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MATHEMATICAL AND NUMERICAL ANALYSIS OF A TURBULENT VISCOSITY MO--ETC(U)

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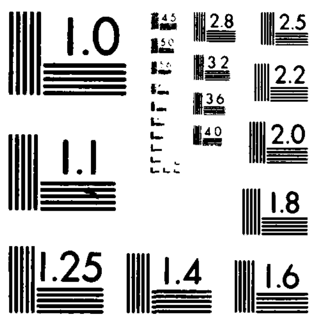


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MRC Technical Summary Report # 2366

# MATHEMATICAL AND NUMERICAL ANALYSIS OF A TURBULENT VISCOSITY MODEL

C. M. Brauner and C. Lainé

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April 1982

Received February 4, 1982

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# MATHEMATICAL AND NUMERICAL ANALYSIS OF A TURBULENT VISCOSITY MODEL

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## Abstract

The results of this paper concern the nonlinear parabolic P.D.E.

$$\begin{cases} \frac{\partial u}{\partial t} - \nu_o \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x} (\nu_T(u) \frac{\partial u}{\partial x}) = f \\ u(0, t) = u(1, t) = 0 \\ u(x, 0) = u_o(x) \end{cases}$$

where u represents the mean velocity field of a turbulent flow in a plane duct, and $\nu_T(u) = G(x) \sqrt{\left|\frac{\partial u}{\partial x}(0, t)\right|}$ is a model of turbulent viscosity based on the length scale of this flow.

First we prove the existence of a solution as well as some results on regularity. Then we demonstrate that u is positive inside the domain, hence the same result holds for $\partial u/\partial x(0, t)$. The latter result is crucial for the uniqueness proof which is developped in the last section.


AMS (MOS) Subject Classifications : 35K55, 76F05

Key Words : Turbulent viscosity model, nonlinear parabolic P.D.E., maximum principle.

Work Unit No. 1 - Applied Analysis

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

## SIGNIFICANCE AND EXPLANATION

The turbulent viscosity assumption is the simplest turbulent closure of the Navier-Stokes equations. Of the possible models, some are of physical interest. In the case of the plane duct, a model based on the length scale, $U_f$, of the flow seems appropriate. Moreover, the introduction of such a model in the Navier-Stokes equations leads to a novel nonlinear parabolic equation, whose mathematical study is the subject of this paper.

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MATHEMATICAL AND NUMERICAL ANALYSIS
OF A TURBULENT VISCOSITY MODEL

C. M. Brauner and C. Lainé

0 - INTRODUCTION

This paper is devoted to some mathematical and numerical results on a particular model of turbulence in a plane duct. In this physical case, the statistical average of the basic Navier-Stokes equations leads to a one dimensional mean velocity field equation. In order to close this equation we adopt a turbulent viscosity model based on the characteristic velocity scale $U_f = (\nu_o |\partial u/\partial x|_{wall})^{1/2}$ (see the Appendix A1). This efficient model induces a nonlinear parabolic problem

$$(0.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \nu_o \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial}{\partial x}\left(\nu_T(u)\dfrac{\partial u}{\partial x}\right) = f \\ \nu_T(u) = G(x)\ \left|\dfrac{\partial u}{\partial x}(0,t)\right|^{1/2} \\ u(0,t) = u(1,t) = 0,\ u(x,0) = u_o(x) \end{cases}$$

whose study is the purpose of this article. In (0.1) $\nu_o > 0$ is the kinematic viscosity of the fluid, while the positive function G adjusts the turbulence level (see details in the Appendix). Several assumptions on G will be made below. The function f represents the pressure gradient.

In section 1, we prove the existence of a solution via a Faedo-Galerkin method and we construct a sequence of approximate solutions $u_m$ in a finite dimensional space. To pass to the limit as $m \to +\infty$, we make use of a compactness argument. In section 2, we prove further results of regularity by the same methods, namely in the space $L^2(0, T ; H^3(\Omega)) \cap L^\infty(0, T ; H^2(\Omega))$. A direct consequence is that u is continuous, and so is $\frac{\partial u}{\partial x}(0, .)$.

The main point of the paper (Section 3) is to prove that u is positive in the case of positive data. This result will be the consequence of a Harnack inequality for parabolic equations due to J. Moser [6], [7]. Then we adapt a proof by Protter and Weinberger [8] to demonstrate that $\frac{\partial u}{\partial x}(0, t) > 0$. The latter result is crucial for the proof of the uniqueness, which is developped in the last section.

Finally, we briefly present some numerical results in Appendix A2.

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## 1 - EXISTENCE OF A SOLUTION

We are looking for a solution to the problem :

$$(1.1) \qquad \frac{\partial u}{\partial t} - \nu_o \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(\nu_T(u) \frac{\partial u}{\partial x}\right) = f$$

with the boundary and initial conditions :

$$(1.2) \qquad u(0, t) = u(1, t) = 0$$

$$(1.3) \qquad u(x, 0) = u_o(x)$$

In (1.1), $\nu_T(u)$ is the "turbulent viscosity" :

$$(1.4) \qquad \nu_T(u) = G(x) \sqrt{\left|\frac{\partial u}{\partial x}(0, t)\right|}$$

Let $\Omega = ]0, 1[$ and T a positive number, we denote by $Q = \Omega \times ]0, T[$, and we assume that :

$$(1.5) \qquad \begin{cases} f \in L^2(Q), \ u_o \in H_o^1(\Omega) \\ G \in C^1(\bar{\Omega}), \ G(x) \geq \nu_1 > 0 \end{cases} \qquad (^\dagger)$$

Let us introduce the following notation : ( , ) is the inner product in $L^2(\Omega)$ and $|| \, . \, ||$ the associated norm, $|| \, . \, ||_\infty$ is the norm in $L^\infty(\Omega)$. Any other norm will be indicated explicitly.

We introduce the eigenfunctions of the operator $-d^2/dx^2$, namely $w_j(x) = \sin j\pi x$, $j = 1, \ldots, m \ldots$. Then $-w_j'' = \lambda_j w_j$, where $\lambda_j = (j\pi)^2$.

We define $u_m(t) = \sum_{j=1}^{m} g_{j_m}(t) w_j$ as the solution, on some interval $[0, t_m]$, of the differential system :

$$(1.6) \qquad \begin{aligned} &(u_m'(t), w_j) - \nu_o \left(\frac{\partial^2 u_m}{\partial x^2}(t), w_j\right) - \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} \left(\frac{\partial}{\partial x}\left(G(x) \frac{\partial u_m}{\partial x}(t)\right), w_j\right) = \\ &= (f(t), w_j) \qquad\qquad 1 \leq j \leq m \end{aligned}$$

---

($^\dagger$) Practically $\nu_1 = G(0) << \nu_o$, i.e. near the wall only the kinematic viscosity occurs.

(1.7) $u_m(0) = u_{o_m}$ = projection of $u_o$ on $V_m$ (†)

### 1.1 - A priori estimates

*Lemma 1.1* : The sequence $u_m$ is bounded independently of m in the space $L^2(0, T ; H^1_o(\Omega)) \cap L^\infty(0, T ; L^2(\Omega))$. Moreover $|\frac{\partial u_m}{\partial x}(0, .)|^{1/4} \frac{\partial u_m}{\partial x}$ is bounded in the space $L^2(\Omega)$.

*Proof* : Multiply Eq. (1.6) by $g_{j_m}(t)$ and sum from j = 1 to m. Then :

$$\frac{1}{2}\frac{d}{dt}||u_m(t)||^2 + \nu_o ||\frac{\partial u_m}{\partial x}(t)||^2 + |\frac{\partial u_m}{\partial x}(0, t)|^{1/2} \int_\Omega G(\frac{\partial u_m}{\partial x}(t))^2 \, dx \leqslant$$

$$\leqslant \frac{c_1}{2} \|f(t)\|^2 + \frac{1}{2c_1} ||u_m(t)||^2_{H^1_o(\Omega)}$$

where $c_1$ is any positive constant. According to Poincaré's inequality, there exists $c_2 > 0$ such that $\nu_o ||\frac{\partial u_m}{\partial x}(t)||^2 \geqslant \frac{\nu_o}{c_2} ||u_m(t)||^2_{H^1_o(\Omega)}$

Then, choosing $c_1 = c_2/\nu_o$, and integrating from 0 to t :

$$||u_m(t)||^2 + \frac{\nu_o}{c_2}\int_o^t || u_m(\tau)||^2_{H^1_o(\Omega)} \, d\tau +$$

$$+ 2\int_o^t |\frac{\partial u_m}{\partial x}(0, \tau)|^{1/2} \int_\Omega G\,(\frac{\partial u_m}{\partial x}(\tau))^2 \, dx \, d\tau \leqslant ||u_{o_m}||^2 + \tag{1.9}$$

$$+ \frac{c_2}{\nu_o}\int_o^t ||f(\tau)||^2 \, d\tau$$

The R.H.S. of (1.9) is bounded by :

$$||u_o||^2 + \frac{c_2}{\nu_o}\int_o^T ||f(t)||^2 \, dt = c_3$$

As $||u_m(t)||^2 \leqslant c_3 \ \forall \ t \in [0, T]$, we infer that $t_m = T$ and we have the estimate in $L^\infty(0, T ; L^2(\Omega))$. Taking t = T in (1.9) gives the estimate in $L^2(0, T ; H^1_o(\Omega))$. Finally as $G \geqslant \nu_1 > 0$ :

---

(†) $V_m$ is the space generated by $w_1, \ldots, w_m$.

$$(1.10) \quad \int_Q \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} \left(\frac{\partial u_m}{\partial x}(t)\right)^2 dx\,dt \leq \frac{c_3}{2\nu_1}$$

and the proof of the lemma is complete.

■

Now we establish a sharper estimate :

*Lemma 1.2* : The sequence $u_m$ is bounded independently of m in the space $L^2(0, T ; H^2(\Omega)) \cap L^\infty(0, T ; H^1_0(\Omega))$. Moreover $\left|\frac{\partial u_m}{\partial x}(0, .)\right|^{1/4} \frac{\partial^2 u_m}{\partial x^2}$ is bounded in the space $L^2(Q)$.

*Proof* : In (1.6) we replace $w_j$ by $- 1/\lambda_j\, w_j''$ ; we multiply (1.6) by $g_{j_m}(t)$ and again sum from j = 1 to m. Clearly :

$$(1.11) \quad \begin{aligned} &\frac{d}{dt}\left\|\frac{\partial u_m}{\partial x}(t)\right\|^2 + 2\nu_o \left\|\frac{\partial^2 u_m}{\partial x^2}(t)\right\|^2 + 2\left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} \int_\Omega G\left(\frac{\partial^2 u_m}{\partial x^2}(t)\right)^2 dx + \\ &+ \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} \int_\Omega \frac{dG}{dx}\frac{\partial u_m}{\partial x}(t)\,\frac{\partial^2 u_m}{\partial x^2}(t)\,dx = 2\left(f(t), \frac{\partial^2 u_m}{\partial x^2}(t)\right)\end{aligned}$$

Integrating from 0 to T, it turns out that :

$$(1.12) \quad \begin{aligned} &\left\|\frac{\partial u_m}{\partial x}(T)\right\|^2 + 2\nu_o \left\|\frac{\partial^2 u_m}{\partial x^2}\right\|^2_{L^2(Q)} + 2\nu_1 \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^2 u_m}{\partial x^2}\right\|^2_{L^2(Q)} \leq \\ &\leq 2\,\|f\|_{L^2(Q)} \cdot \left\|\frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2(Q)} + \left\|\frac{du_{om}}{dx}\right\|^2 + \\ &+ 2\int_Q \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} \left|\frac{dG}{dx}\right| \left|\frac{\partial u_m}{\partial x}(t)\right| \left|\frac{\partial^2 u_m}{\partial x^2}(t)\right| dx\,dt\end{aligned}$$

We can find an upper bound for the R.H.S. in the following way :

$$(1.13) \quad \left\{\begin{aligned} X &= 2\int_Q \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} \left|\frac{dG}{dx}\right| \left|\frac{\partial u_m}{\partial x}(t)\right| \left|\frac{\partial^2 u_m}{\partial x^2}(t)\right| dx\,dt = \\ &= 2\int_Q \left|\frac{dG}{dx}\right| \left[\left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/4} \left|\frac{\partial u_m}{\partial x}(t)\right|\right] \left[\left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/4} \left|\frac{\partial^2 u_m}{\partial x^2}(t)\right|\right] dx\,dt \\ &\leq 2\left\|\frac{dG}{dx}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial u_m}{\partial x}\right\|_{L^2(Q)} \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2(Q)}\end{aligned}\right.$$

and by Young's inequality,

$$(1.14)\quad \begin{cases} X \leqslant \nu_1 \; \| \, |\frac{\partial u_m}{\partial x}(0,.)|^{1/4} \frac{\partial^2 u_m}{\partial x^2} \|^2_{L^2(Q)} + \\ + \frac{1}{\nu_1} \| \frac{dG}{dx} \|^2_\infty \; \| \, |\frac{\partial u_m}{\partial x}(0,.)|^{1/4} \frac{\partial u_m}{\partial x} \|^2_{L^2(Q)} \end{cases}$$

Now taking (1.14) into account and again applying Young's inequality to the quantity $\|f\|_{L^2(Q)} \cdot \|\frac{\partial^2 u_m}{\partial x^2}\|_{L^2(Q)}$, (1.12) becomes :

$$(1.15)\quad \begin{cases} \|\frac{\partial u_m}{\partial x}(T)\|^2 + \nu_o \|\frac{\partial^2 u_m}{\partial x^2}\|^2_{L^2(Q)} + \\ + \nu_1 \; \| \, |\frac{\partial u_m}{\partial x}(0,t)|^{1/4} (\frac{\partial^2 u_m}{\partial x^2}) \; \|^2_{L^2(Q)} \leqslant \frac{1}{\nu_o} \|f\|^2_{L^2(Q)} + \\ + \|\frac{du_{om}}{dx}\|^2 + \frac{1}{\nu_1} \|\frac{dG}{dx}\|^2_\infty \; \| \, |\frac{\partial u_m}{\partial x}(0,.)|^{1/4} \frac{\partial u_m}{\partial x} \|^2_{L^2(Q)} \end{cases}$$

As $u_{om} \to u_o$ in $H^1_o(\Omega)$, by (1.10),

(1.16) $\frac{\partial^2 u_m}{\partial x^2}$ is bounded in $L^2(Q)$.

Now the estimate in $L^\infty(0, T ; H^1_o(\Omega))$ is obtained by integrating (1.11) from 0 to t, $0 < t < T$. Finally, let us note that $\| \, |\frac{\partial u_m}{\partial x}(0,.)|^{1/4} \frac{\partial^2 u_m}{\partial x^2}\|^2_{L^2(Q)}$ is bounded by (1.15) ; so Lemma 1.2 is proved.

■

*Lemma 1.3* : The sequence $u_m'$ is bounded in $L^2(0, T ; H^{-1}(\Omega))$.

*Proof* : Introducing the projector $P_m$ from $L^2(\Omega)$ into $V_m$, we may write :

$$(1.17)\quad u_m' = \nu_o P_m(\frac{\partial^2 u_m}{\partial x^2}) + P_m \left[ |\frac{\partial u_m}{\partial x}(0, t)|^{1/2} \frac{\partial}{\partial x}(G \frac{\partial u_m}{\partial x}) \right] + P_m f$$

It is well known that $P_m$ enjoys the following properties : $\|P_m\|_{\mathcal{L}(H^1_o ; H^1_o)} \leqslant 1$ and, as $P_m$ is self-adjoint, $\|P_m\|_{\mathcal{L}(H^{-1} ; H^{-1})} \leqslant 1$.

As a result of Lemma 1.2, $u_m$ is bounded in $L^2(0, T ; H^2(\Omega))$ and by Sobolev embedding, in $L^2(0, T ; C^1(\bar{\Omega}))$. Therefore, $|\partial u_m/\partial x(0,.)|^{1/2}$ is bounded in $L^4(0,T)$ ; then, as $\frac{\partial}{\partial x}(G \, \partial u_m/\partial x)$ is bounded in $L^\infty(0, T ; H^{-1}(\Omega))$:

(1.18) $\quad |\frac{\partial u_m}{\partial x}(0,.)|^{1/2} \frac{\partial}{\partial x} (G \frac{\partial u_m}{\partial x})$ is bounded in $L^4(0,T ; H^{-1}(\Omega))$,

hence, Lemma 1.3 is proved.

■

1.2 - Passage to the limit

From the above lemmas, it follows that we may extract a subsequence $u_\mu$ such that

(1.19) $\quad u_\mu \to u$ in $L^2(0, T ; H^2(\Omega) \cap H^1_0(\Omega))$ weakly.

By a compactness theorem (see e.g. Lions [4], p. 57-58 with $B_0 = H^2(\Omega)$, $B_1 = H^{-1}(\Omega)$, $B = H^{2-\epsilon}(\Omega)$, $\epsilon > 0$),

we can extract another subsequence, also denoted by $u_\mu$, such that

(1.20) $\quad u_\mu \to u$ in $L^2(0, T ; H^{2-\epsilon}(\Omega))$ strongly.

But we may choose $\epsilon > 0$ small enough that the Sobolev embedding yields $H^{2-\epsilon}(\Omega) \subset C^1(\bar\Omega)$, (in fact $0 < \epsilon < \frac{1}{2}$). Then

(1.21) $\quad u_\mu \to u$ in $L^2(0, T ; C^1(\bar\Omega))$ strongly

and

(1.22) $\quad |\frac{\partial u_\mu}{\partial x}(0,.)|^{1/2} \to |\frac{\partial u}{\partial x}(0,.)|^{1/2}$ in $L^4(0,T)$ strongly.

Let j be fixed in (1.6) with $j < \mu$. As $\mu \to +\infty$, we get

$$(u'(t), w_j) - \nu_0 (\frac{\partial^2 u}{\partial x^2}(t), w_j) - |\frac{\partial u}{\partial x}(0, t)|^{1/2} (\frac{\partial}{\partial x} (G \frac{\partial u}{\partial x}(t)), w_j) = (f(t), w_j)$$

for $j = 1, 2 \ldots$, hence u satisfies the equation (1.1).

So, we have the following existence theorem :

*Theorem 1.1* : Under Assumptions (1.4) and (1.5), Problem (1.1) ... (1.3) possesses a solution in the space $L^2(0, T ; H^2(\Omega)) \cap L^\infty(0, T ; H^1_0(\Omega))$. Moreover $|\frac{\partial u}{\partial x}(0,.)|^{1/4} \frac{\partial u}{\partial x}$ and $|\frac{\partial u}{\partial x}(0,.)|^{1/4} \frac{\partial^2 u}{\partial x^2}$ belong to $L^2(Q)$.

In order to prove the uniqueness, further regularity results must be established.

## 2 - RESULTS ON REGULARITY

### 2.1 - Regularity in Sobolev spaces

*Theorem 2.1* : Let us assume that f, G, and $u_o$ are given functions with

$$f \in L^2(0, T ; H^1_o(\Omega)) \tag{2.1}$$

$$G \in C^2(\bar{\Omega}) \; ; \; G(x) \geq \nu_1 > 0, \; G'(0) = G'(1) = 0 \tag{2.2}$$

$$u_o \in H^2(\Omega) \cap H^1_o(\Omega) \tag{2.3}$$

Then any solution u to problem (1.1) ... (1.3) obtained *via* the preceding section satisfies

$$u \in L^2(0, T ; H^3(\Omega)) \cap L^\infty(0, T ; H^2(\Omega)) \tag{2.4}$$

*Proof* : From the definition of the eigenfunctions $w_j$, we may replace in (1.6) $w_j$ by $- 1/\lambda_j^{\,2} \, w_j^{(\mathrm{IV})}$. Multiply (1.6) by $g_{j_m}(t)$ and sum on j :

$$\begin{aligned} &(u_m'(t), \frac{\partial^4 u_m}{\partial x^4}(t)) - \nu_o (\frac{\partial^2 u_m}{\partial x^2}(t), \frac{\partial^4 u_m}{\partial x^4}(t)) - \\ &- |\frac{\partial u_m}{\partial x}(0, t)|^{1/2} (\frac{\partial}{\partial x} (G \frac{\partial u_m}{\partial x}(t)), \frac{\partial^4 u_m}{\partial x^4}(t)) = (f(t), \frac{\partial^4 u_m}{\partial x^4}(t)) \end{aligned} \tag{2.5}$$

which may be transformed into

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} ||\frac{\partial^2 u_m}{\partial x^2}(t)||^2 + \nu_o ||\frac{\partial^3 u_m}{\partial x^3}(t)||^2 - \\ &- |\frac{\partial u_m}{\partial x}(0, t)|^{1/2} (\frac{\partial}{\partial x} (G \frac{\partial u_m}{\partial x}(t)), \frac{\partial^4 u_m}{\partial x^4}(t)) = (f(t), \frac{\partial^4 u_m}{\partial x^4}(t)) \end{aligned} \tag{2.6}$$

The main term is

$$-\left(\frac{\partial}{\partial x}\left(G\,\frac{\partial u_m}{\partial x}(t)\right),\ \frac{\partial^4 u_m}{\partial x^4}(t)\right) = -\left(\frac{dG}{dx}\,\frac{\partial u_m}{\partial x}(t),\ \frac{\partial^4 u_m}{\partial x^4}(t)\right) -$$

$$-\left(G\,\frac{\partial^2 u_m}{\partial x^2}(t),\ \frac{\partial^4 u_m}{\partial x^4}(t)\right),$$

we compute

$$-\left(\frac{dG}{dx}\,\frac{\partial u_m}{\partial x}(t),\ \frac{\partial^4 u_m}{\partial x^4}(t)\right) = \left(\frac{\partial}{\partial x}\left\{\frac{dG}{dx}\,\frac{\partial u_m}{\partial x}(t)\right\},\ \frac{\partial^3 u_m}{\partial x^3}(t)\right)$$

as $G'(0) = G'(1) = 0$

$$= \left(\frac{d^2G}{dx^2}\,\frac{\partial u_m}{\partial x}(t),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right) + \left(\frac{dG}{dx}\,\frac{\partial^2 u_m}{\partial x^2}(t),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right),$$

and,

$$-\left(G\,\frac{\partial^2 u_m}{\partial x^2}(t),\ \frac{\partial^4 u_m}{\partial x^4}(t)\right) = \left(\frac{\partial}{\partial x}\left(G\,\frac{\partial^2 u_m}{\partial x^2}(t)\right),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right)$$

$$= \left(\frac{dG}{dx}\,\frac{\partial^2 u_m}{\partial x^2}(t),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right) + \left(G\,\frac{\partial^3 u_m}{\partial x^3}(t),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right)$$

Consequently (2.5) may be written :

$$(2.7)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}\left|\left|\frac{\partial^2 u_m}{\partial x^2}(t)\right|\right|^2 + \nu_o \left|\left|\frac{\partial^3 u_m}{\partial x^3}(t)\right|\right|^2 + \\ + \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2}\left(G\,\frac{\partial^3 u_m}{\partial x^3}(t),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right) = \\ = -2\left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2}\left(\frac{dG}{dx}\,\frac{\partial^2 u_m}{\partial x^2}(t),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right) - \\ - \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2}\left(\frac{d^2G}{dx^2}\frac{\partial u_m}{\partial x}(t),\ \frac{\partial^3 u_m}{\partial x^3}(t)\right) - \left(\frac{\partial f}{\partial x}(t), \frac{\partial^3 u_m}{\partial x^3}(t)\right) \end{cases}$$

By integration from 0 to T, it follows that

$$(2.8)\quad \begin{cases} \left|\left|\frac{\partial^2 u_m}{\partial x^2}(T)\right|\right|^2 + 2\,\nu_o \left|\left|\frac{\partial^3 u_m}{\partial x^3}\right|\right|^2_{L^2(Q)} + \\ + 2\int_Q \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} G\left(\frac{\partial^3 u_m}{\partial x^3}(t)\right)^2 dx\,dt = \\ = \left|\left|\frac{\partial^2 u_m}{\partial x^2}(0)\right|\right|^2 - 4\int_Q \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2}\frac{dG}{dx}\,\frac{\partial^2 u_m}{\partial x^2}(t)\,\frac{\partial^3 u_m}{\partial x^3}(t)\,dx\,dt \\ - 2\int_Q \frac{\partial f}{\partial x}(t)\frac{\partial^3 u_m}{\partial x^3}(t)\,dx\,dt - 2\int_Q \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2}\frac{d^2G}{dx^2}\,\frac{\partial u_m}{\partial x}(t)\,\frac{\partial^3 u_m}{\partial x^3}(t)\,dx\,dt \end{cases}$$

Assumption (2.2) about G then enables us to get :

$$
(2.9)\quad \left\{
\begin{aligned}
& \left\|\frac{\partial^2 u_m}{\partial x^2}(T)\right\|^2 + 2\nu_o \left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2(Q)} + 2\nu_1 \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right\|^2_{L^2(Q)} \leqslant \\
& \leqslant \left\|\frac{\partial^2 u_m}{\partial x^2}(0)\right\|^2 + 4\left\|\frac{dG}{dx}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2(Q)} \cdot \\
& \cdot \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2(Q)} + 2\left\|\frac{\partial f}{\partial x}\right\|_{L^2(Q)} \cdot \left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2(Q)} + \\
& + 2\left\|\frac{d^2G}{dx^2}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial u_m}{\partial x}\right\|_{L^2(Q)} \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2(Q)}
\end{aligned}
\right.
$$

By applying Young's inequality to the R.H.S., we have, with $c_1$ being any positive constant,

$$
\begin{aligned}
& 4\left\|\frac{dG}{dx}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^2 u_m}{\partial x^2}\right\|_{L^2(Q)} \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2(Q)} \leqslant \\
& \leqslant \frac{2}{c_1}\left\|\frac{dG}{dx}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^2 u_m}{\partial x^2}\right\|^2_{L^2(Q)} + \\
& + 2c_1\left\|\frac{dG}{dx}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right\|^2_{L^2(Q)}
\end{aligned}
$$

On the same way, for any $c_2 > 0$

$$
\begin{aligned}
& 2\left\|\frac{d^2G}{dx^2}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial u_m}{\partial x}\right\|_{L^2(Q)} \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2(Q)} \leqslant \\
& \leqslant \frac{1}{c_2}\left\|\frac{d^2G}{dx^2}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial u_m}{\partial x}\right\|^2_{L^2(Q)} + \\
& + c_2\left\|\frac{d^2G}{dx^2}\right\|_\infty \left\| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right\|^2_{L^2(Q)}
\end{aligned}
$$

and for any $c_3 > 0$

$$
2\left\|\frac{\partial f}{\partial x}\right\|_{L^2(Q)} \left\|\frac{\partial^3 u_m}{\partial x^3}\right\|_{L^2(Q)} \leqslant \frac{1}{c_3}\left\|\frac{\partial f}{\partial x}\right\|^2_{L^2(Q)} + c_3\left\|\frac{\partial^3 u_m}{\partial x^3}\right\|^2_{L^2(Q)}
$$

Then it is convenient to choose $c_1$, $c_2$ and $c_3$ such that

$$
c_3 = \nu_o\ ,\quad 2c_1\left\|\frac{dG}{dx}\right\|_\infty + c_2\left\|\frac{d^2G}{dx^2}\right\|_\infty = \nu_1
$$

to obtain the following relation :

$$\left|\left|\frac{\partial^2 u_m}{\partial x^2}(T)\right|\right|^2 + \nu_o \left|\left|\frac{\partial^3 u_m}{\partial x^3}\right|\right|^2_{L^2(Q)} + \nu_1 \left|\left| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u_m}{\partial x^3}\right|\right|^2_{L^2(Q)}$$

$$(2.10) \qquad \leqslant \left|\left|\frac{\partial^2 u_m}{\partial x^2}(0)\right|\right|^2 + \frac{2}{c_1}\left|\left|\frac{dG}{dx}\right|\right|_\infty \left|\left| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial^2 u_m}{\partial x^2}\right|\right|^2_{L^2(Q)}$$

$$+ \frac{1}{c_2}\left|\left|\frac{d^2 G}{dx^2}\right|\right|_\infty \left|\left| \left|\frac{\partial u_m}{\partial x}(0,.)\right|^{1/4} \frac{\partial u_m}{\partial x}\right|\right|^2_{L^2(Q)} + \frac{1}{\nu_o}\left|\left|\frac{\partial f}{\partial x}\right|\right|^2_{L^2(Q)}$$

By Lemmas 1.1 and 1.2, and since $u_{o_m} \to u_o$ in $H^2(\Omega)$, the R.H.S. of (2.10) is bounded. Then, the sequence $u_m$ is bounded in $L^2(0, T ; H^3(\Omega))$. The estimate in $L^\infty(0, T ; H^2(\Omega))$ is obtained by integrating (2.7) from 0 to t, $0 < t < T$. Therefore Theorem 2.1 is proved. Note that we have in fact proved the sharper property :

$$\left|\frac{\partial u}{\partial x}(0,.)\right|^{1/4} \frac{\partial^3 u}{\partial x^3} \in L^2(Q)$$

■

2.2 - <u>Continuity results</u>

The application of the "theorem of intermediate derivatives" of LIONS and MAGENES [5] will enable us to establish the continuity of u and $(\partial u/\partial x)(0,.)$ from Theorem 2.1.

<u>*Theorem 2.2*</u> : Under the Assumptions (2.1), (2.2), (2.3), u is continuous in $\bar{Q}$, and $\frac{\partial u}{\partial x}(0,.)$ is continuous in $[0,T]$.

<u>*Proof*</u> : As in (1.17), we write

$$u_m' = \nu_o P_m \left(\frac{\partial^2 u_m}{\partial x^2}\right) + P_m \left[ \left|\frac{\partial u_m}{\partial x}(0,t)\right|^{1/2} \frac{\partial}{\partial x}\left(G \frac{\partial u_m}{\partial x}\right)\right] + P_m f$$

According to the properties of the projector $P_m$ we see that the sequence $u_m'$ is bounded in the space $L^2(0,T ; H^1(\Omega))$, therefore $\partial u/\partial t$ belongs to that space.

Summarizing, we have

$$(2.11) \qquad u \in L^2(0, T ; H^3(\Omega)), \ \frac{\partial u}{\partial t} \in L^2(0, T ; H^1(\Omega))$$

From the theorem of intermediate derivatives (LIONS-MAGENES [5]) we infer that u belongs to $C^0([0, T] ; H^2(\Omega))$ and by Sobolev embedding

(2.12) $$u \in C^0([0, T] ; C^1(\bar{\Omega}))$$

hence the theorem.

■

## 3 - MAXIMUM PRINCIPLE

In order to prove uniqueness, it is crucial to establish the positivity of $\partial u/\partial x(0,.)$. This result will be the consequence of the positivity of u in Q. This latter result comes from a Harnack inequality for parabolic equation due to J. MOSER.

### 3.1 - A Harnack inequality

Let us recall the following result due to J. MOSER [6],[7].

Let $\Omega$ denote an open bounded set in $\mathbb{R}^n$ and $Q = \Omega \times ]0, T[$. We consider the equation in Q

(3.1) $$\frac{\partial u}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_j} (a_{ij} (x, t) \frac{\partial u}{\partial x_i}) = 0$$

where $a_{ij}$ are bounded functions such that :

(3.2) $$\begin{cases} 0 < \lambda \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq \Lambda < +\infty, \forall \xi = (\xi_1, \dots \xi_n) \in \mathbb{R}^n, |\xi|^2 = 1 \\ a_{ij} = a_{ji} \end{cases}$$

Let u be a weak solution of (3.1) such that $\partial u/\partial t$, $\partial u/\partial x_i$, $i=1,\dots,n$, belong to $L^2(Q)$. We suppose that u is non-negative in Q. We consider a compact and connected subdomain of $\Omega$, $A$, and two subintervals

$I^- = \{t, t_1 < t < t_2\}$, $I^+ = \{t, t_3 < t < t_4\}$ where we assume that $0 < t_1 < t_2 < t_3 < t_4 < T$. Define

$$Q^- = I^- \times A , \quad Q^+ = I^+ \times A$$

*Lemma 3.1* : Any non-negative solution u of (3.1) satisfies

$$\operatorname*{ess\,sup}_{Q^-} u \leq c^{(\Lambda + \lambda^{-1})} \operatorname*{ess\,inf}_{Q^+} u \tag{3.3}$$

where $c > 1$ is a constant which depends only on $Q$, $Q^+$, $Q^-$.

3.2 - Positivity of u

Here $\Omega$ is again $]0,1[$. We shall apply the above result of J. MOSER to an auxiliary problem. First we prove that u is non-negative *via* the weak maximum principle.

*Lemma 3.2* : The hypotheses are (1.4), (1.5). Besides, suppose $u_o$ non-negative in $\Omega$ and f non-negative a.e. in Q, then any solution u of Problem (1.1) ... (1.3) is non-negative a.e. in Q.

*Proof* : As in LIONS [4] p. 290, we multiply (1.1) by $u^-(t)$. Then after integration over $\Omega$, it follows that :

$$-\frac{d}{dt}\,||u^-(t)||^2 = 2\,\nu_o\,||\frac{\partial u^-}{\partial x}(t)||^2 + 2\int_\Omega |\frac{\partial u}{\partial x}(0,t)|^{1/2}\,G(\frac{\partial u^-}{\partial x}(t))^2\,dx$$

$$+ 2\int_\Omega f(t)\,u^-(t)\,dx$$

which yields $\frac{d}{dt}\,||u^-(t)||^2 \leq 0$. As $u_o^- = 0$, $||u^-(t)|| = 0$, $\forall\, t$. ■

For u any solution of Problem (1.1) ... (1.3), we introduce now the auxiliary linear problem

$$\left\{\begin{array}{l} \frac{\partial \tilde{u}}{\partial t} - \nu_o\,\frac{\partial^2 \tilde{u}}{\partial x^2} - |\frac{\partial u}{\partial x}(0,t)|^{1/2}\frac{\partial}{\partial x}\,(G(x)\,\frac{\partial \tilde{u}}{\partial x}) = 0 \\ \tilde{u}(0,\,t) = \tilde{u}(1,\,t) = 0 \\ \tilde{u}(x,\,0) = u_o(x) \end{array}\right. \tag{3.4}$$

*Lemma 3.3* : The hypotheses are (2.1), (2.2), (2.3), u given by Theorem 2.1. Also, suppose f non-negative a.e. in Q. Then $\tilde{u} \in C^o(\bar{Q})$ and $u(x,\,t) \geq \tilde{u}(x,\,t)$ $\forall\,(x,\,t) \in Q$.

*Proof* : Problem (3.4) admits a unique solution $\tilde{u} \in L^2(0,\,T\,;\,H^2(\Omega) \cap H_o^1(\Omega))$ $\frac{\partial \tilde{u}}{\partial t} \in L^2(Q)$, then $\tilde{u} \in C^o(\bar{Q})$.

Set $w = u - \tilde{u}$ and w satisfies :

$$(3.5)\quad \begin{cases} \frac{\partial w}{\partial t} - \nu_o \frac{\partial^2 w}{\partial x^2} - (\frac{\partial u}{\partial x}(0,\ t))^{1/2} \frac{\partial}{\partial x} (G(x) \frac{\partial w}{\partial x}) = f \geqslant 0 \\ w(0,\ t) = w(1,\ t) = 0 \\ w(x,\ 0) = 0 \end{cases}$$

Clearly $w \geqslant 0$ by the weak maximum principle.

*Theorem 3.1* : The hypotheses are (2.1), (2.2), (2.3), u given by Theorem 2.1. Besides, suppose f non negative a.e. in Q, and $u_o$ positive in $\Omega$. Then u positive in Q.

*Proof* : By Lemma 3.3 it is sufficient to prove that $\tilde{u} > 0$ in Q. We apply Lemma 3.1 to $\tilde{u}$ with $\Omega = ]0,\ 1[$, $a_{ij}(x,\ t) = a(x,\ t) = G(x)\ (\frac{\partial u}{\partial x}(0,\ t))^{1/2} + \nu_o$, $\lambda = \nu_o$ and $\Lambda = \nu_o + \{ \sup_{x \in \Omega} G(x)\ .\ \sup_{t \in [0,\ T]} (\frac{\partial u}{\partial x}(0,\ t))^{1/2} \}$.

We prove $\tilde{u} \geqslant 0$ in Q, exactly as we did in Lemma 3.2.

Now let $x_o \in \Omega$ be fixed. Since $\tilde{u} \in C^o(\bar{Q})$ and $u_o > 0$ in $\Omega$, there exists a closed interval $A \subset \Omega$ containing $x_o$, and a time $\tau$, such that

(3.6) $\tilde{u}(x,\ t) > 0 \quad \forall\, x \in A, \quad 0 \leqslant t \leqslant \tau$

Next, let $t_* > \tau$ be such that $\tilde{u}(x_o,\ t_*) = 0$. With the notations of Lemma 3.1, we choose $t_1$ and $t_2$ such that $0 < t_1 < t_2 < \tau$. Then we consider $t_3 = t_* - \varepsilon$, $t_4 = t_* + \varepsilon$ ($\varepsilon > 0$ small enough).

So, $\inf_{Q^+} \tilde{u} = 0$. Therefore as a result of (3.3) $\sup_{Q^-} \tilde{u} = 0$ which is indeed inconsistent with (3.6), and there does not exist such a time $t_*$.

■

*Remark 3.1* : The above demonstration may be extended to the case $u_o > 0$ only on a subdomain of $\Omega$.

### 3.3 - Positivity of $\partial u/\partial x(0,.)$

From the positivity of u, we will deduce the positivity of $\partial u/\partial x(0,.)$.

*Theorem 3.2* : The hypotheses are those of Theorem 3.1, then $\frac{\partial u}{\partial x}(0, t)$ is positive for any time $t > 0$.

*Proof* : Here we adapt a proof due to PROTTER and WEINBERGER [8] in the case of strong solutions.

We shall denote by L the operator with bounded coefficients defined by :

$$\begin{cases} L\,v = \frac{\partial v}{\partial t} - a\,\frac{\partial^2 v}{\partial x^2} - b\,\frac{\partial v}{\partial x} \\ \text{with } a(x, t) = \nu_o + (\frac{\partial u}{\partial x}(0, t))^{1/2}\,G(x), \\ \qquad b(x, t) = (\frac{\partial u}{\partial x}(0, t))^{1/2}\,\frac{dG}{dx}(x) \end{cases}$$

Set $P = (0\,;\,t_o)$, $t_o > 0$. We construct a disk K with center at $(x_1, t_1)$ contained in Q, and tangent to $\partial Q$ at P.

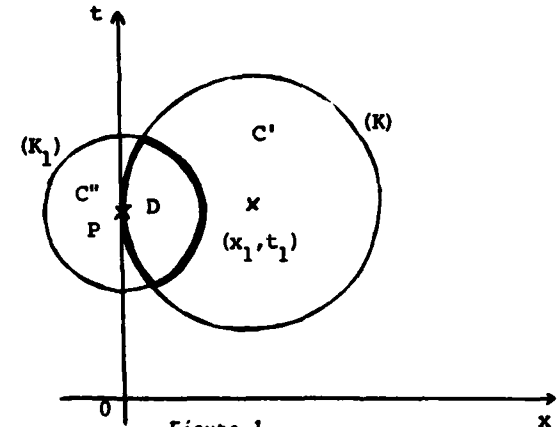


*Figure 1*

We also construct a disk $K_1$ with center at P and radius less than $x_1$. We denote by C' the closed arc portion of $\partial K_1$ contained in K, and by C" the open arc portion $\partial K$ contained in $K_1$. The arcs $C' \cup C"$ form the boundary of a lens-shaped region D.

Since $u \in C^o(\bar{Q})$ and $u > 0$ in Q we have

(i) $u > 0$ on C" (except at P) ; (ii) $u = 0$ at P ; (iii) there exists a sufficiently small $\eta > 0$ such that $u \geqslant \eta$ on C'.

We now introduce the auxiliary function $v(x, t)$:

$v(x, t) = - \exp(-\beta [(x - x_1)^2 + (t - t_1)^2]) + \exp(-\beta R^2)$ and note that

$L v = 2\beta \exp(-\beta[(x - x_1)^2 + (t - t_1)^2]) [2\beta a (x - x_1)^2 - a - b(x - x_1) + t - t_1]$

Thus, for sufficiently large $\beta$ we have $L v > 0$ in D. Set $w = u + \varepsilon v$ for $\varepsilon > 0$. Thus $L w = f + \varepsilon L v > 0$ a.e. in D. Because of fact (iii), we can choose $\varepsilon$ so small that $w > 0$ on C'. Since $v = 0$ on $\partial K$, we have, because of (i) $w > 0$ on C", except at P and $w(P) = 0$.

It is easy to show by the weak maximum principle that $w \geqslant 0$ in D. We deduce that $\frac{\partial w}{\partial x}(P) \geqslant 0$.

On the other hand, we see that $(\partial v/\partial x)(P) < 0$ since :

$\frac{\partial v}{\partial x}(x, t) = 2\beta(x - x_1) \exp(-\beta\left[(x - x_1)^2 + (t - t_1)^2\right])$,

$\frac{\partial v}{\partial x}(P) = - 2\beta x_1 \exp(-\beta R^2) < 0$, and finally, $\frac{\partial u}{\partial x}(P) = \frac{\partial w}{\partial x}(P) - \varepsilon \frac{\partial v}{\partial x}(P) > 0$

*Corollary 3.1* : the hypotheses are those of Theorem 3.2. Moreover, suppose $\frac{du_o}{dx}(0)$ is positive. Then there exists a constant $\alpha > 0$ such that

(3.7) $$\frac{\partial u}{\partial x}(0, t) \geqslant \alpha^2 > 0 \quad \forall t \in [0, T]$$

*Proof* : The corollary is an easy consequence of Theorem 3.2 and of the continuity of $\frac{\partial u}{\partial x}(0, .)$.

■

## 4 - UNIQUENESS

The above results will enable us to prove the uniqueness of a solution to Problem (1.1) ... (1.3), namely :

*Theorem 4.1* : The hypotheses are (2.1), (2.2), (2.3). Besides we assume that f is non-negative a.e. in Q, $u_o$ is positive in $\Omega$ and $\frac{du_o}{dx}(0)$ is positive. Then the solution of (1.1) ... (1.3) is unique.

*Proof* : The proof of the uniqueness is a non standard application of Gronwall's lemma.

Suppose there exist two solutions $u_1$ and $u_2$. Summarizing the results of section 3, there exist 2 positive constants $\alpha$ and K such that :

$$(4.1)\qquad 0 < \alpha \leqslant \left(\frac{\partial u_i}{\partial x}(0,t)\right)^{1/2} \leqslant K, \quad i=1,2 \quad \forall\, t \in [0,T]$$

Now set $u = u_1 - u_2$, and u satisfies

$$(4.2)\qquad \frac{\partial u}{\partial t} - \nu_o \frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial u_1}{\partial x}(0, t)\right)^{1/2} \frac{\partial}{\partial x}\left(G(x)\frac{\partial u}{\partial x}\right) + $$
$$+\left\{ \left(\frac{\partial u_1}{\partial x}(0, t)\right)^{1/2} - \left(\frac{\partial u_2}{\partial x}(0, t)\right)^{1/2}\right\} \frac{\partial}{\partial x}\left(G \frac{\partial u_2}{\partial x}\right)$$

$$(4.3)\qquad u(0, t) = u(1, t) = 0$$

$$(4.4)\qquad u(x, 0) = 0$$

Multiplying (4.2) by $-\frac{\partial^2 u}{\partial x^2}$ yields :

$$(4.5)\qquad \left\{\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|\left|\frac{\partial u}{\partial x}(t)\right|\right|^2 + \nu_o \left|\left|\frac{\partial^2 u}{\partial x^2}(t)\right|\right|^2 + \left(\frac{\partial u_1}{\partial x}(0,t)\right)^{1/2}\left(G\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x}\right) = \\ &-\left(\frac{\partial u_1}{\partial x}(0, t)\right)^{1/2}\left(\frac{dG}{dx}\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) - \\ &-\left(\left(\frac{\partial u_1}{\partial x}(0, t)\right)^{1/2} - \left(\frac{\partial u_2}{\partial x}(0, t)\right)^{1/2}\right)\left(\frac{dG}{dx}\frac{\partial u_2}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) \\ &-\left(\left(\frac{\partial u_1}{\partial x}(0, t)\right)^{1/2} - \left(\frac{\partial u_2}{\partial x}(0, t)\right)^{1/2}\right)\left(G\frac{\partial^2 u_2}{\partial x^2}, \frac{\partial^2 u}{\partial x^2}\right)\end{aligned}\right.$$

Note that :

$$(4.6)\qquad \left\{\begin{aligned} &\left|\left(\frac{\partial u_1}{\partial x}(0, t)\right)^{1/2} - \left(\frac{\partial u_2}{\partial x}(0, t)\right)^{1/2}\right| = \frac{\left|\frac{\partial u_1}{\partial x}(0, t) - \frac{\partial u_2}{\partial x}(0, t)\right|}{\left(\frac{\partial u_1}{\partial x}(0, t)\right)^{1/2} + \left(\frac{\partial u_2}{\partial x}(0, t)\right)^{1/2}} \\ &\leqslant \frac{\left|\frac{\partial u_1}{\partial x}(0, t) - \frac{\partial u_2}{\partial x}(0, t)\right|}{\alpha} \leqslant \frac{\left|\frac{\partial u}{\partial x}(0, t)\right|}{\alpha}\end{aligned}\right.$$

Since

$$(4.7)\qquad \left|\frac{\partial u}{\partial x}(0, t)\right| \leqslant \sqrt{2}\,\left|\left|\frac{\partial u}{\partial x}(t)\right|\right|^{1/2} \left|\left|\frac{\partial^2 u}{\partial x^2}(t)\right|\right|^{1/2},$$

it follows that

$$\left|\left(\frac{\partial u_1}{\partial x}(0,t)\right)^{1/2} - \left(\frac{\partial u_2}{\partial x}(0,t)\right)^{1/2}\right| \leqslant \frac{\sqrt{2}}{\alpha} \left\|\frac{\partial u}{\partial x}(t)\right\|^{1/2} \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^{1/2}$$

and by insertion into (4.5) :

$$(4.8)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x}(t)\right\|^2 + \nu_0 \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2 + \alpha\,\nu_1 \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2 \leqslant \\ \leqslant K \left\|\frac{dG}{dx}\right\|_\infty \left\|\frac{\partial u}{\partial x}(t)\right\| \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\| + \\ + \frac{\sqrt{2}}{\alpha} \left\|\frac{dG}{dx}\right\|_\infty \left\|\frac{\partial u_2}{\partial x}(t)\right\| \left\|\frac{\partial u}{\partial x}(t)\right\|^{1/2} \left\|\frac{\partial^2 u}{\partial x^2}\right\|^{3/2} \\ + \frac{\sqrt{2}}{\alpha} \|G\|_\infty \left\|\frac{\partial^2 u_2}{\partial x^2}(t)\right\| \left\|\frac{\partial u}{\partial x}(t)\right\|^{1/2} \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^{3/2} \end{cases}$$

To estimate the R.H.S. we apply the Young inequality three times : $C_1$, $C_2$, $C_3$ being as usual any positive constants,

$$K \left\|\frac{dG}{dx}\right\|_\infty \left\|\frac{\partial u}{\partial x}(t)\right\| \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\| \leqslant \frac{K}{2C_1} \left\|\frac{dG}{dx}\right\|_\infty \left\|\frac{\partial u}{\partial x}(t)\right\|^2 + \frac{C_1 K}{2} \left\|\frac{dG}{dx}\right\|_\infty \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2$$

$$\left\|\frac{\partial u}{\partial x}(t)\right\|^{1/2} \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^{1/2} \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\| \leqslant \frac{1}{2C_2} \left\|\frac{\partial u}{\partial x}(t)\right\| \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\| + \frac{C_2}{2} \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2 \leqslant \frac{1}{4\,C_2 C_3} \left\|\frac{\partial u}{\partial x}(t)\right\|^2 + \frac{C_3}{2C_2} \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2 + \frac{C_2}{2} \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2$$

Therefore (4.8) becomes, since $\frac{\partial^2 u_2}{\partial x^2} \in L^\infty(0, T\,;\, L^2(\Omega))$ from Theorem 2.1,

$$(4.9)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x}(t)\right\|^2 + \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2 \Big[\nu_0 + \alpha\,\nu_1 - \frac{C_1}{2} K \left\|\frac{dG}{dx}\right\|_\infty - \\ - \frac{\sqrt{2}}{\alpha}\left\{\|G\|_\infty \left\|\frac{\partial^2 u_2}{\partial x^2}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\frac{dG}{dx}\right\|_\infty \left\|\frac{\partial u_2}{\partial x}\right\|_{L^\infty(0,T;L^2)}\right\}\left(\frac{C_3}{2C_2} + C_2\right)\Big] \\ \leqslant \left\|\frac{\partial u}{\partial x}(t)\right\|^2 \Big[\frac{K}{2C_1}\left\|\frac{dG}{dx}\right\|_\infty + \frac{\sqrt{2}}{4\alpha C_3 C_2}\left\{\|G\|_\infty \left\|\frac{\partial^2 u_2}{\partial x^2}\right\|_{L^\infty(0,T;L^2(\Omega))} + \right. \\ \left. + \left\|\frac{dG}{dx}\right\|_\infty \left\|\frac{\partial u_2}{\partial x}\right\|_{L^\infty(0,T;L^2(\Omega))}\right\}\Big] \end{cases}$$

Choosing $C_1$, $C_2$, $C_3$ in order to eliminate the coefficient of $\left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2$ in the L.H.S., it follows

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial x}(t)\right\|^2 \leqslant \text{Cst} \left\|\frac{\partial u}{\partial x}(t)\right\|^2$$

From Gronwall's lemma, $||\frac{\partial u}{\partial x}(t)|| = 0$ , $\forall$ t. Hence $\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x}$ in Q, and the uniqueness is proved.

## APPENDIX

### A1 - THE PHYSICAL PROBLEM

The flow under consideration is incompressible ; the equations describing the turbulent field follow from the statistical average of the Navier-Stokes equations, using the classical decomposition of the velocity field into mean and fluctuating components $U_i = \bar{U}_i$ ... We have then :

$$\text{(A.1)} \qquad \frac{\partial \bar{U}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{U}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \nu_o \frac{\partial^2 \bar{U}_i}{\partial x_i^2} - \frac{\partial}{\partial x_j}(\overline{u_i u_j})$$

$$\text{(A.2)} \qquad \frac{\partial \bar{U}_i}{\partial x_i} = 0$$

To close Equations (A.1) and (A.2), we use the turbulent viscosity assumptions given by :

$$\text{(A.3)} \qquad -\overline{u_i u_j} = \nu_T \left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right) + \frac{1}{3}\bar{q}^2 \delta_{ij}$$

where $\overline{u_i u_j}$ is the Reynolds stresses tensor. In (A.1) the kinetic energy $\bar{q}^2 = \overline{u_i u_i}$ will be absorbed into $\bar{P}$ and so will not need to be calculated explicitly.

In particular, we consider the turbulent fluid in motion between two parallel planes at a distance 2 D. Let $U(x_2, t)$ be the mean velocity of the fluid at a point $x_2 \in [0,2D]$, at a time t. The physical simplifications of the general equations lead to :

$$\text{(A.4)} \qquad \frac{\partial U}{\partial t} - \nu_o \frac{\partial^2 U}{\partial x_2^2} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_1} - \frac{\partial}{\partial x_2}(\overline{u_1 u_2})$$

In this case, the turbulent viscosity $\nu_T$ is chosen as a function of the characteristic scale $u_f$, and so is defined by,

$$\text{(A.5)} \qquad \nu_T = G(x)\sqrt{\left|\frac{\partial U}{\partial x_2}\right|_{wall}} ,$$

where G(x) is a polynomial function adjusting the turbulence level. Consequently, with $-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_1} = f$ being given, and using the turbulent viscosity model defined in (A.5), we have

$$\text{(A.6)}\qquad \frac{\partial U}{\partial t} - \nu_o \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(G(x)\sqrt{\left|\frac{\partial U}{\partial x}(0,\ t)\right|}\ \frac{\partial U}{\partial x}\right) + f$$

where x henceforth only denotes the spatial variable $x_2$, which is here a dimensionless variable.

## A2 - SOME NUMERICAL RESULTS

We are interested in computing the established solution of Problem (1.1)... (1.3) :

$$\text{(A.1)}\qquad \begin{cases} -\nu_o\ \bar{U}_{yy} = (G(y)\ |\bar{U}_y(0)|^{1/2}\ \bar{U}_y)_y + f & \text{on } \Omega = ]0,\ 2D[ \\ \bar{U}(0) = \bar{U}(2D) = 0 \end{cases}$$

First we reduce (A.1) to dimensionless equations by introducing the characteristic length and velocity of the flow, defined by :

$$\text{(A.2)}\qquad L_o = 2D \quad , \quad u_f = \sqrt{-\frac{1}{\rho}\frac{d\bar{p}}{dx}D}$$

Using a new set of dimensionless variables,

$$\text{(A.3)}\qquad u = \bar{U} / u_o \quad , \quad x = y/_{2D} \quad , \ R_f = (u_f . 2D)/\nu_o ,$$

the mean velocity field becomes,

$$\text{(A.4)}\qquad \begin{cases} -u_{xx} = R_f^{1/2}\ (G(x)\ |u_x(0)|^{1/2}\ u_x)_x + 2\ R_f & \text{on } ]0,\ 1[ \\ u(0) = u(1) = 0 \end{cases}$$

in which we choose G(x) according to a modified Van Driest model ([9] p. 194)

The computation has been realized with a classical fixed point method ; also, we have tested an optimal control method, to point out the capability of this algorithm to solve nonlinear equations with a very large nonlinear/linear ratio (in the experimental cases $R_f$ is larger than 5000 !).

We have used a non regular subdivision of ]0,1[ in order to have a better approximation of the variables in the wall regions, where the gradients are large. Namely we take a $x^2$ - subdivision. The internal approximation is realized with the Lagrange finite element P1. The algorithm efficiency depends on the physical meaning of initialization. The "parabolic" profile corresponding to the laminar solution leads to good convergence of the method (note that the sequence is alternating).

Physically, this model has a good efficiency since it does not demand any parameter adjustment. Moreover, it is able to predict the mean velocity field into the whole duct. Other tested models only work in a subregion of the duct, whose border belongs to the so-called "logarithmic zone". In this case, inhomogenous boundary conditions should be prescribed. Figure 2 shows a comparison between the experimental results of COMTE-BELLOT [1] and our numerical results for a Reynolds number $R_f = 5160$.

AKNOWLEGMENTS - This paper was completed when the first author was visiting the Mathematical Research Center of the University of Wisconsin-Madison. We are indebted to Professor M.G. Crandall who brought J. Moser's work to our attention.

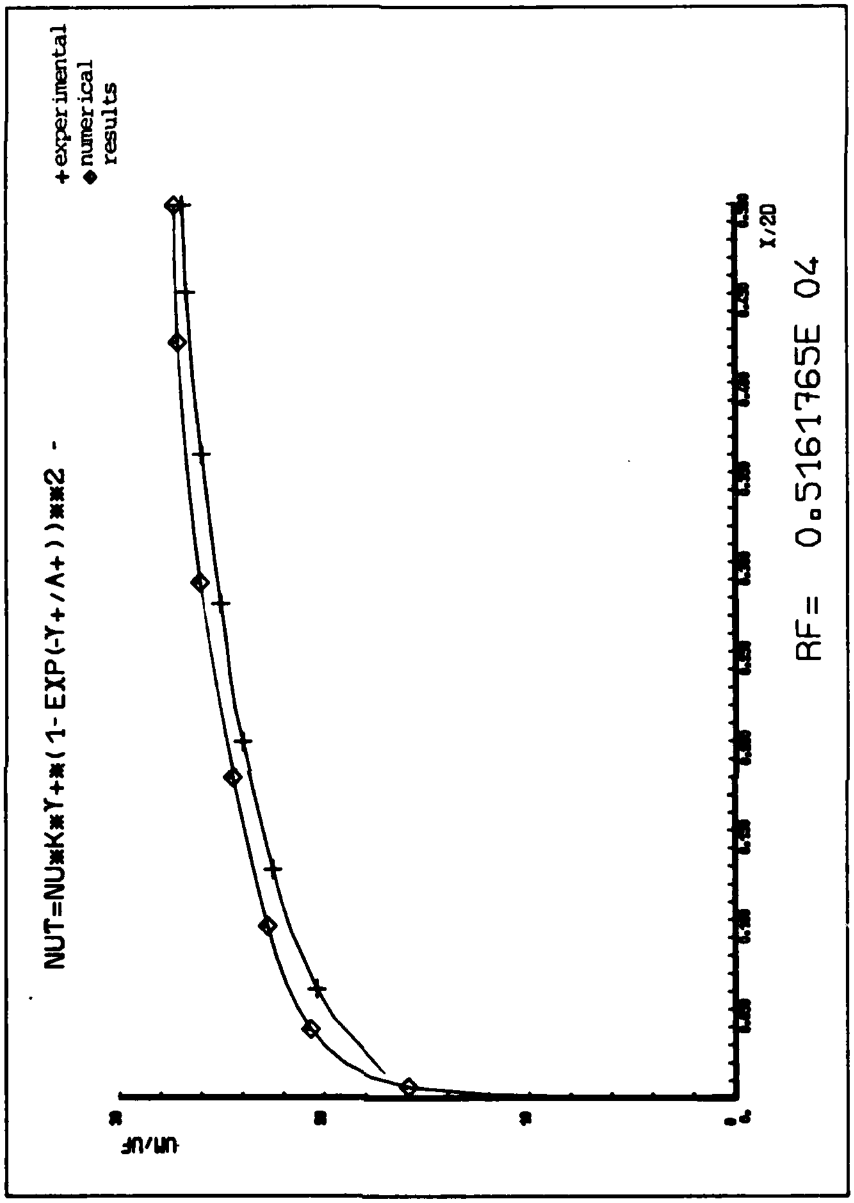
+ experimental
◆ numerical results
NUT=NU*K*Y+*(1-EXP(-Y+/A+))**2
RF= 0.5161765E 04
X/2D
UN/UF

*Figure 2*

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>2366 | 2. GOVT ACCESSION NO.<br>AD-A116 176 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>MATHEMATICAL AND NUMERICAL ANALYSIS OF A TURBULENT VISCOSITY MODEL | | 5. TYPE OF REPORT & PERIOD COVERED<br>Summary Report - no specific reporting period<br>6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>C. M. Brauner and C. Lainé | | 8. CONTRACT OR GRANT NUMBER(s)<br>DAAG29-80-C-0041 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Mathematics Research Center, University of Wisconsin<br>610 Walnut Street<br>Madison, Wisconsin 53706 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>1 - Applied Analysis |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>U. S. Army Research Office<br>P.O. Box 12211<br>Research Triangle Park, North Carolina 27709 | | 12. REPORT DATE<br>April 1982<br>13. NUMBER OF PAGES<br>22 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED<br>15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Turbulent viscosity model, nonlinear parabolic P.D.E., maximum principle.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)


The results of this paper concern the nonlinear parabolic P.D.E.

$$\begin{cases} \frac{\partial u}{\partial t} - \nu_0 \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(\nu_T(u) \frac{\partial u}{\partial x}\right) = f \\ u(0, t) = u(1, t) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$


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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) where u represents the mean velocity field of a turbulent flow in a plane duct and $\nu_T(u) = G(x) \sqrt{\left|\frac{\partial u}{\partial x}(0, t)\right|}$ is a model of turbulent viscosity based on the length scale of this flow.

First we prove the existence of a solution as well as some results on regularity. Then we demonstrate that u is positive inside the domain, hence the same result holds for $\partial u/\partial x(0, t)$. The latter result is crucial for the uniqueness proof which is developped in the last section.

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