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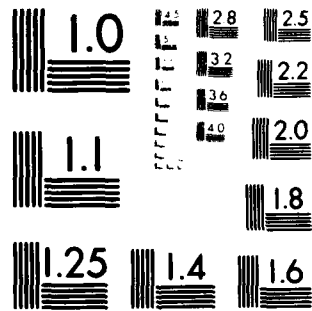
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AN ANALYTIC, ONE-PARAMETER FAMILY OF
SELF-ADJOINT BOUNDARY CONDITIONS FOR
SCHRÖDINGER OPERATORS ON AN INTERVAL

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AN ANALYTIC, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CONDITIONS
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ABSTRACT

A one-parameter family of real, homogeneous boundary conditions on the interval $[0,1]$, under which the operator $-\frac{d^2}{dx^2}$ is self-adjoint, is constructed. The relation between such boundary conditions and Lagrangian planes in \mathbb{R}^4 is used and the resulting circle of boundary conditions is seen to include Dirichlet, Neumann, periodic, antiperiodic, and several other well-known examples.

AMS(MOS) Subject Classifications: 34B10; 34B25

Key Words: Deformation of Boundary Conditions; Self-adjointness.

Work Unit #1 - Applied Analysis

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AN ANALYTIC, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CONDITIONS
FOR SCHRÖDINGER OPERATORS ON AN INTERVAL

Robert L. Sachs

Consider the operator $L \equiv -\frac{d^2}{dx^2}$ acting on reasonably nice functions which satisfy the pair of real linear homogenous boundary conditions

$$(1) \quad a_i y(0) + b_i y'(0) + c_i y(1) + d_i y'(1) = 0, \quad i = 1, 2.$$

By definition, this operator is self-adjoint if and only if the bilinear form:

$$(2) \quad B(y, z) \equiv y(0) z'(0) - y'(0) z(0) - y(1) z'(1) + y'(1) z(1)$$

vanishes identically for all u, v satisfying the boundary conditions (1). In

terms of column vectors $Y \equiv (y(0), y'(1), y'(0), y(1))^T$, $Z \equiv$

$(z(0), z'(1), z'(0), z(1))^T$, (2) is equivalent to

$$(3) \quad Y^T J Z = 0 \quad \text{where } J \text{ is the usual } 4 \times 4 \text{ symplectic matrix}$$

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{where } 0, I \text{ are } 2 \times 2 \text{ matrices. If } Y, Z \text{ are in the span of}$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix}, \quad \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{pmatrix} \quad \text{then (3) is equivalent to } (A^T B^T) J \begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad \text{where}$$

$$A \equiv \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}, \quad B \equiv \begin{pmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix} \quad \text{and this leads immediately to the condition}$$

$$(4) \quad A^T B - B^T A = 0$$

which is clearly no stronger than the requirement

$$(5) \quad A + iB \text{ is unitary.}$$

We shall construct a one-parameter family of unitary matrices $U(t)$

connecting the matrix representing periodic boundary conditions:

$$(6) \quad y(0) = y(1); \quad y'(0) = y'(1)$$

with the matrix representing Dirichlet boundary conditions:

$$(7) \quad y(0) = y(1) = 0.$$

(6) is equivalent to $y \in \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ which leads to the unitary matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ while Dirichlet boundary conditions (7) are

equivalent to the unitary matrix $\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$. Thus we seek a one-parameter family $U(t)$ of unitary matrices with

$$(8) \quad U(0) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}; \quad U(1) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$

Such a family is easily found in the form

$$(9) \quad U(t) = U(0)e^{iSt} \quad \text{where} \quad e^{iS} = U(0)^{-1}U(1) = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

i.e.

$$(10) \quad iS = \log \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \log(U(0)^{-1}U(1)). \\ = \log M$$

Now $M = U(0)^{-1}U(1)$ has eigenvalues $\lambda = e^{-\pi i/4}(e^{\pm\pi i/3}) = e^{\pi i/12}, e^{-7\pi i/12}$ and, diagonalizing M , we find

$$(11) \quad M = P \begin{pmatrix} e^{\pi i/12} & 0 \\ 0 & e^{-7\pi i/12} \end{pmatrix} P^{-1} \quad \text{where } P, P^{-1} \text{ are given by the } 2 \times 2 \text{ matrices}$$

$$(12) \quad P = \begin{pmatrix} \frac{\sqrt{3}-1}{\sqrt{2}} e^{\pi i/4} & -\frac{(\sqrt{3}+1)}{\sqrt{2}} e^{\pi i/4} \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{e^{-\pi i/4}}{\sqrt{6}} & \frac{\sqrt{3}+1}{2\sqrt{3}} \\ \frac{-e^{-\pi i/4}}{\sqrt{6}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \end{pmatrix}$$

$$\text{Thus } e^{iSt} = P \begin{pmatrix} e^{\pi i/12t} & 0 \\ 0 & e^{-7\pi i/12t} \end{pmatrix} P^{-1}$$

and the desired path is

$$(13) \quad U(t) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} e^{iSt}$$

which, by a tedious computation, is precisely the matrix

$$(14) \quad U(t) =$$

$$\begin{pmatrix} \frac{e^{-\pi i/4t}}{\sqrt{2}} \left(\cos(\pi/3t) - \frac{\sin \pi/3t}{\sqrt{3}} \right) & \frac{e^{-\pi i/4t}}{\sqrt{2}} \left(i \cos \frac{\pi}{3}t + \frac{(-2+i)}{\sqrt{3}} \sin \frac{\pi}{3} \right) \\ \frac{e^{-\pi i/4t}}{\sqrt{2}} \left(i \cos(\pi/3t) + \frac{(2+i)}{\sqrt{3}} \sin \left(\frac{\pi}{t} \right) \right) & \frac{e^{-\pi i/4t}}{\sqrt{2}} \left(\cos \pi/3t - \frac{1}{\sqrt{3}} \sin \pi/3t \right) \end{pmatrix}$$

We see easily that U has period 24 and that $U(t + 12) = -U(t)$, indeed $U(t+6) = iU(t)$ so that, in terms of the corresponding boundary conditions, we need only consider $U(t)$, $0 \leq t < 12$. Listing $U(j)$, $j = 0, \dots, 11$ and the corresponding boundary conditions, we have

$$(15) \left\{ \begin{array}{ll} U(0) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, & U(1) = \begin{pmatrix} 0 & +i \\ 1 & 0 \end{pmatrix} \\ \\ U(2) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, & U(3) = \begin{pmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} \end{pmatrix} \\ \\ U(4) = \begin{pmatrix} 0 & \frac{-1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}, & U(5) = \begin{pmatrix} \frac{-1+i}{2} & \frac{-1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{pmatrix} \\ \\ U(6) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, & U(7) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \\ U(8) = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, & U(9) = \begin{pmatrix} \frac{-1+i}{2} & \frac{-1-i}{2} \\ \frac{-1-i}{2} & \frac{-1+i}{2} \end{pmatrix} \\ \\ U(10) = \begin{pmatrix} 0 & \frac{-1-i}{\sqrt{2}} \\ \frac{-1+i}{\sqrt{2}} & 0 \end{pmatrix}, & U(11) = \begin{pmatrix} \frac{-1-i}{2} & \frac{-1-i}{2} \\ \frac{1+i}{2} & \frac{-1-i}{2} \end{pmatrix} \end{array} \right.$$

The corresponding boundary conditions, found by reversing the procedure above, are as follows:

t	Boundary conditions
0	$y(0) = y(1) , y'(0) = y'(1)$
1	$y(0) = 0 , y(1) = 0$
2	$y(0) = 0 , y'(1) = 0$
3	$y(0) = -y'(1), y'(0) = y(1)$
4	$y(0) = -y'(0), y'(1) = y(1)$
5	$y(0) = -y'(0), y'(1) = -y(1)$
6	$y(0) = -y(1) , y'(0) = -y'(1)$
7	$y'(0) = 0 , y'(1) = 0$
8	$y'(0) = 0 , y(1) = 0$
9	$y(0) = y'(1) , y'(0) = -y(1)$
10	$y(0) = y'(0) , y'(1) = -y(1)$
11	$y(0) = y'(0) , y'(1) = y(1)$

Thus our one parameter family in fact includes periodic, Dirichlet, antiperiodic, Neumann, and several other well-known boundary conditions.

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