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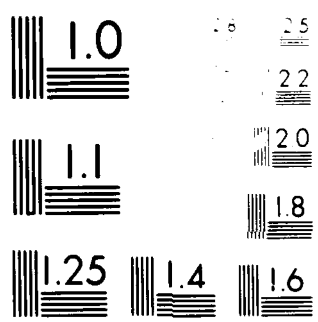
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SINGULARLY PERTURBED HYPERBOLIC EVOLUTION PROBLEMS WITH INFINITE DELAY AND AN APPLICATION TO POLYMER RHEOLOGY

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ABSTRACT

We prove an existence theorem locally in time for quasilinear hyperbolic equations, in which the coefficients are allowed to depend on the history of the dependent variable. Singular perturbations, which change the type of the equation to parabolic, are included, and continuous dependence of the solutions on the perturbation parameter is shown. It is demonstrated that, for a substantial number of constitutive models suggested in the literature, the stretching of filaments of polymeric liquids is described by equations of the kind under study here.

AMS(MOS) Subject Classifications: 35L15, 45K05, 47D05, 76A10.

Key Words: Quasilinear Hyperbolic Equations, Differential Delay Equations, Semigroups, Singular Perturbations, Viscoelastic Liquids.

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SIGNIFICANCE AND EXPLANATION

In an earlier report [23], I proved an existence theorem locally in time for a class of parabolic differential-delay equations modelling the stretching of filaments of polymeric liquids. For many materials, however, the term determining the parabolic nature of the equation seems to be very small. The question was therefore raised whether an existence theorem could still be proved in the limit when this term tends to zero. This paper gives a partial affirmative answer to that question. For a broad class of constitutive assumptions suggested in the rheological literature, the problem can be described by quasilinear hyperbolic equations. We prove an existence theorem for these, and we also prove continuous dependence on small parabolic perturbations. The results are based on theorems due to Kato. In one respect the problem studied here is different from the one in [23]: Instead of a filament pulled at its ends, we study an infinite filament under the influence of a longitudinal body force. It is hoped that similar results can eventually be obtained for the former - physically more relevant - problem and also for more general flow geometries.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SINGULARLY PERTURBED HYPERBOLIC EVOLUTION PROBLEMS WITH INFINITE DELAY AND AN APPLICATION TO POLYMER RHEOLOGY

Michael Renardy

1. Introduction.

In a recent paper [23], I proved an existence theorem (locally in time) for solutions to a class of quasilinear parabolic differential-delay equations that can be used to model the stretching of filaments of polymeric liquids. Such equations arise, if the constitutive law is such that, besides an "elastic" part which is a functional of the strain history, the stress has also a Newtonian part. For many materials, e.g. molten polyethylene, however, this latter contribution is small. This warrants a theory that can treat the Newtonian contribution as a perturbation rather than as the "leading" term in the equation.

In the present paper, I shall give a partial solution to this problem. Mathematically, we are concerned with differential equations of hyperbolic type with a small perturbation changing the type to parabolic. A mathematical theory applicable to such problems was developed by Kato [12], [14-16]. (My results in [23] were based on the theory of Sobolevskii [25]). Since Kato's theory is more easily applicable to pure Cauchy problems than for mixed initial-boundary value problems (some results concerning the latter are in [16]), we confine our attention to the former class of problems here. Physically, this means that rather than a filament pulled at its ends, we will study the deformation of infinite filaments subjected to longitudinal body forces. It is hoped that

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further research will lead to similar results for the boundary value problem and also for more general (in particular more than one-dimensional) flow geometries. It is clear that the results we obtain apply to other one-dimensional problems in continuum mechanics, e.g., those discussed in [6], [9].

In Chapter 2, I quote those results of Kato's theory that are needed in this paper. One of Kato's results will be mildly generalized. In Chapter 3, these results are applied to a class of singularly perturbed quasilinear hyperbolic differential-delay equations, which have the following form

$$\rho u_{\underbrace{tt\cdots t}_{n}} = \eta \cdot \left(f \cdot u_{x\underbrace{t\cdots t}_{n-1}}\right)_x + h \cdot u_{xx\underbrace{t\cdots t}_{n-2}} + k + \phi, \quad x \in R. \tag{1.1}$$

Here η is a small non-negative constant, ϕ is a given function of x and t, and f,h,k are functionals of the histories of derivatives of u, which are of lower order than those displayed. It is assumed that f and h take posi-tive values. Under appropriate assumptions, we prove that the initial history problem associated to (1.1) has a unique solution locally in time, and, more-over, that this solution depends continuously on η, including the limit $\eta \to 0$. In representing differential-delay equations as abstract evolution problems, we follow the method outlined in [23] rather than the classical approach [11].

Chapter 4 deals with the problem of stretching filaments of polymeric liquids. We use a one-dimensional approximation to this problem based on the thinness of the filament [23]. Various constitutive models suggested in the rheological literature [1-5], [8], [10], [13], [17-22], [27] are discussed. It is shown that, for all these models, an equation of the form (1.1) is obtained.

The diversity of the models studied here illustrates the fact that - at least yet - there is no particular constitutive law successfully describing all the phenomena in polymer rheology. Whether or not one constitutive law can fully describe a substantial number of materials, is not yet known. As pointed out in [26], a "theory of theories" is needed. The approach towards this goal taken here, however differs from that in [26]. Whereas there the main emphasis is laid on results obtainable from general principles of continuum mechanics or thermodynamics, I found it worthwhile to look at the models suggested by rheologists, and I tried to work out common mathematical structures. If these exist, they can be hoped to persist also in a more accurate description, to which the models are more or less successful approximations.

One crucial feature that we have found common to all the models under study here is that, under appropriate assumptions on the kernels of the memory integrals, it is possible to rewrite the equations in such a form that the highest order derivatives occur only by their present values (in this context, cf. [24]). More specifically, for some $n \in \mathbb{N}$, the nth time derivative of the stress depends linearly on the nth and $(n+1)$st time derivatives of the strain with coefficients depending on the histories only of lower order derivatives. One might regard this as a very general formulation of the ancient idea that polymers combine "elastic" and "viscous" effects. The oldest models suggested a linear superposition. What we have here may be called a "quasilinear" superposition.

It is essential in our development that the integral kernels occuring in the constitutive equation are sufficiently smooth everywhere, in particular, that they are bounded. This assumption has also been made by other authors [9], [30], [31]. Both molecular theories and experiments suggest, however, that the kernels may have a singularity (see the remark at then end of Ch. 4). Further research needs to be done on this point.

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2. Abstract Hyperbolic Equations.

In this chapter, I summarize the results from Kato's theory that will be needed in the following. One of Kato's theorems will be generalized.

We study an evolution problem of the form

(2.1) $\quad \dot{u} = A(t,u)u + f(t,u), \quad 0 \leq t \leq T, \quad u(0) = \phi,$

where u takes values in a Banach space X and $A(t,u)$ is a linear operator depending on t and u. Our assumptions will involve further Banach spaces Y and Z such that $Y \subset Z \subset X$ with continuous and dense embeddings. It is assumed that Y, Z and X are reflexive and separable. Let W denote an open set in Y.

First, we quote Theorem I from [12] in a simplified form (with assumption N being obsolete) It is assumed that the following estimates hold for $t,t',\cdots \in [0,T]$ and $w,w',\cdots \in W$ (K denotes a generic constant independent of t and w):

(S1) There is an isomorphism $S(t,w): Y \to X$ satisfying

$$\|S(t,w)\|_{Y,X} \leq K, \quad \|S^{-1}(t,w)\|_{X,Y} \leq K$$

$$\|S(t',w') - S(t,w)\|_{Y,X} \leq K(|t-t'| + \|w-w'\|_Z)$$

(A1) $A(t,w)$ generates a quasi-contraction semigroup in X uniformly with respect to t,w: $\|e^{A(t,w)\tau}\|_{X,X} \leq e^{K\tau}$

(A2) $S(t,w)\,A(t,w)\,S^{-1}(t,w) = A(t,w) + B(t,w)$

where $B(t,w) \in B(X)$, $\|B(t,w)\|_{X,X} \leq K$

(A3) $A(t,w) \in B(Y,Z)$ with $\|A(t,w)\|_{Y,Z} \leq K$

and $\|A(t,w') - A(t,w)\|_{Y,X} \leq K\|w'-w\|_X$.

The mapping $t \mapsto A(t,w) \in B(Y,X)$ is norm-continuous.

(A4) There is some $y_0 \in W$ such that

$$A(t,w)y_0 \in Y \quad \text{and} \quad \|A(t,w)y_0\|_Y \leq K$$

(f1) $f(t,w) \in Y$, $\|f(t,w)\|_Y \leq K$, $\|f(t,w') - f(t,w)\|_X \leq K\|w'-w\|_X$.

Moreover, the mapping $t \mapsto f(t,w) \in X$ is continuous.

Theorem I in [12] reads as follows.

Theorem 2.1:

Let (S), (A1)-(A4) and (f1) hold. Then there is a positive ρ and a positive $T' \leq T$ such that for $\|\phi-y_0\|_Y \leq \rho$, equation (2.1) has a unique solution $u \in C^0([0,T'];W) \cap C^1([0,T'];X)$. ρ and T' depend only on K and the distance of y_0 from the boundary of W.

The second theorem stated in this chapter is a continuous dependence result. It generalizes Theorem II of [12] insofar as it allows S to depend on w. We adopt the following assumptions:

(S2) There is an open set $W' \subset Z$ such that $W \subset W'$ and the following holds. The definition of $S(t,w) \in B(Y,X)$ can be extended to $w \in W'$. Moreover, we have uniformly on $[0,T] \times W'$:

$$\|S(t,w)\|_{Y,X} \leq K, \quad \|D_w S(t,w)\|_{Z,B(Y,X)} \leq K, \quad \|D_t S(t,w)\|_{Y,X} \leq K$$

$$\|S(t,w') - S(t,w)\|_{Y,X} \leq K\|w'-w\|_Z , \quad \|D_w S(t,w') - D_w S(t,w)\|_{Z,B(Y,X)}$$

$$\leq K\|w'-w\|_Z , \quad \|D_t S(t,w') - D_t S(t,w)\|_{Y,X} \leq K\|w'-w\|_Z .$$

Here D_w and D_t denote the derivatives w.r. to t and w.

(A5) $\|B(t,w') - B(t,w)\|_X \leq K\|w'-w\|_Y$

(A6) $\|A(t,w) - A(t,w')\|_{Y,Z} \leq K\|w'-w\|_Y$

(f2) $\|f(t,w') - f(t,w)\|_Y \leq K\|w'-w\|_Y .$

Let us now consider a sequence of evolution problems $(n \in \mathbb{N})$

(2.2) $\dot{u}^n = A(t,u^n)u^n + f(t,u^n), \quad 0 \leq t \leq T, \; u^n(0) = \phi^n.$

Theorem 2.2.:

Assume (A1)-(A6), (f1), (f2) are satisfied uniformly in n and assume (S1), (S2). (The operator S shall not depend on n.) Moreover, assume $\|\phi^n-y_0\|_Y < \rho$, $\|\phi-y_0\|_Y < \rho$ with ρ as of Theorem 2.1. Finally, assume that for $t,w \in [0,T] \times W$

$$A^n(t,w) \to A(t,w) \quad \text{strongly in} \quad B(Y,Z)$$

$$B^n(t,w) \to B(t,w) \quad \text{strongly in} \quad B(X)$$

$$f^n(t,w) \to f(t,w) \quad \text{in} \quad Y$$

as $n \to \infty$. If $\phi^n \to \phi$ in the Y-norm as $n \to \infty$, then there is a $T'' < T$ such that (2.2) has a solution $u^n \in C^1([0,T''];X) \cap C^0([0,T''];W)$ for any n. Moreover, $u^n(t) \to u(t)$ in Y, uniformly for $t \in [0,T'']$, where u is a solution of (2.1).

Proof:

The proof essentially follows the same line of argument as that in [15]. Theorem 2.1 yields the existence of solutions to (2.2) and the limiting equation (2.1) on some interval $[0,T']$ with T' independent of n. It is moreover proved precisely as in [15] that $u^n \to u$ uniformly in t in the X-norm. To prove convergence in the Y-norm, we rely on Theorem IV of [14]. This involves estimating a number of terms. Most of those estimates go as in [15] or are straightforward, and my exposition will focus only on those terms that present difficulties. As in [15], it is used that u^n solves the linear equation

$$(2.3) \qquad \dot{u}^n = A^n u^n + f^n,$$

with $A^n = A^n(t,u^n)$ and $f^n = f^n(t,u^n)$. The limit u solves the linear equation

$$(2.4) \qquad \dot{u} = Au + f$$

with $A = A(t,u)$, $f = f(t,u)$. From [14], Theorem IV, we have the estimate

$$\begin{aligned}\|u^n-u\|_{\infty,Y} &\le K(\|\phi^n-\phi\|_Y + \|f^n-f\|_{1,Y}) \\ &+ K(\|(S^n(0)-S(0))\phi\|_X + \|(S^n-S)f\|_{1,X} + \|(S^n-S)u\|_{\infty,X} \\ &+ K(\|(B^n-B)Su\|_{1,X} + \|(C^n-C)Su\|_{1,X}) \\ &+ K(\|(U^n-U)(\delta \oplus \psi \oplus g)\|_{\infty,X}.\end{aligned}$$

Here S^n denotes $S(t,u^n)$, B^n denotes $B(t,u^n)$. The symbol C stands for $\dot{S}S^{-1}$. The U,U^n are the evolution operators associated with A,A^n. Finally, ψ denotes $S(0)\phi$, and g stands for $Sf + (C-B)Su$. The indices 1 and ∞ indicate the L^1- and L^∞-norms on the interval $[0,T"]$. On the right hand side of (2.5), the term $\|\phi^n-\phi\|_Y$ converges to zero by assumption and we are left with seven more contributions. Of these, the first, fifth and seventh have been dealt with in [15], and no change in the argument is needed here. The second, third and fourth term are estimated in terms of $\|\phi^n-\phi\|_Y$ and $\|u^n-u\|_{\infty,Z}$ by virtue of (S1). Now, note that

$$\|u^n-u\|_{\infty,Z} \le K(\|\phi^n-\phi\|_Z + \|\dot{u}^n-\dot{u}\|_{1,Z}) \le K(\|\phi^n-\phi\|_Y$$

$$+ \|A^nu^n + f^n - Au - f\|_{1,Z}).$$

The last term will be estimated below. We may thus focus on the term

$$\|(C^n-C)Su\|_{1,X} = \|(\dot{S}^n(S^n)^{-1} - \dot{S}S^{-1})Su\|_{1,X}$$

$$\le \|(\dot{S}^n-\dot{S})u\|_{1,X} + \|\dot{S}^n((S^n)^{-1}-S^{-1})Su\|_{1,X}.$$

By (S2), $\dot{S}^n$ is bounded in $B(Y,X)$, and by (S1), $\|(S^n)^{-1}-S^{-1}\|_{X,Y}$ can be estimated by $\|u^n-u\|_{\infty,Z}$. This takes care of the second term. For the first term, observe that

$$\dot{S}^n(t) = \frac{d}{dt}S(t,u^n(t)) = \dot{S}(t,u^n(t)) + D_uS(t,u^n(t))\dot{u}^n(t)$$

$$= \dot{S}(t,u^n(t)) + D_uS(t,u^n(t))(A^n(t,u^n)u^n + f^n(t,u^n))$$

and likewise

$$\dot{S}(t) = \dot{S}(t,u) + D_uS(t,u)\,(A(t,u)u + f(t,u)).$$

We have

$$\|(\dot{S}(t,u^n) - \dot{S}(t,u))u\|_{1,X} \le \|\dot{S}(t,u^n) - \dot{S}(t,u)\|_{1,Y,X}\|u\|_{\infty,Y}$$

$$\le K\|u^n-u\|_{1,Z} \le K\|u^n-u\|_{1,Y}$$

and

$$\|(D_uS(t,u^n(t)) - D_uS(t,u(t)))(A^n(t,u^n(t))u^n(t) + f^n(t,u^n(t)))\|_{1,B(Y,X)}$$

$$\leq \|(D_uS(t,u^n(t)) - D_uS(t,u(t))\|_{1,Z,B(Y,X)}\|A^nu^n + f^n\|_{\infty,Z}$$

$$\leq K\|u^n-u\|_{1,Z} \leq K\|u^n-u\|_{1,Y}.$$

Finally,

$$\|D_uS(t,u(t))(A^nu^n+ f^n-Au-f)\|_{1,B(Y,X)} \leq \|D_uS(t,u(t))\|_{\infty,Z,B(Y,X)}$$

$$\cdot \|A^nu^n+f^n-Au-f\|_{1,Z}.$$

This last term can be estimated by

$$\|A^n(t,u)u - A(t,u)u\|_{1,Z} + \|f^n(t,u) - f(t,u)\|_{1,Z}$$

$$+ \|(A^n(t,u^n) - A^n(t,u))u\|_{1,Z} + \|A^n(t,u^n)(u^n-u)\|_{1,Z}$$

$$+ \|f^n(t,u^n) - f^n(t,u)\|_{1,Z}.$$

It follows from Assumptions (A6), (f2) that the last three terms can be estimated by $\|u^n-u\|_{1,Y}$. The first two contributions converge to zero because of the assumptions of the theorem. This concludes the proof.

3. Application to Delay Equations.

In this chapter, we shall apply the preceding results to differential-delay equations of the following form

$$
\begin{aligned}
u_{(n-1)} &= \eta\left(f(\hat{u}_x,\hat{u}_{xt},\cdots,\hat{u}_{x(n-4)})u_{x(n-2)}\right)_x \\
&\quad + h(\hat{u}_x,\hat{u}_{xt},\cdots,\hat{u}_{x(n-3)};\eta)u_{xx(n-3)} \\
&\quad + k(\hat{u}_x,\cdots,\hat{u}_{x(n-3)};\hat{u}_{xx},\cdots,\hat{u}_{xx(n-4)};\eta) + \tilde{\phi}(t,x).
\end{aligned}
\tag{3.1}
$$

Here, the index (k) stands for k-fold differentiation with respect to the time variable t, and the hat denotes the past history: $\hat{u}_x(t,x)(S) = u_x(t+S,X)$ for $S \in (-\infty,0]$. The $\tilde{f},\tilde{h}$ and $\tilde{k}$ are smooth functionals on a history space, the topology of which will be specified later.

Differentiating (3.1) with respect to x, we obtain

$$
\begin{aligned}
u_{x(n-1)} &= \eta\left(f(\hat{u}_x,\hat{u}_{xt},\cdots,\hat{u}_{x(n-4)})u_{x(n-2)}\right)_{xx} \\
&\quad + h(\hat{u}_x,\hat{u}_{xt},\cdots,\hat{u}_{x(n-3)};\eta)u_{xxx(n-3)} + \frac{\partial h}{\partial x}\cdot u_{xx(n-3)} \\
&\quad + \left(k(\hat{u}_x,\cdots,\hat{u}_{x(n-3)};\hat{u}_{xx},\cdots,\hat{u}_{xx(n-4)};\eta)\right)_x + \phi(t,x),
\end{aligned}
\tag{3.2}
$$

where $\phi = \frac{\partial\tilde{\phi}}{\partial x}$. In the following, we need only be concerned with equations of the form (3.2). For applying the results of Chapter 2, it is convenient to rewrite (3.2) as a system of equations. Let us put $v_k = u_{x(k)}$, $w_k = u_{xx(k)}$. We thus obtain the following system equivalent to (3.2)

$$v_{kt} = v_{k+1} \qquad k = 0,1,\cdots,n-3$$

$$w_{kt} = w_{k+1} \qquad k = 0,1,\cdots,n-4$$

(3.3) $$v_{n-2,t} = \eta\big(f(\hat{v}_0,\cdots,\hat{v}_{n-4})v_{n-2}\big)_{xx} + h(\hat{v}_0,\cdots,\hat{v}_{n-3};\eta)w_{n-3,x} + \frac{\partial h}{\partial x}\cdot w_{n-3}$$
$$+ \big(k(\hat{v}_0,\cdots,\hat{v}_{n-3};\hat{w}_0,\cdots,\hat{w}_{n-4};\eta)\big)_x + \phi$$

$$w_{n-3,t} = v_{n-2,x}.$$

It will be advantageous to make some further substitutions. Let us put $v'_{n-2} = fv_{n-2}$, $w'_{n-3} = f\sqrt{h}\, w_{n-3} + \frac{f}{\sqrt{h}}\cdot k$. Then we obtain the following system

$$v_{kt} = v_{k+1} \qquad k = 0,1,\cdots,n-4$$

$$v_{n-3,t} = \frac{v'_{n-2}}{f}$$

$$w_{kt} = w_{k+1} \qquad k = 0,1,\cdots,n-5$$

$$w_{n-4,t} = \frac{1}{f\sqrt{h}}\, w'_{n-3} - \frac{k}{h}$$

(3.4) $$v'_{n-2,t} = \eta\, f\, v'_{n-2,xx} + \sqrt{h}\, w'_{n-3,x} + \Big(\frac{1}{2}\, f\, \frac{\partial h}{\partial x} - h\,\frac{\partial f}{\partial x}\Big)$$
$$\Big(\frac{1}{f\sqrt{h}}\, w'_{n-3} - \frac{k}{h}\Big) - \sqrt{h}\cdot k\cdot\frac{\partial}{\partial x}\Big(\frac{f}{\sqrt{h}}\Big) + \frac{\partial f}{\partial t}\cdot\frac{v'_{n-2}}{f} + \phi$$

$$w'_{n-3,t} = \sqrt{h}\, v'_{n-2,x} - \frac{\sqrt{h}}{f}\,\frac{\partial f}{\partial x}\, v'_{n-2} + \frac{\partial}{\partial t}\,(f\sqrt{h})\,\cdot$$
$$\Big(\frac{w'_{n-3}}{f\sqrt{h}} - \frac{k}{h}\Big) + \frac{\partial}{\partial t}\Big(\frac{fk}{\sqrt{h}}\Big)$$

Next, let us define the history spaces, in which (3.4) will be analyzed.

<u>Definition 3.1:</u>

For a given Banach space Z, let $\tilde{H}^1(Z)$ denote the space of all functions $(-\infty,0] \to Z$, which are the sum of a constant element $z_0 \in Z$ and a function $z(t)$, which is square integrable and has a square integrable derivative (in the Bochner sense). Analogously, let $\tilde{L}^2(Z)$ be the space of all functions $(-\infty,0] \to Z$, which are the sum of a constant and a square integrable function.

In particular, $\tilde{H}^1$ shall denote $\tilde{H}^1(\mathbf{R})$ and $\tilde{H}^1_m$ shall denote $\tilde{H}^1(H^m(\mathbf{R}))$ where $H^m(\mathbf{R})$ is the Sobolev space of all functions $\mathbf{R} \to \mathbf{R}$, which have n square integrable derivatives. Analogously, let $\tilde{L}^2_m = \tilde{L}^2(H^m(\mathbf{R}))$.

Remarks:

1. In [23], I used the space $C_b^{\lim}$ (the space of bounded continuous functions having a limit at $-\infty$. The reason why this space cannot be used here is that it is not reflexive as required by Kato's theory.
2. The choice of the space $\tilde{H}^1$ seems to impose rather restrictive conditions on the given history. However, as in [23], one can allow histories in more general spaces, e.g. "fading memory" spaces [7], by reducing the problem to one that has a history in $\tilde{H}^1$, but is equivalent to the given one for $t > 0$. The only modification necessitated by this is that f,h,k must be allowed to depend explicitly on x and t. This modification presents no major difficulties.

As in [23], we define a shift operator T_S on $\tilde{H}_1 : T_S\phi(t) = \phi(t+s)$ for $s \in (-\infty,0]$.

Our assumption on f,h,k in (3.3) are as follows:

(i) The mappings $f:(\tilde{H}^1)^{n-3} \to \mathbf{R}$, $h:(\tilde{H}^1)^{n-3} \times \mathbf{R} \to \mathbf{R}$ and $k:(\tilde{H}^1)^{2n-3} \times \mathbf{R} \to \mathbf{R}$ are smooth (i.e., sufficiently often continuously differentiable) and the induced operators $\hat{f},\hat{h},\hat{k}$ defined by $\hat{f}(\phi)(S) = f(T_S\phi)$ map into $\tilde{H}^1$ and depend again smoothly on their arguments. Moreover, f and h take strictly positive values: $f > \varepsilon > 0$, $h > \varepsilon > 0$. k vanishes if its arguments are zero. Moreover, the Frechet-derivative Df is a linear operator from $(\tilde{L}^2)^{n-3}$ into $\mathbf{R}$, which depends smoothly on the arguments of f (in the topology of $\tilde{H}^1$), and the corresponding operator $D\hat{f}$ maps into $\tilde{L}^2$ and again depends smoothly on the arguments of f. Analogous conditions hold for h and k.

The following lemma is easily proved.

Lemma 3.2:

If (i) holds, with a sufficient degree of differentiability, then $\hat{f}$ defines in a natural way (acting pointwise in the space variable x) a smooth operator from $(\tilde{H}^1_m)^{n-3}$ into $\tilde{H}^1_m + \mathbb{R}$. Here m is a given integer greater or equal to 1. The same holds for $\hat{h}, \hat{k}$, and an analogous statement also holds for $D\hat{f}$, $D\hat{h}$, $D\hat{k}$ (regarded as linear operators $(\tilde{L}^2)^{n-3}$ or $(\tilde{L}^2)^{2n-3} + \tilde{L}^2$).

Remark:

Since we are concerned with existence theorems locally in time, it is clearly sufficient that condition (i) holds in a neighborhood of the prescribed initial condition.

Let us also note that $\frac{\partial h}{\partial x} = \sum_{i=1}^{n-2} D_i h(\hat{v}_0, \cdots, \hat{v}_{n-3}; \eta)\hat{w}_{i-1}$, where D_i denotes the Fréchet derivative w.r. to the ith argument. Analogous manipulations are possible for the other x- and t derivatives of f, h and k that occur in (3.4). As in [23], let us assume that the given initial history up to time $t=0$ satisfies the equations (this can be achieved by appropriately changing ϕ. We can then rewrite equation (3.4) in the following abstract form

$$\hat{v}_{kt} = \hat{v}_{k+1} \qquad k = 0,1,\cdots,n-4$$

$$\hat{w}_{kt} = \hat{w}_{k+1} \qquad k = 0,1,\cdots,n-5$$

$$\hat{v}_{n-3,t} = \frac{\hat{v}'_{n-2}}{\hat{f}} \tag{3.5}$$

$$\hat{w}_{n-4,t} = \frac{\hat{w}'_{n-3}}{f\sqrt{\hat{h}}} - \frac{\hat{k}}{\hat{h}}$$

$$\hat{v}'_{n-2,t} = \eta\, \hat{f}\, \hat{v}'_{n-2,xx} + \sqrt{\hat{h}}\, \hat{w}'_{n-3,x} + \cdots + \hat{\phi}$$

$$\hat{w}'_{n-3,t} = \sqrt{\hat{h}}\, \hat{v}'_{n-2,x} + \cdots$$

where $\hat{f},\hat{h}$ etc. are defined as in Lemma 3.2. Now let the spaces X,Y be as follows: $X = (\tilde{H}_1^1 \cap \tilde{L}_4^2)^{2n-5}$, $Y = (\tilde{H}_m^1 \cap \tilde{L}_{m+3}^2)^{2n-5}$, where m is an odd number greater or equal to 3. Finally, let $Z = (\tilde{H}_{m-2}^1 \cap \tilde{L}_{m+1}^2)^{2n-5}$. When identifying (3.5) with the abstract system (2.1), we incorporate in A only those terms that contain derivatives w.r. to x, everything else is included in f. With this identification, the conditions (A3), (A4), (f1), (A6) and (f2) are rather obvious, (note that, for $m \geq 3$, $\tilde{H}_m^1 \cap \tilde{L}_{m+3}^2$ is a Banach algebra) provided that the following holds:

(ii) $\hat{\phi}$ takes values in $\tilde{H}_m^1$ (it follows automatically that it is continuous into $\tilde{H}_m^1$), and the initial condition at $t=0$ lies in Y.

For verifying the remaining conditions, we have to study the operator $A(\hat{f},\hat{h})$, defined by

$$A(\hat{f},\hat{h})(\hat{v},\hat{w}) = (\eta \hat{f} \hat{v}_{xx} + \sqrt{\hat{h}}\, \hat{w}_x, \sqrt{\hat{h}}\, \hat{v}_x).$$

The operator S is defined by $S = \left(\hat{f} \frac{\partial^2}{\partial x^2} - \lambda\right)^{\frac{m-1}{2}}$ for $\lambda \in \mathbb{R}$ large enough. With this choice of S, conditions (S1), (S2) are obvious.

Moreover, one sees easily that

$$\left(\left(\hat{f} \frac{\partial^2}{\partial x^2} - \lambda\right)A(\hat{f},\hat{h}) - A(\hat{f},\hat{h})\left(\hat{f} \frac{\partial^2}{\partial x^2} - \lambda\right)\right)(\hat{v},\hat{w})$$

yields an expression involving only first and second order derivatives of $\hat{v},\hat{w},\hat{f}$ and $\hat{h}$. From this it is not difficult to conclude (A2) and (A5). (Note that $T^nA-AT^n = T^{n-1}(TA-AT) + T^{n-2}(TA-AT)T + \cdots$ and apply this with $T = \left(\hat{f} \frac{\partial^2}{\partial x^2} - \lambda\right)$). For (A1), we have to show that $\mathrm{Re}(A(\hat{v},\hat{w}),(\hat{v},\hat{w}))_{(\tilde{H}_1^1)^2} \le C((\hat{v},\hat{w}),(\hat{v},\hat{w}))_{(\tilde{H}_1^1)^2}$, which follows from a simple integration by parts in the x-variable.

We have thus proved:

Theorem 3.2.

Let (i), (ii) be satisfied. Then there is a $T > 0$ such that (3.5) has a solution $\hat{U} = (\hat{v}_0, \hat{v}_1, \cdots, \hat{w}'_{n-2}) \in C^1([0,T];X) \cap C^0([0,T];Y)$. $\hat{U}$ depends continuously on $\eta \in [0,\eta_0]$ in the norm of Y.

4. Stretching of Filaments of Viscoelastic Liquids.

We are studying the motion of an infinitely extended filament of an incompressible viscoelastic liquid under the influence of a longitudinal body force. It was shown in [23] that, if the filament is thin, this problem can be modelled by a one-dimensional approximation, where only longitudinal motions need to be studied. Let $u(x,t)$ denote the position of a fluid particle at time t, which is at the position x in certain reference state. For simplicity, this reference configuration is chosen to be one in which the filament has uniform thickness. I showed in [23] that the evolution of u is governed by the following equation

$$\rho u_{tt} = \frac{\partial}{\partial x}(u_x \pi^{11} - u_x^{-2}\pi^{22}) + \phi. \tag{4.1}$$

In this equation ρ denotes the density of the fluid (i.e., a constant), ϕ is the given body force, and π^{11}, π^{22} are the longitudinal and transverse components of the convected extra stress tensor (i.e., not including the pressure, which was eliminated in the derivation of (4.1)). The tensor π is related by a constitutive law to the right Cauchy-Green tensor γ (in our notation we follow [22]). In the approximation leading to (4.1), γ is given by

$$\gamma = \begin{pmatrix} u_x^2 & 0 & 0 \\ 0 & u_x^{-1} & 0 \\ 0 & 0 & u_x^{-1} \end{pmatrix}.$$

In the following, we discuss various constitutive laws that have been suggested in the rheology literature and the corresponding equations (4.1) that they lead to. It will be shown that all these equations can be transformed to the form (3.1). In particular, we shall check the positivity of the functions f and h. (It will always be understood that u is the sum of a given function $u_0(x,t)$ and a function tending to zero appropriately as

$x \to \pm\infty$, moreover, u_x is always assumed uniformly positive. In comparison to Chapter 3, the variable called u there will be identified with $u - u_0(x,t)$ in this chapter.) The notation used in the original papers cited here is often different from ours, and we have transcribed the constitutive laws appropriately. Tables of some (but not all) constitutive assumptions discussed here can be found in [2] and [22].

a) <u>The rubberlike liquid of Green and Tobolsky [10] and Lodge [19], [20] and modifications of Ward and Jenkins [27] and Lodge [21].</u>

In these theories, the constitutive law has the following form

$$\pi = -\eta \frac{\partial}{\partial t}(\gamma^{-1}) + \int_{-\infty}^{t} a(t-s)\gamma^{-1}(s)ds - \int_{-\infty}^{t} b(t-s)\gamma^{-1}(t)\gamma(s)\gamma^{-1}(t)ds$$

$$+ \int_{-\infty}^{t} c(t-s)\gamma^{-1}(s)\gamma(t)\gamma^{-1}(s)ds + \int_{-\infty}^{t} d(t-s)(\gamma(t):\gamma^{-1}(s))\gamma^{-1}(s)ds.$$

The first term is a Newtonian contribution, the second is the one given by the rubberlike liquid theory [10], [19], [20]. The third term accounts for a modification suggested by Ward and Jenkins [27], and the last two represent corrections of Lodge [21] (Lodge finds $c=2d$ from a molecular theory, but we shall make no use of this). With this constitutive law, Equation (4.1) assumes the following form

$$\rho u_{tt} = 3\eta \frac{\partial^2}{\partial x \partial t}\left(-\frac{1}{u_x}\right) + \frac{\partial}{\partial x}\{u_x^3(t) \cdot \int_{-\infty}^{t}(c(t-s)+d(t-s))u_x^{-4}(s)ds$$

$$+ u_x(t) \cdot \int_{-\infty}^{t} a(t-s)u_x^{-2}(s)ds + \int_{-\infty}^{t} b(t-s)+d(t-s))u_x^{-1}(s)ds$$

$$- u_x^{-2}(t) \cdot \int_{-\infty}^{t} a(t-s)u_x(s)ds - u_x^{-3}(t) \cdot \int_{-\infty}^{t} (b(t-s)+c(t-s)+2d(t-s))u_x^2(s)ds\}$$

$$+ \phi .$$

In order to obtain the form (3.1), we differentiate this once with respect to time. This yields

$$\rho u_{ttt} = 3\eta \frac{1}{u_x^2} \cdot u_{xxtt} - 6\eta \frac{u_{xx}}{u_x^3} u_{xtt} + u_{xxt} \cdot \{-12\eta \frac{u_{xt}}{u_x^3} + 3u_x^2 \cdot$$

$$\int_{-\infty}^{t} (c(t-s) + d(t-s))u_x^{-4}(s)ds + \int_{-\infty}^{t} a(t-s)u_x^{-2}(s)ds + 2u_x^{-3} \cdot$$

$$\int_{-\infty}^{t} a(t-s)u_x(s)ds + 3u_x^{-4}(t) \cdot \int_{-\infty}^{t} (b(t-s)+c(t-s)+2d(t-s))u_x^2(s)ds\} + \cdots .$$

Here, as always in the following, the dots indicate terms involving only lower order derivatives of u and the derivatives of ϕ (ϕ is always assumed "smooth enough"). We have assumed here that the kernels have derivatives in L^1 so that

$$\frac{d}{dt} \int_{-\infty}^{t} a(t-s)f(s)ds = \int_{-\infty}^{t} a'(t-s)f(s)ds + a(0)f(t).$$

The equation above clearly has the form (3.1), and the coefficient of u_{xxt} is positive if the kernels are positive and η is small enough.

b) The model of Kaye [17] and Bernstein, Kearsley and Zapas [1].

In this model, the constitutive law has the form

$$\pi = \int_{-\infty}^{t} a(t-s,I_1,I_2)\gamma^{-1}(s)ds - \int_{-\infty}^{t} b(t-s,I_1,I_2)\gamma^{-1}(t)\gamma(s)\gamma^{-1}(t)ds.$$

I_1 and I_2 are the invariants of $\gamma^{-1}(t)\gamma(s)$: $I_1 = tr(\gamma(t)\gamma^{-1}(s))$ and $I_2 = tr(\gamma^{-1}(t)\gamma(s))$. In our special problem, we thus have $I_1 = u_x^2(t)u_x^{-2}(s) + 2u_x^{-1}(t)u_x(s)$ and $I_2 = u_x^2(s)u_x^{-2}(t) + 2u_x^{-1}(s)u_x(t)$. Both are thus functions of the single variable $I = \frac{u_x(t)}{u_x(s)}$, and we shall use the obvious notation $a(t-s,I)$, $b(t-s,I)$. The dynamic equation (4.1) assumes the form

$$\rho u_{tt} = \frac{\partial}{\partial x} \{u_x(t) \int_{-\infty}^{t} a(t-s,I)u_x^{-2}(s)ds + \int_{-\infty}^{t} b(t-s,I)u_x^{-1}(s)ds$$

$$-u_x^{-2}(t) \int_{-\infty}^{t} a(t-s,I)u_x(s)ds - u_x^{-3}(t) \cdot \int_{-\infty}^{t} b(t-s,I)u_x^2(s)ds\} + \phi.$$

By differentiation with respect to time, we find

$$\rho u_{ttt} = u_{xxt}\{\int_{-\infty}^{t} a(t-s,I)u_x^{-2}(s)ds + 2u_x^{-3}(t) \int_{-\infty}^{t} a(t-s,I)u_x(s)ds$$

$$+ 3u_x^{-4} \int_{-\infty}^{t} b(t-s,I)u_x^2(s)ds + u_x(t) \int_{-\infty}^{t} \frac{\partial a}{\partial I}(t-s,I)u_x^{-3}(s)ds$$

$$+ \int_{-\infty}^{t} \frac{\partial b}{\partial I}(t-s,I)u_x^{-2}(s)ds - u_x^{-2}(t) \int_{-\infty}^{t} \frac{\partial a}{\partial I}(t-s,I)ds$$

$$- u_x^{-3}(t) \int_{-\infty}^{t} \frac{\partial b}{\partial I}(t-s,I)u_x(s)ds\} + \cdots .$$

A Newtonian term can be added to this as before. Suppose the kernels a,b are positive. Then the coefficient of u_{xxt} is positive in two cases:

α). If $\frac{\partial a}{\partial I}$, $\frac{\partial b}{\partial I}$ are small, i.e., if the model is considered a perturbation of the Ward-Jenkins model.

β) If $\frac{\partial a}{\partial I_1}, \frac{\partial a}{\partial I_2}, \frac{\partial b}{\partial I_1}, \frac{\partial b}{\partial I_2}$ are positive. It would be interesting if this condition has a physical interpretation.

c) The Bird-Carreau model [3], [5].

In this model, we have

$$\pi = (1 + \frac{\varepsilon}{2}) \int_{-\infty}^{t} a(t-s,I(s))\gamma^{-1}(s)ds - \frac{\varepsilon}{2} \int_{-\infty}^{t} a(t-s,I(s))\gamma^{-1}(t)\gamma(s)\gamma^{-1}(t)ds$$

where $I(s) = tr(\dot{\gamma}(s)\gamma^{-1}(s)\dot{\gamma}(s)\gamma^{-1}(s)) = 6\,\frac{u_{xt}^2(s)}{u_x^2(s)}$.

This leads to the equation

$$\rho u_{tt} = (1 + \frac{\varepsilon}{2}) \frac{\partial}{\partial x}\{u_x \int_{-\infty}^{t} a(t-s,I(s))u_x^{-2}(s)ds$$

$$- u_x^{-2}(t) \int_{-\infty}^{t} a(t-s,I(s))u_x(s)ds\}$$

$$+ \frac{\varepsilon}{2} \frac{\partial}{\partial x}\{ \int_{-\infty}^{t} a(t-s,I(s))u_x^{-1}(s)ds$$

$$- u_x^{-3}(t) \int_{-\infty}^{t} a(t-s,I(s))u_x^2(s)ds\} + \phi.$$

Differentiating this w.r. to time, we find

$$\rho u_{ttt} = u_{xxt}[(1+\frac{\varepsilon}{2})\{\int_{-\infty}^{t} a(t-s,I(s)u_x^{-2}(s)ds + 2u_x^{-3}(t)\int_{-\infty}^{t} a(t-s,I(s))u_x(s)ds\}$$

$$+\frac{\varepsilon}{2}\cdot 3u_x^{-4}\int_{-\infty}^{t} a(t-s,I(s))u_x^2(s)ds]$$

$$+\frac{\partial}{\partial t}[(1+\frac{\varepsilon}{2})\{u_x(t)\int_{-\infty}^{t}\frac{\partial a}{\partial I}(t-s,I(s))I_x(s)u_x^{-2}(s)ds$$

$$-u_x^{-2}\int_{-\infty}^{t}\frac{\partial a}{\partial I}(t-s,I(s))I_x(s)u_x(s)ds\} + \frac{\varepsilon}{2}\{\int_{-\infty}^{t}\frac{\partial a}{\partial I}(t-s,I(s))\cdot$$

$$I_x(s)u_x^{-1}(s)ds - u_x^{-3}\int_{-\infty}^{t}\frac{\partial a}{\partial I}(t-s,I(s))I_x(s)u_x^2(s)ds\}] + \cdots \quad .$$

A second differentiation yields (the kernels are assumed to be twice differentiable w.r. to time

$$\rho u_{tttt} = u_{xxtt}[(1+\frac{\varepsilon}{2})\{\int_{-\infty}^{t} a(t-s,I(s))u_x^{-2}(s)ds$$

$$+ 2u_x^{-3}\int_{-\infty}^{t} a(t-s,I(s))u_x(s)ds\}$$

$$+\frac{\varepsilon}{2}\ 3u_x^{-4}\int_{-\infty}^{t} a(t-s,I(s))u_x^2(s)ds] + \cdots \quad .$$

For a positive kernel, the coefficient of u_{xxtt} is positive.

d) <u>The Carreau Model B [4].</u>

In this model, it is assumed that

$$\pi = (1+\frac{\varepsilon}{2})\int_{-\infty}^{t}\exp(-\int_{s}^{t} f(I(r))dr)\gamma^{-1}(s)ds$$

$$-\frac{\varepsilon}{2}\int_{-\infty}^{t}\exp(-\int_{s}^{t} f(I(r))ds)\gamma^{-1}(t)\gamma(s)\gamma^{-1}(t)ds,$$

where I has the same meaning as in the Bird-Carreau model. We thus obtain the following equation

$$\rho u_{tt} = \frac{\partial}{\partial x}\{\int_{-\infty}^{t}\exp(-\int_{s}^{t} f(I(r))dr)[(1+\frac{\varepsilon}{2})(u_x(t)u_x^{-2}(s) - u_x^{-2}(t)u_x(s))$$

$$+\frac{\varepsilon}{2}(u_x^{-1}(s) - u_x^{-3}(t)u_x^2(s))ds]\} + \phi.$$

The integral converges, if f takes strictly positive values. Differentiating with respect to time, we obtain

$$\rho u_{ttt} = u_{xxt}\{ \int_{-\infty}^{t}\exp(-\int_{s}^{t} f(I(r))dr)[(1+\frac{\varepsilon}{2})(u_x^{-2}(s) + 2u_x^{-3}(t)u_x(s))$$

$$+\frac{\varepsilon}{2}\,3u_x^{-4}(t)u_x^2(s)]ds\} + \frac{\partial}{\partial t}\{-\int_{-\infty}^{t}\exp(-\int_{s}^{t} f(I(r))dr)$$

$$\cdot \int_{s}^{t} f'(I(r))I_x(r)dr[(1+\frac{\varepsilon}{2})(u_x(t)u_x^{-2}(s) - u_x^{-2}(t)u_x(s)) +$$

$$+\frac{\varepsilon}{2}(u_x^{-1}(s) - u_x^{-3}(t)u_x^2(s))]ds\} + \cdots \ .$$

The second differentiating with respect to time yields

$$\rho u_{tttt} = u_{xxtt}\{ \int_{-\infty}^{t}\exp(-\int_{s}^{t} f(I(r))dr\ [(1+\frac{\varepsilon}{2})(u_x^{-2}(s) + 2u_x^{-3}(t)u_x(s))$$

$$+\frac{\varepsilon}{2}\,3u_x^{-4}(t)u_x^2(s)]ds - 12\,\frac{u_{xt}(t)}{u_x^2(t)}\int_{-\infty}^{t}\exp(-\int_{s}^{t} f(I(r))dr)f'(I(t))$$

$$\cdot\ [(1+\frac{\varepsilon}{2})(u_x(t)u_x^{-2}(s) - u_x^{-2}(t)u_x(s))$$

$$+\frac{\varepsilon}{2}(u_x^{-1}(s) - u_x^{-3}(t)u_x^2(s))]ds\} + \cdots \ .$$

The coefficient of u_{xxtt} is positive under the restriction that $f'(I)\sqrt{I}$ is not too big.

e). The Leonov model [18].

This model does not explicitly give the stress as a functional of the strain history. Instead it is given by a system of equations as follows

$$\pi = \sum_k W_1^{(k)}(I_{1k},I_{2k})\ c_k^{-1} - W_2^{(k)}(I_{1k},I_{2k})\gamma^{-1}c_k\gamma^{-1}$$

$$-\eta\ W(I_{11},I_{21})\ \frac{\partial}{\partial t}\ (\gamma^{-1})$$

$$\frac{\partial}{\partial t}(c_k^{-1}) = -f_k(I_{1k},I_{2k})(c_k^{-1}\gamma c_k^{-1} - \frac{1}{3}\ I_{1k}c_k^{-1})$$

$$-g_k(I_{1k},I_{2k})(\frac{1}{3}\ I_{2k}\ c_k^{-1} - \gamma^{-1})$$

where $I_{1k} = \mathrm{tr}(c_k^{-1}\gamma)$, $I_{2k} = \mathrm{tr}(\gamma^{-1}c_k)$. The tensors c_k satisfy the restriction $\det c_k = 1$ (it can be shown that $\det c_k$ is an inv .iant of the evolution equation). In Leonov's paper, the analogue of c_k^{-1} is called c_k, we have changed this for consistency of notation. The $W_1^{(k)}, W_2^{(k)}$, W, f_k, g_k are positive scalar functions, they are not independent in Leonov's model. It is convenient to introduce $d_k = c_k^{-1}\gamma$. With this, the constitutive equation becomes

$$\pi = \sum_k W_1^{(k)}(I_{1k}, I_{2k})d_k\gamma^{-1} - W_2^{(k)}(I_{1k}, I_{2k})d_k^{-1}\gamma^{-1}$$

$$-\eta\, W(I_{11}, I_{21})\, \frac{\partial}{\partial t}(\gamma^{-1})$$

(4.2)

$$\frac{\partial}{\partial t}(d_k) = -f_k(I_{1k}, I_{2k})(d_k^2 - \frac{1}{3} I_{1k}d_k)$$

$$-g_k(I_{1k}, I_{2k})(\frac{1}{3} I_{2k}d_k - 1) + d_k\gamma^{-1}\dot{\gamma} .$$

I_{1k} and I_{2k} are the first and second invariants of d_k, and we have $\det d_k = 1$. If f_k, g_k have positive values, then, for $\dot{\gamma} = 0$, the solution $d_k = \mathrm{id}$ is an exponentially asymptotically stable solution of (4.2) when this equation is restricted to $\{d_k | \det d_k = 1\}$. Consequently, if $\gamma^{-1}\dot{\gamma} \to 0$ as $t \to -\infty$, then on some interval $(-\infty, T)$ there is a unique solution d_k which converges to the identity as $t \to -\infty$. Whether this solution can be continued up to $t = 0$ depends on the form of f_k, g_k and the history of γ. We shall assume that (4.2) has a solution up to $t = 0$. Then this solution is a smooth functional of the histories of γ and $\dot{\gamma}$: $d_k = F_1\,(\hat{\gamma}, \hat{\dot{\gamma}})$. From (4.2) or, resp., its differentiated version, we also find functional relationships of the form $\dot{d}_k = F_2(\hat{\gamma}, \hat{\dot{\gamma}})$, $\ddot{d}_k = F_3(\hat{\gamma}, \hat{\dot{\gamma}}) + d_k\gamma^{-1}\ddot{\gamma}$. For the filament problem, γ is a diagonal matrix, and so is d_k. Let us denote the 11- and 22-components of d_k by d_{k1}, d_{k2}. The dynamic equation (4.1) reads now as follows

$$\rho u_{tt} = \frac{\partial}{\partial x}\{u_x^{-1}[\sum_k W_1^{(k)}(I_{1k},I_{2k})d_{k1} - W_2^{(k)}(I_{1k},I_{2k})d_{k1}^{-1}$$

$$- W_1^{(k)}(I_{1k},I_{2k})d_{k2} + W_2^{(k)}(I_{1k},I_{2k})d_{k2}^{-1}]$$

$$+ 3\eta\ W(I_{11},I_{21})\ \frac{\partial}{\partial t}\ (-\frac{1}{u_x})\} + \phi.$$

Retaining only terms of the highest differentiation orders, we obtain by two-fold differentiation

$$\rho u_{tttt} = 3\eta\ W(I_{11},I_{21})\ \frac{1}{u_x^2}\ u_{xxttt} + u_{xttt}\ O(\eta) + u_{xxtt}\{O(\eta)$$

$$+ u_x^{-2}(W_1^{(k)}d_{k1} + 3\ W_2^{(k)}d_{k1}^{-1} + 2W_1^{(k)}d_{k2}) + u_x^{-2}[2W_{11}^{(k)}(d_{k1} - d_{k2})^2$$

$$+ 2W_{22}^{(k)}(d_{k2}^{-1} - d_{k1}^{-1})^2 + 2(W_{12}^{(k)} + W_{21}^{(k)})(d_{k1} - d_{k2})(d_{k2}^{-1} - d_{k1}^{-1})]\}$$

$$+ \cdots .$$

Here $W_{ij}^{(k)}$ stands for $\frac{\partial W_i^{(k)}}{\partial I_{jk}}$. For small η, the coefficient of u_{xxtt} is positive in particular if $W_1^{(k)}, W_2^{(k)}, W_{11}^{(k)}, W_{22}^{(k)}$ are positive and $(W_{12}^{(k)} + W_{21}^{(k)})^2 < 4\ W_{11}^{(k)}W_{22}^{(k)}$. This corresponds to inequality [1.33] in Leonov's paper.

f). <u>The models of Johnson and Segalman [13] and Chang, Bloch and Tschoegl [28].</u>

This model is described by the following system

$$\pi = \int_{-\infty}^{t} a(t-s)G(s,t)\gamma^{-1}(s)G^T(s,t)ds$$

$$\frac{\partial G}{\partial t} = -\alpha\gamma^{-1}(t)\dot{\gamma}(t)G$$

$$G(t,t) = id$$

$$\frac{\partial G}{\partial s} = \alpha\ G\ \gamma^{-1}(s)\dot{\gamma}(s).$$

The parameter α ranges between 0 and $^1/_2$. For our problem, γ is diagonal, whence $\gamma(t'),\gamma(t'')$ commute for any t',t''. The equations for G can therefore be solved as follows.

$$G(s,t) = \exp(-\alpha \int_s^t \gamma^{-1}(r)\dot{\gamma}(r)dr) = \gamma^{\alpha}(s)\gamma^{-\alpha}(t).$$

This leads to an equation very similar to the ones studied in part a, and the discussion follows closely the one given there. We leave the details to the reader. For α in the range $(0, 1/2)$, the coefficient of u_{xxtt} turns out to be positive. If α is allowed bigger than $1/2$, the type of the equation may change from hyperbolic to elliptic.

The model of Chang, Bloch and Tschoegl is, for this particular problem, equivalent to that of Johnson and Segalman.

g) <u>The model of Curtiss and Bird [8].</u>

This model proposes the following constitutive law

$$\pi = -\int_{-\infty}^{t} \int_{\Omega_{\gamma(t)}} a(t-s)[1 + v^T(\gamma(s)-\gamma(t))v]^{-3/2} \frac{vv^T}{\sqrt{v^T\gamma^2(t)v}} dv.$$

$$- \eta \int_{-\infty}^{t} \int_{\Omega_{\gamma(t)}} b(t-s)[1 + v^T(\gamma(s)-\gamma(t))v]^{-3/2} v^T\dot{\gamma}(t)v \cdot \frac{vv^T}{\sqrt{v^T\gamma^2(t)v}} dv.$$

Here $\Omega_{\gamma(t)}$ is the set $\Omega_{\gamma(t)} = \{v | v^T\gamma(t)v = 1\}$. With Ω denoting the unit sphere, this yields for our problem

$$\pi^{11} = -\int_{-\infty}^{t}\int_{\Omega} a(t-s)[1 + (u_x^{-2}(t)u_x^2(s)-1)w_1^2 + (u_x(t)u_x^{-1}(s)-1) \cdot$$
$$(w_2^2 + w_3^2)]^{-3/2} w_1^2u_x^{-2}(t)dw$$

$$- \eta \int_{-\infty}^{t}\int_{\Omega} b(t-s)[1 + (u_x^{-2}(t)u_x^2(s)-1)w_1^2 + (u_x(t)u_x^{-1}(s)-1) \cdot$$
$$(w_2^2+w_3^2)]^{-3/2} w_1^2u_x^{-3}(t)\dot{u}_x(t)(2w_1^2-w_2^2-w_3^2)dw.$$

For π^{22}, we have the same exprssion with $w_1^2u_x^{-2}$ replaced by $w_2^2u_x$. When inserting this into the dynamic equation (4.1), we can again achieve the form (3.1) by differentiating with respect to time. The term involving u_{xxtt} has a positive coefficient proportional to η, the coefficient of u_{xxt} is, up to terms of $O(\eta)$

$$\int_{-\infty}^{t} \int_{\Omega} a(t-s)[\cdots]^{-3/2}(w_1^2-w_2^2)dw\; u_x^{-2}(t)$$

$$+ \frac{3}{2} \int_{-\infty}^{t} \int_{\Omega} a(t-s)[\cdots]^{-5/2}(-2u_x^{-4}(t)u_x^2(s)w_1^2 + u_x^{-1}(t)u_x^{-1}(s)(w_2^2+w_3^2) \cdot$$

$$\cdot (w_2^2-w_1^2)dw.$$

It can easily be checked that this coefficient is positive in a neighborhood of the rest state.

Remark:

When we differentiated equations with respect to time, we have always assumed that the integral kernels were sufficiently smooth. Some of the kernels suggested in the literature have singularities at $t = 0$ (see e.g. [8], where $a(t) = \sum_{\alpha \text{ odd}} e^{-\alpha^2 t}$). A mathematical theory accomodating such kernels would be of interest. Experimental data on polymer melts (see e.g. [28]) also seem to suggest that the integral kernel may be singular at $t = 0$.

REFERENCES

1. B. Bernstein, E. A. Kearsley and L. J. Zapas, A Study of Stress Relaxation with Finite Strain, Trans. Soc. Rheol. 7 (1963), 391-410.

2. R. B. Bird, et.al., Dynamics of Polymeric Liquids (2 vol.), J. Wiley, New York 1977.

3. R. B. Bird and P. J. Carreau, Chem. Eng. Sci. 23 (1968), 427-434.

4. P. J. Carreau, Rheological Equations from Molecular Network Theories, Trans. Soc. Rheol. 16 (1972), 99-128.

5. I. J. Chen and D. C. Bogue, Time-Dependent Stress in Polymer Melts and Review of Viscoelastic Theory, Trans. Soc. Rheol. 16 (1972), 59-78.

6. B. D. Coleman, M. E. Gurtin and J. Herrera, Waves in Materials with Memory, Arch. Rat. Mech. Anal. 19 (1965), 1-19 and 239-298.

7. B. D. Coleman and W. Noll, An Approximation Theorem for Functionals, with Applications in Continuum Mechanics, Arch. Rat. Mech. Anal. 6 (1960), 355-370.

8. C. F. Curtiss and R. B. Bird, A Kinetic Theory for Polymer Melts, J. Chem. Phys. 74 (1981), 2016-2033.

9. C. M. Dafermos and J. A. Nohel, A Nonlinear Hyperbolic Volterra Equation in Viscoelasticity, MRC TSR#2095, Univ. of Wisconsin-Madison 1980.

10. M. S. Green and A. V. Tobolsky, A New Approach to the Theory of Relaxing Polymeric Media, J. Chem. Phys. 14 (1946), 80-100.

11. J. K. Hale, Theory of Functional Differential Equations, Springer, Berlin-Heidelberg-New York 1977.

12. T. J. R. Hughes, T. Kato and J. E. Marsden, Well-posed Quasi-linear Second-order Hyperbolic Systems with Applications to Nonlinear Elastodynamics and General Relativity, Arch. Rat. Mech. Anal. 63 (1976), 273-284.

13. M. W. Johnson and D. Segalman, A Model for Viscoelastic Fluid Behaviour which Allows Non-affine Deformation, J. Non-Newtonian Fluid Mech. 2 (1977), 255-270.

14. T. Kato, Linear Evolution Equations of "Hyperbolic" Type II, J. Math. Soc. Japan 25 (1973), 648-666.

15. T. Kato, Quasi-linear Equations of Evolution with Application to Partial Differential Equations, in: Spectral Theory of Differential Equations, W. N. Everitt (ed.), Springer Lectures Notes in Math. 448 (1975), 25-70.

16. T. Kato, Linear and Quasilinear Equations of Evolution of Hyperbolic Type, C. I. M. E. 1976.

17. A. Kaye, Co A Note No. 134, The College of Aeronautics, Cranfield, Bletchley, England, 1962.

18. A. I. Leonov, Nonequilibrium Thermodynamics and Rheology of Viscoelastic Polymer Media, Rheol. Acta. 15 (1976), 85-98.

19. A. S. Lodge, Proc. Int. Congr. Rheol. 2nd, 1953 (V. G. W. Harrison, ed.) 229, Butterworth, London.

20. A. S. Lodge, A Network Theory of Flow Birefringence and Stress in Concentrated Polymer Solutions, Trans. Faraday Soc. 52 (1956), 120-130.

21. A. S. Lodge, Constitutive Equations from Molecular Network Theories for Polymer Solutions, Rheol. Acta 7 (1968), 379-392.

22. A. S. Lodge, Body Tensor Fields in Continuum Mechanics, Academic Press, New York - San Francisco - London 1974.

23. M. Renardy, A Semigroup Approach to Equations with Infinite Delay and Applications to a Problem in Viscoelasticity, MRC TSR#2287, Univ. of Wisconsin-Madison 1981 (to appear in J. Diff. Eq.).

24. J. C. Saut and D. D. Joseph, Fading Memory, in preparation.

25. P. E. Sobolevskii, Equations of Parabolic Type in a Banach Space, Amer. Math. Soc. Transl. 49 (1966), 1-62.

26. C. A. Truesdell and W. Noll, The Non-linear Field Theories of Mechanics, Handbuch der Physik III/3, Springer, Berlin-Heidelberg-New York 1965.

27. A. F. K. Ward and G. M. Jenkins, Normal Thrust in Dynamic Torsion for Rubberlike Materials, Rheol. Acta 1 (1958), 110-114.

28. W. V. Chang, R. Bloch and V. W. Tschoegl, On the Theory of Viscoelastic Behaviour of Soft Polymers in Moderately Large Deformations, Rheol. Acta 15 (1976), 367-378.

29. H. M. Laun, Description of the Non-linear Shear Behaviour of a Low Density Polyethylene Melt by Means of an Experimentally Determined Strain Dependent Memory Function, Rheol. Acta 17 (1978), 1-15.

30. M. Slemrod, A Hereditary Partial Differential Equation with Applications in the Theory of Simple Fluids, Arch. Rat. Mech. Anal. 62 (1976), 303-321.

31. E. F. Infante and J. A. Walker, A Stability Investigation for an Incompressible Simple Fluid with Fading Memory, Arch. Rat. Mech. Anal. 72 (1980), 203-218.

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We prove an existence theorem locally in time for quasilinear hyperbolic equations, in which the coefficients are allowed to depend on the history of the dependent variable. Singular perturbations, which change the type of the equation to parabolic, are included, and continuous dependence of the solutions on the perturbation parameter is shown. It is demonstrated that, for a substantial number of constitutive models suggested in the literature, the stretching of filaments of polymeric liquids is described by equations of the kind under study here.

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