-\infty}^{\infty} (-)^{n_k} \exp in_k \xi J_{n_k}(r_k \xi) = \exp -ir_k \xi \sin \theta \dots (149)$$

see Watson<sup>10</sup>, §2.22, page 22.

And so

$$p(\theta) = \frac{1}{2\pi} \lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} \frac{3\xi}{\xi} i \frac{3}{3\xi} \exp -i\xi \sin \theta \sum_k r_k = \frac{1}{2\pi} \lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} s \cos \theta \exp -is\xi \sin \theta d\xi \dots (150)$$

where  $s = \sum_k r_k$

$$\begin{aligned} \therefore p(\theta) &= s \cos \theta \delta(s \sin \theta) = \cos \theta \delta(\sin \theta) \\ &= \sum_n (-)^n \delta(\theta - n\pi) \dots (151) \end{aligned}$$

ignoring all delta functions except that within the valid domain of  $\theta$ , ie that at  $\theta = 0$ , leaves:

$$p(\theta) = \delta(\theta) \dots (152)$$

which is the correct result, for the resultant is certain to lie along the real axis at  $\theta = 0$ .

This is a particularly simple example and unfortunately there do not appear to be any more step angle probability density functions which are zero outside an acute segment, which lead to a series which can be summed to give a simple function. For example if the step density is uniform within a segment  $-\alpha \leq \phi \leq \alpha$  and zero elsewhere then  $B_n(k) = \sin n_k \alpha / n_k \alpha$  and so

$$p(\theta) = \frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_{-\alpha}^{\alpha} \frac{d\xi}{\xi} \left[ i \frac{1}{3\theta} \prod_{k=1}^N \left\{ \sum_{n_k=-\infty}^{\infty} (-)^{n_k} \frac{\sin n_k \alpha}{n_k \alpha} \exp i n_k \phi J_{n_k}(r_k \xi) \right\} \right]. \quad (153)$$

In general the sum in (148) can be written as an integral, because

$$B_n(k) = \int_{-\pi}^{\pi} p_k(\phi) \exp - i n_k \phi d\phi$$

and so

$$\begin{aligned} \sum_{n_k=-\infty}^{\infty} (-)^{n_k} B_n(k) \exp i n_k \phi J_{n_k}(r_k \xi) &= \int_{-\pi}^{\pi} p_k(\phi) d\phi \sum_{n_k=-\infty}^{\infty} (-)^{n_k} \exp i n_k (\theta - \phi) J_{n_k}(r_k \xi) \\ &= \int_{-\pi}^{\pi} p_k(\phi) \exp[-i r_k \xi \sin(\theta - \phi)] d\phi \end{aligned} \quad (154)$$

and this leads to the third and last example in this section, the case of a Gaussian step angle density with small variance. Let  $P_k(\phi)$  be the theta density of section 4 (99). Then

$$\begin{aligned} \int_{-\pi}^{\pi} P_k(\phi) \exp - i r_k \xi \sin(\theta - \phi) d\phi &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp - \frac{(\phi - 2n\pi)^2}{2\sigma^2} \exp[-i r_k \xi \sin(\theta - \phi)] d\phi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp - \frac{\phi^2}{2\sigma^2} \exp[-i r_k \xi \sin(\theta - \phi)] d\phi. \end{aligned} \quad (155)$$

It will now be assumed that the density function is sharply peaked around  $\phi = 0$ , i.e.  $\sigma^2$  is small so that  $p(\phi)$  will also be sharply peaked around  $\theta = 0$  and will thus be very small outside the positive half plane. Hence making the approximation  $\sin(\theta - \phi) \approx \sin \theta - \phi \cos \theta$

$$\int_{-\pi}^{\pi} P_k(\phi) \exp(-ir_k \xi \sin(\theta - \phi)) d\phi \sim \frac{1}{\sigma\sqrt{2\pi}} \exp(-ir_k \xi \sin \theta) \int_{-\infty}^{\infty} \exp(-\frac{t^2}{2\sigma^2}) \exp(ir_k \xi \cos \theta) dt$$

$$= \exp(-ir_k \xi \sin \theta) \exp\left\{-r_k^2 \sigma^2 \cos^2 \theta \frac{\xi^2}{2}\right\}. \quad (156)$$

And so

$$p(\theta) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \left[ i \frac{\partial}{\partial \xi} \left\{ \exp(-is\xi \sin \theta) \exp(-t^2 \sigma^2 \cos^2 \theta \frac{\xi^2}{2}) \right\} \right], \quad (157)$$

where  $s = \sum_{k=1}^N r_k$  and  $t^2 = \sum_{k=1}^N r_k^2$ .

The integration is straightforward and leads to:

$$p(\theta) \sim \frac{1}{\sqrt{2\pi}} \frac{s}{t\sigma} \frac{1}{\cos^2 \theta} \exp\left(-\tan^2 \theta \frac{s^2}{2t^2 \sigma^2}\right), \quad (158)$$

which for  $\theta$  small gives:

$$p(\theta) \sim \frac{1}{\sqrt{2\pi}} \frac{s}{t\sigma} \exp\left(-\theta^2 \frac{s^2}{2t^2 \sigma^2}\right). \quad (159)$$

Note that in (158) there is an approximately Gaussian function centred on  $\theta = 0$  and a second centred on  $\theta = \pm\pi$ . The second is positive whereas according to the analysis of this section it should be negative. The reason for this is that in making the approximation  $\sin \phi \approx \phi$  and  $\cos \phi \approx 1$  the periodic properties of  $\sin \phi$  and  $\cos \phi$  are destroyed. Near  $\phi = \pm\pi$ ,  $\sin \phi \approx -\phi$  and  $\cos \phi \approx -1$  and if this is inserted into (156) it will be found that  $-p(\theta)$  results in the negative half plane.

### 5.3 The phase density function on the whole complex plane

In order to obtain  $p(\theta)$  over the whole plane it is necessary to integrate (130) and this seems to be, in general, intractable. One method is to note that the random walk has a finite span, and  $p(a, \theta) \equiv 0$  for  $a > |r_1 + \dots + r_N|$ . Then

$\frac{1}{2} \left[ \sum r_k \cos u_k + \sum r_k \cos u_k \right]$  can be expanded as a Fourier series on the interval

$-\sum r_k$  to  $+\sum r_k$ , and the result will be valid for all  $u_k$  with the sole exception of  $u_k = 0$  or  $n\pi$  where the Fourier series will give  $\frac{1}{2} \sum r_k$  instead of  $\sum r_k$ .

Alternatively, an integral representation could be used. The technique of expanding a function on the span of a random walk has been employed before by, for example, Slater<sup>17</sup>.

The resulting series for  $p(\theta)$  will not be given here; it seems to be beyond all hope of summation and practical usefulness. As pointed out in the Introduction, the main interest is in computing the first order term in  $p(\theta)$  for a small deviation from uniformity in the step density. As shown in section 4, for a Gaussian step angle

density (the practical case of interest) the step angle density can be very accurately described by the constant and first harmonic (fundamental) in the Fourier expansion of  $P(\phi)$ , provided that  $\sigma^2 > 2\pi$ . Fortunately, to obtain  $p(\phi)$  in this case it is not necessary to integrate (130), and it will be shown that the behaviour of  $p(\phi)$  can be deduced from the odd-numbered harmonics alone, from equation (148). Substituting  $(1 + b \cos \phi)/2\pi$  for  $P(\phi)$  (and thus assuming equal length steps) into equation (148) leads to an awkward analysis, however, and a more elegant method for obtaining  $p(\phi)$  for small non-uniformity is the following:

Assume that each step has equal (and arbitrary) length. This simplification is not strictly necessary but gives tidier expressions, and is easily removed. Let the angle probability density function\* of each step be

$$P_k(\phi) = \exp(a \cos \phi) / 2\pi I_0(a) \quad (160)$$

where  $I_0(a)$  is the modified Bessel function of the first kind (order zero). Then  $P_k(\phi) > 0$  and  $\int_{-\pi}^{\pi} P_k(\phi) d\phi = 1$ .

Note that the probability density functions for each step are 'in phase'. The corresponding Fourier series is

$$P_k(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} B_n \exp in\phi,$$

$$\text{where } B_n = \frac{1}{2\pi I_0(a)} \int_{-\pi}^{\pi} \exp(a \cos \phi - in\phi) d\phi$$

and so

$$B_n = I_n(a) / I_0(a) \quad (161)$$

Hence

$$P_k(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{I_n(a)}{I_0(a)} \exp in\phi \quad (162)$$

So in equation (148) it is necessary to find the sum

$$S = \sum_{n=-\infty}^{\infty} (-)^n \frac{I_n(a)}{I_0(a)} J_n(r_k \xi) \exp in\phi \quad (163)$$

This is easily found by Neumann's addition theorem (Watson<sup>10</sup> §11.2 (1)).

Since

$$J_0(\omega) = \sum_{n=-\infty}^{\infty} J_n(z) J_n(z) \exp in\phi \quad (164)$$

\* This function has been plotted in Fig 8 for  $a = 0$  to 1.0 in steps of 0.1. This function is Von Mises density function (Ref 4, equation (12), also Ref 19, section 3.4.9, page 57).

where  $\omega = \sqrt{Z^2 + z^2 - 2Zz \cos \phi}$  and  $Z, z$  have general complex values, putting  $Z = ia$ ,  $z = \xi$ ,  $\phi = \theta + \pi/2$  gives

$$S = \frac{1}{I_0(a)} J_0\left(\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}\right) \quad (165)$$

Hence if  $q(\theta)$  is the 'odd numbered harmonic content' of  $p(\theta)$ , then

$$\begin{aligned} 2q(\theta) &= \frac{1}{2\pi} \left[ \frac{1}{I_0(a)} \right]^N \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \frac{i\partial}{\partial \theta} \left[ J_0\left(\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}\right) \right]^N \\ &= \frac{1}{2\pi} \left[ \frac{1}{I_0(a)} \right]^N Na \cos \theta \int_{-\infty}^{\infty} \left[ J_0\left(\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}\right) \right]^{N-1} \frac{J_1\left(\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}\right)}{\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}} d\xi \quad \dots (166) \end{aligned}$$

It is now assumed that (i)  $a$  is small  $< 0.1$  say and (ii)  $N$  is large  $> 10$  say. Under these conditions it can be shown (see Watson<sup>10</sup> §13.48, page 421) that the significant part of the integrand is that for which  $\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}$  is small, and then

$$\begin{aligned} \left[ J_0\left(\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}\right) \right]^{N-1} J_1\left(\sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta}\right) / \sqrt{\xi^2 - a^2 + 2ia\xi \sin \theta} \\ \sim \frac{1}{2} \exp\left[-\frac{(N-1)}{4} (\xi^2 - a^2 + 2ia\xi \sin \theta)\right] \quad (167) \end{aligned}$$

In connection with this result it is easy to see that for  $|z|$  small  $J_0(z) \sim 1 - \frac{z^2}{4}$

$$\text{hence } [J_0(z)]^N \sim \left(1 - \frac{z^2}{4}\right)^N \sim \exp\left(-\frac{Nz^2}{4}\right)$$

So

$$2q(\theta) \sim \frac{1}{2\pi} \left[ \frac{1}{I_0(a)} \right]^N \frac{Na \cos \theta}{2} \int_{-\infty}^{\infty} \exp\left[-\frac{(N-1)}{4} (\xi^2 - a^2 + 2ia\xi \sin \theta)\right] d\xi \quad (168)$$

Also, for small  $a$

$$I_0(a) \sim 1 + a^2/4 \quad (169)$$

Hence

$$\left[ I_0(a) \right]^N \sim (1 + a^2/4)^N \sim \exp(Na^2/4) \quad a \text{ small} \quad (170)$$

And so, if  $\frac{1}{2}$  is neglected in comparison with  $N$ :

$$2q(\theta) \sim \frac{1}{2\pi} \frac{Na \cos \theta}{2} \int_{-\infty}^{\infty} \exp\left[-\frac{N}{4} (\xi^2 + 2ia\xi \sin \theta)\right] d\xi \quad (171)$$

$$q(\theta) \sim \frac{a \cos \theta}{4} \sqrt{\frac{N}{\pi}} \exp\left(-\frac{Na^2 \sin^2 \theta}{4}\right), \quad (172)$$

and this is an interesting result.

So far it has been assumed that  $N$  is large and  $a$  is small. Now suppose that  $Na^2$  is large, and then:

$$q(\theta) \sim \frac{a}{4} \sqrt{\frac{N}{\pi}} \exp\left(-\frac{Na^2 \theta^2}{4}\right). \quad (173)$$

Note that this is a  $\delta$  convergent sequence, hence

$$\lim_{N \rightarrow \infty} q(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-)^n \delta(\theta - n\pi), \quad (174)$$

which gives delta functions of amplitude  $+1/2$  at  $\theta = 0$  and  $-1/2$  at  $\theta = -\pi$ .

Since  $p(\theta) > 0$  the even harmonics must supply a delta of amplitude at least  $+1/2$  at  $\theta = -\pi$ , and hence also a delta of amplitude at least  $+1/2$  at  $\theta = 0$ . Since

$\int_{-\pi}^{\pi} p(\theta) d\theta = 1$  the amplitudes of the deltas at  $\theta = 0$  and  $\pi$  are each  $+1/2$ , hence as  $N \rightarrow \infty$

$$p(\theta) \rightarrow \sum_{n=-\infty}^{\infty} \delta(\theta - 2n\pi). \quad (175)$$

Therefore, for a small departure from uniformity no matter how small the phase probability density function of the resultant tends towards a delta function as the number of steps increases without limit. When the phase density function of each step are 'in phase' the effect of the random walk is to amplify the non-uniformity in the density of each step.

The first harmonic in  $q(\theta)$  will next be obtained.

Now

$$q(\theta) = \frac{a}{4} \sqrt{\frac{N}{\pi}} \cos \theta \exp\left(-\frac{Na^2 \sin^2 \theta}{4}\right) \quad (176)$$

$$= \frac{a}{4} \sqrt{\frac{N}{\pi}} \sum_{k=0}^{\infty} \left(-\frac{Na^2}{4}\right)^k \frac{1}{k!} \cos \theta \sin^{2k} \theta, \quad (177)$$

and also

$$\cos^{2k} \theta = \frac{2\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k + 1)} \left\{ \frac{1}{2} + \frac{k}{k+1} \cos 2\theta + \dots \right\}, \quad (178)$$

see Whittaker and Watson<sup>16</sup>, page 191, example 13.

Hence

$$\sin^{2k}\theta = \frac{2\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k + 1)} \left\{ \frac{1}{2} - \frac{k}{k + 1} \cos 2\theta + \dots \right\}, \quad (179)$$

ie

$$\begin{aligned} \cos \theta \sin^{2k}\theta &= \frac{\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k + 1)} \left\{ \cos \theta - \frac{2k}{k + 1} \cos \theta \cos 2\theta + \dots \right\}, \\ &= \frac{\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k + 2)} \cos \theta + \text{higher harmonics}. \end{aligned} \quad (180)$$

Hence

$$q(\theta) \sim \frac{a}{4} \sqrt{\frac{N}{\pi}} \cos \theta \sum_{k=0}^{\infty} \left( -\frac{Na^2}{4} \right)^k \frac{\Gamma(k + \frac{1}{2})}{k! \Gamma(\frac{1}{2}) \Gamma(k + 2)} + \text{higher harmonics}. \quad (181)$$

$$= \frac{a}{4} \sqrt{\frac{N}{\pi}} \cos \theta {}_1F_1\left(\frac{1}{2}, 2, -\frac{Na^2}{4}\right) + \text{higher harmonics}. \quad (182)$$

where  ${}_1F_1$  is a confluent hypergeometric function (Whittaker and Watson<sup>16</sup>, §16.1, page 338). Evidently, for small  $Na^2$ , we have

$$q(\theta) \sim \frac{a}{4} \sqrt{\frac{N}{\pi}} \cos \theta \left[ 1 - \frac{Na^2}{16} + \dots \right], \quad (183)$$

$$p(\theta) = \frac{1}{2\pi} \left[ 1 + \frac{a}{2} \sqrt{N\pi} \cos \theta \left\{ 1 - \frac{Na^2}{16} + \dots \right\} + \text{higher harmonics} \right]. \quad (184)$$

Hence the behaviour of  $p(\theta)$  is given by (184) for small values of  $Na^2$  (say for  $Na^2 < 1$ ) and by

$$p(\theta) = \frac{a}{2} \sqrt{\frac{N}{\pi}} \exp - \frac{Na^2 \theta^2}{4}, \quad (185)$$

for large values of  $Na^2$  ( $Na^2 > 10$ ). In each case  $N$  is large and  $a$  small.

The step phase density is

$$P_k(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{I_n(a)}{I_0(a)} \exp in\phi. \quad (186)$$

$$= \frac{1}{2\pi} \left[ 1 + \frac{2I_1(a)}{I_0(a)} \cos \phi + \dots \right], \quad (186)$$

since  $I_{-n}(a) = I_n(a)$ . For small  $a$

$$\frac{I_1(a)}{I_0(a)} \sim \frac{a}{2} \quad (187)$$

and so

$$P_k(\phi) \sim \frac{1}{2\pi} [1 + a \cos \phi] + O(a^2). \quad (188)$$

For the theta density of section 4 for  $\sigma^2 > 2\pi$  :

$$P_k(\phi) \sim \frac{1}{2\pi} \left[ 1 + 2 \exp - \frac{\sigma^2}{2} \cos \phi \right] + 0(\exp - \sigma^2) \quad (189)$$

And so the results above are valid for a theta density with large variance if we put

$$a = 2 \exp - \frac{\sigma^2}{2} \quad (190)$$

The analysis presented here is not particularly rigorous and so it is checked in the next section by comparing results (184) and (185) against the phase probability density function obtained by numerically computing random walks with a Gaussian step phase density.

## 6 NUMERICAL TESTS

In principle it is a straightforward matter to numerically compute the phase probability density of the resultant of a non-isotropic random walk using a Monte-Carlo technique. Suppose that it is desired to compute the phase density of an N-step walk with theta phase density. A quasi-random number generator with a Gaussian density supplies the phase angle of each step. The fact that one computes the changes in x and y coordinates of the resultant (with the addition of each step) from  $\Delta x = r \cos \theta$  and  $\Delta y = r \sin \theta$  ensures that the tails of the Gaussian are wrapped back on to the interval  $[-\pi, \pi]$  because of the periodicity of  $\sin \theta$  and  $\cos \theta$ . The theta probability density is therefore generated. When N steps have been added together, the phase of the resultant is computed and stored. This process is repeated many times and when sufficient walks have been computed a histogram is constructed from the stored phases. This histogram is then an approximation over finite intervals of the probability density of the phase of the resultant of the walk. This is the 'Monte-Carlo' method.

In this section histograms generated using the above method are compared with the analytic results of the previous section. There are certain numerical difficulties which should be mentioned. In particular the quasi-random number generators found in the libraries of most computers, and used to generate the numbers, are usually not entirely satisfactory. This is despite the fact that these generators pass the standard tests for 'randomness' of the numbers generated. In practice the higher order moments of these numbers are far from ideal and if a large number of random numbers is required then correlations between the numbers distort the random walk results. Great care is necessary. It was found necessary to re-randomise the output of the particular generator used by accepting or rejecting the numbers according to an independent table of 0s and 1s. This is easily constructed by tossing a coin. The length of such a table is important. It should not be too long nor too short and experiments must be carried out to determine the optimum. In this case about 100 binary values proved optimum. Greater lengths produced greater smoothing of the probability density of the numbers, but introduced skewness. The random number generator used produced numbers uniformly distributed on the interval  $[0, 1]$ . Gaussian variables were produced from this by the method of Box and Muller<sup>18</sup>, the so-called 'direct method'.

In the computations which follow the histograms were each constructed from 4096 walks. Each walk consists of from 1 to 10000 steps. Thus in the most extreme case about  $4 \times 10^7$  random numbers were required. Each histogram has either 32 or 64 sub-intervals covering the interval  $[-\pi, \pi]$ . Thus, on the average, each sub-interval contains the phase of the resultant of 128 or 64 walks. If the number of sub-intervals is equal to some power of 2 this makes the computation of the harmonic content easy via a discrete (fast) Fourier transform.

The objective of the numerical tests is to check the validity of formulae (159), (184) and (185) and they are treated in that order.

#### 6.1 Gaussian step density of small variance

For equal length steps formula (159) gives:

$$p(\theta) \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{N}}{\sigma} \exp\left(-\frac{\theta^2 N}{2\sigma^2}\right), \quad (191)$$

where  $\sigma^2$  is the variance of the phase distribution of each step, and is assumed to be small enough so that most of the Gaussian distribution is contained within  $\pm\pi/2$ . A value of  $(\pi/6)^2$  was chosen for  $\sigma^2$  and the resulting phase density of the resultant, was computed for 1, 2, 5, 10, 20 and 50 steps. The case of 1 step was chosen as a check on the random walk generator. It will be noted that (159) is valid for any number of steps. The results are shown in Figs 9 to 14. In each case the histogram is computed by the Monte-Carlo method and the super-imposed Gaussian curve is that given by (191) with the appropriate value of  $N$ . It will be seen that in each case formula (191) fits the computed histograms quite closely. Note that, for a constant value of  $\sigma^2$ , the probability density becomes 'sharper' as  $N$  increases.

Table 1 lists the variance of the phase density of the resultant computed from the histogram and also that given by (191). Again there is close agreement. The proportional error becomes larger as  $N$  increases and is caused by the finite and fixed number of sub-intervals in the histogram.

Table 1

Number of steps	$\sigma^2$ from histogram	$\sigma^2$ from (191)
1	0.270	0.274
2	0.136	0.137
5	0.0542	0.0548
10	0.0286	0.0274
20	0.0145	0.0137
50	0.00645	0.00548

In all cases the step phase density has a variance of  $(\pi/6)^2$

### 6.2 Gaussian step density of large variance ( $Na^2 < 1$ )

When the underlying step phase density has a large variance and the number of steps  $N$  is such that  $Na^2 < 1$  where  $a = 2 \exp(-\sigma^2/2)$ ,  $\sigma^2$  being the step phase variance, then the phase density of the resultant is given by (184)

ie

$$p(\theta) \sim \frac{1}{2\pi} \left[ 1 + \frac{a}{2} \sqrt{N\pi} \cos \theta \left( 1 - \frac{Na^2}{16} + \dots \right) + \text{higher harmonics} \right] \quad (184)$$

This formula has been tested by generating random walks having 2, 4, 6, 8, 10, 15 and 20 steps for values of  $Na^2$  of 1, 0.1, and 0.01; a total of 21 cases. Although the analysis is only valid for large  $N$  the computed histograms show remarkable agreement with (184) for only 2 steps. It was therefore decided to include walks with a small number of steps as well as a moderate number: it might be expected that  $N = 10$  would be the lower limit on the validity of (184).

Figs 15, 16 and 17 show the histograms obtained for  $N = 2, 10$  and  $20$  and  $Na^2 = 1$ . Figs 18, 19 and 20 show histograms for  $N = 2, 10$  and  $20$  for  $Na^2 = 0.1$  and Figs 21, 22 and 23 for  $N = 2, 10$  and  $20$ ,  $Na^2 = 0.01$ . In each case the superimposed constant plus first harmonic is that obtained from the histogram by a discrete Fourier transform. It represents the 'filtered' histogram with all frequencies greater than the fundamental cut off.

Table 2 lists the amplitude of the first harmonic for each computed case. Those cases marked with a star are those in Figs 15 to 24.

Table 2

Number of steps	$Na^2$		
	1.0	0.1	0.01
2*	0.904	0.301	0.109
4	0.861	0.277	0.080
6	0.851	0.288	0.105
8	0.844	0.282	0.100
10*	0.844	0.300	0.099
15	0.840	0.282	0.101
20*	0.832	0.278	0.084

Values of the fundamental (first harmonic) given by (184) are 0.831, 0.278 and 0.0886 for  $Na^2 = 1, 0.1$  and  $0.01$  respectively. Note that the amplitude of the first harmonic is taken to be  $\frac{a}{2} \sqrt{N\pi} \left( 1 - \frac{Na^2}{16} + \dots \right)$  and the factor of  $\frac{1}{2}\pi$  has not been included.

There is reasonable agreement between (184) and the Monte-Carlo computations, even for small values of  $N$ . It will be noted that at  $Na^2 = 1$  there is also a significant second harmonic component, of amplitude approximately 1/5 of the first harmonic.  $Na^2 = 1$ , therefore, represents the approximate upper limit of the validity of (184).

### 6.3 Gaussian step density of large variance ( $Na^2 > 10$ )

In this case we have formula (185)

$$p(\theta) \sim \frac{a}{2} \sqrt{\frac{N}{\pi}} \exp - \frac{Na^2 \theta^2}{4} \quad , \quad (185)$$

and this represents the second of the two asymptotic limits of  $p(\theta)$  when the underlying step phase density has a large variance. It has been tested by computing random walks for  $Na^2 = 10$ ,  $N = 10, 10^2$  and  $10^3$  (Figs 24, 25 and 26), for  $Na^2 = 20$ ,  $N = 10^2, 10^3$  and  $10^4$  (Figs 27, 28 and 29), and for  $Na^2 = 100$ ,  $N = 10^2, 10^3$  and  $10^4$  (Figs 30, 31 and 32). In each case the superimposed Gaussian density function is obtained from (185) with the appropriate value of  $Na^2$ . There is reasonable agreement in all cases. Table 3 lists the variance obtained from the histograms. The variance according to (185) should be 0.2, 0.1 and 0.02 for  $Na^2 = 10, 20$  and  $100$  respectively. There is thus reasonable quantitative agreement which seems to be better for the larger values of  $Na^2$ , as is to be expected. The values for  $N = 10^4$  are probably distorted by imperfections in the random number generator which show up for large numbers of steps.

Table 3

Number of steps	$Na^2$		
	10	20	100
10	0.207	-	-
$10^2$	0.273	0.112	0.020
$10^3$	0.270	0.110	0.021
$10^4$	-	0.084	0.018

## 7 CONCLUSIONS

It has been demonstrated that a theory of the phase probability density function of the resultant of a non-isotropic two-dimensional random walk is possible without recourse to the central limit theorem. Although the most general case seems to be totally intractable, in the case of a Gaussian step phase probability density function (theta density) the following results have been obtained:

- (1) If the variance of the phase probability density of each step ( $\sigma^2$ ) is small,  $\sigma^2 < \pi/3$  say, then for  $N$  equal steps the phase probability density of the resultant is given by:

$$p(\theta) \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{N}}{\sigma} \exp\left(-\frac{Na^2}{2\sigma^2}\right),$$

and this is valid for any number of steps.

- (2) (a) If the variance  $\sigma^2$  of the step phase density is large  $\sigma^2 > 2\pi$  say, and the number of steps  $N$  is such that  $Na^2 < 1$  where  $a = 2 \exp(-\sigma^2/2)$  then the phase probability density of the resultant is given by

$$p(\theta) \sim \frac{1}{2\pi} \left[ 1 + \frac{a}{2} \sqrt{N\pi} \cos \theta \left( 1 - \frac{Na^2}{16} + \dots \right) \right].$$

This formula, derived for  $N$  large, is also reasonably good for  $N$  small ( $\geq 2$ ).

- (b) If  $N$  is such that  $Na^2 > 10$  then

$$p(\theta) \sim \frac{a}{2} \sqrt{\frac{N}{\pi}} \exp\left(-\frac{Na^2\theta^2}{4}\right)$$

and this is valid for large  $N$ .

These three cases are of physical interest and are useful results.

In conclusion it is pointed out that if Pearson's drunk<sup>2</sup> is slightly biased in his direction of stagger (perhaps he is trying to make for home) then if he takes enough steps it is very probable that he will go in the desired direction, provided he does not collapse of course.

## REFERENCES

- | <u>No.</u> | <u>Author</u>                  | <u>Title, etc</u>   |
|------------|--------------------------------|---|
| 1          | K. Pearson                     | The problem of the random walk.<br>Nature, Vol 72, No.1865, p 294, 27 July 1905   |
| 2          | K. Pearson                     | The problem of the random walk.<br>Nature, Vol 72, No.1867, p 342 10 August 1905  |
| 3          | A. Wintner                     | On the shape of the angular case of Cauchy's distribution curves.<br><i>Ann. Math. Stat.</i> , <u>18</u> , 589-593 (1947)   |
| 4          | R. von Mises                   | Über die Ganzzahligkeit der Atomgewichte und verwandte Fragen.<br><i>Physikal Zeitschr</i> , <u>19</u> , 490-500 (1918)   |
| 5          | P. Lévy                        | L'addition des variables aléatoires définies sur une circonférence.<br><i>Bull. Soc. Math. France</i> , <u>67</u> , 1-41 (1939)   |
| 6          | J.A. Greenwood<br>D. Durand    | The distribution of length and components of the sum of n random unit vectors.<br><i>Ann. Math. Stat.</i> , <u>26</u> , 233-246 (1955)  |
| 7          | J.W. Strutt<br>(Lord Rayleigh) | On the resultant of a number of unit vibrations, whose phases are random over a range not limited to an integral number of periods.<br><i>Scientific Papers</i> , <u>6</u> , 627-641. Also <i>Phil. Mag.</i> , <u>37</u> , 498-515 (1919) |
| 8          | W.B. Davenport<br>W.L. Root    | An introduction to the theory of random signals and noise.<br>McGraw-Hill (1958)  |
| 9          | I.M. Gel'fand<br>G.E. Shilov   | Generalised functions.<br>Vol 1, Properties and operations.<br>English translation Academic Press (1964)  |
| 10         | G.N. Watson                    | A treatise on the theory of Bessel functions.<br>CUP, Second edition (1958)   |
| 11         | J.C. Kluyver                   | A local probability problem.<br>Proceedings section of <i>Sci. Königl Akad. von Wetensch, Amsterdam</i> , <u>8</u> , 341-350 (1906)   |
| 12         | J.W. Strutt<br>(Lord Rayleigh) | On the problem of random vibrations and of random flights in one, two or three dimensions.<br><i>Scientific papers</i> , <u>6</u> , 604-626. Also <i>Phil. Mag.</i> , <u>37</u> , 321-347 (1919)  |
| 13         | W.N. Bailey                    | Some infinite integrals involving Bessel functions.<br><i>Proc. London Math. Soc.</i> , <u>40</u> , 37-48 (1936)  |

REFERENCES (concluded)

- | <u>No.</u> | <u>Author</u>                 | <u>Title, etc</u>   |
|------------|-------------------------------|---|
| 14         | K. Pearson                    | A mathematical theory of random migration.<br>No.15 of Mathematical contributions to the theory of evolution in<br>Drapers Company Research Memoirs Biometric Series III, pp 1-60<br>(1906) |
| 15         | M. Slack                      | The probability distributions of sinusoidal oscillations combined<br>in random phase.<br><i>IEEE</i> , <u>93</u> , Part 3, 76-86 (1946)   |
| 16         | E.T. Whittaker<br>G.N. Watson | A course of modern analysis.<br>CUP, Fourth edition (reprinted) (1952)  |
| 17         | N.B. Slater                   | General forms of Nielsen's Bessel-function identity.<br><i>Quart. J. Math.</i> , Oxford series 2, <u>13</u> , 111-118 (1962)  |
| 18         | G.E.P. Box<br>M.E. Muller     | A note on the generation of random normal deviates.<br><i>Ann. Math. Stat.</i> , <u>29</u> , 610-611 (1958)   |
| 19         | K.V. Mardia                   | Statistics of directional data.<br>Academic Press (1972)  |

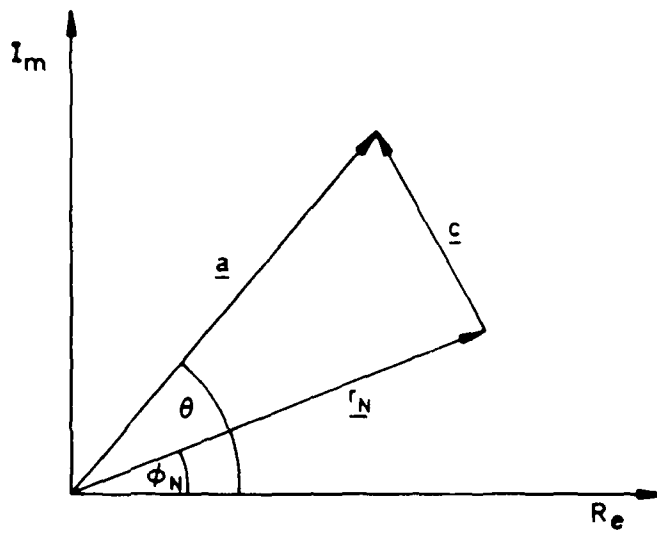


Fig 1 The random walk resultant  $\underline{a}$  minus the last step  $\underline{r_N}$  at a fixed angle  $\phi_N$

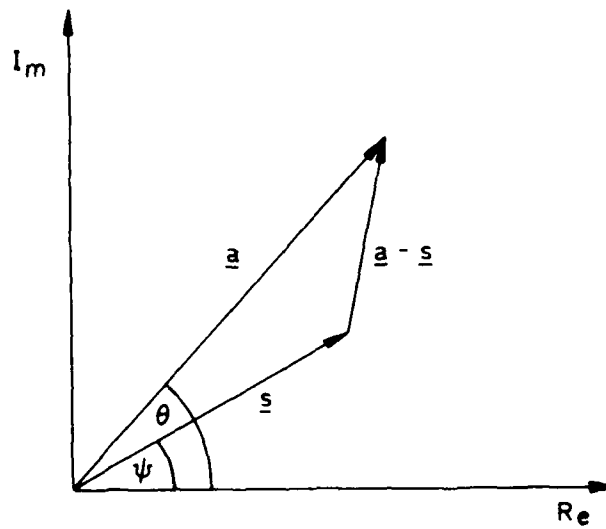


Fig 2 The random walk resultant  $\underline{a}$  minus the resultant of a fixed walk  $\underline{s}$

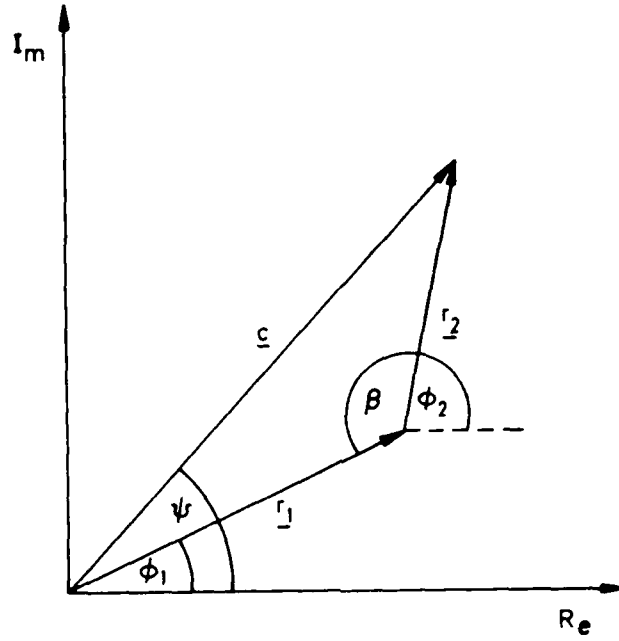


Fig 3 A 2-step walk

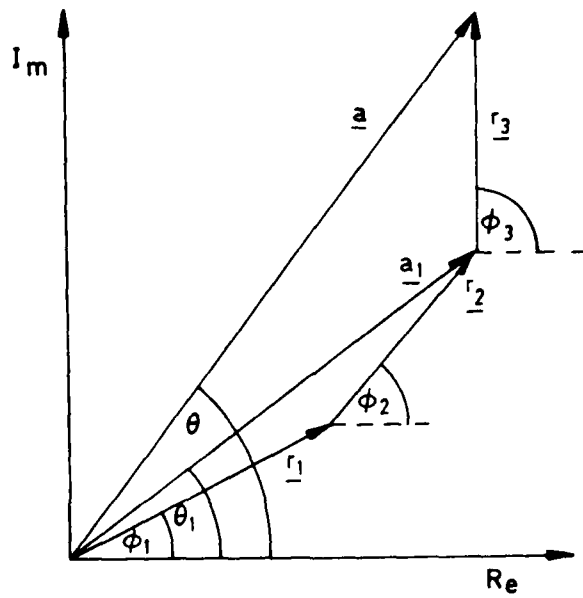


Fig 4 A 3-step walk

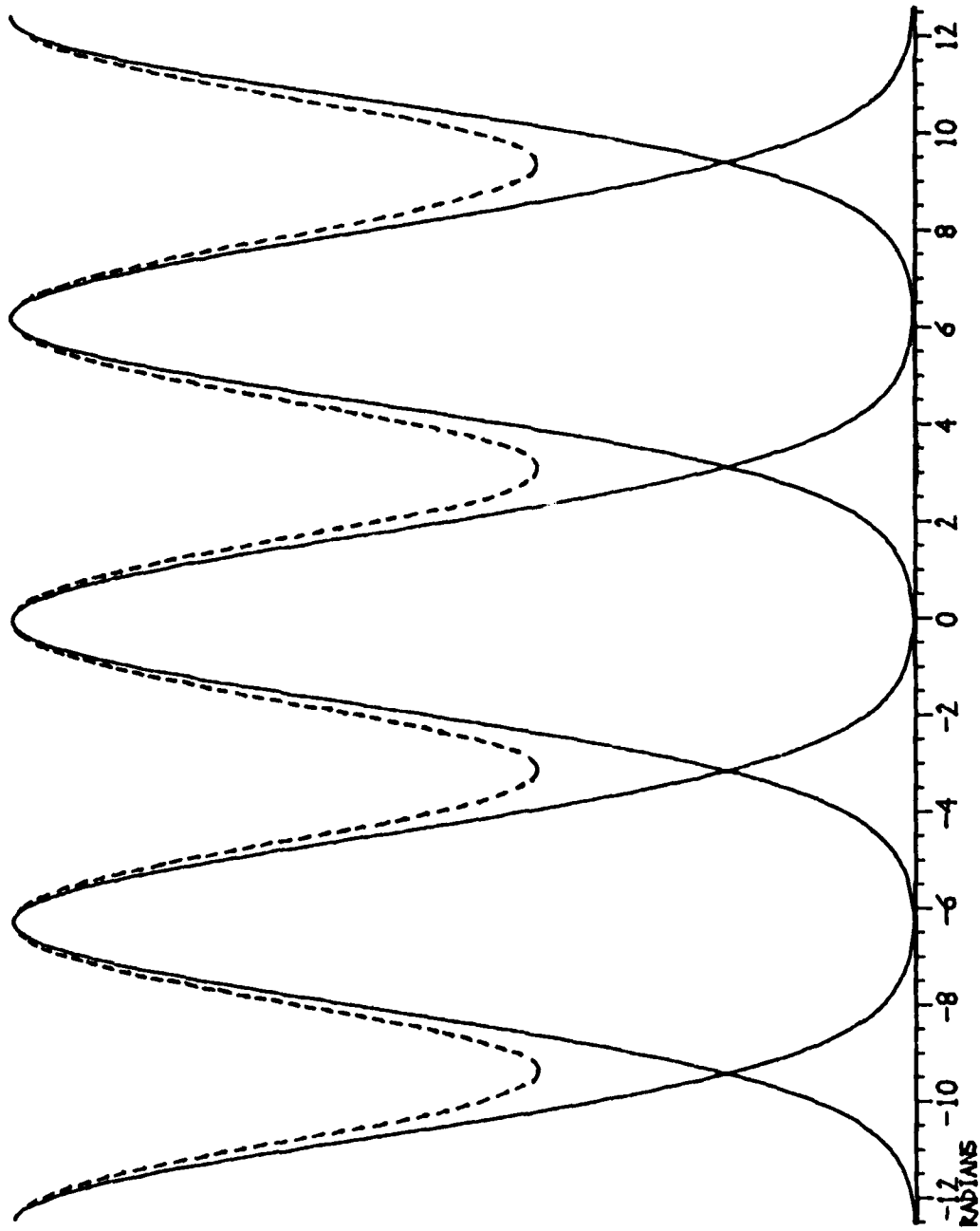


Fig 5 Periodically repeated gaussian density function

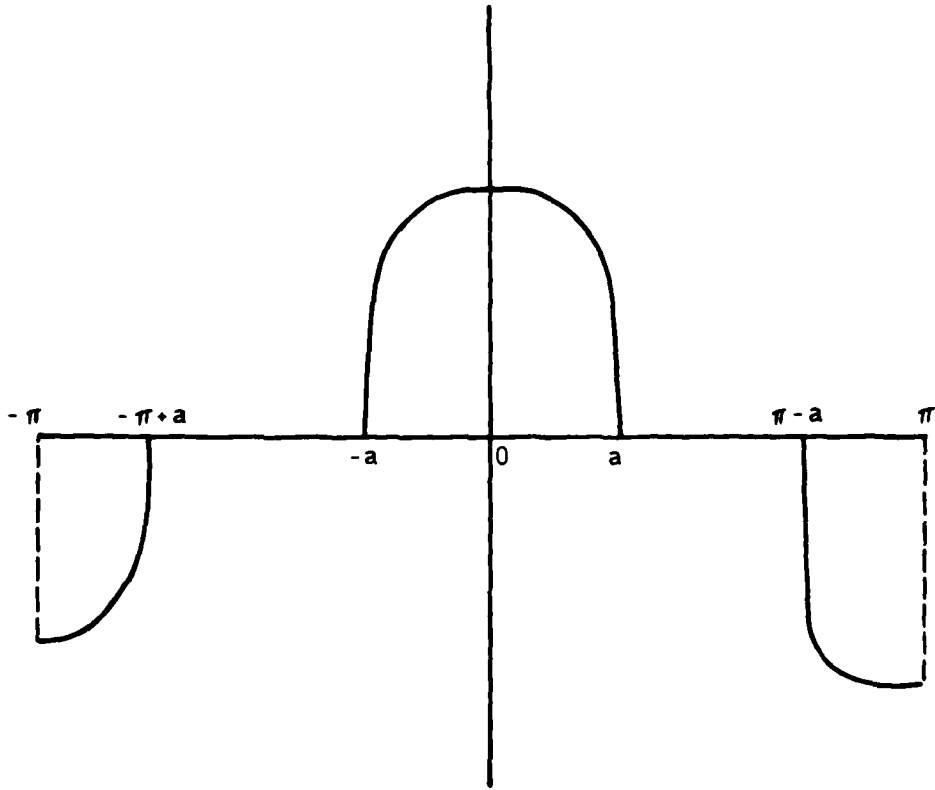


Fig 6 Function containing odd numbered harmonics only

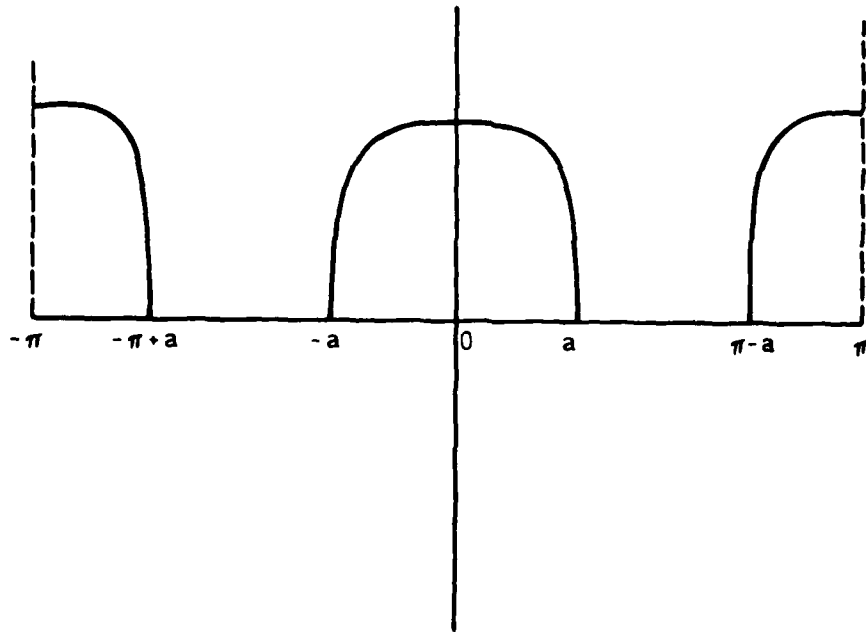


Fig 7 Function containing even numbered harmonics only

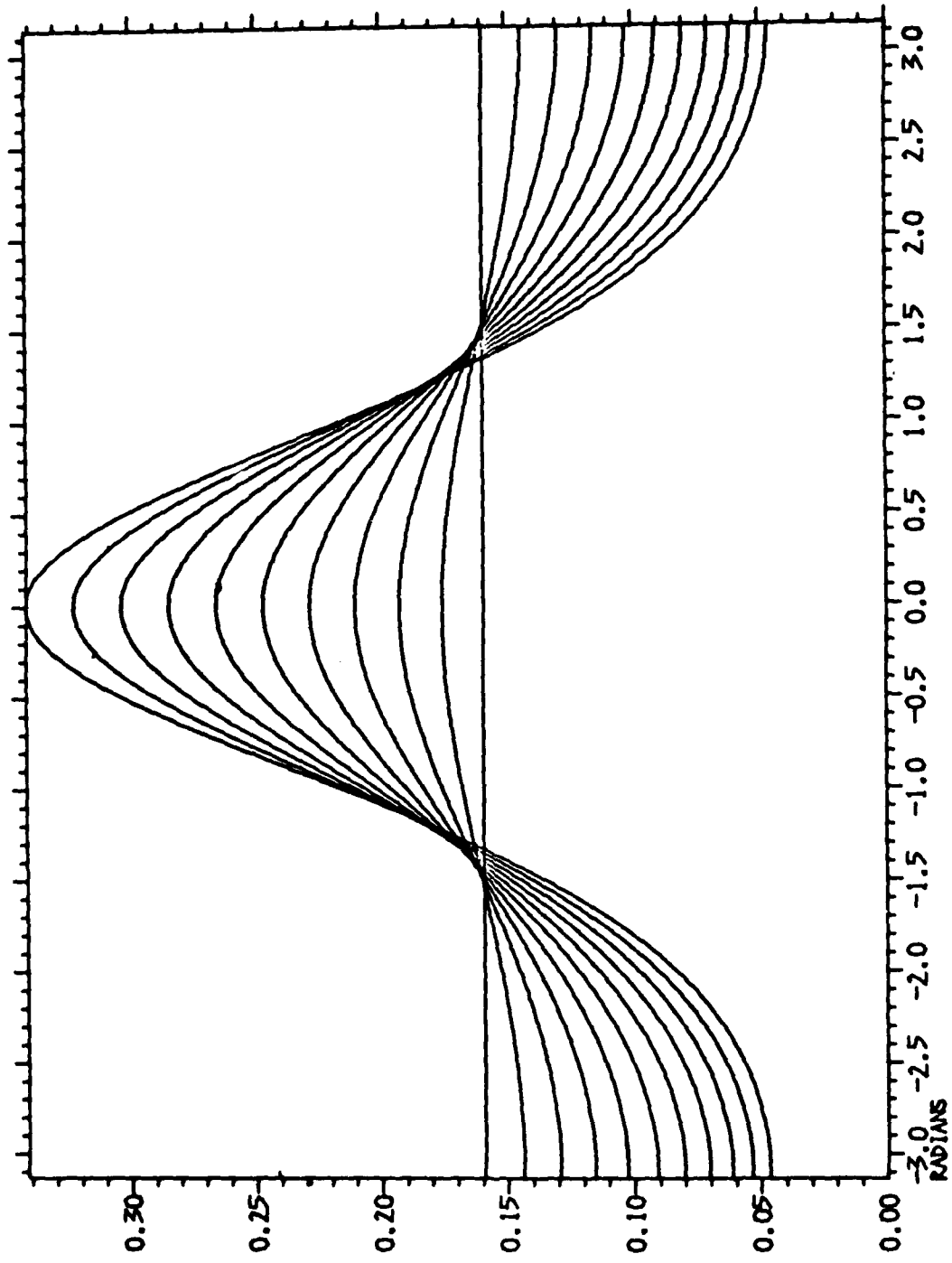


Fig 8

Fig 8 Mises phase probability density function

Fig 9

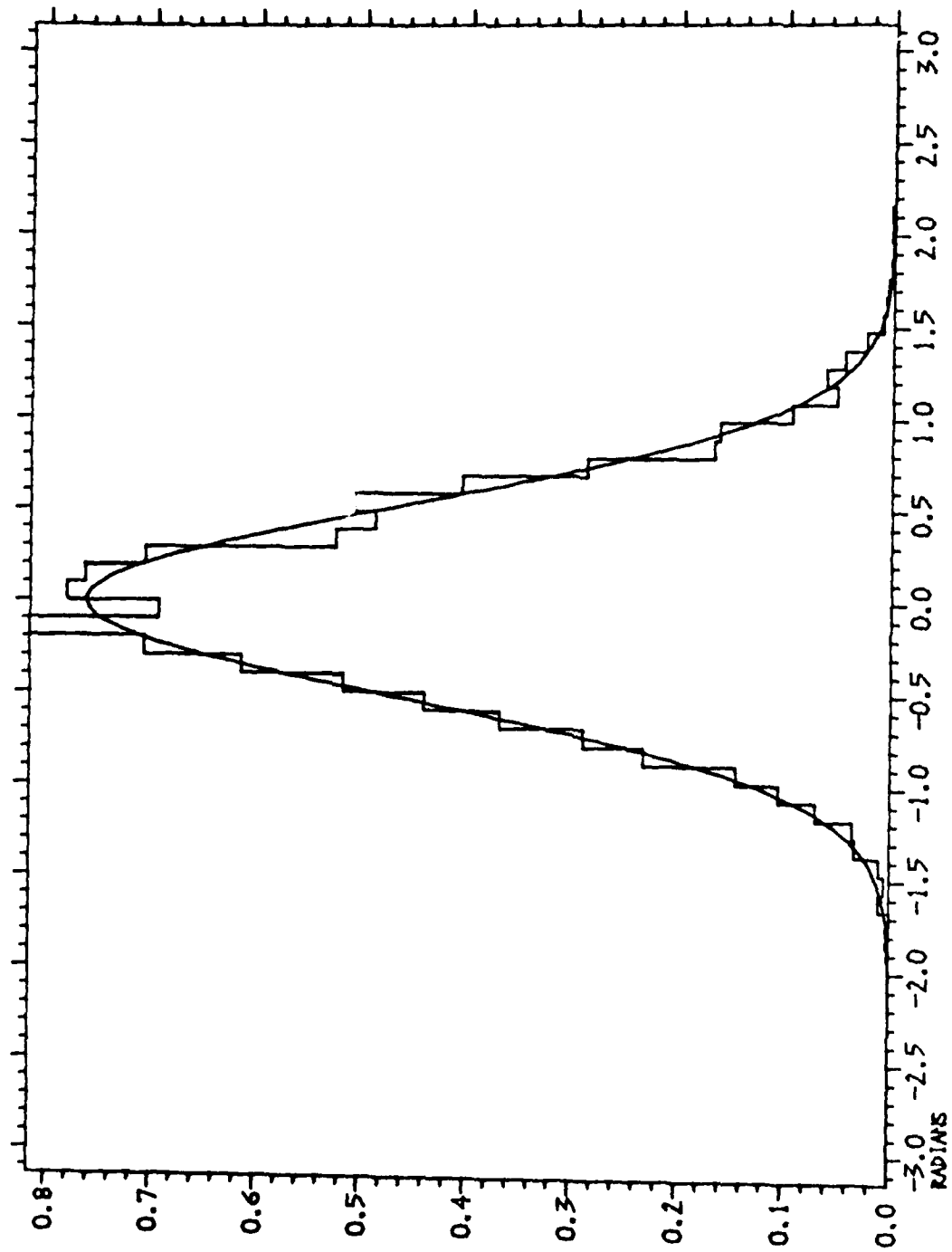


Fig 9 Random walk phase probability density function. No. of steps = 1

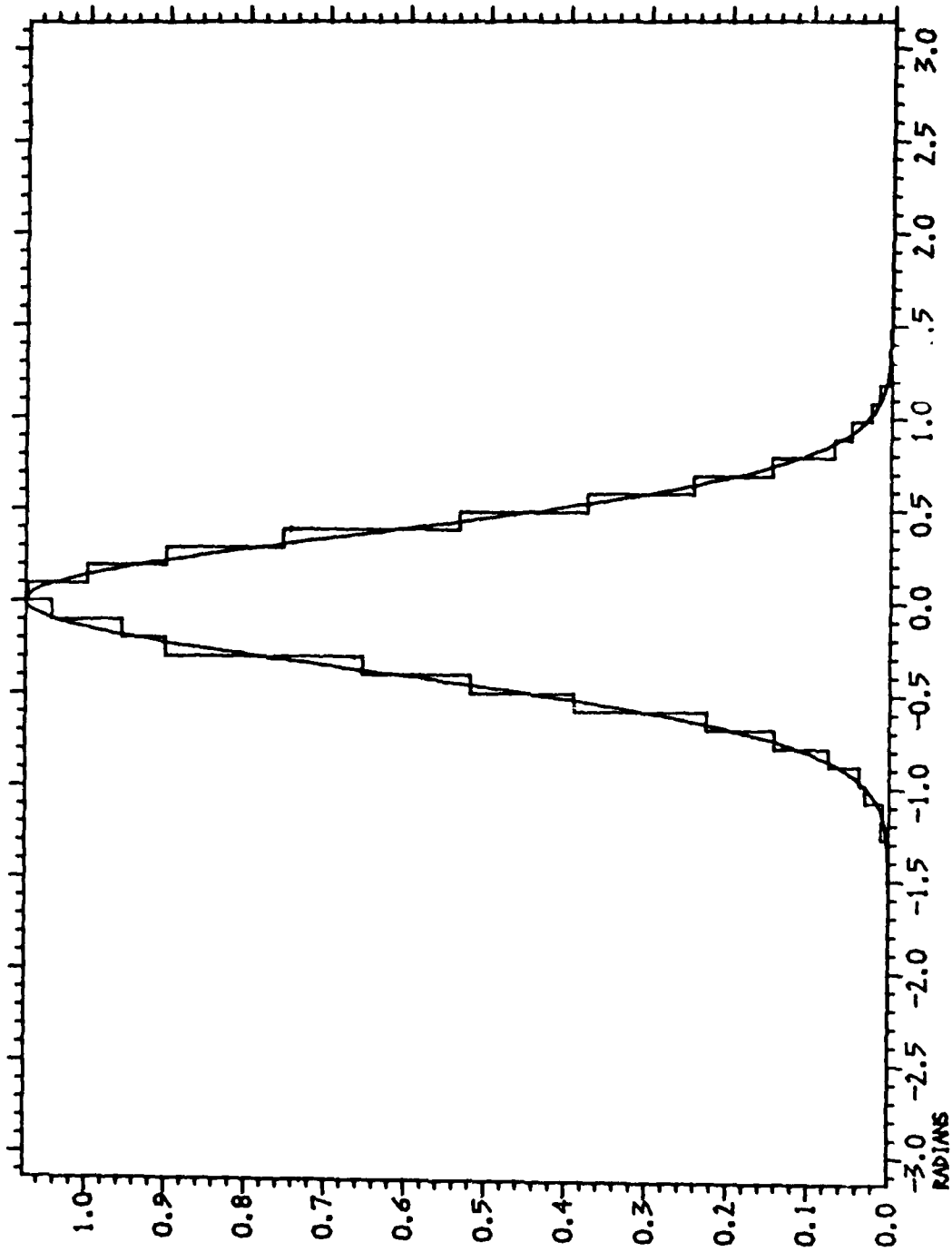


Fig 10 Random walk phase probability density function. No. of steps = 2

Fig 11

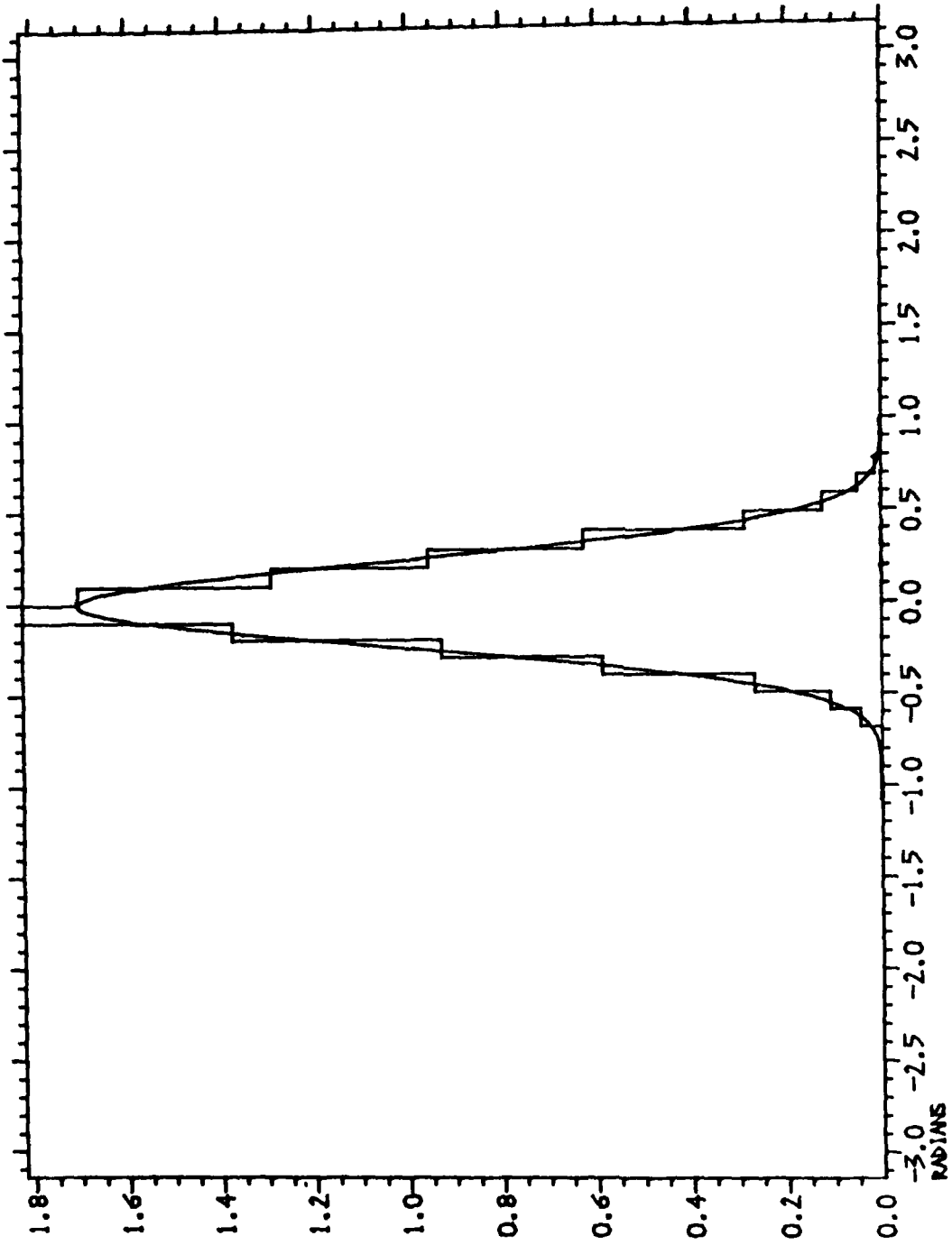


Fig 11 Random walk phase probability density function. No. of steps = 5

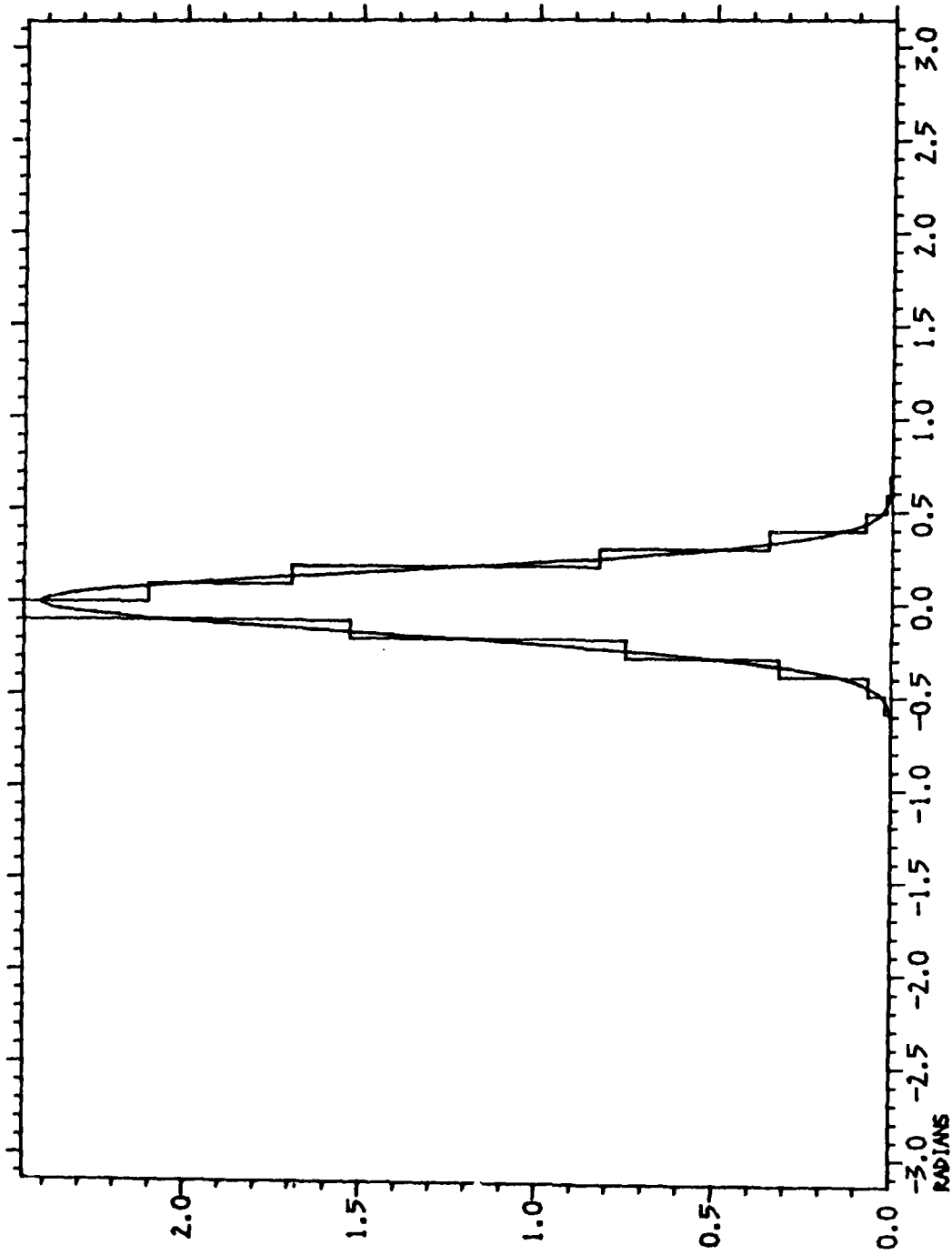


Fig 12 Random walk phase probability density function. No. of steps = 10

Fig 13

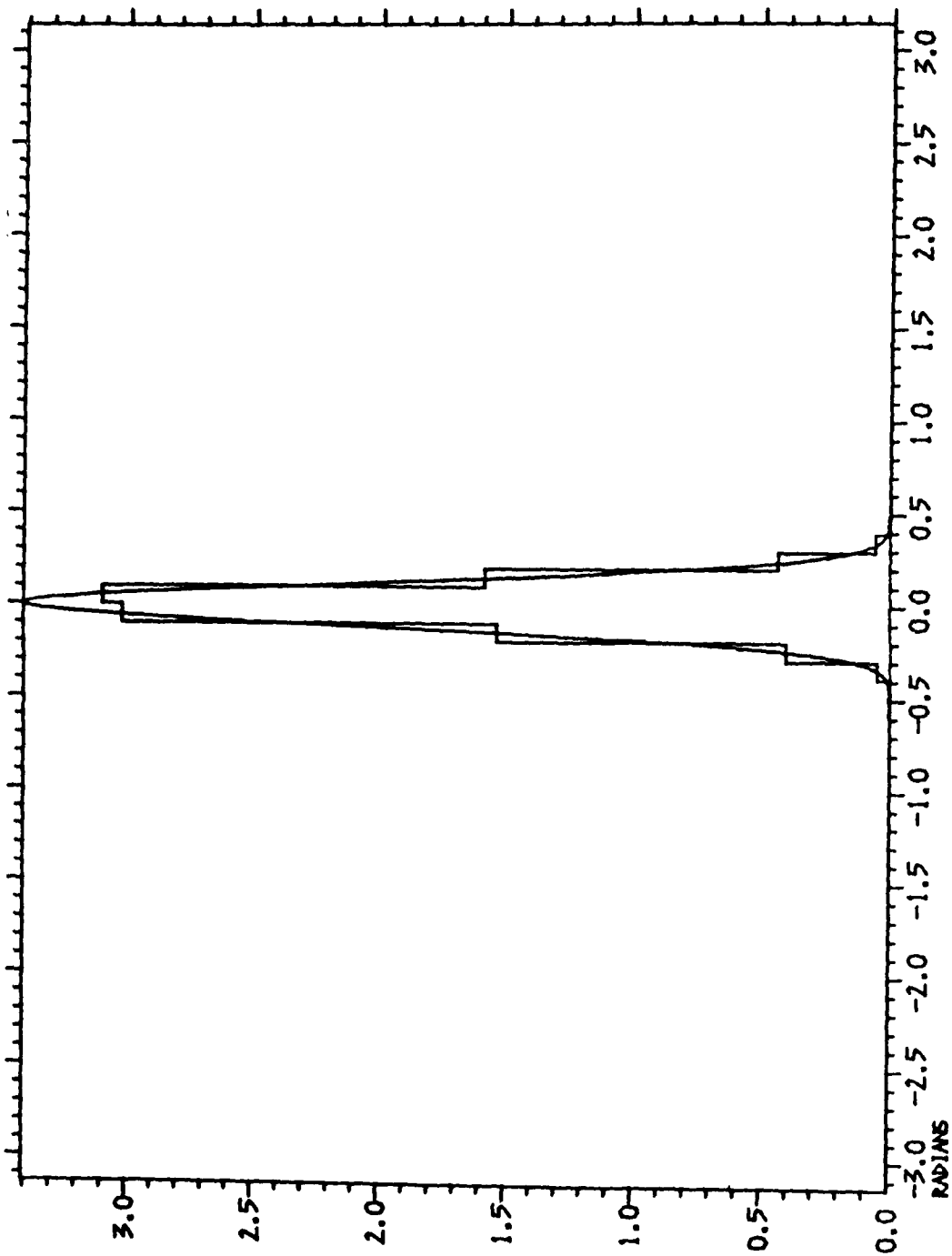


Fig 13 Random walk phase probability density function. No. of steps = 20

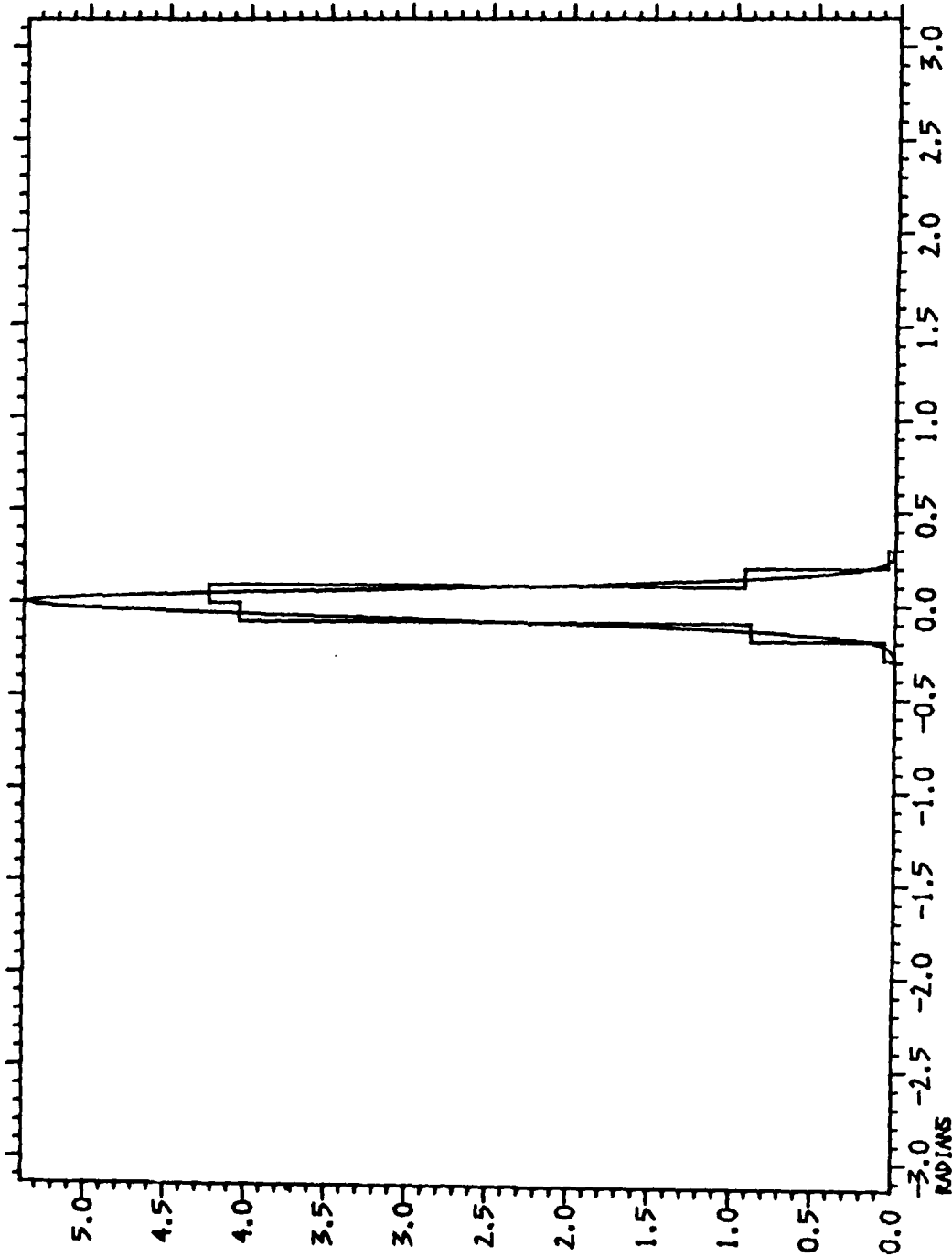


Fig 14 Random walk phase probability density function. No. of steps = 50

Fig 15

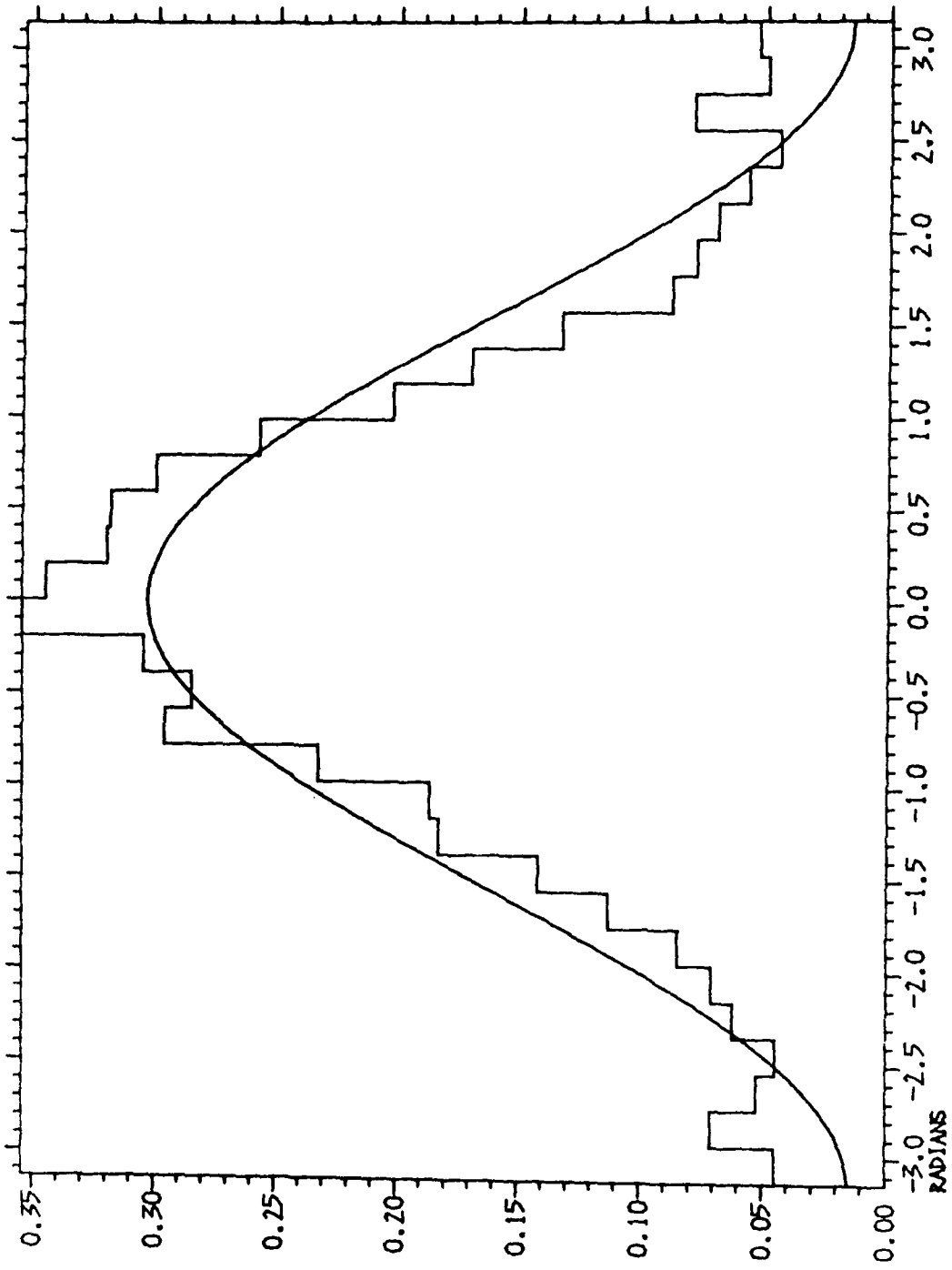


Fig 15 Random walk phase probability density function.  $N = 2$ ,  $N\alpha^2 = 1$

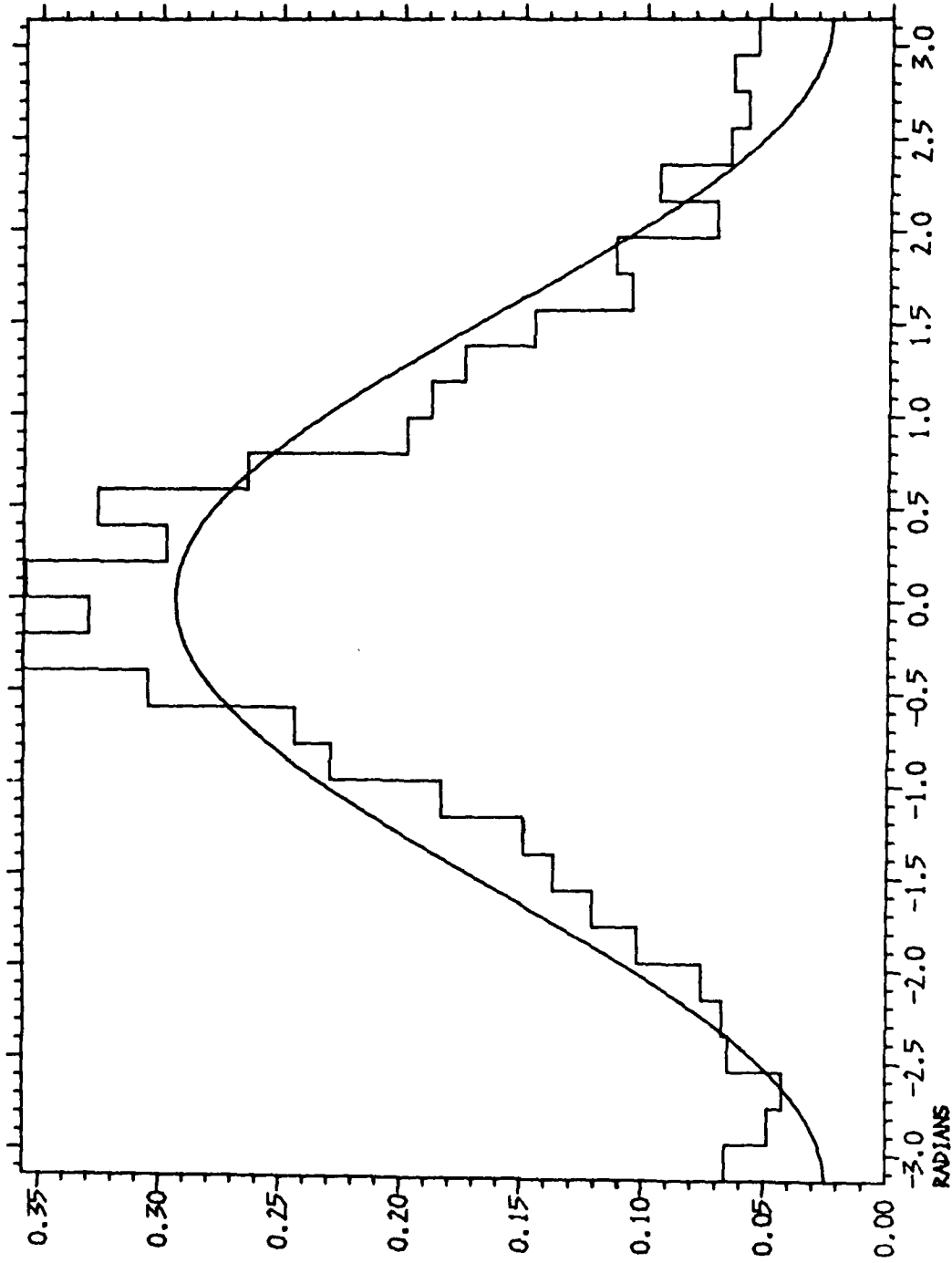


Fig 16 Random walk phase probability density function.  $N = 10$ ,  $Na^2 = 1$

Fig 17

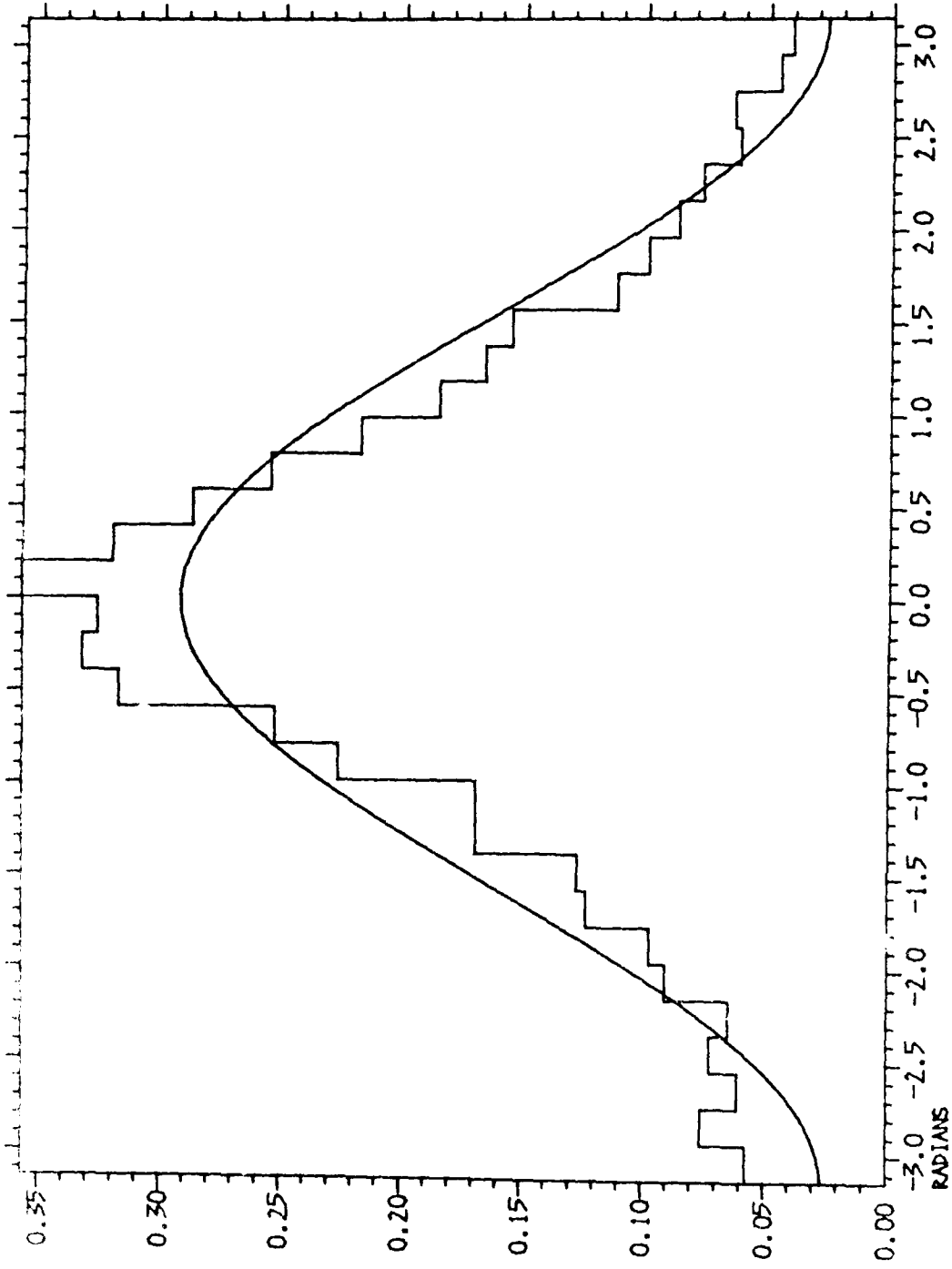


Fig 17 Random walk phase probability density function.  $N = 20$ ,  $Na^2 = 1$

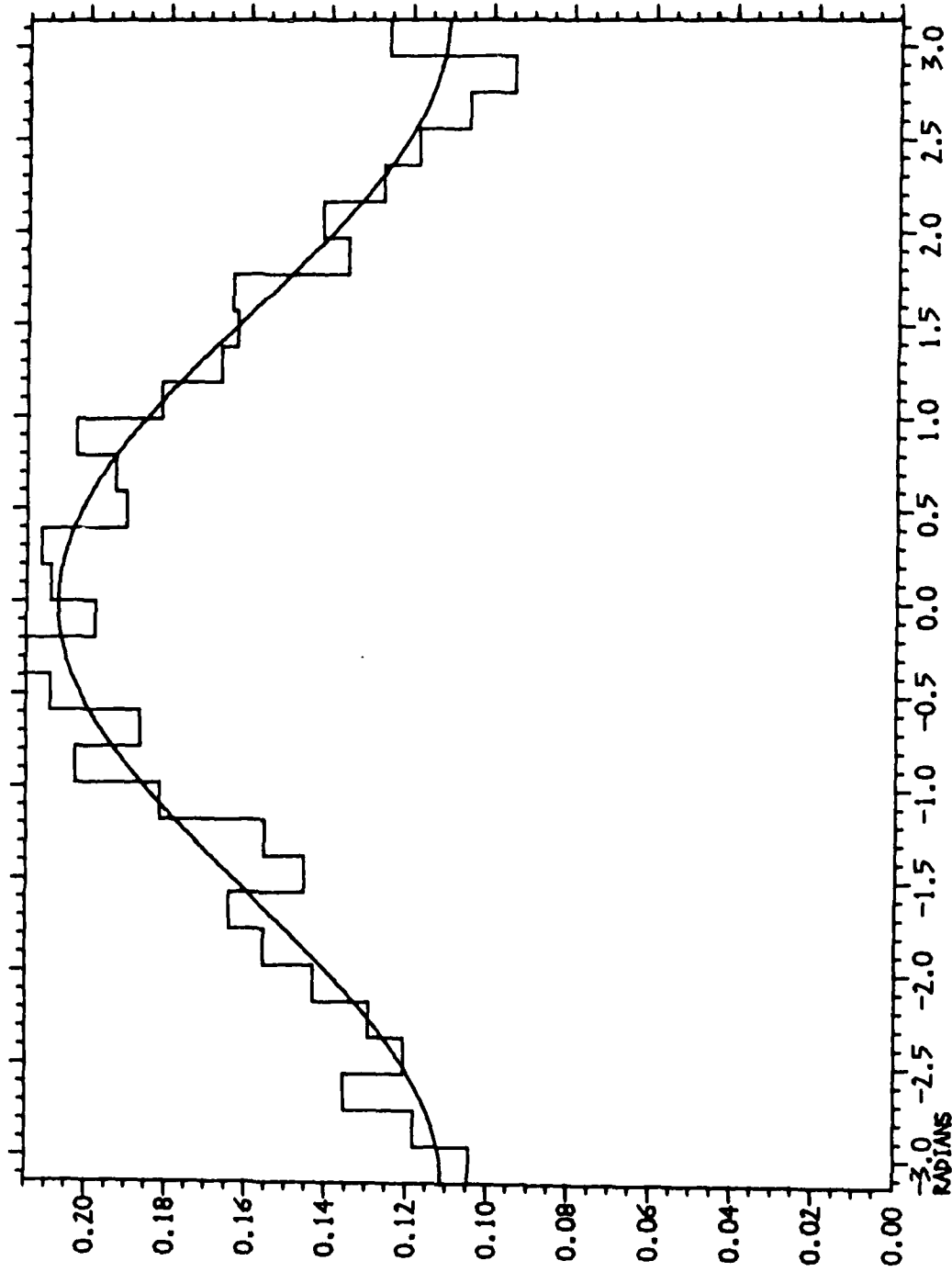


Fig 18 Random walk phase probability density function.  $N = 2$ ,  $Na^2 = 0.1$

Fig 19

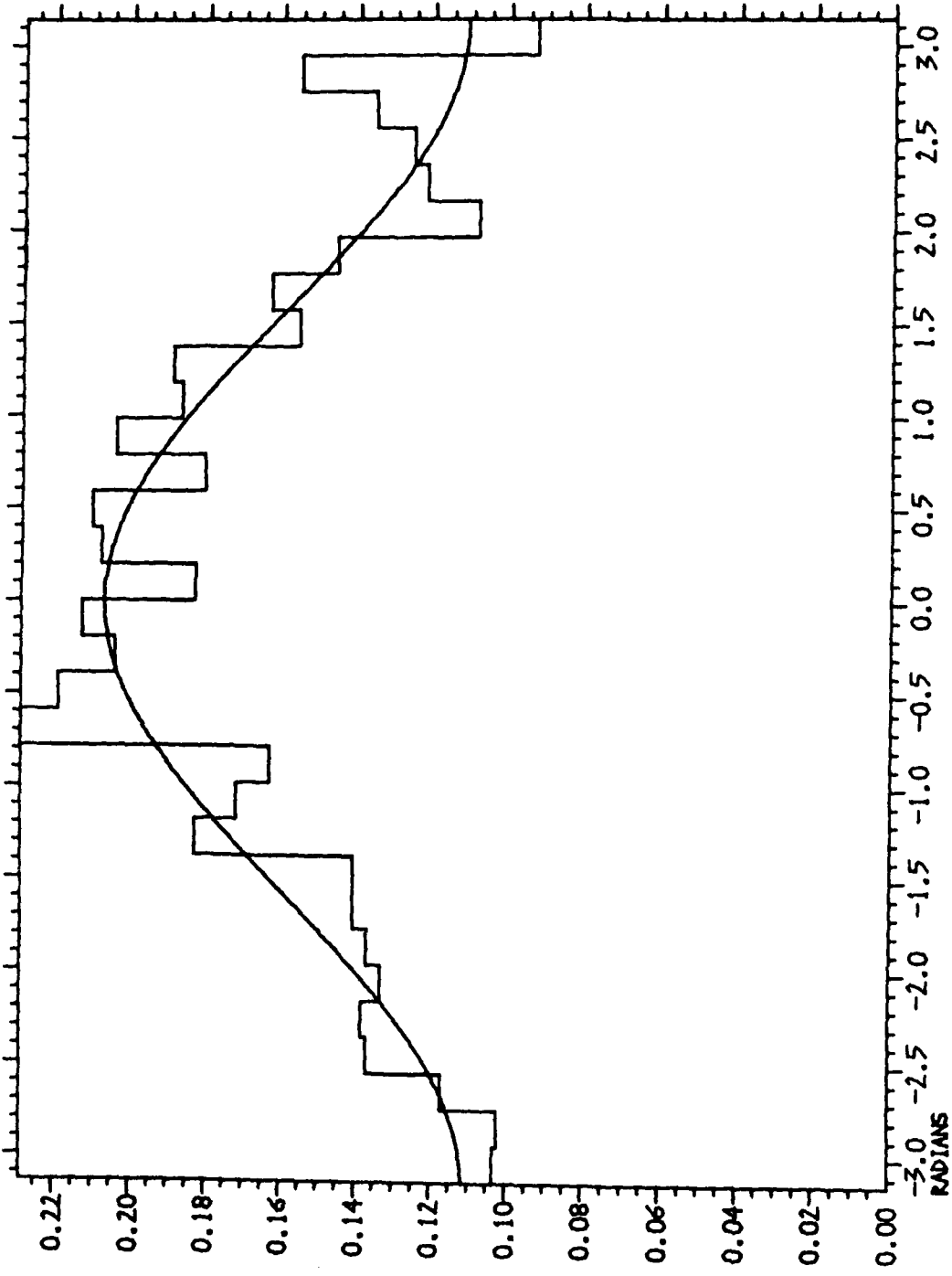


Fig 19 Random walk phase probability density function.  $N = 10$ ,  $N\sigma^2 = 0.1$

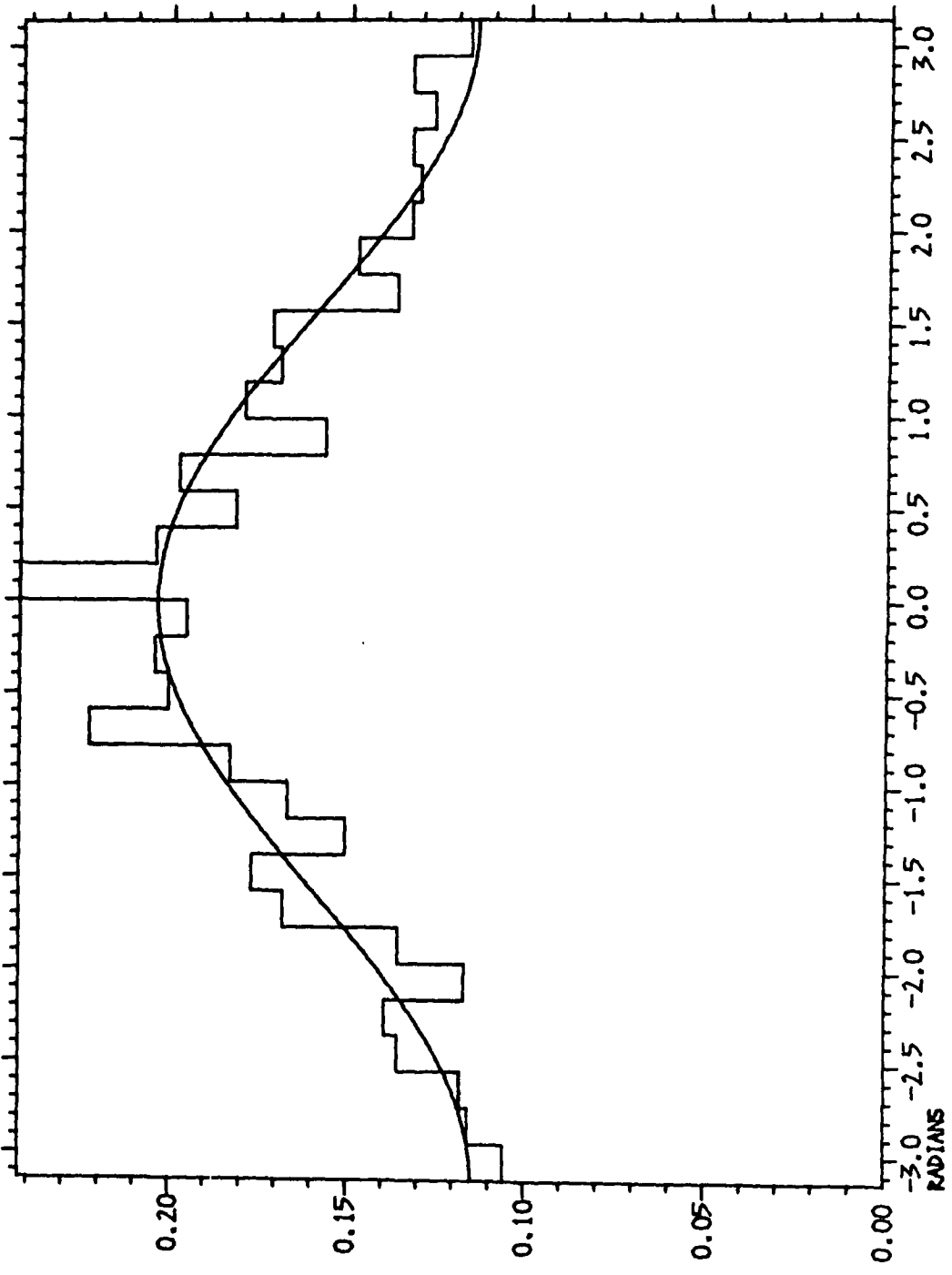


Fig 20 Random walk phase probability density function.  $N = 20$ ,  $N\sigma^2 = 0.1$

Fig 21

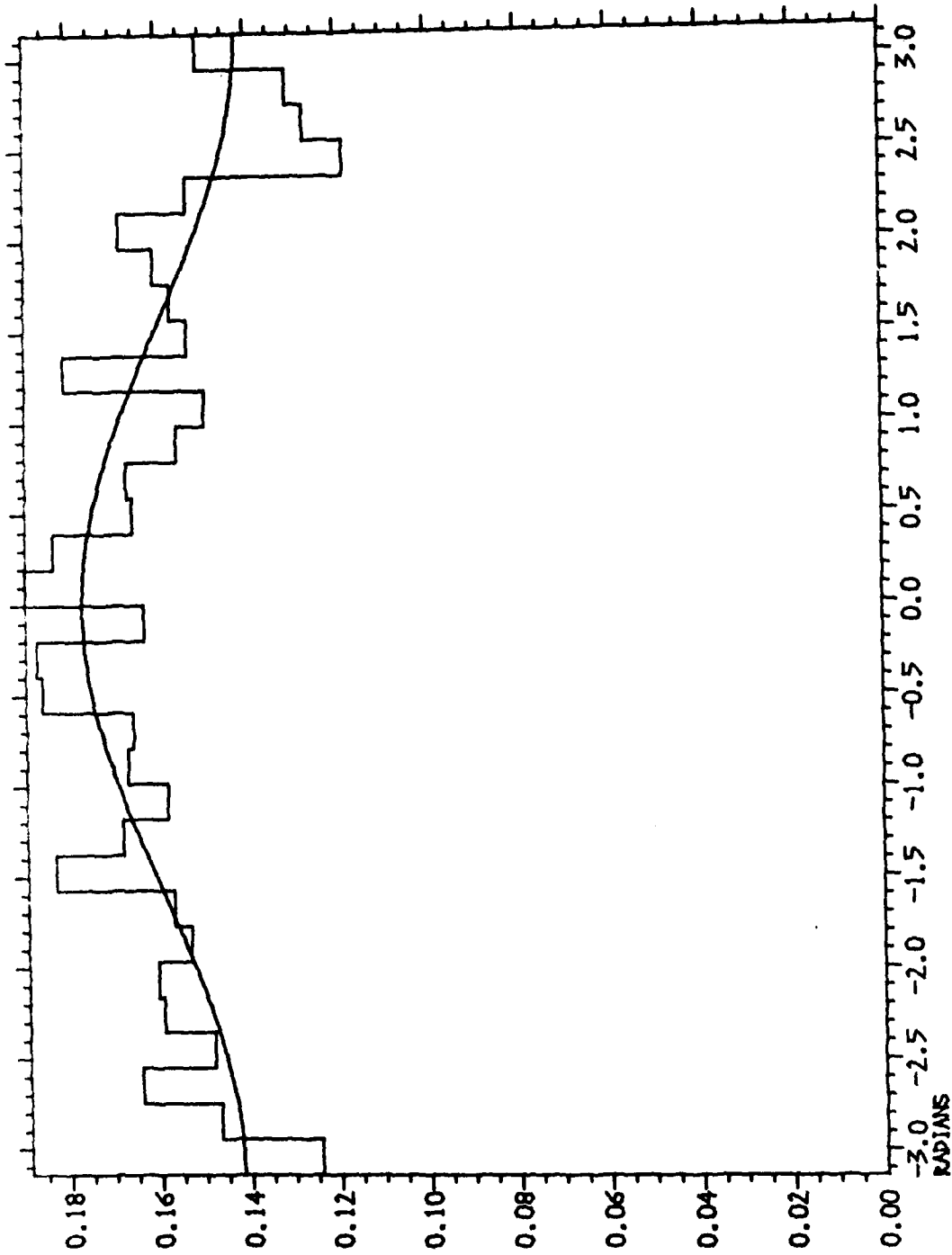


Fig 21 Random walk phase probability density function.  $N = 2$ ,  $\text{Na}^2 = 0.01$

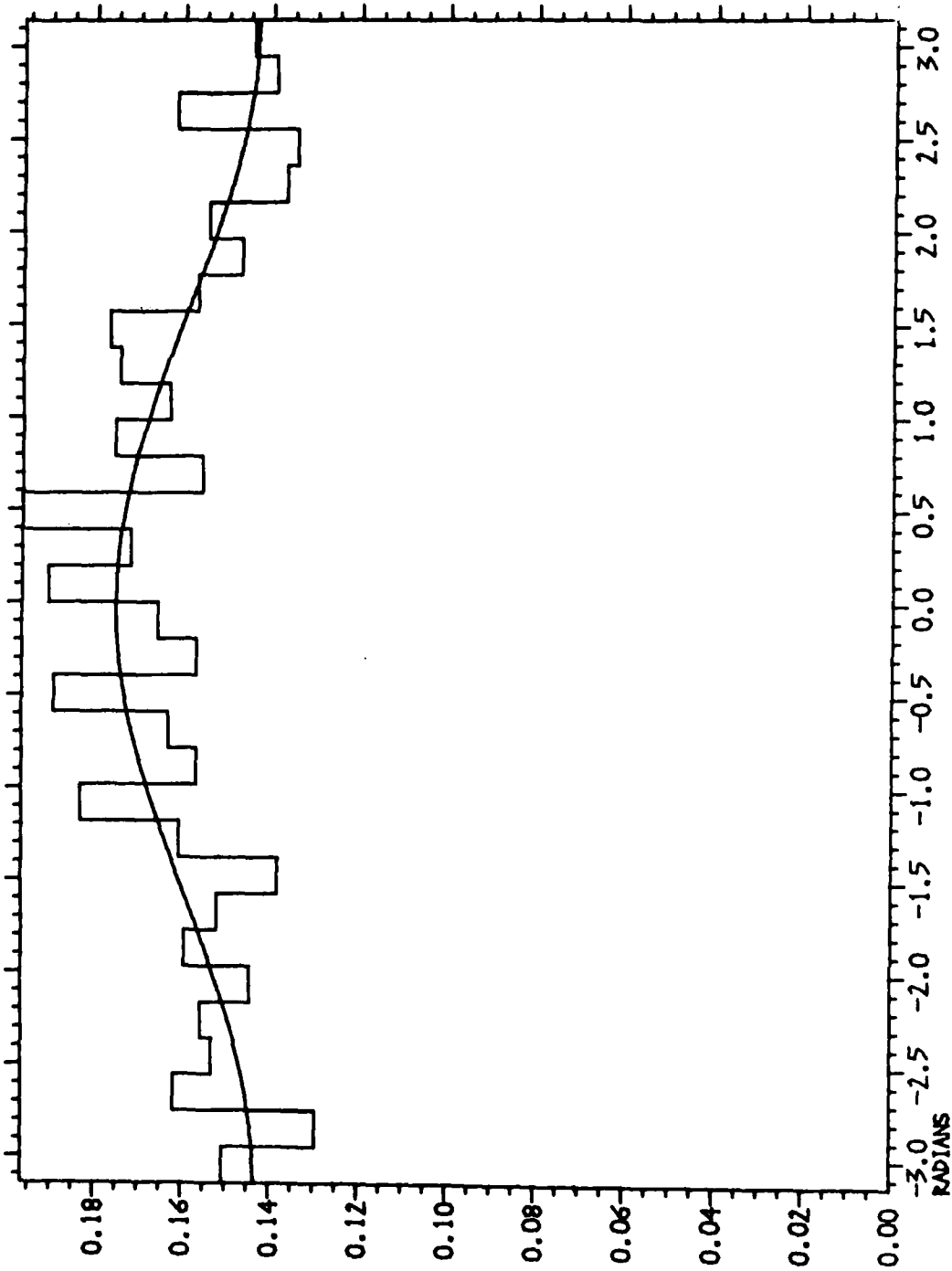


Fig 22 Random walk phase probability density function.  $N = 10$ ,  $\text{Na}^2 = 0.01$

Fig 23

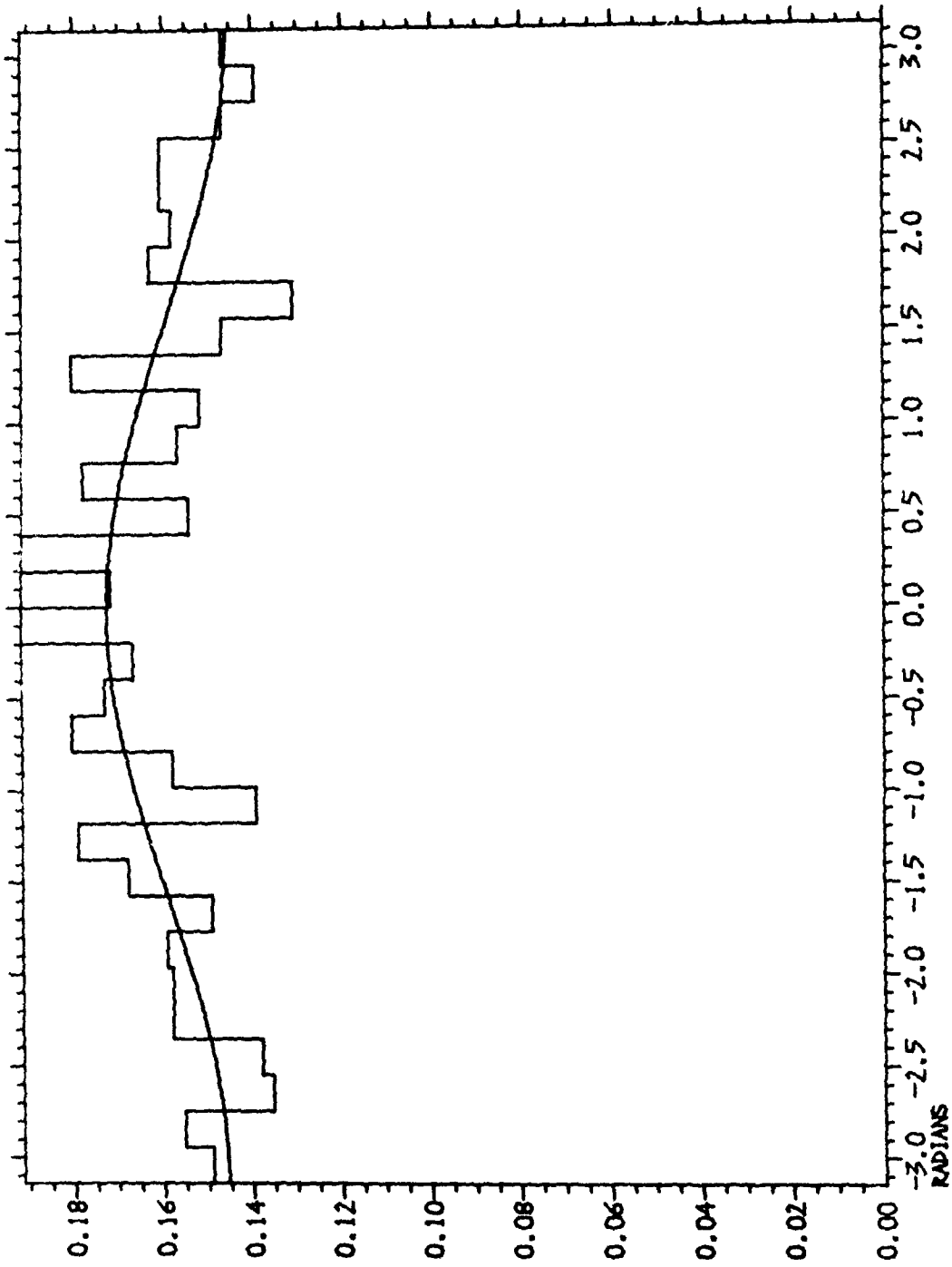


Fig 23 Random walk phase probability density function.  $N = 20$ ,  $Na^2 = 0.01$

Fig 24

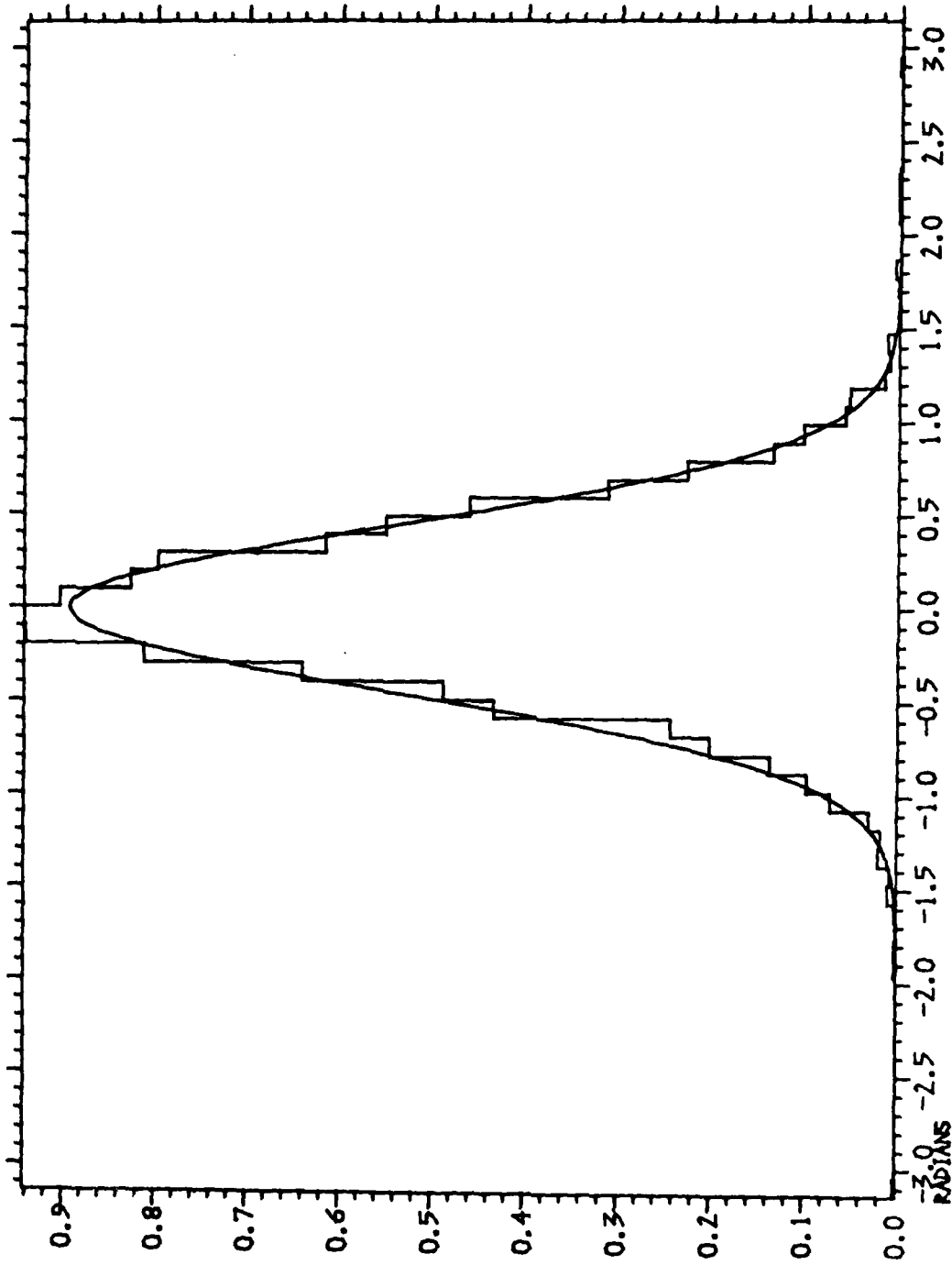


Fig 24 Random walk phase probability density function,  $N = 10$ ,  $Na^2 = 10$

Fig 25

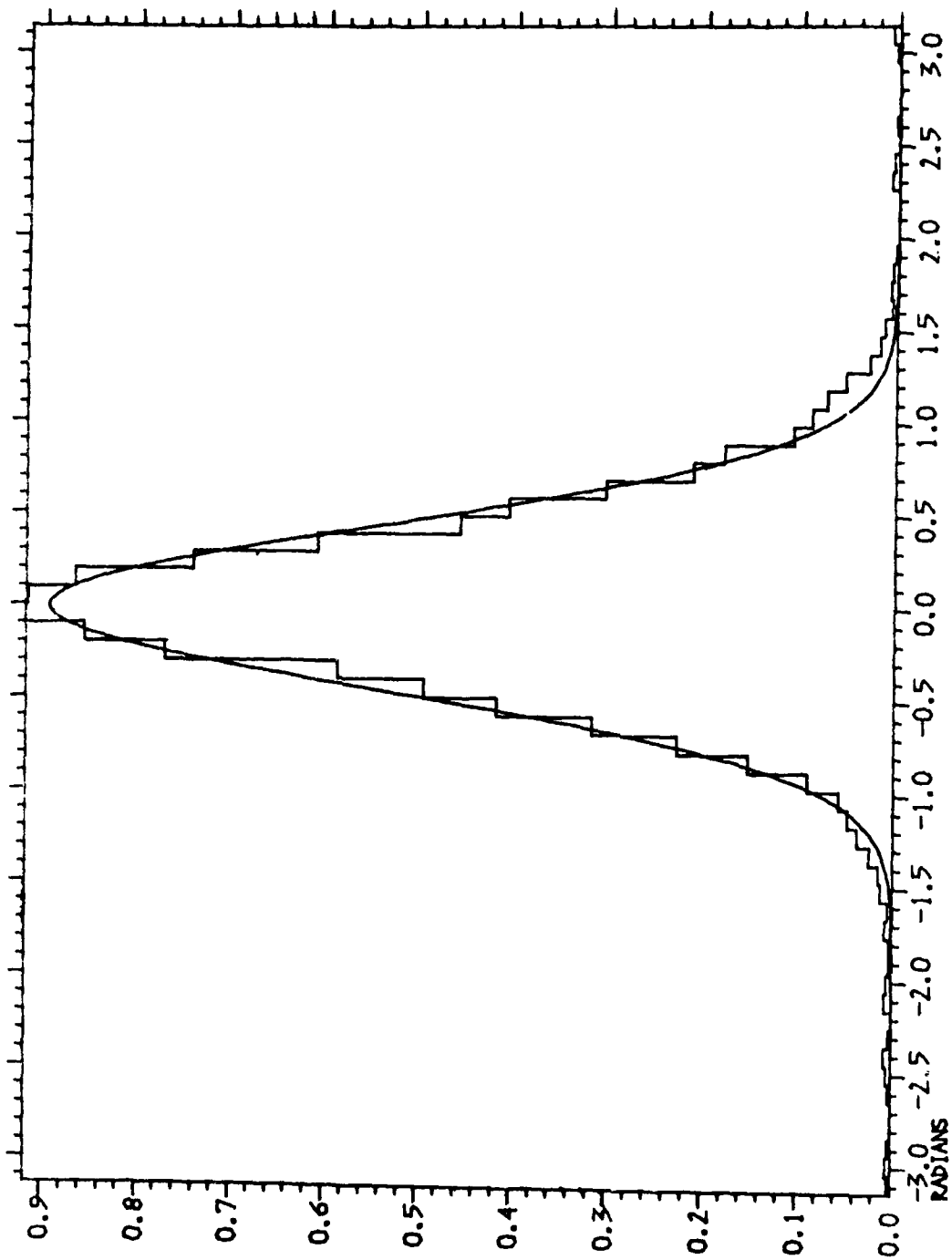


Fig 25 Random walk phase probability density function.  $N = 100$ ,  $N\sigma^2 = 10$

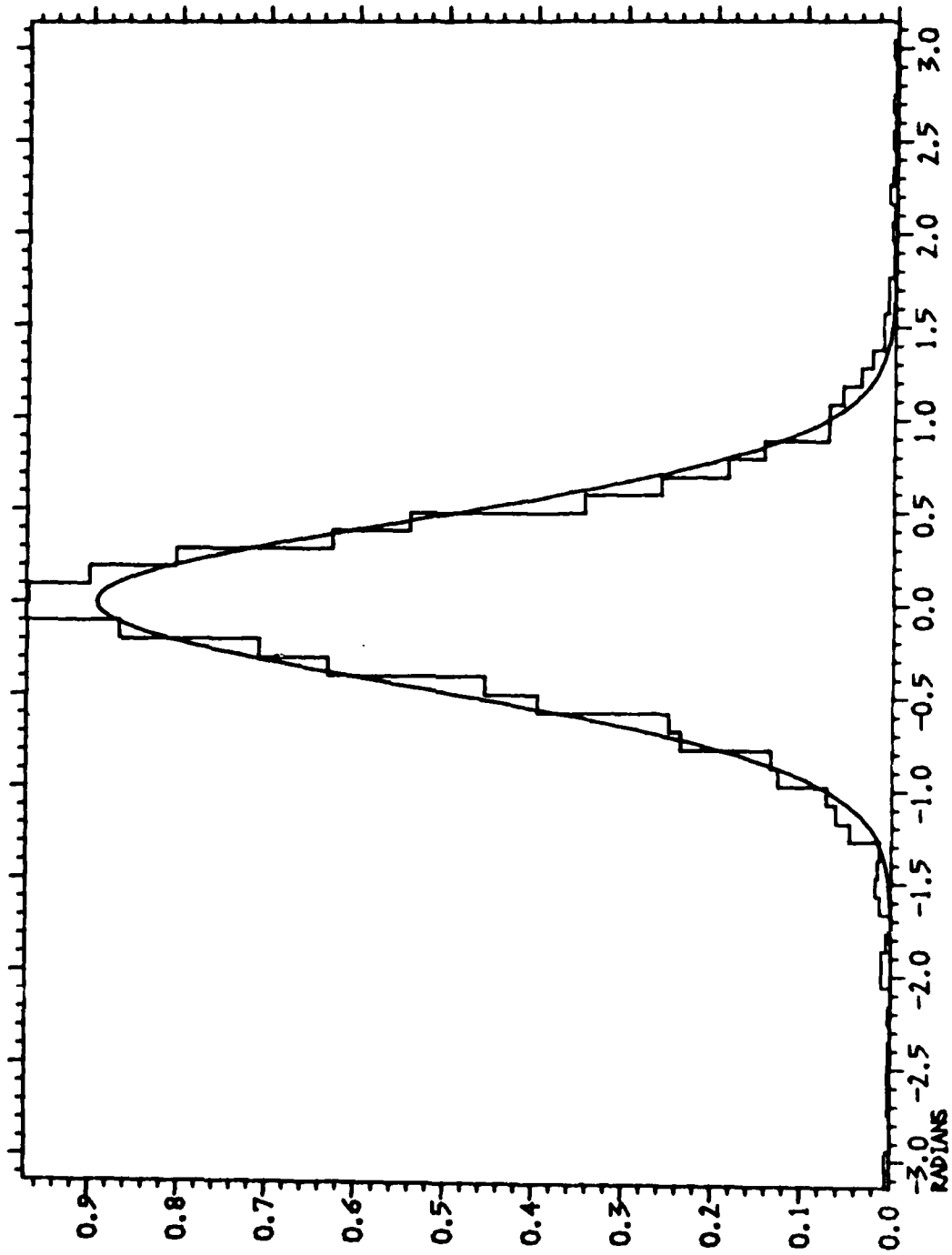


Fig 26 Random walk phase probability density function.  $N = 1000$ ,  $Na^2 = 10$

Fig 27

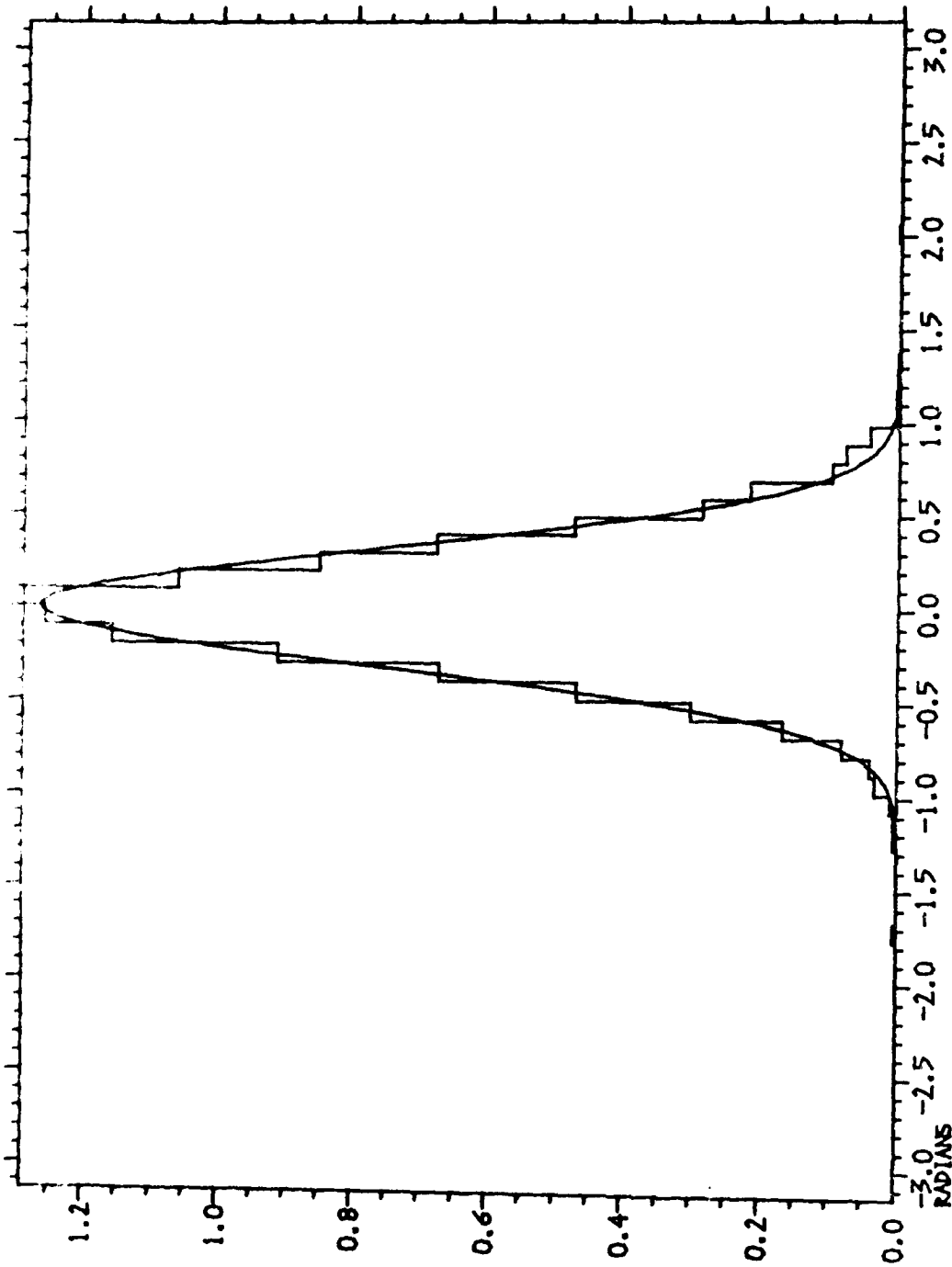


Fig 27 Random walk phase probability density function.  $N = 100$ ,  $\text{Na}^2 = 20$

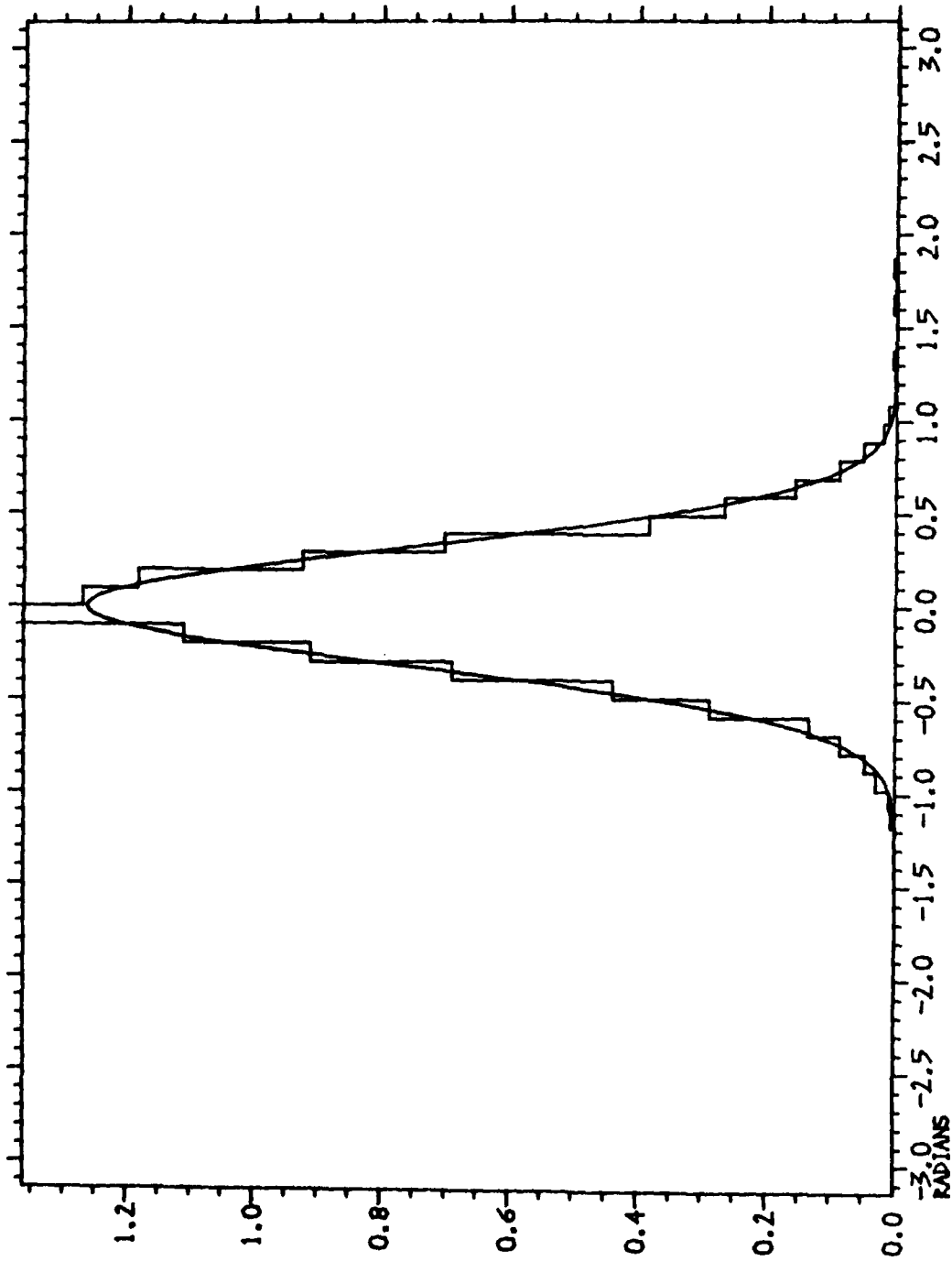


Fig 28 Random walk phase probability density function.  $N = 1000$ ,  $Na^2 = 20$

Fig 29

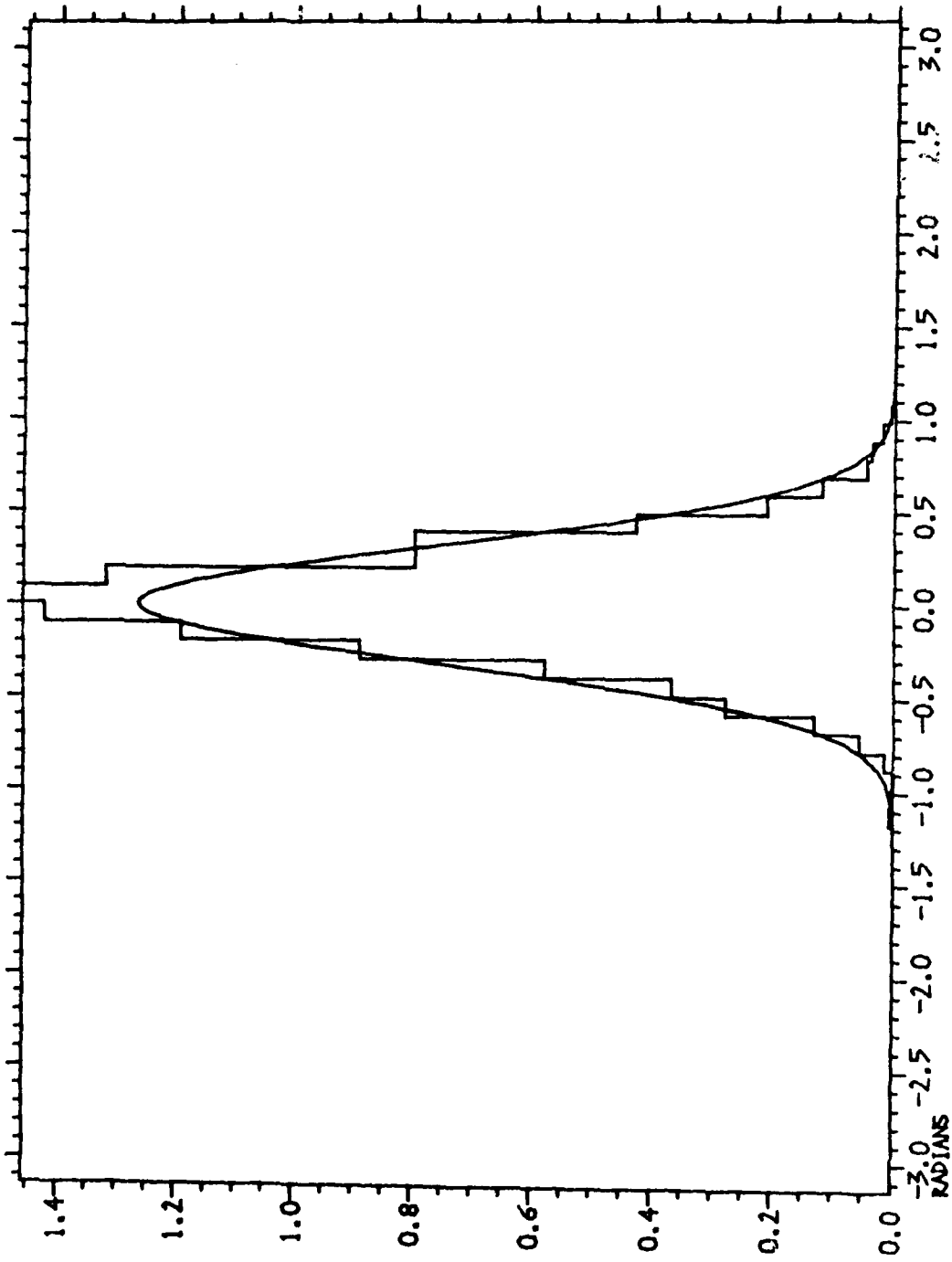


Fig 29 Random walk phase probability density function,  $N = 10000$ ,  $N\sigma^2 = 20$

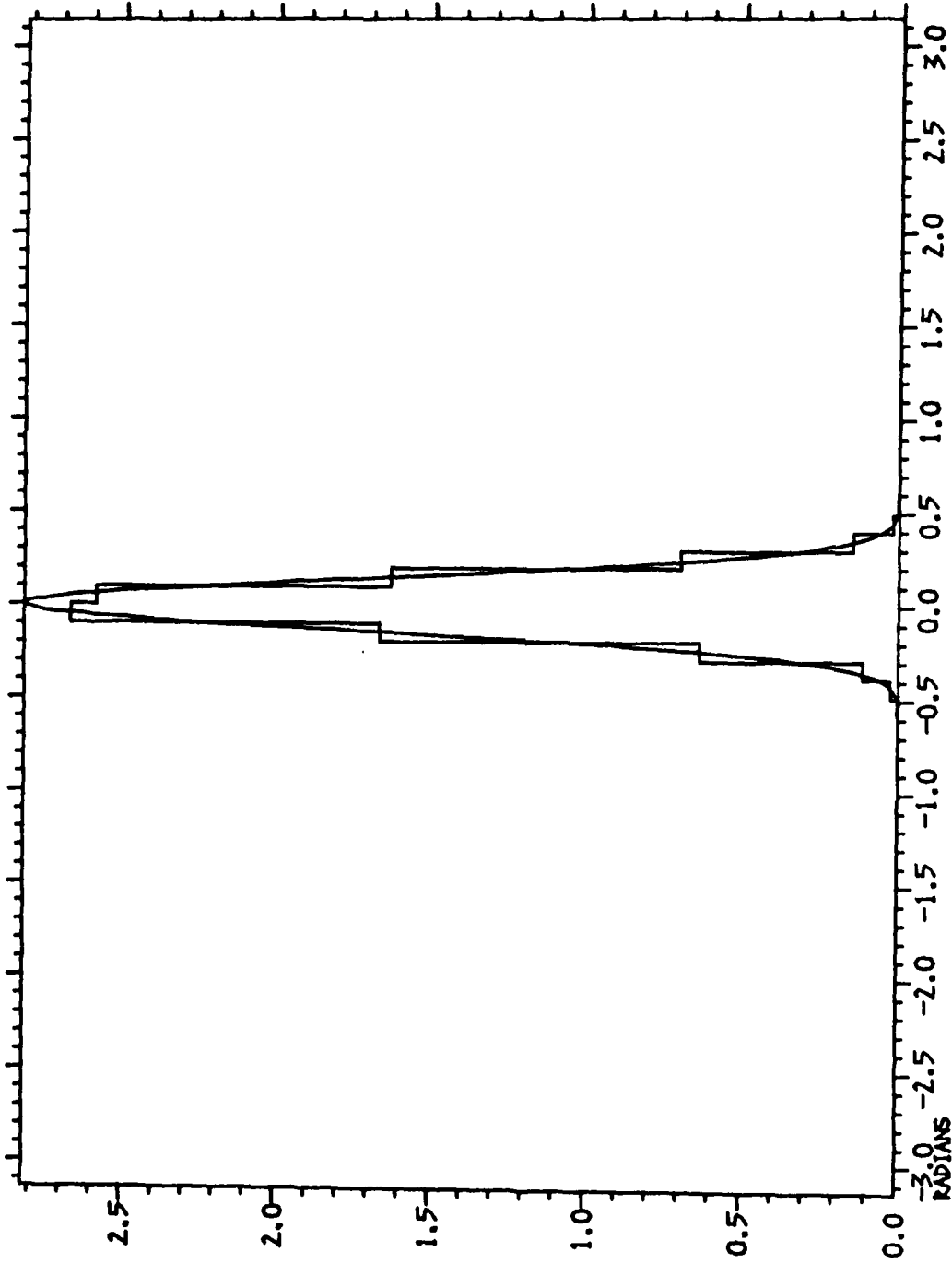


Fig 30 Random walk phase probability density function,  $N = 100$ ,  $Na^2 = 100$

Fig 31

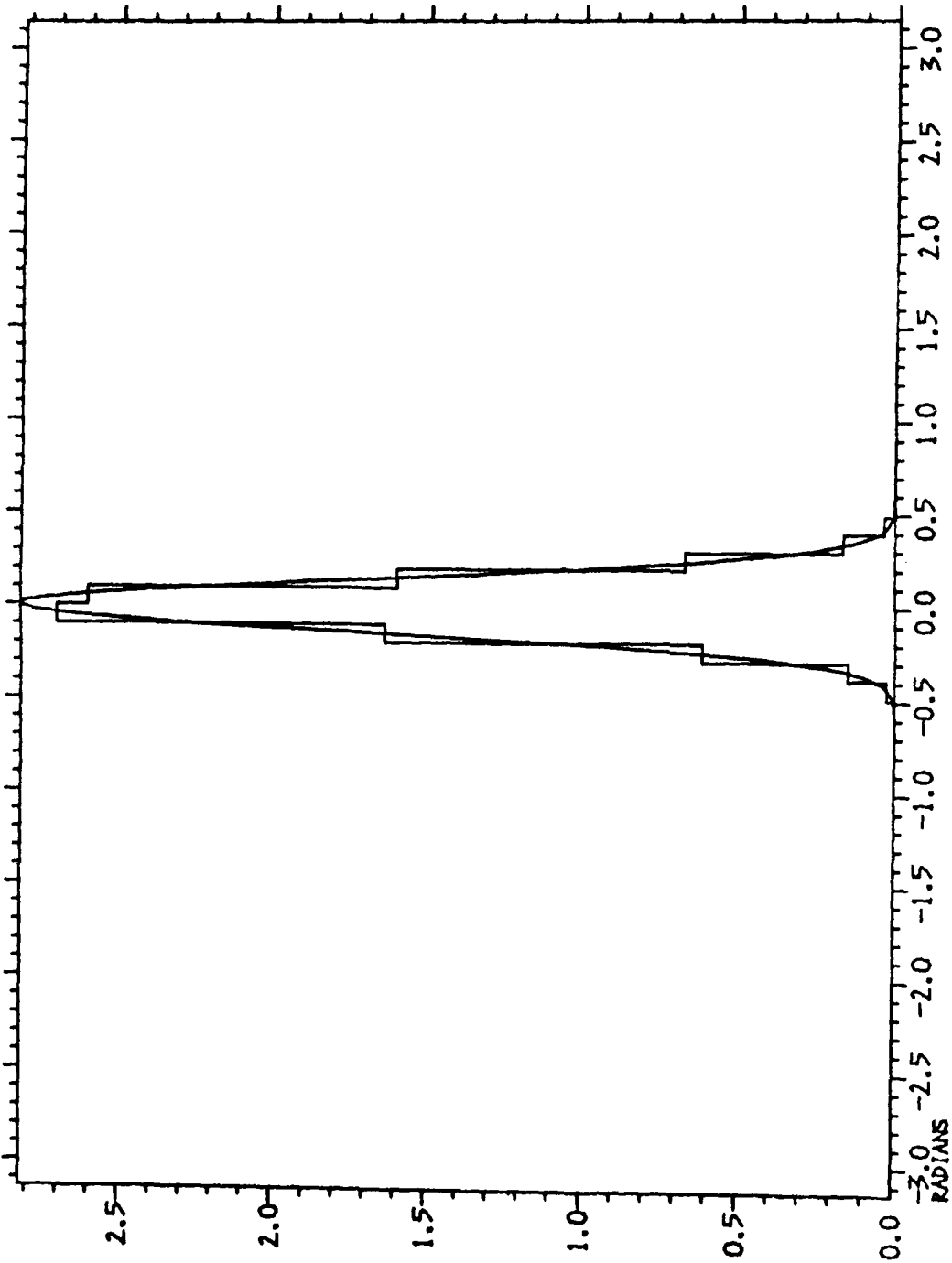


Fig 31 Random walk phase probability density function,  $N = 1000$ ,  $\sigma_a^2 = 100$

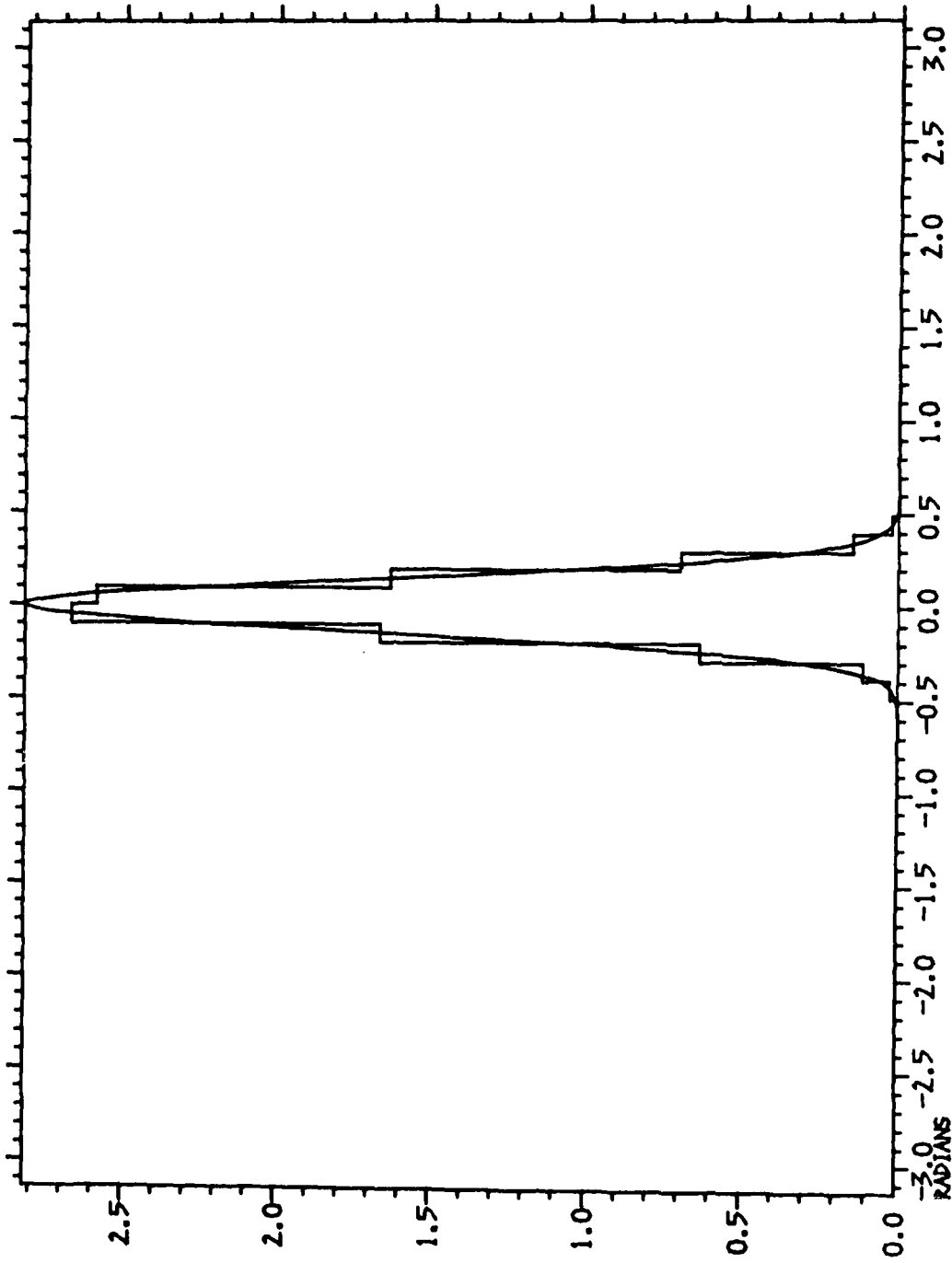


Fig 30 Random walk phase probability density function,  $N = 100$ ,  $N\sigma^2 = 100$

Fig 31

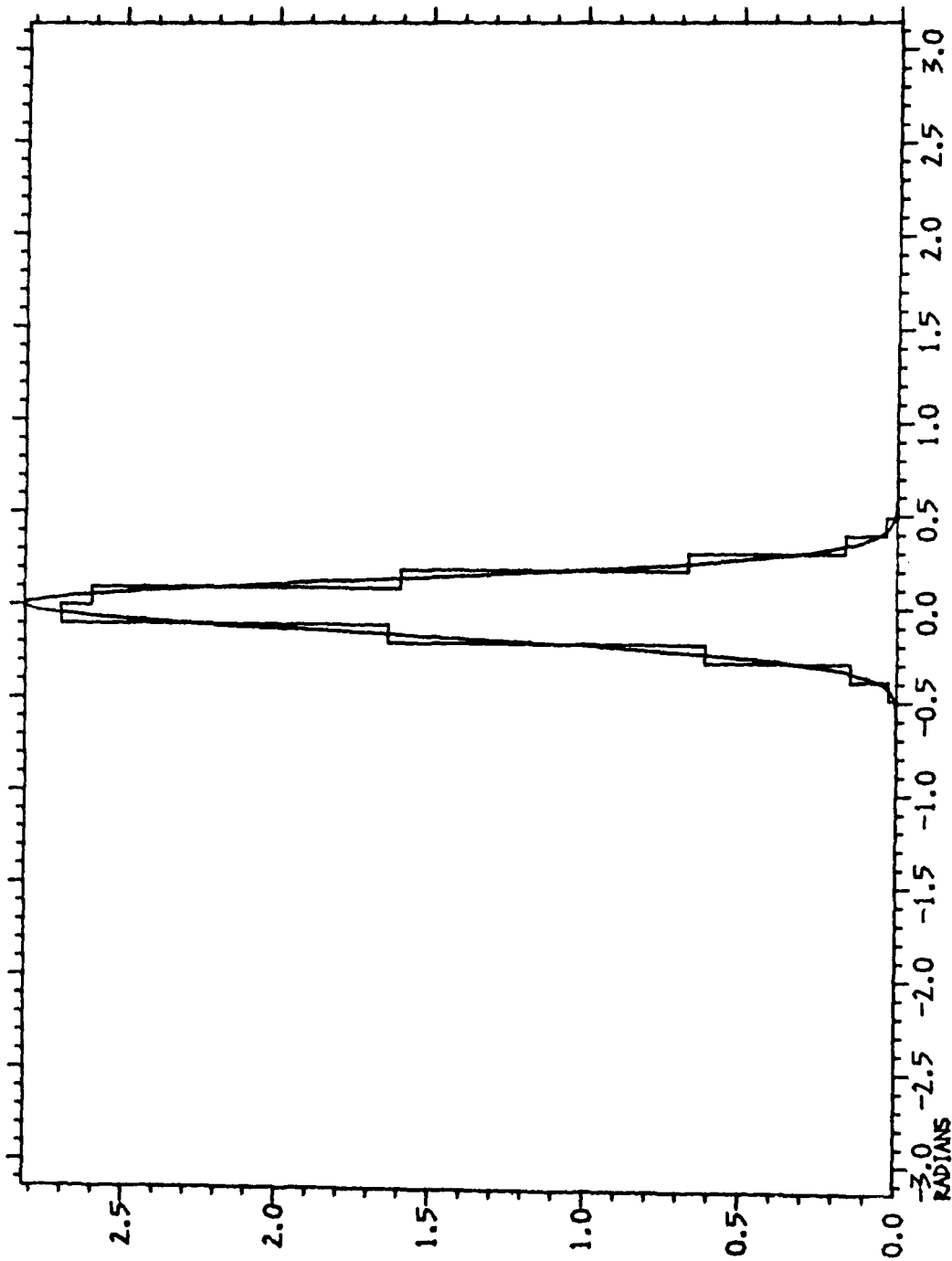


Fig 31 Random walk phase probability density function,  $N = 1000$ ,  $Na^2 = 100$

# REPORT DOCUMENTATION PAGE

Overall security classification of this page

UNCLASSIFIED

As far as possible this page should contain only unclassified information. If it is necessary to enter classified information, the box above must be marked to indicate the classification, e.g. Restricted, Confidential or Secret.

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16. Descriptors (Keywords)      (Descriptors marked * are selected from TEST) Random processes. Random walk. Random functions. Generalised functions. Bessel functions.			
17. Abstract  The problem of determining the phase probability density function of the resultant of an N-step non-isotropic random walk in two dimensions is examined. A formula is obtained for the joint probability density function of angle and radius of the resultant for arbitrary step angle probability density and for any number of steps. The theory of generalised functions concentrated on smooth manifolds is applied to the problem. Asymptotic solutions are obtained for the case where the phase probability density of each step is the same Gaussian function periodically wrapped on to the interval $(-\pi, +\pi)$ . In particular solutions are obtained for small phase variance for any number of steps and for large variance for both small and large numbers of steps. Throughout the paper a physical point of view is taken.			

$-\pi \quad + \pi$

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