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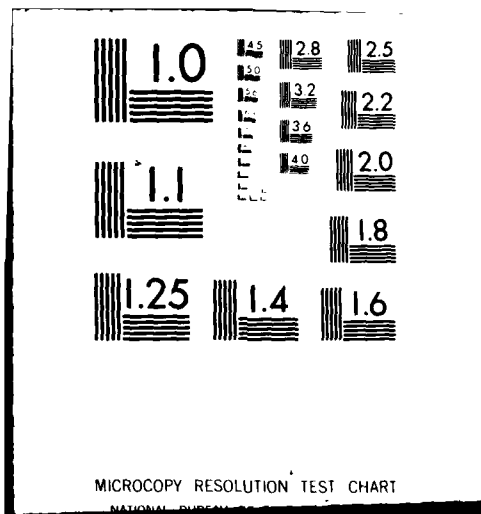
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A METHOD FOR DETERMINING PERIODS IN TIME SERIES

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Technical Report No. N-23

April 1981

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REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
Technical Report N-23	AD-A116 134	

4. TITLE *(and Subtitle)*	5. TYPE OF REPORT & PERIOD COVERED
A Method for Determining Periods in Time Series	Technical
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
H. Joseph Newton and Marcello Pagano	DAAG29-80-C-0070 ONR N0001481MP10001

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Texas A&M University Institute of Statistics College Station, TX 77843	

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
U.S. Army Research Office	April, 1981
P.O. Box 12277	13. NUMBER OF PAGES
Research Triangle Park, NC 27709	32

14. MONITORING AGENCY NAME & ADDRESS(*if different from Controlling Office*)	15. SECURITY CLASS. *(of this report)*
	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

NA

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Univariate time series; spectral density function; Newton's method; Autoregressive processes.

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A METHOD FOR DETERMINING PERIODS

IN TIME SERIES .

H. Joseph Newton

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Marcello Pagano*

Abstract

A method for estimating periods corresponding to peaks in the spectral density of univariate time series is proposed. The large sample distribution of the estimator is derived under the assumption that the observed process is from a known finite order autoregressive process. A simulation study is performed to illustrate the results and the method is applied to a series of hormone levels data.

KEY WORDS: Univariate time series; Spectral density function; Newton's
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1. INTRODUCTION

An important use of time series analysis has traditionally been to determine if rhythmic patterns exist in a time series Y. Udny Yule in 1927 proposed the (second order) autoregressive process as a model for periodic phenomena. The purpose of this paper is to investigate the autoregressive method for determining the period of such cycles.

Let $\{Y(t), t\epsilon Z\}$, Z the set of integers, be a zero mean covariance stationary time series with autocovariance function $R(v) = E(Y(t)Y(t+v))$, $v\epsilon Z$ and spectral density function

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} R(v)e^{-iv\omega} \ , \quad \omega\epsilon[0, 2\pi] \ .$$

If we assume that f is bounded above and below by positive constants, then (Masani (1966)) we can write as a limit in mean square

$$\sum_{j=0}^{\infty} \alpha(j)Y(t-j) = \epsilon(t) \ , \quad t\epsilon Z$$

where $\alpha(0) \equiv 1$, the $\alpha(\cdot)$ are real numbers, and $\{\epsilon(t), t\epsilon Z\}$ is a white noise series of zero mean independent, identically distributed random variables with common positive variance σ^2 .

Thus Y can be written as an infinite order autoregressive process $(AR(\infty))$ and f can be written

$$f(\omega) = \frac{\sigma^2}{2\pi} \ \frac{1}{|g(e^{i\omega})|^2} \ , \quad \omega\epsilon[0, 2\pi] \ ,$$

where the complex polynomial $g(z) = \sum_{j=0}^{\infty} \alpha(j)z^j$ has no zeros in or on the unit circle.

We seek an estimator of a period $\lambda = 2\pi/\omega$ corresponding to a peak in f, i.e. estimate λ such that $f'(\frac{2\pi}{\lambda}) = 0$, $f''(\frac{2\pi}{\lambda}) < 0$.

In section 2 we derive an estimator $\hat{\lambda}$ of λ based on an estimated AR($\hat{p}$) approximant to f, i.e. use (1.1) with g truncated at some suitable degree $\hat{p}$. Section 3 derives the large sample distribution of $\hat{\lambda}$ under the assumption of known finite autoregressive order. The results of a Monte Carlo simulation study illustrating the known order, unknown but finite order, and infinite order cases are given in section 4 as well as the analysis of a real time·series.

2. THE AUTOREGRESSIVE METHOD

Given a sample realization Y(1), ..., Y(T) from Y, let $\rho_T(v) = R_T(v)/R_T(0)$, v = 0, ..., M, where

$$R_T(v) = \frac{1}{T} \sum_{t=1}^{T-v} Y(t)Y(t+v), \quad v = 0, \ldots, M ,$$

and M is an integer larger than the approximating order $\hat{p}$. Then $\hat{p}$ is chosen as the value of m minimizing Parzen's (1974) CAT criterion

$$CAT(m) = \frac{1}{T} \sum_{j=1}^{m} (\frac{T-j}{T})\hat{\sigma}_j^{-2} - (\frac{T-m}{T})\hat{\sigma}_m^{-2} , \quad m = 0, \ldots, M,$$

and the $\hat{\sigma}_j^2$ are found by the Yule-Walker equations

$$\sum_{\ell=0}^{j} \hat{\alpha}_j(\ell) \, \rho_T(\ell-v) = \delta_v \hat{\sigma}_j^2 , \quad v = 0, \ldots, j \leq M ,$$

and δ_v is the Kronecker delta.

Thus the estimated AR($\hat{p}$) spectral density $\hat{f}_{\hat{p}}$ is given by

$$\hat{f}_{\hat{p}}(\omega) = \frac{\hat{\sigma}_{\hat{p}}^2}{2\pi} \frac{1}{\left| \sum_{j=0}^{\hat{p}} \hat{\alpha}_{\hat{p}}(j)e^{ij\omega} \right|^2} , \quad \omega \in [0, 2\pi] .$$

To estimate the period λ from $\hat{f}_{\hat{p}}$, we need only find the value of ω maximizing a function of the form

$$s(\omega) = \frac{1}{\left| \sum_{j=0}^{p} \alpha(j) e^{ij\omega} \right|^2} \quad , \quad \omega \varepsilon [0, 2\pi] \;,$$

given p, $\alpha(0) = 1$, $\alpha(1)$, ..., $\alpha(p)$. But this is equivalent to minimizing $s^{-1}(\omega) = \left| \sum_{j=0}^{p} \alpha(j) e^{ij\omega} \right|^2$ which is the spectral density of a moving average process of order p (MA(p)) with parameters α and white noise variance 2π. Thus we can write

$$s^{-1}(\omega) = \sum_{v=-p}^{p} Ri(v) e^{-iv\omega} = Ri(0) + 2 \sum_{v=1}^{p} Ri(v) \cos v\omega$$

where the inverse autocovariance function (Cleveland (1972)) $Ri(\cdot)$ consists of the autocovariances corresponding to the MA(p) process and are given by

$$Ri(v) = 2\pi \sum_{j=0}^{p-v} \alpha(j) \alpha(j+v) \;, \quad v = 0, \ldots, p \;.$$

Then given an initial approximation ω_0 to a value ω^* of ω minimizing $h(\omega) = \sum_{v=1}^{p} Ri(v) \cos v\omega$ we use Newton's method to find ω^* as the limit of the sequence

$$\omega_{n+1} = \omega_n - \frac{h(\omega_n)}{h'(\omega_n)} = \omega_n + \frac{\sum_{v=1}^{p} Ri(v) \cos v\omega_n}{\sum_{v=1}^{p} v Ri(v) \sin v\omega_n}$$

The initial value ω_0 is obtained in practice as the frequency of a relative maximum of $\hat{f}_{\hat{p}}$ evaluated over a suitable grid of frequencies.

Then if K maxima of $\hat{f}_{\hat{p}}$ are found, the estimated frequencies are labelled $\hat{\omega}_1$, ..., $\hat{\omega}_K$ and corresponding periods by $\hat{\lambda}_1$, ..., $\hat{\lambda}_K$.

3. LARGE SAMPLE DISTRIBUTION OF AN ESTIMATED PERIOD

We assume that Y is in fact an AR(p) process with known p. Then (Parzen (1961))

$$\sqrt{T}\ (\hat{\underset{\sim}{\alpha}}\ \underset{\sim}{\alpha}) \xrightarrow{\ \mathcal{D}\ } N\ (0,\Lambda)$$

where $\hat{\underset{\sim}{\alpha}}^T = (\hat{\alpha}(1),\ \ldots,\ \hat{\alpha}(p))$, $\underset{\sim}{\alpha}^T = (\alpha(1),\ \ldots,\ \alpha(p))$, $\Lambda = \sigma^2 \Gamma_p^{-1}$ has

(j,k) element given by

$$A_{jk} = \sum_{\ell=0}^{v} [\alpha(j-1-\ell)\alpha(k-1-\ell) - \alpha(p+1+\ell-j)\alpha(p+1+\ell-k)]$$

$j,k = 1,\ \ldots,\ p$, $v = \min\ (j-1,k-1)$, and Γ_p is the $p \times p$ covariance matrix

of p consecutive Y's. .

From this fact we have

Lemma

Let $\underset{\sim}{Ri}^T = (Ri(1),\ \ldots,\ Ri(p))$, $\hat{Ri}^T = (\hat{Ri}(1),\ \ldots,\ \hat{Ri}(p))$, and C be the

$p \times p$ matrix having v^{th} row $\underset{\sim}{C}_v^T = (C_v(1),\ \ldots,\ C_v(p))$ where $\frac{1}{2\pi}C_v(j) = \gamma_{v+j}\alpha(v+j)$

$+\gamma_{j-v}\alpha(j-v)$, and $\gamma_k = 1$ if $k = 0,\ \ldots,\ p$ and 0 otherwise.

Then

$$\sqrt{T}(\hat{Ri}-\underset{\sim}{Ri}) \xrightarrow{\ \mathcal{D}\ } N(0,C\Lambda C^T)\ .$$

Proof

$$\sqrt{T}\ (\hat{Ri}(v) - Ri(v)) = \sqrt{T}\ 2\pi \sum_{j=0}^{p-v} [\hat{\alpha}(j)\hat{\alpha}(j+v) - \alpha(j)\alpha(j+v)]$$

$$= \sqrt{T}\ 2\pi \sum_{j=0}^{p-v} \{[\hat{\alpha}(j)-\alpha(j)]\hat{\alpha}(j+v)+\alpha(j)[\hat{\alpha}(j+v)-\alpha(j+v)]\}$$

$$\approx \sqrt{T}\ 2\pi \sum_{j=0}^{p-v} \{[\hat{\alpha}(j)-\alpha(j)]\alpha(j+v)+\alpha(j)[\hat{\alpha}(j+v)-\alpha(j+v)]\}$$

$$= \sum_{j=1}^{p} C_v(j)z(j)\ ,$$

where $z(j) = \sqrt{T}\ (\hat{\alpha}(j)-\alpha(j))$, $j = 1,\ \ldots,\ p$ and the symbol $\approx$ means

"has the same asymptotic distribution as". Thus $\sqrt{T}\ (\hat{Ri}-\underset{\sim}{Ri}) \approx C\ \sqrt{T}\ (\hat{\underset{\sim}{\alpha}}-\underset{\sim}{\alpha})$

and the lemma follows.

From this we obtain

Theorem

Let $\hat{\omega}$ be the estimator obtained by the method of section 2 of the true peak frequency ω_0. Then

$$\sqrt{T}\ (\hat{\omega}-\omega_0) \xrightarrow{D} N\ (0,\ \underline{b}^T CAC^T \underline{b})$$

where $b(v) = v\ \text{Sin}\ v\omega_0/h''(\omega_0)$, and $h''(\omega) = -\sum_{v=1}^{p} v^2 Ri(v)\ \text{Cos}\ v\omega$.

Proof

Since $\sum_{v=1}^{p} v\hat{R}i(v)\ \text{Sin}\ v\omega = 0$ and $\hat{\omega}$ is a consistent estimator of ω_0 since $\hat{f}_p$ converges almost surely to f_p, we can write

$$\frac{\sum_{v=1}^{p} v\ (\hat{R}i(v)-Ri(v))\ \text{Sin}\ v\omega_0}{\hat{\omega}-\omega_0} = \frac{\sum_{v=1}^{p} v\hat{R}i(v)\ \text{Sin}\ v\omega_0}{\hat{\omega}-\omega_0} = \frac{\sum_{v=1}^{p} v\hat{R}i(v)[\text{Sin}\ v\hat{\omega}-\text{Sin}\ v\omega_0]}{\hat{\omega}-\omega_0}$$

$$\xrightarrow{P} \frac{\sum_{v=1}^{p} vRi(v)[\text{Sin}\ v\hat{\omega}-\text{Sin}\ v\omega_0]}{\hat{\omega}-\omega_0}$$

since

$$\frac{\sum_{v=1}^{p} v\hat{R}i(v)[\text{Sin}\ v\hat{\omega}-\text{Sin}\ v\omega_0]}{\hat{\omega}-\omega_0}$$

$$= \frac{\sum_{v=1}^{p} vRi(v)[\text{Sin}\ v\hat{\omega}-\text{Sin}\ v\omega_0]}{\hat{\omega}-\omega_0} + \frac{\sum_{v=1}^{p} v(\hat{R}i(v)-Ri(v))[\text{Sin}\ v\hat{\omega}-\text{Sin}\ v\omega_0]}{\hat{\omega}-\omega_0}$$

and $\left| \dfrac{\sum_{v=1}^{p} v(\hat{R}i(v)-Ri(v))(\text{Sin}\ v\hat{\omega}-\text{Sin}\ v\omega_0)}{\hat{\omega}-\omega_0} \right|$

$$\leq \max_{v=1,\ldots,p} |\hat{R}i(v)-Ri(v)| \sum_{v=1}^{p} \frac{|\text{Sin } v\hat{\omega}-\text{Sin } v\omega_0|}{|\hat{\omega}-\omega_0|}$$

$$\xrightarrow{P} 0 .$$

Thus

$$\frac{\sum_{v=1}^{p} v(\hat{R}i(v)-Ri(v)) \text{ Sin } v\omega_0}{\hat{\omega}-\omega_0} \xrightarrow{P} h''(\omega_0)$$

and

$$\sqrt{T} (\hat{\omega}-\omega_0) \approx \frac{\sum_{v=1}^{p} v\sqrt{T} (\hat{R}i(v)-Ri(v) \text{ Sin } v\omega_0}{h''(\omega_0)}$$

$$= \dot{\underset{\sim}{b}}^T \sqrt{T} (\hat{\underset{\sim}{R}i}-\underset{\sim}{R}i) .$$

Corollary

1) $\sqrt{T} (\hat{\lambda}-\lambda_0) \xrightarrow{\mathcal{D}} N(0, \frac{\lambda_0^4}{4\pi^2} \underset{\sim}{b}^T CAC^T \underset{\sim}{b})$

2) If $\hat{\lambda}_1, \ldots, \hat{\lambda}_K$ are estimators of $\lambda_1, \ldots, \lambda_K$,

$$\sqrt{T} (\hat{\underset{\sim}{\lambda}}-\underset{\sim}{\lambda}) \xrightarrow{\mathcal{D}} N(0,DBCAC^T B^T D^T)$$

where $D = \text{Diag} (\lambda_1^2/2\pi, \ldots, \lambda_k^2/2\pi)$, and the jth row of the k x p matrix B is the $\underset{\sim}{b}^T$ corresponding to ω_j .

We note that since $b(v) = v \text{ Sin } v\omega_0/h'(\omega_0)$ that $\sigma_{\hat{\lambda}}^2$, the asymptotic variance of $\hat{\lambda}$ is inversely related to the square of the second derivative of the reciprocal of the function to be maximized. Thus the sharper the peak in f, the more precisely λ_0 can be estimated.

4. SIMULATION STUDY AND WORKED EXAMPLE

To illustrate the results of sections 2 and 3 above for known, unknown but finite, and infinite order autoregressive processes we performed a simulation study in two parts: 1) the known and unknown order case, and 2) the infinite order case.

4.1 Simulation of Known and Unknown but Finite Order Case

We simulated on an Amdahl 470V/6 computer at Texas A&M University 20 series each of length 80, 160, and 240 for each of the five autoregressive models given in table 1. Each of the 300 series was obtained by

$$Y(t) = -\sum_{j=1}^{P}\alpha(j)Y(t-j)+\varepsilon(t), \quad t = p+1, \ldots, T$$ where $\varepsilon(1), \ldots, \varepsilon(T)$ are iid $N(0,\sigma^2)$ variates obtained using the Box-Muller method (Kennedy and Gentle (1980), p. 202) from $U(0,1)$ variates generated by a composite generator (Kennedy and Gentle (1980), p. 162), and starting values $Y(1)$, $\ldots, Y(p)$ were obtained as a $N_p(0,\Gamma_p)$ random variate where Γ_p is obtained by the algorithm of Pagano (1973).

Estimators of the spectral density for each series were determined in four ways:

1) p,YW: the Yule-Walker equations were solved via Levinson's algorithm (Levinson (1947)) to determine estimates of the parameters of the AR(p) process.

2) p,LS: least squares estimators of $\alpha(1), \ldots, \alpha(p), \sigma^2$ were obtained via a Gram-Schmidt orthogonalization procedure applied to the regression model $Y = -X\alpha+\varepsilon$ where $Y^T = (Y(p+1), \ldots, Y(T))$, $\alpha^T = (\alpha(1), \ldots, \alpha(p))$, $X_{jk} = Y(p+j-k)$, $j = 1, \ldots, T-p$, $k=1, \ldots, p$, and $\varepsilon^T = (\varepsilon(p+1), \ldots, \varepsilon(T))$.

3) $\hat{p}$,YW: The CAT criterion determined order (with $\hat{\sigma}_j^2$ obtained via Levinson's algorithm) parameters estimated by the Yule-Walker equations.

4) $\hat{p}$,LS: Same as $\hat{p}$,YW except parameters estimated by least squares.

Figure A contains plots of the true log spectra for each of the five models, while table 1 lists the true periods (obtained from the coefficients by the algorithm of section 2) for each model as well as $T\sigma_{\hat{\lambda}}^2$. In each case the peak of interest is the one for smallest frequency. We note that models 1, 2, and 5 are very similar except for order and location of peak while model 4 has a single broad peak (reflected in the size of $T\sigma_{\hat{\lambda}}^2$). Model 3 is of particular interest since it appears that h"(8.951) $\approx$ 0 (note how this is reflected in $T\sigma_{\hat{\lambda}}^2$).

Thus it appears that these models are representative of a wide class of models and that one pathological case (model 3) has been included.

The purpose of this part of the simulation study is twofold:

1) In the p,YW and p,LS case to see if the methodology of section 2 agrees with the theory of section 3 and to compare the performance of YW and LS, particularly on model 3.

2) To investigate the distribution of $\hat{\lambda}$ in the finite but unknown order case.

Table 2 contains results for the p,YW and p,LS cases while figures B-F display the 20 p,YW and p,LS estimated spectra for the three sample sizes for each of the five models. A comparison of the spectra for the YW and LS estimators shows they are almost indistinguishable except that LS has a wider spectral range in models 2, 3, and 5 and that the peak in model 3 is estimated somewhat differently by the two methods. This is reflected in table 2 where it is seen that LS decomposes this peak into two peaks in 4, 4, and 3 series (T = 80, T = 160, T = 240) while YW does

this in 3, 1, and 1 series. Thus except for this pathological model we find no large difference in the two estimation procedures. Also table 2 shows the expected adherence of the simulated data to the theory of section 3.

Table 3 and figures G-K report the results of the $\hat{p}$,YW analysis of the 300 simulated autoregressive processes. The results indicate that the method estimates periods remarkably well. In fact, even though 45% of the series had the wrong order determined, the results of table 3 are remarkably similar to those of table 2.

4.2   Simulation of Infinite Order Case

Twenty series of lengths 80, 160, and 240 were generated from the MA(8) model $Y(t) = \varepsilon(t) + .309\varepsilon(t-1) - .0748\varepsilon(t-2) - .0113\varepsilon(t-3) - .0853\varepsilon(t-4) - .0552\varepsilon(t-5) + .0084\varepsilon(t-6) + .4621\varepsilon(t-7) + .288\varepsilon(t-8)$. Figure L contains the true MA(8) log spectrum and the estimated autoregressive spectra for each of the 60 simulated series.   Table 4 describes the analysis of each of the 60 series.

Inspection of table 4 raises two points:

1)   The AR orders chosen appear to increase with sample size.

2)   This increase leads to estimated models which make two (or even three in one case) peaks out of the single broad peak.

Thus it appears that if the process Y cannot be adequately represented as a finite order AR process (in particular if the process has very broad peaks), the proposed method may not be satisfactory.

We note however that if instead of using approximating AR schemes one used approximating MA schemes, this would make virtually no change in

the algorithm and only the matrix A in the asymptotic covariance of $\hat{\lambda}$ would need to be changed in theorem 1 to the asymptotic covariance matrix of maximum likelihood estimators of MA parameter estimators. We note however that these methods cannot be extended to the mixed AR-MA processes since neither their spectra nor its reciprocal can be written as a finite degree trigonometric polynomial. In this paper we have confined our attention to approximating AR schemes because of their computational attractiveness and widespread use.

4.3 Analysis of Hormone Levels Data

We consider the levels of luteinizing hormone (LH) in a cow as measured at 10 minute intervals for a 24 hour period (Rahe et al (1980)).

A plot of the data is given in figure M. Figure N displays the log spectra of the $\hat{p}$,YW determined AR(13) model

$$Y(t) - .2736\ Y(t-1) + .0769\ Y(t-2) + .0046\ Y(t-3) + .1576\ Y(t-4)$$
$$- .0131\ Y(t-5) + .1486\ Y(t-6) - .4795\ Y(t-7) - .1942\ Y(t-8)$$
$$+ .0546\ Y(t-9) + .0876\ Y(t-10) - .1546\ Y(t-11 + .0756\ Y(t-12)$$
$$- .1926\ Y(t-13) = \varepsilon(t)\ .$$

This model gives $\hat{\lambda} = 7.278$ time intervals with an estimated standard error $s_{\hat{\lambda}} = .116$. Thus a large sample confidence interval for λ is given by (70.46 min, 75.10 min) .

REFERENCES

Cleveland, W.S. (1972), "The Inverse Autocorrelations of a Time Series
and Their Applications," Technometrics, 14, 277-293.

Kennedy, William J., and Gentle, James E. (1980), Statistical Computing,
New York: Marcell Dekker, Inc.

Levinson, N. (1947), "The Wiener RMS (Root Mean Square) Error Criterion
in Filter Design and Prediction," Journal of Mathematical Physics,
25, 261-278.

Masani, P. (1966), "Recent Trends in Multiple Prediction Theory," in
Multivariate Analysis, ed. P.R. Khrishnaiah, New York: John Wiley
and Sons.

Pagano, Marcello (1973), "When is an Autoregressive Scheme Stationary,"
Communications in Statistics, 1, 533-544.

Parzen, Emanuel (1961), "An Approach to Time Series Analysis," Annals
of Mathematical Statistics, 32, 383-421.

Parzen, Emanuel (1974), "Recent Advances in Time Series Modeling,"
IEEE Transactions on Automatic Control, AC-19, 723-730.

Rahe, C.H., Owens, R.E., Fleeger, J.L., Newton, H.J., and Harms, P.G. (1980),
"Pattern of Plasma Luteinizing Hormone in the Cyclic Cow: Dependence
Upon the Period of the Cycle," Endocrinology, 107, 498-503.

1. Autoregressive Models used in Simulation Study

Model Number	p	λ	$T\sigma^2_{\lambda}$	Coefficients				
1	7	5.900	3.120	-.6381	-.0140	.0295	.0272	-.1587
				-.4581	.3572			
2	7	7.398	4.240	-.2004	.1158	.1679	.1056	.1357
				.0566	-.3899			
3	9	8.951	924.0	-.4335	.3740	.1071	.3213	.2071
				.2089	.1942	-.2828	.3656	
4	4	6.900	13.44	-.1477	.0601	.3565	.1964	
5	10	7.586	4.80	-.3190	-.0175	.1803	.0340	.1832
				.0423	-.1393	-.2463	.1365	.1974

2. Simulation Results for Known Order Case[a]

		Model 1	Model 2	Model 3	Model 4	Model 5
	p	7	7	9	4	10
	λ	5.900	7.398	8.951	6.900	7.586
	$\sigma^2_{\hat{\lambda},80}$	.039	.053	11.550	.168	.036
	$\sigma^2_{\hat{\lambda},160}$	.020	.026	5.775	.084	.018
	$\sigma^2_{\hat{\lambda},240}$	.013	.018	3.850	.056	.012
	$\hat{\lambda}_{AVE}$	5.912,5.937	7.410,7.412	8.776,8.82	6.897,6.891	7.480,7.504
	$s^2_{\hat{\lambda}}$	.056, .041	.198, .173	.624, .736	.158, .148	.077, .071
T=80	CIC[b]	.950, .900	.750, .750	1.000,1.000	1.000, .950	.850, .800
	NB[c]			3 ,4		
	$\hat{\lambda}_{AVE}$	5.983,5.914	7.474,7.474	8.997.8.961	6.901.6.906	7.598,7.606
T=160	$s^2_{\hat{\lambda}}$	.041, .019	.034, .033	.500, .504	.052, .054	.030, .028
	CIC	1.000, .950	.900, .900	1.000,1.000	.950, .950	.850, .950
	NB			1 ,4		
	$\hat{\lambda}_{AVE}$	5.871,5.874	7.431,7.428	8.889,8.82	6.832,6.832	7.556,7.564
T=240	$s^2_{\hat{\lambda}}$	.023, .021	.021, .021	.575, .652	.042, .041	.009, .009
	CIC	.900, .900	1.000,1.000	1.000,1.000	1.000,1.000	1.000,1.000
	NB			1 ,3		

a First number in a pair is for Yule-Walker estimator, second is for Least Squares estimator.

b Confidence Interval Coverage. (Proportion of the 20 $\hat{\lambda}$ within $2\sigma_{\hat{\lambda}}$ of λ).

c Number of Series producing dual peaks in model 3. These series not included in calculation of $\hat{\lambda}_{AVE}$, $s^2_{\hat{\lambda}}$, CIC.

3. Simulation Results for Unknown Order Case[a]

		Model 1	Model 2	Model 3	Model 4	Model 5
	P	7	7	9	4	10
	λ	5.900	7.398	8.951	6.900	7.586
	$\sigma^2_{\lambda,80}$	.039	.053	11.550	.168	.036
	$\sigma^2_{\lambda,160}$	.020	.026	5.775	.084	.018
	$\sigma^2_{\lambda,240}$	.013	.018	3.850	.056	.012
T=80	$\hat{P}_{AVE}, s^2_{\hat{p}}$	7.050, .366	6.650,9.621	7.650,5.713	3.850,1.082	7.150,10.239
	$\hat{P}_{MIN}, \hat{P}_{MAX}, PCC^c$	6,8,.65	0,13,.40	3,13,.45	3,7,.50	0,11,.15
	$\hat{\lambda}, s^2_{\hat{p}}$	5.944, .061	7.384[b], .310	8.634, .832	6.786, .534	7.578[b], .151
	CIC, NB	.95	1.00	1.00 ,3	.85	.80
T=160	$\hat{P}_{AVE}, s^2_{\hat{p}}$	7.450, .471	7.950,3.418	9.600, .568	5.050,2.050	9.800,4.695
	$\hat{P}_{MIN}, \hat{P}_{MAX}, PCC^c$	7,9,.65	7,13,.70	9,11,.55	3,8,.35	3,14,.60
	$\hat{\lambda}, s^2_{\lambda}$	5.919, .018	7.479,.049	8.923, .557	6.811, .395	7.632, .057
	CIC, NB	1.00	.85	1.00 ,3	.65	.85
T=240	$\hat{P}_{AVE}, s^2_{\hat{p}}$	8.000,2.947	7.550,1.520	10.30,3.800	4.900,2.305	10.45, .787
	$\hat{P}_{MIN}, \hat{P}_{MAX}, PCC^c$	7,13,.60	7,11,.80	9,15,.50	3,8,.60	10,13,.75
	$\hat{\lambda}, s^2_{\lambda}$	5.878, .045	7.449, .026	9.055, .592	6.830, .010	7.550, .012
	CIC, NB	.85	.95	1.00 ,3	.80	.95

a Yule-Walker estimators used.

b Series having no peaks not included in calculations.

c Proportion choosing correct order.

4. Simulation Results for MA(8) Model [a]

	T = 80		T = 160		T = 240	
Series Number	$\hat{p}$	Periods	p	Periods	p	Periods
1	14	9.204,5.888	14	8.552,5.912	15	8.159
2	7	7.141	7	7.578	9	7.473
3	8	6.407	13	8.546,5.815	19	7.736,5.869
4	7	7.881	12	8.470,6.293	7	7.409
5	0	---	7	7.964	7	7.529
6	8	7.238	14	9.357,5.997	9	7.540
7	2	5.382	7	7.559	14	8.871,6.278
8	7	7.452	7	7.324	7	7.351
9	7	8.015	7	7.110	14	9.117,6.351
10	7	7.312	7	7.817	8	7.209
11	7	7.386	7	7.097	10	7.896
12	4	6.328	7	7.541	7	7.585
13	7	7.678	7	7.133	7	7.111
14	7	7.508	7	7.408	14	8.824,5.906
15	7	6.916	8	6.589	15	7.563,5.807
16	7	7.216	14	8.500,6.280	14	7.788
17	2	5.021	7	7.750	18	9.785,7.393,5.529
18	8	6.847	7	7.255	15	9.181,6.367
19	7	7.365	7	7.060	9	8.405
20	7	6.787	7	7.335	14	8.937,6.019
Mean [b]	6.500	6.993	8.650	7.368	11.600	7.621
Variance [b]	8.158	.591	8.134	.115	15.832	.133

[a] YW estimators used, true period is 7.334.

[b] These calculations for periods do not include peaks or multi-peaked series.

FIGURE A.   True Log Spectra of Five Autoregressive Models given in
            Table 1
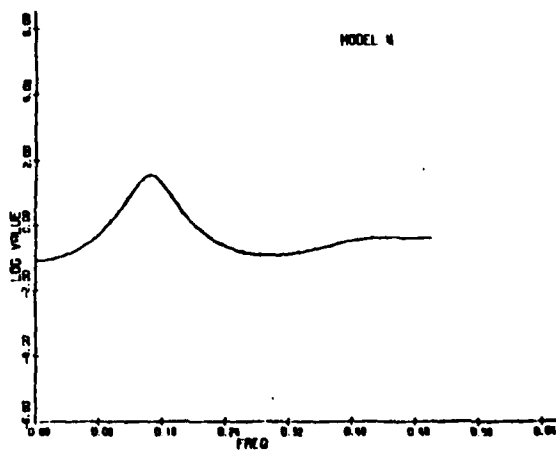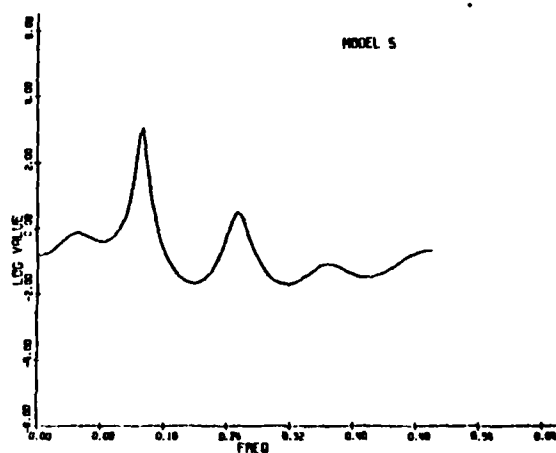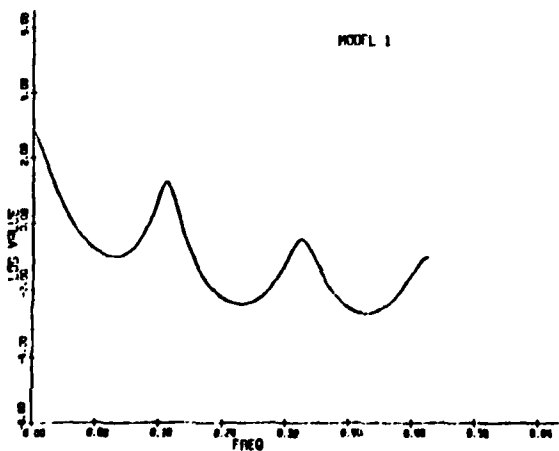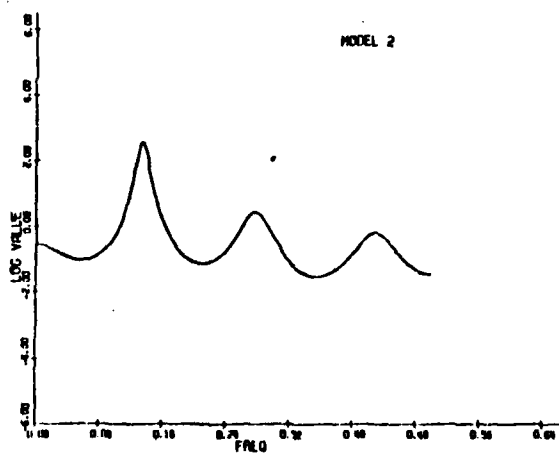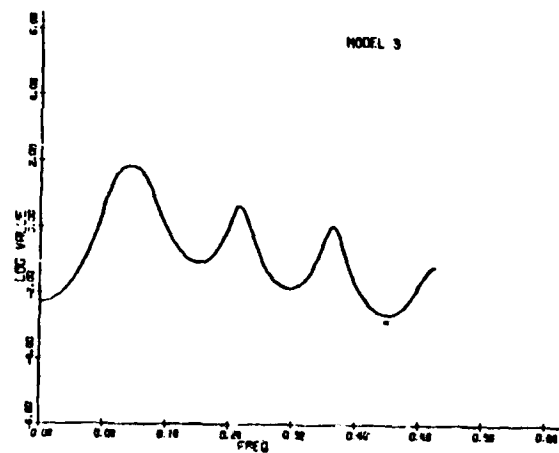
FIGURE B.   Log of the p,YW and p,LS Estimated Spectra for Model 1 for
Three Sample Sizes
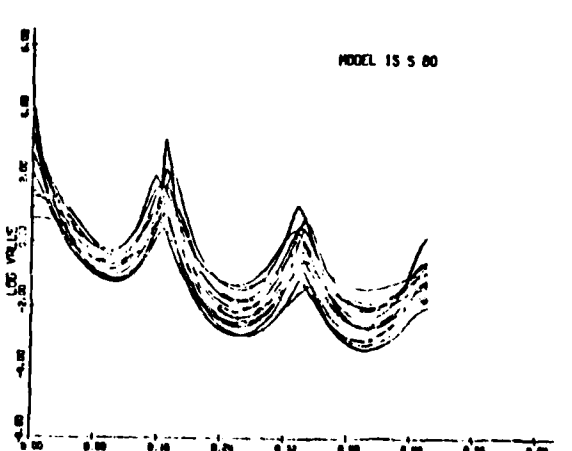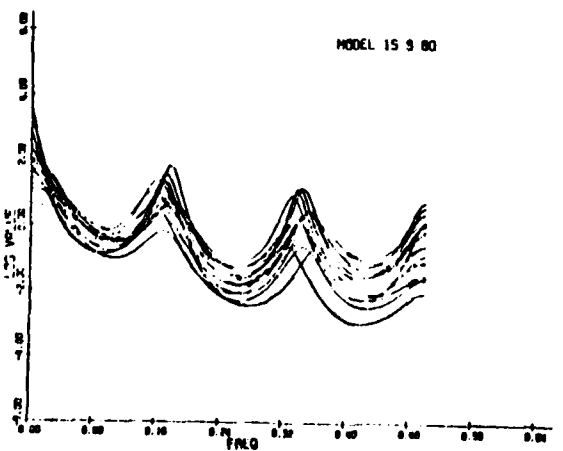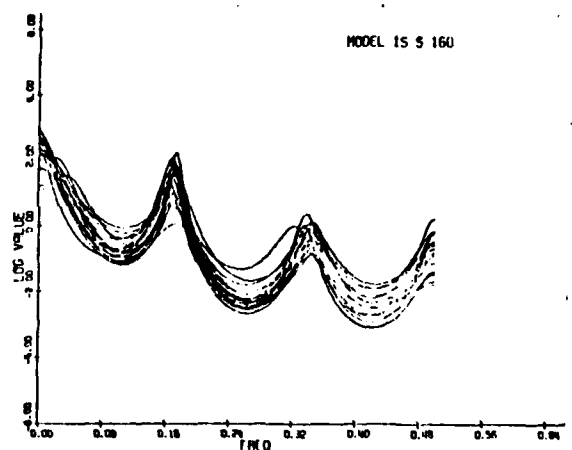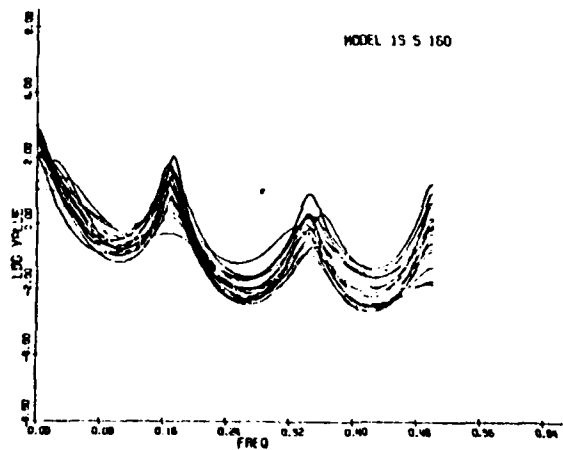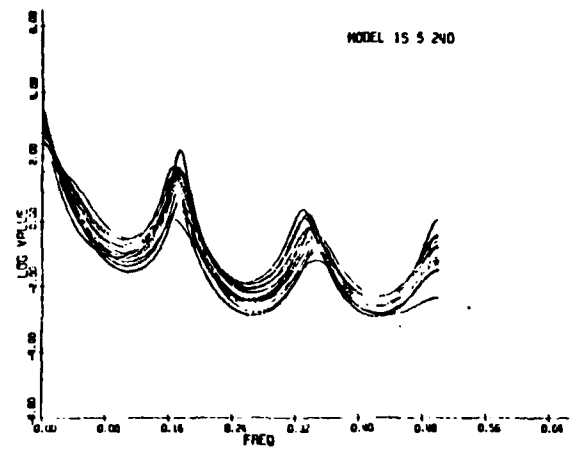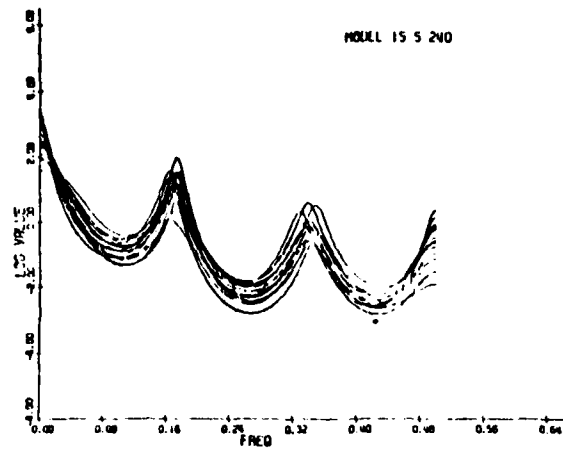
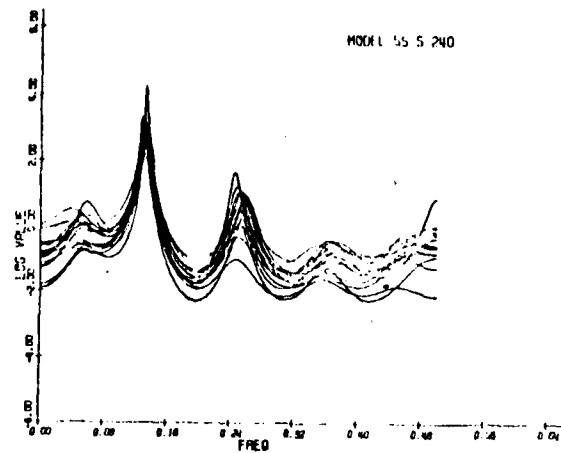p,YW   p,LS

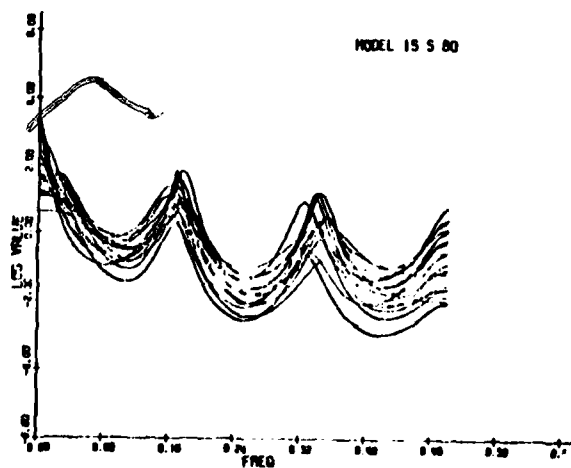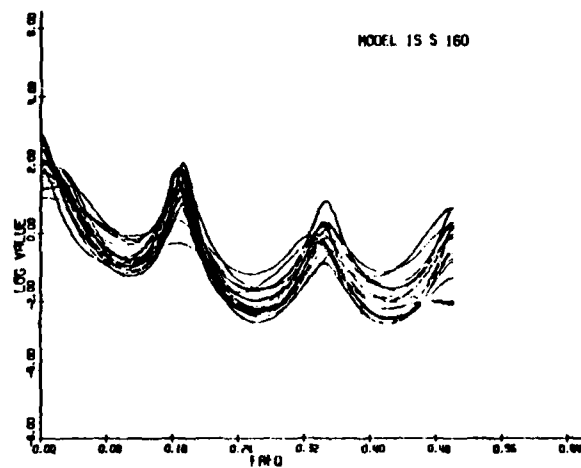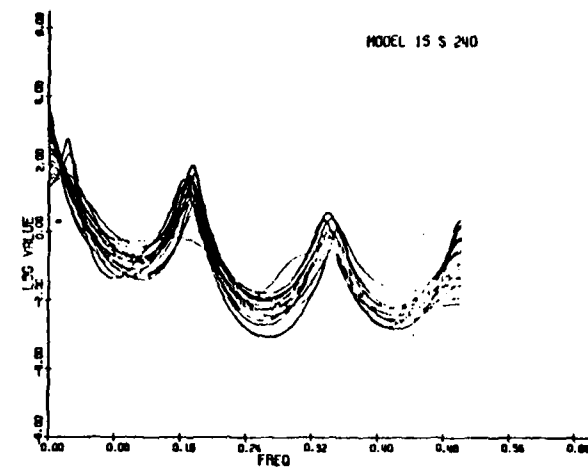FIGURE C. Log of the p,YW and p,LS Estimated Spectra for Model 2 for
Three Sample Sizes
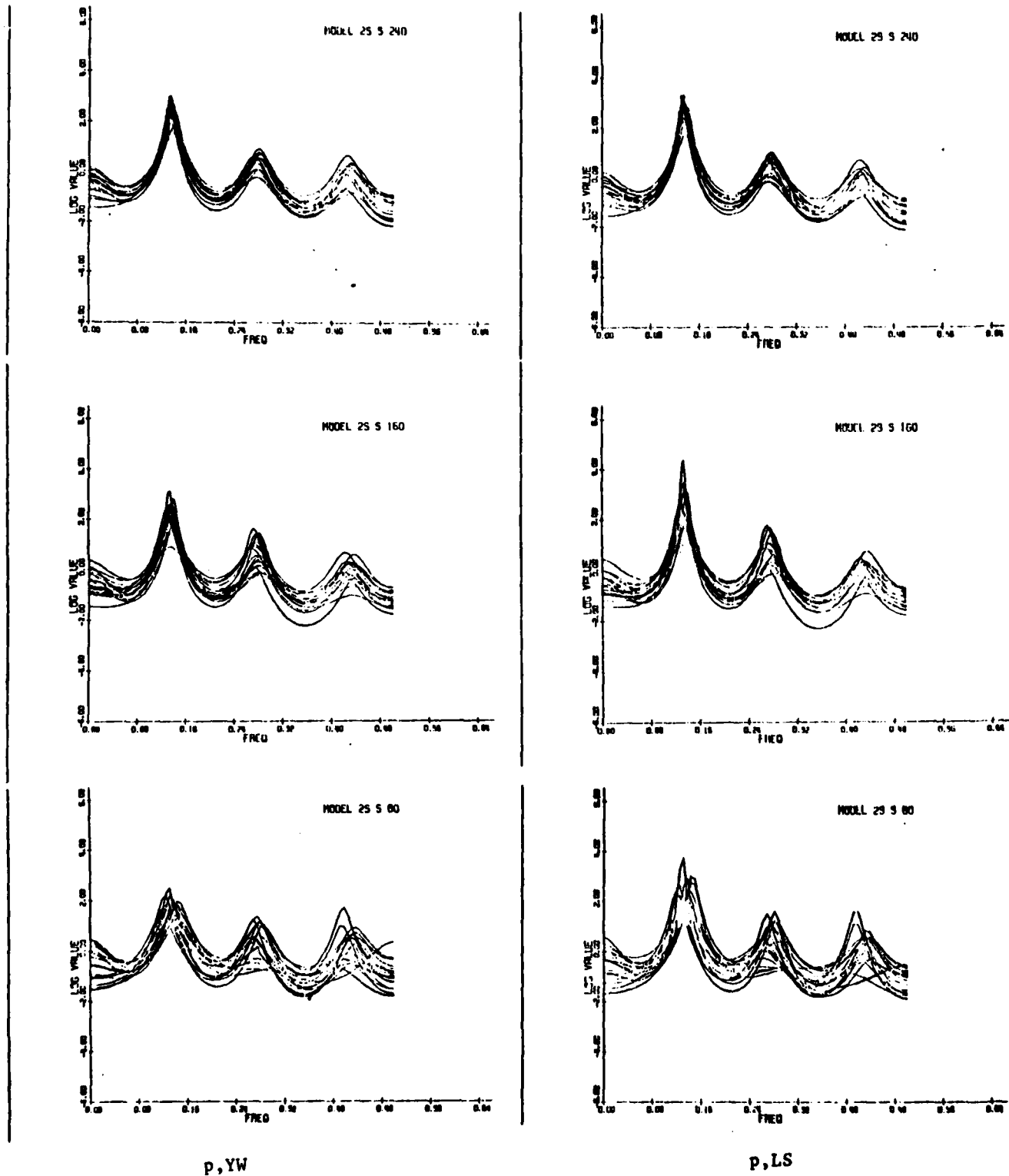
p,YW   p,LS

FIGURE D.   Log of the p,YW and p,LS Estimated Spectra for Model 3 for
Three Sample Sizes
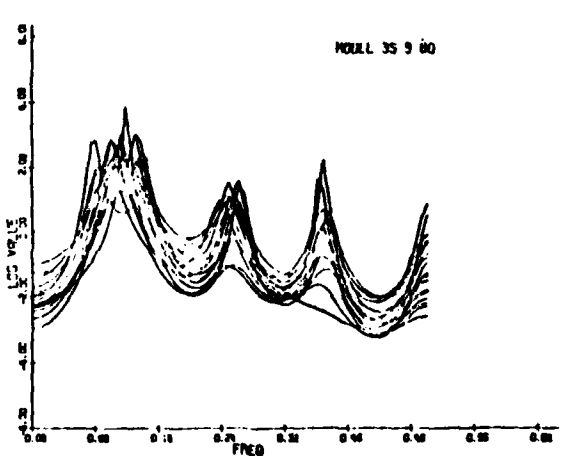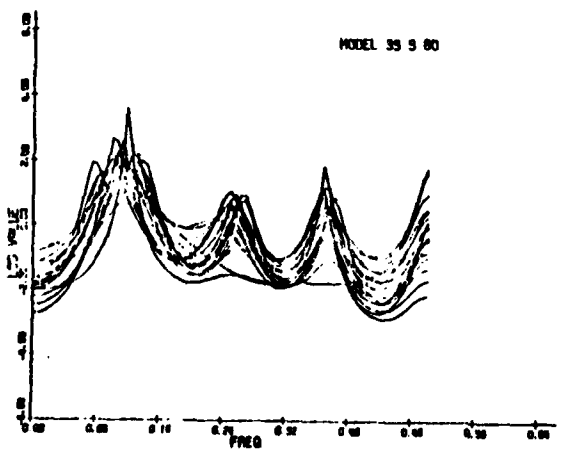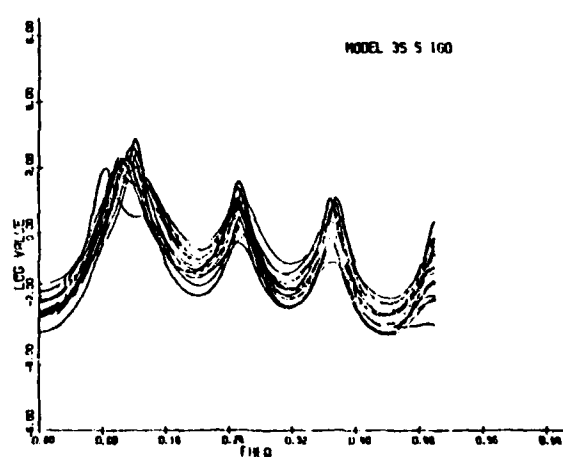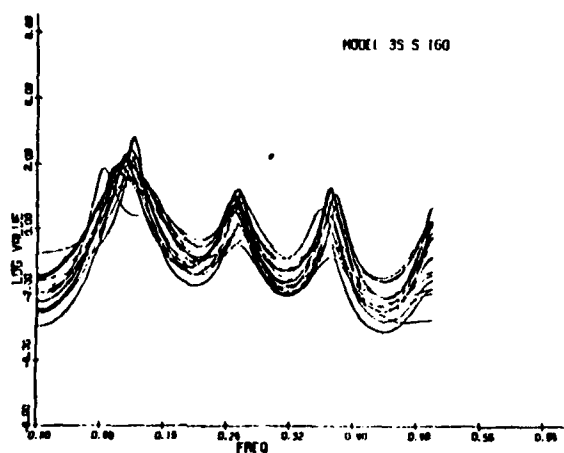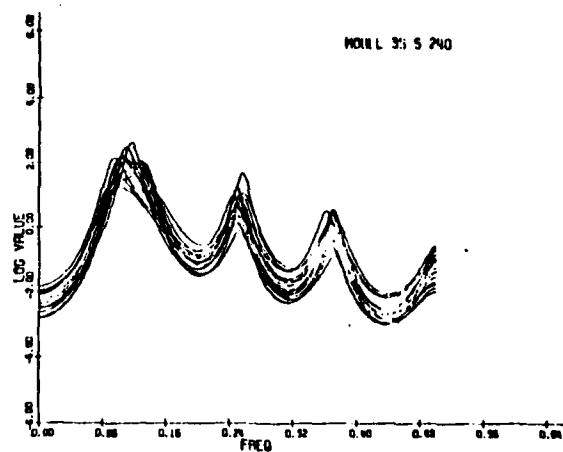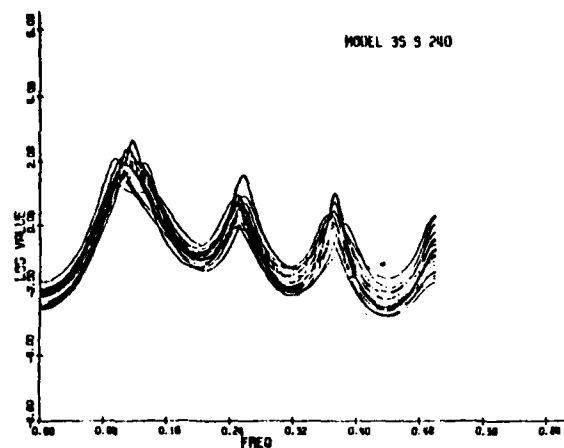
p,YW

p,LS

FIGURE E. Log of the p,YW and p,LS Estimated Spectra for Model 4 for
Three Sample Sizes
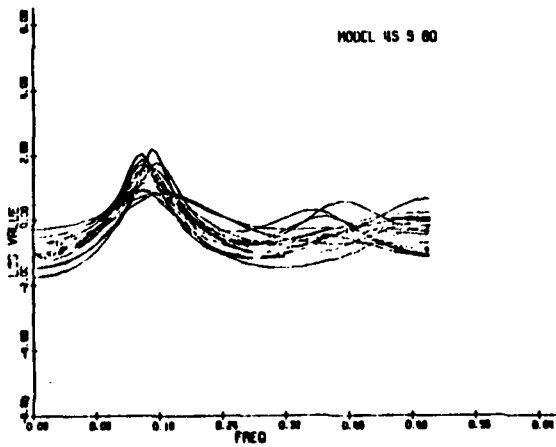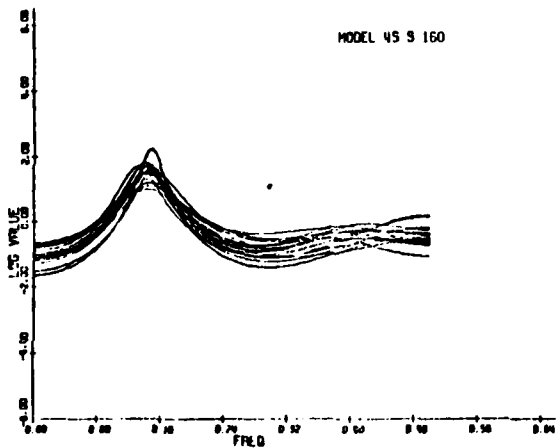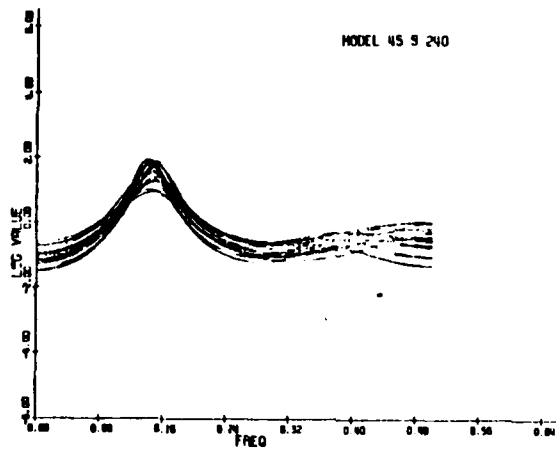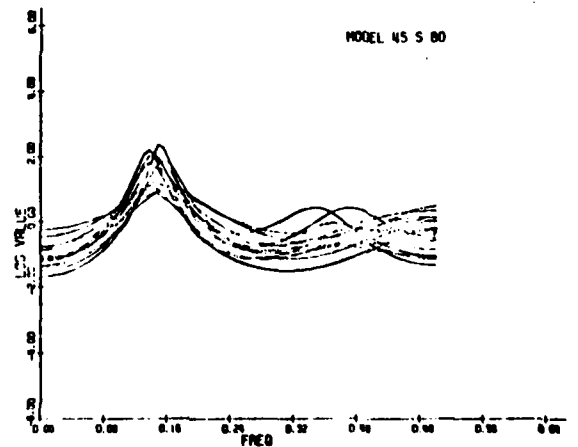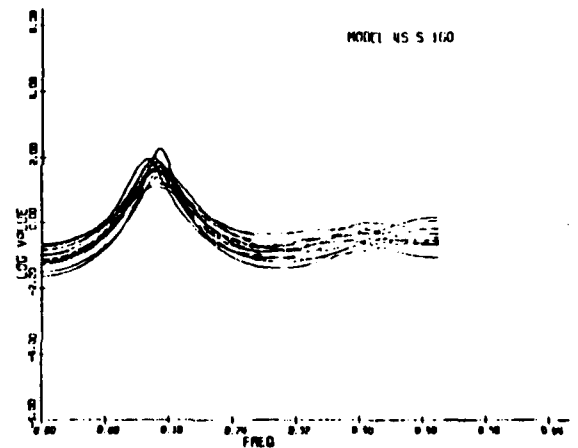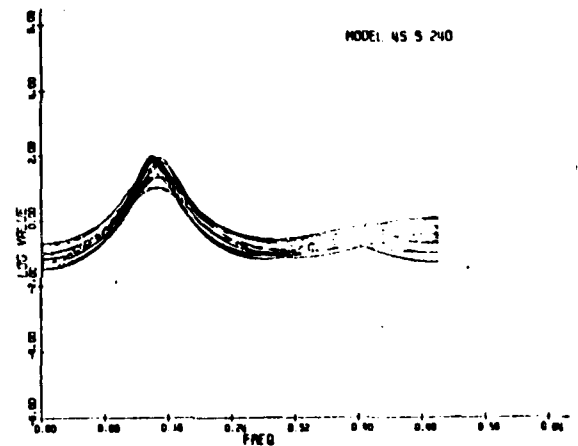
p,YW   p,LS

FIGURE F. Log of the p,YW and p,LS Estimated Spectra for Model 5 for
Three Sample Sizes
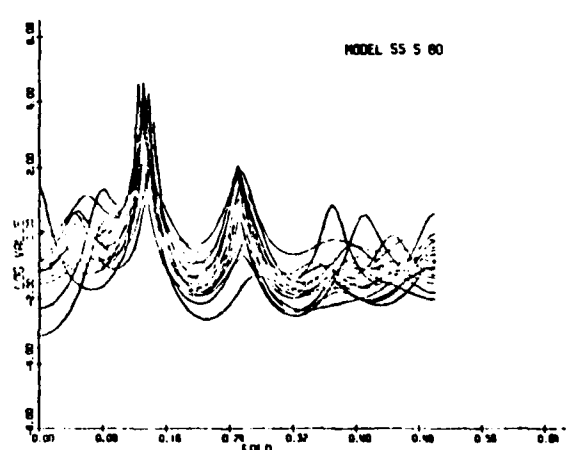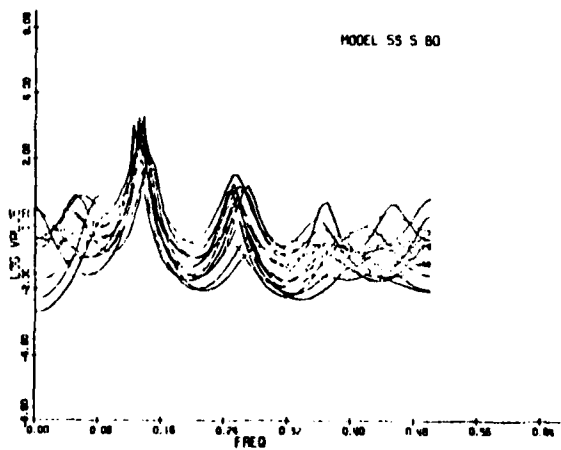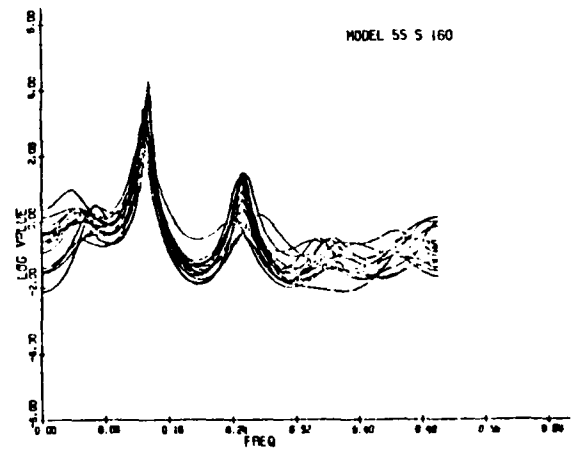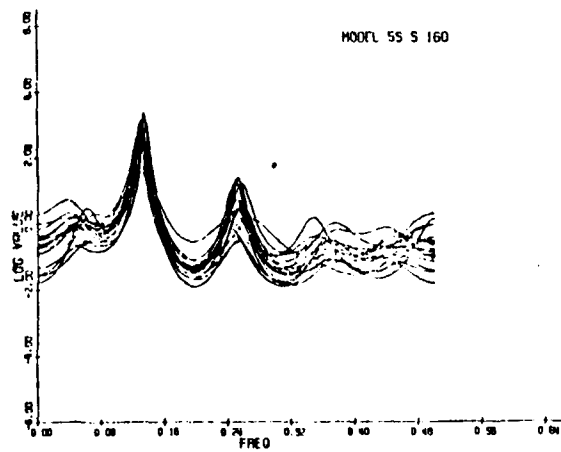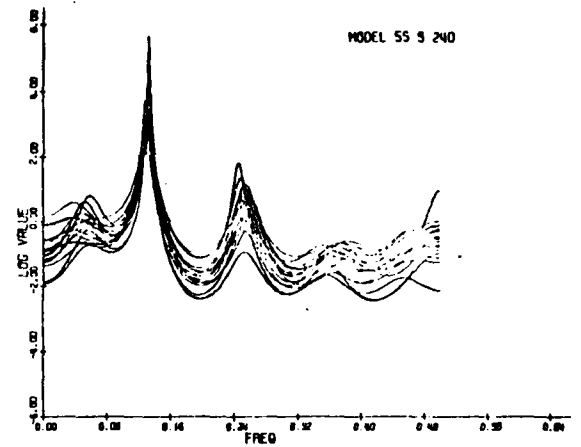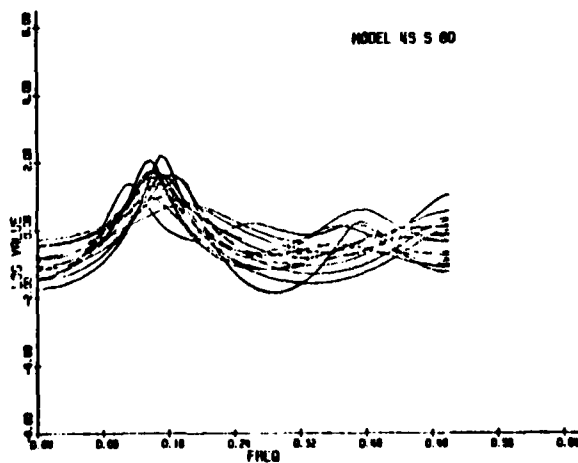

p,YW   p,LS

FIGURE G. Log of the $\hat{p}$,YW Estimated Spectra for Model 1 for Three Sample
Sizes

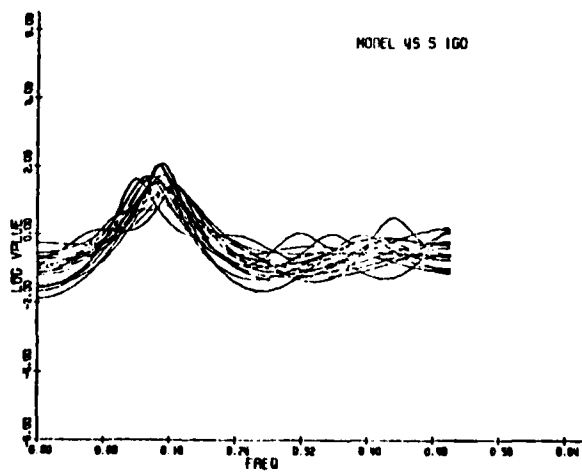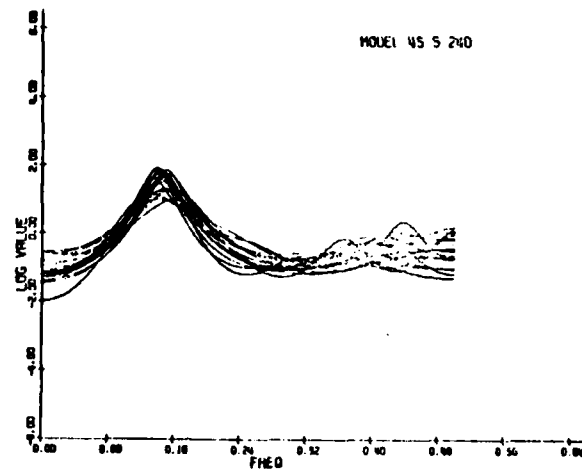FIGURE II. Log of the p̂,YW Estimated Spectra for Model 2 for Three Sample
Sizes
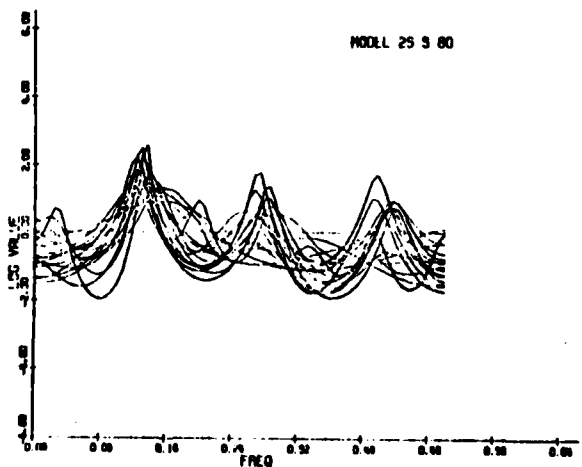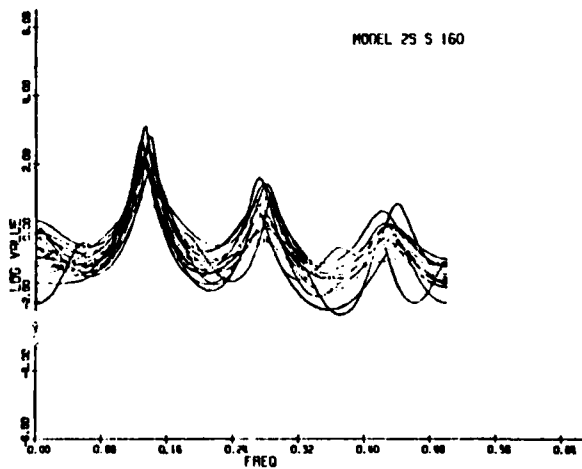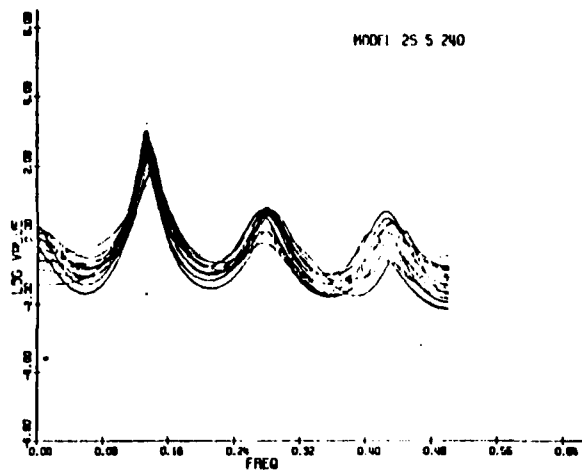
FIGURE I. Log of the $\hat{p}$,YW Estimated Spectra for Model 3 for Three Sample
Sizes
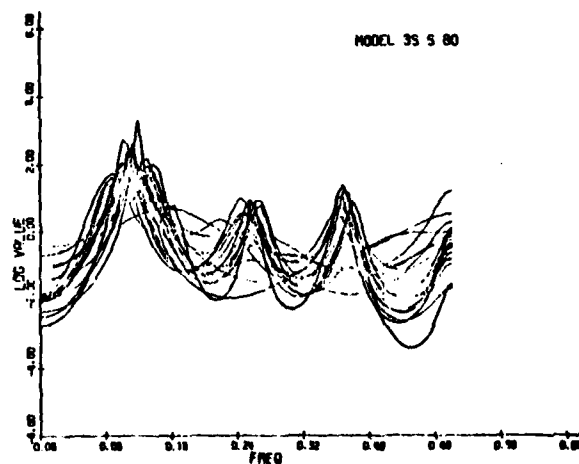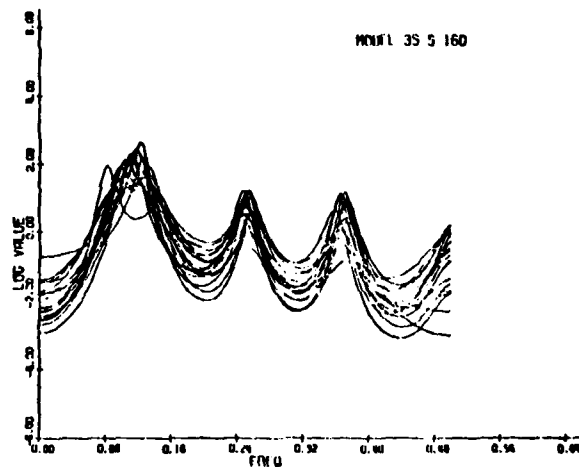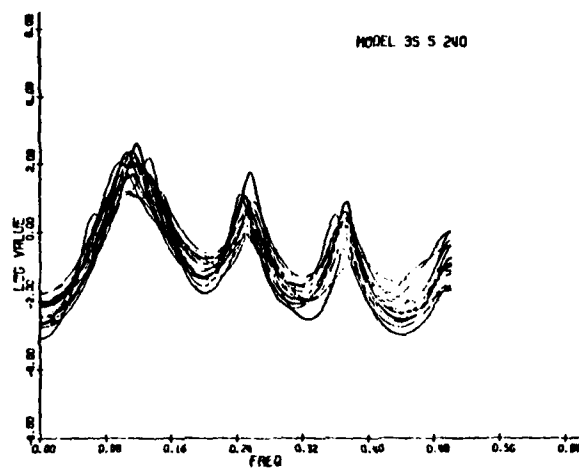
FIGURE J. Log of the $\hat{\rho}$,YW Estimated Spectra for Model 4 for Three Sample Sizes

FIGURE K. Log of the $\hat{p}$,YW Estimated Spectra for Model 5 for Three Sample
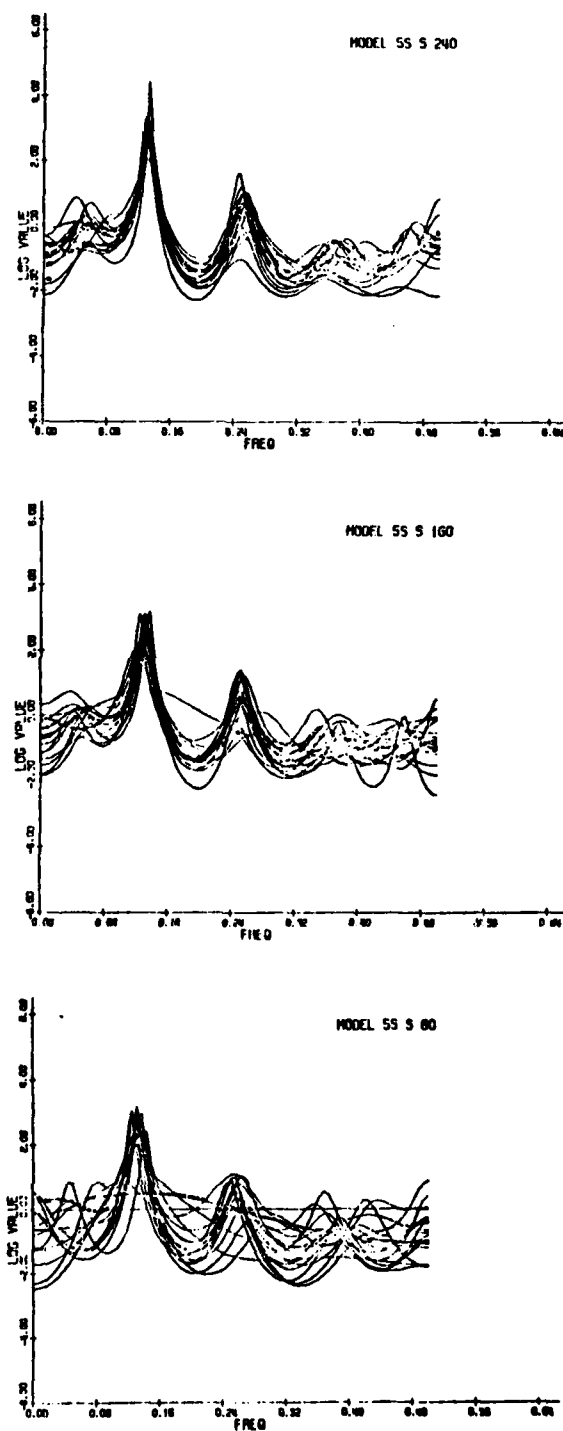   Sizes

FIGURE L.   Log of True MA(B) Spectrum and $\hat{p}$,YW Estimated Spectra for Three
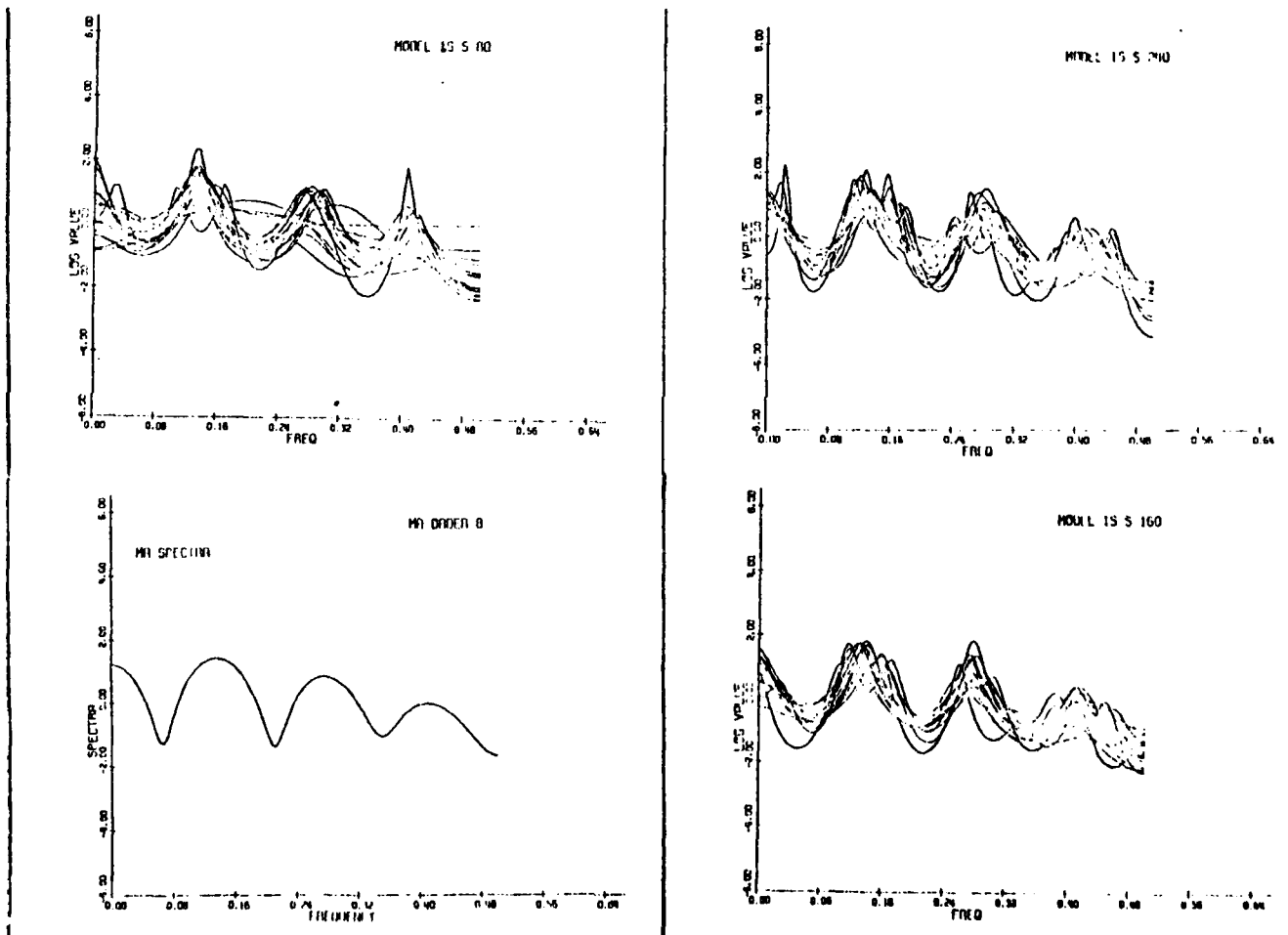   Sample Sizes

FIGURE M.   Levels of Luteinizing Hormone in a Cow Measured at Ten Minute
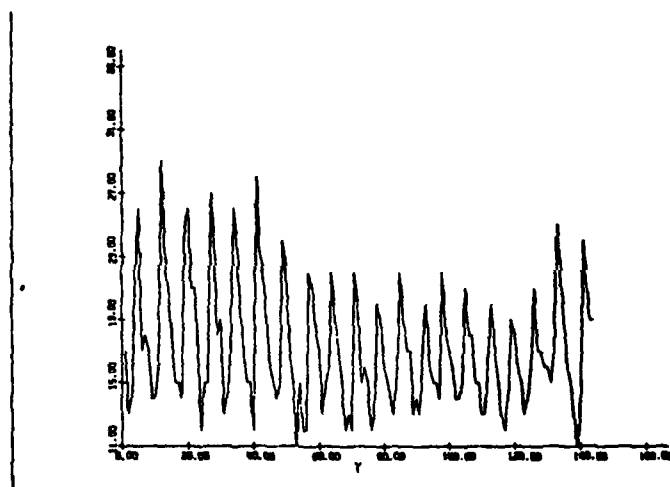   Intervals for 24 Hours

FIGURE N.   Log of $\hat{p}$, YW Estimated Spectra for Hormone Series