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BIFURCATION AND OPTIMAL STOCHASTIC CONTROL.(U)
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BIFURCATION AND OPTIMAL STOCHASTIC CONTROL

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P. L. Lions*

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ABSTRACT

Two questions concerning bifurcation theory and optimal stochastic control are considered. First, in a few examples, we give the interpretation of a bifurcation in terms of optimal stochastic control. Next, we introduce the analogue of the lowest eigenvalue for the nonlinear operator associated with the Hamilton-Jacobi-Bellman equations of Optimal Stochastic Control.

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SIGNIFICANCE AND EXPLANATION

We consider here two questions related to bifurcation analysis in nonlinear elliptic equations. We first study the interpretation in terms of optimal stochastic control of a bifurcation in some particular nonlinear equation and we prove that, roughly speaking, it corresponds to the appearance of critical sensitivity of the cost function (naturally associated to the equation by classical Optimal Stochastic Control Theory). The second question is related to the study of spectral properties of the nonlinear operator arising in Optimal Stochastic Control: we prove that the nonlinearity produces two constants, that we call demi-eigenvalues, which play the same role as the first eigenvalue of a linear elliptic operator (namely existence of constant sign eigenfunctions, uniqueness properties, bifurcation analysis, etc.).

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BIFURCATION AND OPTIMAL STOCHASTIC CONTROL

P. L. LIONS*

Introduction:

We consider here two questions concerning Bifurcation Theory and Optimal Stochastic Control.

The first one concerns the interpretation in terms of Optimal Stochastic Control of a bifurcation (in semilinear second-order elliptic equations). Let us give a typical example: let $\mathcal{O}$ be a bounded, connected, smooth domain in $\mathbb{R}^N$. We consider nonnegative solutions of

(1) $$-\Delta u + \lambda u^p = \lambda u \quad \text{in} \quad \mathcal{O},\ u \in C^2(\overline{\mathcal{O}}),\ u \geq 0 \quad \text{in} \quad \mathcal{O},\ u = 0 \quad \text{on} \quad \partial\mathcal{O}$$

where $\lambda > 0$, $p > 1$.

It is well-known (see for example P. H. Rabinowitz [43], H. Berestycki [6], P. L. Lions [37]) that, if we denote by λ_1 the first eigenvalue of $-\Delta$ (with Dirichlet boundary conditions), we have:

i) for $0 < \lambda \leq \lambda_1$, the unique solution of (1) is $u \equiv 0$;

ii) *for* $\lambda > \lambda_1$, *there exist exactly two solutions of* (1): 0 *and* u_λ *where* $u_\lambda(x) > 0$ *in* $\mathcal{O}$.

In other words, at $\lambda = \lambda_1$, there is bifurcation of the curve (λ, u_λ) from the trivial branch of solutions $(\lambda, 0)$ (this is by the way an immediate consequence of the general result concerning bifurcation from a simple eigenvalue - see M. G. Crandall and P. H. Rabinowitz [4]).

To give a stochastic interpretation of the solutions of (1), we introduce the following Optimal Stochastic Control problems:

(2) $$u_\lambda^1(x) = \inf_{\xi(\cdot)\in K_0} E\left[\int_0^{\tau_x} \lambda(p-1)\xi(t,\omega)^p \exp\{\lambda t - \lambda \int_0^t p\xi(s,\omega)^{p-1} ds\} dt\right] ,$$

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(3) $$u_\lambda^2(x) = \inf_{\xi(\cdot)\in K_1} E[\int_0^{\tau_x} \lambda(p-1)\xi(t,\omega)^p \exp\{\lambda t - \lambda \int_0^t p\xi(s,\omega)^{p-1} ds\} dt$$

where (Ω, F, F_t, P) denotes a probability space with a right-continuous filtration of complete sub-σ-algebras of F - and E denotes the expectation -, where the state of the system is given by the process $x + B_t$ where $B_t/\sqrt{2}$ is a Brownian motion with respect to F_t, where $\xi(t,\omega)$ is the control required to be in K_0, K_1 which are detailed below and where τ_x is the first exit time of $x + B_t$ from $\bar{\mathcal{O}}$ (or $\mathcal{O}$). Finally K_0 (resp. K_1) is the set of bounded progressively measurable processes ξ such that: $0 \leq \xi$ a.e. in $\mathbb{R}_+ \times \Omega$ (resp. $0 \leq \delta \leq \xi$ a.e. in $\mathbb{R}_+ \times \Omega$, for some $\delta > 0$ depending on ξ).

Remark: These optimal stochastic control problems are explained below in more details. Let us point out for the moment that the quantity minimized in (2) or (3) is not necessarily finite but takes its values in $[0,+\infty]$, and that nevertheless u_λ^1, u_λ^2 are finite for every $x \in \bar{\mathcal{O}}$ and for every $\lambda > 0$.

Our main result on this simple example states that we have:

i) If $0 < \lambda \leq \lambda_1$, then $u_\lambda^1 \equiv u_\lambda^2 \equiv 0$;

ii) If $\lambda > \lambda_1$, then $\equiv u_\lambda^1 < u_\lambda^2 \equiv u_\lambda$ in $\mathcal{O}$.

Of course formally (1) is the Hamilton-Jacobi-Bellman equation associated with the control problems (2) or (3) (see W. H. Fleming and R. Rishel [20], A. Bensoussan and J. L. Lions [5], N. V. Krylov [26] for a general presentation of Hamilton-Jacobi-Bellman equations), thus it is natural that u_λ^1, u_λ^2 are solutions of (1). A more interesting phenomenon is that, since K_0 is in some sense the closure of K_1, when the bifurcation occurs, the cost function (that is the quantity minimized in (2) or (3)) becomes highly sensitive on the values of the control $\xi(t,\omega)$.

The second question that we consider below is concerned with the existence of analogues of eigenvalues and eigenfunctions for the nonlinear operator of Hamilton-Jacobi-Bellman equations namely:

$$A\varphi = \sup_{i \geq 1} (A_i\varphi) \quad , \quad \text{for } \varphi \in \mathcal{D}(\mathcal{O})$$

where $A_i = -a^i_{k\ell}(x)\,\partial^2_{k\ell} + b^i_k(x)\,\partial_k + c^i(x)$ (*) is a sequence of uniformly elliptic operators with smooth coefficients. We denote by $\lambda_1(A_i)$ the first eigenvalue of the operator A_i with Dirichlet boundary conditions (corresponding to a unique - up to a multiplicative constant - positive eigenfunction).

We introduce here two constants $\underline{\lambda}_1$, $\overline{\lambda}_1$ *such that:*

i) $\underline{\lambda}_1 \le \inf_{i\ge 1} \lambda_1(A_i) \le \sup_{i\ge 1} \lambda_1(A_i) \le \overline{\lambda}_1$

ii) *If* $\lambda < \underline{\lambda}_1$, *and if* $(f_i(x))_{i\ge 1}$ *is a sequence of smooth functions then there exists a unique solution* $u \in W^{2,\infty}(\mathcal{O})$ *of:*

(4) $$\sup_{i\ge 1} (A_i u - f_i) = \lambda u \quad \text{a.e. in } \mathcal{O}, \; u = 0 \text{ on } \partial\mathcal{O} .$$

iii) *If* $\lambda < \overline{\lambda}_1$, *and if* $(f_i(x))_{i\ge 1}$ *is a sequence of nonnegative smooth functions then there exists a unique nonnegative solution* $u \in W^{2,\infty}(\mathcal{O})$ *of* (4).

iv) *There exist* $\varphi_1, \psi_1 \in W^{2,\infty}(\mathcal{O})$ *satisfying:*

(5) $$A\varphi_1 = \sup_{i\ge 1} (A_i\varphi_1) = \underline{\lambda}_1\varphi_1 \quad \text{a.e. in } \mathcal{O}, \; \varphi_1 < 0 \text{ in } \mathcal{O}, \; \varphi_1 = 0 \text{ on } \partial\mathcal{O}$$

(6) $$A\psi_1 = \sup_{i\ge 1} (A_i\psi_1) = \overline{\lambda}_1\psi_1 \quad \text{a.e. in } \mathcal{O}, \; \xi_1 > 0 \text{ in } \mathcal{O}, \; \psi_1 = 0 \text{ on } \partial\mathcal{O} .$$

v) *Let* $(\varphi,\lambda) \in W^{2,\infty}(\mathcal{O}) \times \mathbb{R}$ *satisfy:*

$$\varphi = \lambda\varphi \quad \text{a.e. in } \mathcal{O}, \; \varphi = 0 \text{ on } \partial\mathcal{O} .$$

If $\varphi \le 0$ *then* $\lambda = \underline{\lambda}_1$ *and* $\varphi = \theta\varphi_1$ *for some* $\theta > 0$, *and if* $\varphi \ge 0$ *then* $\lambda = \overline{\lambda}_1$ *and* $\varphi = \theta\psi_1$ *for some* $\theta > 0$.

From this list of results, it is clear that $\underline{\lambda}_1$, $\overline{\lambda}_1$ play the role of eigenvalues and we call them *demi-eigenvalues* (in particular because of some relation with a result of H. Berestycki [7] concerning nonlinear Sturm-Liouville problems).

Let us also give a simple example showing the relevance of $\underline{\lambda}_1$, $\overline{\lambda}_1$ for bifurcation problems: consider the equation

(7) $$Au + \lambda|u|^{p-1}u = \lambda u \quad \text{a.e. in } \mathcal{O}, \; u \in W^{2,\infty}(\mathcal{O}), \; u = 0 \text{ on } \partial\mathcal{O} .$$

(*) In everything that follows, we use the implicit summation convention.

We prove by a simple application of the above results:

i) If $\lambda \leq \underline{\lambda}_1$, the unique solution of (7) is $u \equiv 0$,

ii) If $\underline{\lambda}_1 < \lambda \leq \overline{\lambda}_1$, the only solutions of (7) with constant sign are $u \equiv 0$ and $\underline{u}_\lambda$ where $\underline{u}_\lambda$ is a negative solution of (7).

iii) If $\lambda > \overline{\lambda}_1$, there are exactly three solutions of (7) with a constant sign namely: $u \equiv 0$, $\underline{u}_\lambda$ the negative solution of (7) and $\overline{u}_\lambda$ the positive solution of (7).

Finally let us mention that $\underline{\lambda}_1$, $\overline{\lambda}_1$ have very natural stochastic interpretations (in terms of Optimal Stochastic Control) and claims i) - v) above extend the results on the solvability of Hamilton-Jacobi-Bellman equations obtained by P. L. Lions [31], L. C. Evans and P. L. Lions [17], but heavily rely on these works (for the obtention of a priori estimates).

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I. Optimal stochastic control problems associated with bifurcations.

I.1. An example:

Let $\mathcal{O}$ be a bounded, connected, smooth domain in $\mathbb{R}^N$. Let us consider the following equation:

(1) $$-\Delta u + \lambda u^p = \lambda u \quad \text{in} \quad \mathcal{O},\ u \in C^2(\overline{\mathcal{O}}),\ u > 0 \quad \text{in} \quad \mathcal{O},\ u = 0 \quad \text{on} \quad \partial\mathcal{O} ,$$

where $\lambda > 0$, $p > 1$. We recall that for $\lambda \leq \lambda_1$ $(= \lambda_1(-\Delta))$ (1) has a unique solution $u \equiv 0$ and that for $\lambda > \lambda_1$, (1) has exactly two solutions: $u \equiv 0$ and u_λ which is the unique positive solution of (1) (see for example H. Berestycki [6], H. Amann and T. Laetsch [3], P. H. Rabinowitz [42]).

We now introduce the optimal stochastic control problems that are associated with (1). This is based upon the remark that (1) is equivalent to

(1') $$\begin{cases} \sup\limits_{0 \leq \xi} [-\Delta u + \lambda(p\xi^{p-1} - 1)u - \lambda(p-1)\xi^p] = 0 \quad \text{in} \quad \mathcal{O} ; \\ u > 0 \quad \text{in} \quad \mathcal{O},\ u \in C^2(\overline{\mathcal{O}}),\ u = 0 \quad \text{on} \quad \partial\mathcal{O} ; \end{cases}$$

indeed remark that if u solves (1), then: $0 \leq u \leq 1$ and (1') follows from the convexity of $(\xi \mapsto |\xi|^p)$. Next, if u solves (1'), then the supremum is obtained at each point x of $\mathcal{O}$ for $\xi = u(x)$ and this yields (1).

Let (Ω, F, F_t, P) be a probability space with a right-continuous filtration of complete sub-σ-algebras F_t of F and with some adapted Brownian motion $\tilde{B}_t$. We set $B_t = \sqrt{2}\,\tilde{B}_t$. The <u>state of the system</u> we want to control is given by $x + B_t$ (for $x = \overline{\mathcal{O}}$). Let ξ be a bounded nonnegative progressively measurable process that we will call the <u>control</u>, for such a control ξ we introduce the following cost function $J(x,\xi) \in [0,+\infty]$.

$$J(x,\xi) = E[\int_0^{\tau_x} \lambda(p-1)\xi^p(t,\omega)\exp\{+ \lambda t - \lambda p \int_0^t \xi^{p-1}(s,\omega)ds\}dt \quad ; \tag{8}$$

where τ_x is the first exit time of $x + B_t$ from $\overline{\mathcal{O}}$ (or $\mathcal{O}$).

Next, let K_0 be the set of bounded nonnegative progressively measurable processes ξ and let K_1 be the subset of K_0 consisting of processes ξ satisfying:

$$\xi(t,\omega) \geq \delta \text{ a.e. in } \mathbb{R}_+ \times \Omega$$

for some $\delta > 0$ depending eventually on ξ.

Then we introduce for all $x \in \overline{\mathcal{O}}$

$$u_\lambda^1(x) = \inf_{\xi \in K_0} J(x,\xi) \quad , \tag{2}$$

$$u_\lambda^2(x) = \inf_{\xi \in K_1} J(x,\xi) \quad . \tag{3}$$

Of course, $u_\lambda^i \geq 0$ in $\mathcal{O}$, $u_\lambda^i = 0$ on $\partial\mathcal{O}(i = 1,2)$ and in view of the heuristic dynamic programming principle (see [20] for example) one would expect that u_λ^i solve (1) (i = 1,2) - provided these functions are at least finite. In fact, we prove:

<u>Theorem I.1</u>:

i) <u>If</u> $0 < \lambda \leq \lambda_1$, <u>then</u> $u_\lambda^1(x) = u_\lambda^2(x) = 0$, <u>for all</u> $x \in \overline{\mathcal{O}}$.

ii) <u>If</u> $\lambda > \lambda_1$, <u>then</u> $u_\lambda^1(x) = 0$, $u_\lambda^2(x) = u_\lambda(x)$ <u>for all</u> $x \in \overline{\mathcal{O}}$ (<u>we recall that</u> u_λ <u>is the unique positive solution of</u> (1).

Remark I.1: As it will be clear from the proof, in the definition of u_λ^2 we may replace K_1 by

$$K_2 = \{\xi \in K_0,\ \forall\alpha > 0, \exists\delta > 0 \quad \xi(t\wedge\tau_x^\alpha(\omega),\omega) \geq \delta \text{ a.e. in } \mathbb{R}_+ \times \Omega\}$$

where τ_x^α is the first exit time of $x + B_t$ from $\mathcal{O}^\alpha = \{x \in \mathcal{O},\ \mathrm{dist}(x,\partial\mathcal{O}) > \alpha\}$.

We will give after the proof of Theorem I.1 a few remarks on the existence of optimal Markovian controls for (3) (0 is an optimal control for (2)) and on the Cauchy problem associated with (1).

Proof of Theorem I.1: Since $J(x,0) = 0$, it is clear that $u_\lambda^1 \equiv 0$, $\forall\lambda > 0$. Now, to prove i) we need to prove that $u_\lambda^2 \equiv 0$ if $\lambda \leq \lambda_1$. We recall first the well-known stochastic characterization of λ_1 (see for more general results P. L. Lions [32]).

Lemma I.1: The first eigenvalue λ_1 is given by:

$$\lambda_1 = \sup(\lambda > 0,\ \sup_{x\in\overline{\mathcal{O}}} E[e^{\lambda\tau_x}] < +\infty) \quad .$$

Now, if we take the constant control $\xi(t,\omega) = \varepsilon > 0$, we have:

$$J(x,\varepsilon) = \lambda(p-1)\varepsilon^p\, E \int_0^{\tau_x} \exp\{\lambda t\ (1 - p\varepsilon^{p-1})\}dt$$

and for ε small enough, we find:

$$J(x,\varepsilon) = \frac{(p-1)\varepsilon^p}{1-p\varepsilon^{p-1}}\, E[\exp(\lambda(1 - p\varepsilon^{p-1})\tau_x) - 1] \quad .$$

Now, in view of Lemma I.1, if $\lambda < \lambda_1$ the expectation is bounded independently of ε and we conclude since:

$$0 \leq u_\lambda^2(x) \leq J(x,\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0 \quad .$$

If $\lambda = \lambda_1$, we remark that by Ito's formula we have:

$$J(x,\varepsilon) = \lambda(p-1)\varepsilon^p v_\alpha(x)$$

where $\alpha = p\varepsilon^{p-1}$, and v_α is the solution of:

(9) $\qquad -\Delta v_\alpha = 1 + \lambda_1(1 - \alpha)v_\alpha$ in $\mathcal{O}$, $v_\alpha \in C^2(\overline{\mathcal{O}})$, $v_\alpha = 0$ on $\partial\mathcal{O}$.

But in view of the following Lemma, we conclude since we have

$$0 \leq u_\lambda^2(x) \leq J(x,\varepsilon) \leq C\varepsilon^p \|v_\alpha\|_{L^\infty(\mathcal{O})} \leq C\varepsilon \quad .$$

Lemma 1.2: Let v_α be the solution of (9); as α goes to 0, then αv_α converges in $C^2(\bar{\mathcal{O}})$ to $\theta\varphi_1$ where φ_1 is the normalized eigenfunction associated with λ_1:

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 \ \text{in} \ \mathcal{O}, \ \varphi_1 \in C^2(\bar{\mathcal{O}}), \ \varphi_1 > 0 \ \text{in} \ \mathcal{O}, \ \varphi_1 = 0 \ \text{on} \ \partial\mathcal{O} \\ \|\varphi_1\|_{L^2(\mathcal{O})} = 1 \ , \end{cases}$$

and where θ is given by: $\theta = \frac{1}{\lambda_1} \int_{\mathcal{O}} \varphi_1 dx$.

Proof of Lemma 1.2: If we denote by $w_\alpha = \alpha v_\alpha$, we have

$$-\Delta w_\alpha = \alpha + \lambda_1(1-\alpha)w_\alpha \ \text{in} \ \mathcal{O}, \ w_\alpha > 0 \ \text{in} \ \mathcal{O}, \ w_\alpha \in C^2(\bar{\mathcal{O}}), \ w_\alpha = 0 \ \text{on} \ \partial\mathcal{O} \quad .$$

Multiplying this equation by φ_1 and integrating by parts over $\mathcal{O}$, we find:

$$\lambda_1 \int_{\mathcal{O}} w_\alpha\varphi_1 \, dx = \alpha \int_{\mathcal{O}} \varphi_1 dx + \lambda_1(1-\alpha) \int_{\mathcal{O}} w_\alpha\varphi_1 dx$$

or

(10) $$\lambda_1 \int_{\mathcal{O}} w_\alpha\varphi_1 dx = \int_{\mathcal{O}} \varphi_1 dx \quad .$$

This proves in particular that w_α is bounded in $L^1_{loc}(\mathcal{O})$ and more precisely we have: $\int_{\mathcal{O}} w_\alpha\varphi_1 dx \leq C$ (indep. of α). We may now (for example) apply the method of H. Brezis and R. E. L. Turner [10] to obtain

$$\|w_\alpha\|_{L^\infty(\mathcal{O})} \leq C \quad (\text{indep. of } \alpha)$$

and by L^p and Schauder estimates this yields

$$\|w_\alpha\|_{C^{2,\beta}(\bar{\mathcal{O}})} \leq C \quad (0 < \beta < 1) \quad .$$

Now if w_α converges in $C^2(\bar{\mathcal{O}})$ to some w, obviously

$$-\Delta w = \lambda_1 w \ \text{in} \ \mathcal{O}, \ w \geq 0 \ \text{in} \ \bar{\mathcal{O}}, \ w = 0 \ \text{on} \ \partial\mathcal{O}, \ w \in C^2(\bar{\mathcal{O}}) \quad ;$$

and from (10) we deduce $\lambda_1 \int_{\mathcal{O}} w\varphi_1 dx = \int_{\mathcal{O}} \varphi_1 dx$. Therefore $w = \theta\varphi_1$ and $\theta = \frac{1}{\lambda_1} \int_{\mathcal{O}} \varphi_1 dx$.

We next turn to the proof of ii): we first prove that we have

$$u_\lambda(x) \leq u_\lambda^2(x) = \inf_{\xi \in K_1} J(x,\xi) \quad .$$

Indeed let $\xi \in K_1$ be such that $J(x,\xi) < +\infty$ (x is fixed in $\bar{\mathcal{O}}$). Because of the definition of K_1, this implies:

$$\int_0^\infty E[1_{(\tau_x > t)} \exp\{\lambda t - \lambda p \int_0^t \xi^{p-1}(s,\omega)ds\}]dt < +\infty \quad .$$

Therefore there exists $t_n \xrightarrow[n]{} +\infty$ such that:

$$E[1_{(\tau_x > t_n)} \exp\{\lambda t - \lambda p \int_0^{t_n} \xi^{p-1}(s,\omega)ds\}] \xrightarrow[n]{} 0 \quad .$$

We now apply Ito's formula to $u_\lambda(x + B_t)\exp\{\lambda t - \lambda p \int_0^t \xi^{p-1}(s,\omega)ds\}$ between 0 and $\tau_x \wedge t_n$, and we obtain

$$u_\lambda(x) = E[u_\lambda(x + B_{\tau_x \wedge t_n})\exp\{\lambda\tau_x \wedge t_n - \lambda p \int_0^{\tau_x \wedge t_n} \xi^{p-1}(s,\omega)ds\}]$$

$$+ E[\int_0^{\tau_x \wedge t_n} (\lambda p\, \xi^{p-1}(t,\omega)u_\lambda(x + B_t) - \lambda\, u_\lambda^p(x + B_t)) \cdot$$

$$\cdot \exp\{\lambda t - \lambda p \int_0^t \xi^{p-1}(s,\omega)ds\}dt] \quad .$$

But: $p\, \xi^{p-1}u_\lambda - u_\lambda^p \leq (p-1)\xi^p$, and we deduce

$$u_\lambda(x) \leq J(x,\xi) + E[u_\lambda(x + B_{\tau_x \wedge t_n})\exp(\lambda\tau_x \wedge t_n - \lambda p \int_0^{\tau_x \wedge t_n} \xi^{p-1}(s,\omega)ds)] \quad .$$

But the second term may be bounded by

$$\|u_\lambda\|_{L^\infty(\mathcal{O})} E[1_{(\tau_x > t_n)} \exp(\lambda t_n - \lambda p \int_0^{t_n} \xi^{p-1}(s,\omega)ds)]$$

and this goes to 0 as n goes to ∞.

Now, for α small enough, there exists u_λ^α solution of

$$-\Delta u_\lambda^\alpha + \lambda(u_\lambda^\alpha)^p = \lambda u_\lambda^\alpha \text{ in } \mathcal{O}^\alpha,\ u_\lambda^\alpha > 0 \text{ in } \mathcal{O}^\alpha,\ u_\lambda^\alpha = 0 \text{ on } \partial\mathcal{O}^\alpha$$

(indeed the first eigenvalue of $-\Delta$ in $\mathcal{O}^\alpha$ converges to λ_1 as α goes to 0). In addition it is easy to show that, extending u_λ^α to $\mathcal{O}$ by 0, we have:

$$u_\lambda^\alpha \xrightarrow[\alpha \to 0]{} u_\lambda \text{ in } C(\bar{\mathcal{O}}) \quad .$$

But the above proof shows that

$$u_\lambda^\alpha(x) \leq \inf_{\xi \in K_1} J^\alpha(x,\xi) \quad , \quad \forall x \in \mathcal{O}^\alpha$$

where $J^\alpha(x,\xi) = E[\int_0^{\tau_x^\alpha} \lambda(p-1)\xi^p(t,\omega)\exp\{\lambda t - \lambda p \int_0^t \xi^{p-1}(s,\omega)ds\}dt]$. Now if $\xi \in K_2$ then $\xi(t \wedge \tau_x^\alpha(\omega),\omega) \in K_1$ and we deduce from the above inequality:

$$u_\lambda^\alpha(x) \leq E[\int_0^{\tau_x^\alpha} \lambda(p-1)\xi^p(t,\omega)\exp\{\lambda t - \lambda p \int_0^t \xi^{p-1}(s,\omega)ds\}dt]$$

$$\leq J(x,\xi) \leq +\infty$$

for all $x \in \bar{\mathcal{O}}_\alpha$, on the other hand if $x \notin \bar{\mathcal{O}}_\alpha$, this inequality is trivially true. Thus, taking $\alpha \to 0$, we obtain:

$$u_\lambda(x) \leq \inf_{\xi \in K_2} J(x,\xi) \quad . \tag{11}$$

We now prove that:

$$u_\lambda(x) = J(x,\xi_x) = \inf_{\xi \in K_1} J(x,\xi) \tag{12}$$

where $\xi_x(t,\omega) = u_\lambda(x + B_{t\wedge\tau_x})$.

We first remark that $\lambda_1(-\Delta - \lambda + \lambda p u_\lambda^{p-1}) > 0$. Indeed from the equation (1), we deduce:

$$\lambda_1(-\Delta - \lambda + \lambda u_\lambda^{p-1}) = 0$$

and this yields the above inequality in view of well-known comparison principles for eigenvalues. But this implies (by an extension of Lemma I.1 which can be found for example in [32])

$$\sup_{\substack{x \in \bar{\mathcal{O}} \\ \theta \text{ stopping time}}} E[\exp(\delta\theta\wedge\tau_x + \lambda\theta\wedge\tau_x - \lambda p \int_0^{\theta\wedge\tau_x} u_\lambda^{p-1}(x + B_s)ds)] < \infty \tag{13}$$

for some $\delta > 0$.

Thus, applying Ito's formula to $u_\lambda(x + B_t)\exp(\lambda t - \lambda p \int_0^t \xi_x^{p-1}ds)$ between 0 and $T \wedge \tau_x$, we find:

$$u_\lambda(x) = E \int_0^{\tau_x \wedge T} \lambda(p-1)\xi_x^p \exp\{\lambda t - \lambda p \int_0^t \xi_x^{p-1} ds\} +$$

$$+ E[u_\lambda(x + B_T)1_{(T<\tau_x)} \exp\{\lambda T - \lambda p \int_0^T \xi_\lambda^{p-1} ds\}] \quad .$$

Therefore

$$0 \le J(x,\xi_x) - u_\lambda(x) \le C\, E \int_{\tau_x \wedge T}^{\tau_x} \exp(\lambda t - \lambda p \int_0^t \xi_x^{p-1} ds)dt +$$

$$+ C\, E[1_{(T<\tau_x)} \exp\{\lambda T - \lambda p \int_0^T \xi_\lambda^{p-1} ds\}]$$

and the first term may be bounded by:

$$C \int_T^\infty E[1_{(\tau_x > t)} \exp(\lambda t - \lambda p \int_0^t \xi_x^{p-1} ds)]dt \le C \int_T^\infty e^{-\delta t}\, dt = \frac{C}{\delta} e^{-\delta T} \quad \text{(because of (13))} \quad ;$$

while the second term is bounded by $C\, e^{-\delta T}$.

Letting $T \to \infty$ *we find:* $u_\lambda(x) = J(x,\xi_x)$, $\forall x \in \bar{\mathcal{O}}$.

Now to obtain (12), we argue as follows: let $\alpha > 0$, we introduce

$$\xi^\alpha(t,w) = \xi_x(t,\omega) \qquad \text{if} \quad t \le \tau_x^\alpha(\omega)$$

$$= \left(\frac{1}{p}\right)^{1/(p-1)} \qquad \text{if} \quad t > \tau_x^\alpha(\omega) \quad .$$

Obviously $\xi^\alpha \in K_1$, now by similar computations as above we show:

$$0 \le J(x,\xi^\alpha) - u_\lambda(x) \le E[\int_{\tau_x^\alpha \wedge T}^{\tau_x} C \exp[\lambda t - \lambda p \int_0^t (\xi^\alpha)^{p-1} ds]dt] \quad .$$

And this last term may be bounded by:

$$C\, E[(\tau \;\; - \tau_x^\alpha)\exp(\lambda\tau_x^\alpha - \lambda p \int_0^{\tau_x^\alpha} \xi_x^{p-1}\, ds)] +$$

$$+ C\, E[1_{(T<\tau_x^\alpha)} \int_T^{\tau_x^\alpha} \exp(\lambda t - \lambda p \int_0^t \xi_x^{p-1}\, ds)dt] \quad .$$

But we estimate the first term by

$$C\, E[|\tau_x - \tau_x^\alpha|^{q'}]^{1/q'} \quad E[\exp(\lambda q \tau_x^\alpha - \lambda pq \int_0^{\tau_x^\alpha} \xi_x^{p-1}\, ds)]^{1/q}$$

$q' = \frac{q}{q-1}$ and $q \in (1, +\infty)$ is determined such that:

$$E[\exp(\lambda q\, \tau_x^\alpha - \lambda pq \int_0^{\tau_x^\alpha} \xi_x^{p-1}\, ds)] \le C \quad \text{(ind. of } x, \alpha) \quad ,$$

this is possible because of (13), choosing $q - 1$ small enough. Now since $E[e^{\lambda\tau_x}] < \infty$ for $\lambda < \lambda_1$ and since $\tau_x^\alpha(\omega) \underset{\text{a.e.}}{\xrightarrow{\alpha\to 0}} \tau_x(\omega)$; we see that: $E[|\tau_x - \tau_x^\alpha|^{q'}]^{1/q'} \xrightarrow[\alpha\to 0]{} 0$.

Next, we estimate the second term by:

$$C\, E[1_{(T\le\tau_x)} \int_T^{\tau_x} \exp(\lambda t - \lambda p \int_0^t \xi_x^{p-1} ds)dt] \le C \int_T^\infty E[1_{(\tau_x > t)} \exp(\lambda t - \lambda p \int_0^t \xi_x^{p-1} ds)]dt$$

$$\le \frac{C}{\delta} e^{-\delta T} \quad \text{(in view of (13))} \quad .$$

And we have shown: $\lim_{\alpha\to 0} J(x,\xi^\alpha) = u_\lambda(x)$. This proves (12) and completes the proof of Theorem I.1.

Actually, we proved more than Theorem I.1, namely we prove the

Corollary I.1: *If* $\lambda > \lambda_1$, *then we have:*

$$u_\lambda(x) = \inf_{\xi\in K_1} J(x,\xi) = \inf_{\xi\in K_2} J(x,\xi) = J(x,\xi_x)$$

where ξ_x *is the optimal control given by:* $\xi_x(t,\omega) = u_\lambda(x + B_t(\omega))$.

Remark I.2: A feedback control like ξ_x is called a Markovian control; thus we proved the existence of an optimal Markovian control.

Remark I.3: We would like to make a few comments on the Cauchy problem associated with (1) namely:

$$(14)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \lambda u^p = \lambda u \quad \text{in } \mathcal{O}\times(0,+\infty),\ u \in C^2(\overline{\mathcal{O}}\times(0,+\infty)) \\ u(x,0) = u_0(x) \quad \text{in } \overline{\mathcal{O}},\ u \in C(\overline{\mathcal{O}}\times[0,+\infty)),\ u = 0 \text{ on } \partial\mathcal{O}\times[0,+\infty) \end{cases}$$

where $u_0 \in C_0(\overline{\mathcal{O}}) = \{v \in C(\overline{\mathcal{O}}),\ v = 0 \text{ on } \partial\mathcal{O}\}$ and $u_0 \ge 0$. It is well-known that there exists a unique solution of (14) and it is a simple exercise on Ito's formula to check that we have: $\forall x \in \overline{\mathcal{O}}$, $\forall t \ge 0$

$$u(x,t) = \inf_{\xi\in K_0} [J(x,t,\xi) + E[u_0(x + B_t)1_{(t<\tau_x)}\exp(\lambda t - \lambda p \int_0^t \xi^{p-1}(s,\omega)ds)]]$$

where $J(x,t,\xi) = E[\int_0^{\tau_x\wedge t} \lambda(p-1)\xi^p \exp(\lambda s - \lambda p \int_0^s \xi^{p-1} d\sigma)ds]$. Now, if $\lambda > \lambda_1$ and if $u_0 \not\equiv 0$, then it is well-known (see for example [6]) that $u(x,t) \xrightarrow[t\to\infty]{} u_\lambda(x)$ in $C^2(\overline{\mathcal{O}})$.

Therefore, in view of Theorem I.1, if $\xi \in K_0$, $\xi \notin K_1$ (or K_2)

$$\lim_{t\to\infty} E[u_0(x + B_t)1_{(t<\tau_x)} \exp(\lambda t - \lambda p \int_0^t \xi^{p-1}(s)ds)] + J(x,\xi) > \varphi_\lambda(x)$$

(indeed, formally, this is the case when ξ vanishes "a lot" and in this case the term $E[\exp(\lambda t - \lambda p \int_0^t \xi^{p-1}(s)ds)]$ becomes large.

Remark I.4: Everything we said in this section remains trivially valid if we replace $-\Delta$ by a general uniformly elliptic second-order operator

$$A = -a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$$

where $a_{ij} \in C(\bar{\mathcal{O}})$, b_i, $c \in L^\infty(\mathcal{O})$ and $c \geq 0$, $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$ $\forall x \in \bar{\mathcal{O}}$, $\forall \xi \in \mathbb{R}^N$ for some $\nu > 0$. Then we just have to replace $x + B_t$ by the diffusion process associated with A. We can treat as well Neumann boundary conditions or even more general ones as $\frac{\partial u}{\partial n} + \gamma(x)\, u = 0$ on $\partial\mathcal{O}$, where $\gamma \in C_+(\partial\mathcal{O})$, n is the unit outward normal to ∂ at the point x of $\partial\mathcal{O}$.

I.2 Interpretation of solutions of semilinear elliptic equations.

Our goals in this section are first to extend the results of the previous section and second to give a stochastic interpretation of some solutions of semilinear elliptic equations. But as we will speak here only of optimal stochastic control problems and not of differential games problems, the only nonlinearities which we can treat here are either convex of concave (we hope to come back on this point in a future study). To simplify we will look for solutions of the following three types of equations:

(15) $\quad -\Delta u + \lambda f(u) = \lambda u$ in $\mathcal{O}$, $u \in C^2(\bar{\mathcal{O}})$, $u > 0$ in $\mathcal{O}$, $u = 0$ on $\partial\mathcal{O}$,

where $\lambda > 0$, $f(0) = f'(0) = 0$, $f \in C^1(\mathbb{R})$, f is strictly convex and $\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$;

(16) $\quad -\Delta u = \lambda f(u)$ in $\mathcal{O}$, $u \in C^2(\bar{\mathcal{O}})$, $u > 0$ in $\mathcal{O}$, $u = 0$ on $\partial\mathcal{O}$

where $\lambda > 0$, $f(0) > 0$, $f \in C^1(\mathbb{R})$ and f satisfies

- either f is concave and $\lim_{t\to+\infty} f(t)t^{-1} \leq 0$
- or f is convex.

The first case (equation (15)) is very similar to the case treated in the preceding section. It is known (see [6], [3]) that for $\lambda \leq \lambda_1$ the only solution of (15) is $u \equiv 0$, while for $\lambda > \lambda_1$, there are exactly two solutions 0 and u_λ of (15) and u_λ is the unique positive solution of (15).

With the same notations as in the preceding section, we introduce: $\forall \xi \in K_0$

$$J(x,\xi) = E[\int_0^{\tau_x} \lambda\{f'(\xi(t,\omega))\xi(t,\omega) - f(\xi(t,\omega))\}\exp[\lambda t - \lambda \int_0^t f'(\xi(x,\omega))dt] \quad ;$$

since $f'(t) - f(t)t \geq 0$ for $t \geq 0$, we see that: $0 \leq J(x,\xi) \leq +\infty$.

Exactly as in the preceding section, we find:

<u>Theorem I.2</u>: <u>If</u> $f \in C^1(\mathbb{R})$, $f(0) = f'(0) = 0$, f <u>is strictly convex and</u> $\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty$ <u>then we have</u>:

i) <u>If</u> $0 < \lambda \leq \lambda_1$:

$$\inf_{\xi \in K_1} J(x,\xi) = \inf_{\xi \in K_0} J(x,\xi) = 0, \ \forall x \in \overline{\mathcal{O}} \ .$$

ii) <u>If</u> $\lambda > \lambda_1$:

$$\inf_{\xi \in K_0} J(x,\xi) = 0, \ \forall x \in \overline{\mathcal{O}}$$

$$\inf_{\xi \in K_2} J(x,\xi) = \inf_{\xi \in K_1} J(x,\xi) = u_\lambda(x) = J(x,\xi_x), \ \forall x \in \overline{\mathcal{O}}$$

<u>where</u> ξ_x <u>is the optimal control given by</u>: $\xi_x(t,\omega) = u(x + B_t(\omega))$.

We skip the proof of this result since it is absolutely identical to the proof of Theorem I.1. Let us also mention that Remarks I.2-4 are still valid here.

We now turn to the case of (16) when f is assumed to satisfy:

(17) $$f \in C^1(\mathbb{R}),\ f(0) > 0,\ f \text{ is concave and } \lim_{t \to +\infty} f(t)t^{-1} < \frac{\lambda_1}{\lambda} \ .$$

Then it is well-known (see H. Berestycki [6], P. L. Lions [37], H. Amann and T. Laetsch [3]) that (16), in this case, has a unique solution u (which is positive in $\mathcal{O}$). We denote by $g(t) = f(t) - f(0)$ and by $h(t) = g(t) - g'(t)t$. We keep the notations of section I.1 and we introduce for $\xi \in K_0$ the following cost function:

$$J(x,\xi) = E[\int_0^{\tau_x} (\lambda f(0) + \lambda h(\xi(t,w)))\exp\{\lambda \int_0^t f'(\xi(s,w))ds\}dt] \quad ;$$

then by a proof identical to the proof of Theorem I.1 (even simpler) one finds:

Theorem I.3: Under assumption (17), we have:

$$u(x) = \inf_{\xi \in K_0} J(x,\xi) = J(x,\xi_x), \ \forall x \in \bar{\mathcal{O}} \quad ;$$

where ξ_x is the optimal control defined by: $\xi_x(t,w) = u(x + B_t(\omega))$.

Let us also mention that analogues of Remarks I.2-4 are still valid here.

We finally consider the case of (16) when f is assumed to satisfy:

(18) $$f \in C^1(\mathbb{R}),\ f(0) > 0,\ f'(0) \geq 0,\ f \text{ is convex} \quad .$$

Then it is well-known (see M. G. Crandall and P. H. Rabinowitz [12], I. M. Guelfand [23], D. D. Joseph and T. S. Lundgren [24], C. Bandle [4], F. Mignot and J. P. Puel [39], P. L. Lions [37]) that there exists a constant $\bar{\lambda} \in (0,+\infty]$ such that:

i) If $\lambda < \bar{\lambda}$, then (16) has a minimum positive solution u_λ. In addition, we have:

(19) $$\lambda_1(-\Delta - \lambda f'(u_\lambda)) > 0 \quad ;$$

ii) If $\lambda > \bar{\lambda}$, then (16) has no solution;

iii) If $\bar{\lambda} < \infty$, $\lambda = \bar{\lambda}$ and if (16) has a solution, then (16) has a unique positive solution $u_{\bar{\lambda}}$. In addition we have:

(20) $$\lambda_1(-\Delta - \bar{\lambda} f'(u_{\bar{\lambda}})) = 0,\ u_{\bar{\lambda}} = \lim_{\lambda \uparrow \bar{\lambda}} u_\lambda \quad .$$

In addition, in [12], [39] sufficient conditions are given insuring that $\bar{\lambda} < \infty$ and that (16) has a solution for $\lambda = \bar{\lambda}$. Finally let us mention that in [12] and in D. G. deFigueiredo, R. D. Nussbaum and P. L. Lions [18], [19], various conditions on f are given insuring the existence, for $\lambda < \bar{\lambda}$, of a solution different from u_λ. Nevertheless, u_λ is the only "stable" solution of (16) (the precise meaning of the stability is explained in H. Fujita [21], P. L. Lions [35], [36]).

We keep the notations of the preceding section and we consider again the set K_0 of bounded progressively measurable processes ξ such that:

$$\xi(t,\omega) \geq 0 \quad \text{a.e. in } \mathbf{R}_+ \times \Omega \ .$$

But, here in order to be able to define the cost function we have to restrict our controls ξ to the set (depending on x):

$$K_0^x = \{\xi \in K_0,\ E[\int_0^{\tau_x} \exp\{\int_0^t \lambda f'(\xi(s,\omega))ds\}dt] < \infty\} \ .$$

Then we define the cost function $J(x,\xi)$ for $\xi \in K_0^x$:

$$J(x,\xi) = E[\int_0^{\tau_x} \lambda\{f(\xi(t,\omega)) - f'(\xi(t,\omega))\xi(t,\omega)\}\exp\{\int_0^t \lambda f'(\xi(s,\omega))ds\}dt]$$

obviously $J(x,\xi)$ is well defined since ξ is bounded $(\xi \in K_0)$. Finally, we look for

$$u(x) = \sup_{\xi \in K_0^x} J(x,\xi) \ .$$

We have:

Theorem I.4: Under assumption (18) and if $\lambda < \bar{\lambda}$, then we have:

$$u_\lambda(x) = u(x) = \sup_{\xi \in K_0^x} J(x,\xi) = J(x,\xi_x),\ \forall x \in \bar{\mathcal{O}}$$

where ξ_x is the optimal control (in K_0^x) defined by: $\xi_x(t,\omega) = u_\lambda(x + B_t(\omega))$. In addition if $\bar{\lambda} < \infty$, and if (16) has a solution for $\lambda = \bar{\lambda}$ then we have:

$$u_{\bar{\lambda}}(x) = u(x) = \sup_{\xi \in K_0^x} J(x,\xi) = \lim_{\lambda \uparrow \bar{\lambda}} J(x,\xi_x^\lambda),\ \forall x \in \bar{\mathcal{O}}$$

where ξ_x^λ is the control (in K_0^x) defined by: $\xi_x^\lambda(t,\omega) = u_\lambda(x + B_t(\omega))$.

We see that if $\lambda < \bar{\lambda}$, ξ_x is an optimal Markovian control, while for $\lambda = \bar{\lambda}$ (if (16) has a solution) then ξ_x^λ define the so-called ϵ-optimal Markovian control. Let us also mention that analogues of Remarks I.2-4 hold here.

Proof of Theorem I.4: We first show that: $u_\lambda(x) \geq u(x)$, $\forall x \in \bar{\mathcal{O}}$. Indeed let $\xi \in K_0^x$, just as in the proof of Theorem I.1, there exists $t_n \underset{n}{\to} \infty$ such that:

$$E[1_{(\tau_x > t_n)} \exp\{\int_0^{t_n} \lambda f'(\xi(s,\omega))ds\}] \underset{n\to\infty}{\to} 0 \ .$$

Now, applying Ito's formula to $u_\lambda(x + B_t)\exp(\int_0^t \lambda f'(\xi)ds)$ between 0 and $\tau_x \wedge t_n$, we find:

$$u_\lambda(x) = E[u_\lambda(x + B_{t_n})\, 1_{(\tau_x > t_n)} \exp\{\int_0^{t_n} \lambda f'(\xi)ds\}] +$$

$$+ E[\int_0^{t_n \wedge \tau_x} \{\lambda f(u_\lambda(x + B_s)) - \lambda f'(\xi(s))u_\lambda(x + B_s)\} \cdot \exp\{\int_0^s \lambda f'(\xi)d\sigma\}ds] \quad .$$

But the first term goes to 0, because of the choice of t_n. And the second term is larger than:

$$E[\int_0^{t_n \wedge \tau_x} (\lambda f(\xi(s)) - \lambda f'(\xi(s))\xi(s))\exp\{\int_0^s \lambda f'(\xi)d\sigma\}ds]$$

and this quantity converges, as $n \to \infty$, to $J(x,\xi)$ since $\xi \in K_0^x$.

Next, we define $\xi_x(t,\omega) = u_\lambda(x + B_t(\omega))$; since we have:

$$\lambda_1(-\Delta - \lambda f'(u_\lambda)) > 0 \quad ,$$

a trivial application of Ito's formula shows i) that $\xi_x \in K_0^x$, and ii) that $u_\lambda(x) = J(x,\xi_x)$. This proves the case $\lambda < \bar\lambda$ of Theorem I.4.

Now if $\lambda = \bar\lambda < \infty$ and if $u_{\bar\lambda}$ exists, then clearly:

$$u_{\bar\lambda}(x) = \lim_{\lambda \uparrow \bar\lambda} J(x,\xi_x^\lambda) = \lim_{\lambda \uparrow \bar\lambda} u_\lambda(x)$$

and $\xi_x^\lambda \in K_0^x$ (for $\lambda = \bar\lambda$) since $\lambda_1(-\Delta - \bar\lambda f'(u_\lambda)) > \lambda_1(-\Delta - \bar\lambda f'(u_{\bar\lambda})) = 0$. Thus:

$$u_{\bar\lambda}(x) \leq u(x), \ \forall x \in \mathcal{O} \ .$$

But on the other hand, if $\xi \in K_0^x$ for $\lambda = \bar\lambda$ then $\xi \in K_0^x$ for every $\lambda < \bar\lambda$ thus:

$$J(x,\xi) \leq \lim_{\lambda \uparrow \bar\lambda} u_\lambda(x) = u_{\bar\lambda}(x) \quad \text{and} \quad u(x) = u_{\bar\lambda}(x) \quad ,$$

and this completes the proof of Theorem I.4.

We would like to conclude this section by a *general* remark: we have seen in the three above cases a stochastic interpretation of some solution of semilinear elliptic equations. It seems that, in general, one can give a stochastic interpretation of stable solutions of semilinear elliptic equations (in particular if one has $\lambda_1(-\Delta - f'(u)) > 0$ for the solution u).

II. Semi-eigenvalues for the Hamilton-Jacobi-Bellman operator.

II.1. Notations and assumptions.

Let $\mathcal{O}$ be a bounded, connected, smooth domain in $\mathbf{R}^N$. Let $(A_i)_{i\geq 1}$ be a sequence of uniformly elliptic second-order operators:

$$A_i = -a^i_{k\ell}(x)\partial^2_{k\ell} + b^i_k(x)\partial_k + c^i(x)$$

where $a^i_{k\ell}$, b^i_k, c^i satisfy:

(21) $$\sup_{\substack{i\geq i\\ k,\ell}}\{\|a^i_{k\ell}\|_{W^{2,\infty}(\mathcal{O})} + \|b^i_k\|_{W^{2,\infty}(\mathcal{O})} + \|c^i\|_{W^{2,\infty}(\mathcal{O})}\} < \infty \ ;$$

(22) $$\exists\, \nu > 0,\ \forall i \geq 1,\ \forall x \in \overline{\mathcal{O}},\ \forall \xi \in \mathbf{R}^N:\ a^i_{k\ell}(x)\xi_\ell,\xi_\ell \geq \nu|\xi|^2 \ .$$

We will be concerned with the following type of equations:

(23) $$\sup_{i\geq 1}\{A_i u - f_i\} = 0 \ \text{ a.e. in } \mathcal{O},\ u \in W^{2,\infty}(\mathcal{O}),\ u = 0 \ \text{ on } \partial\mathcal{O} \ ;$$

where (f_i) are given functions satisfying:

(24) $$\sup_{i\geq 1}\|f_i\|_{W^{2,\infty}(\mathcal{O})} < +\infty \ .$$

This problem arises in connection with the general problem of Optimal Control of solutions of stochastic differential equations via the argument of Dynamic Programming: these equations are known as Hamilton-Jacobi-Bellman equations.

Let us briefly describe the associated Optimal Stochastic Control problem: we define an admissible system A as the collection of i) a probability space (Ω, F, F_t, P) with a right continuous filtration of complete sub σ-algebras F_t of F, ii) a Brownian motion B_t adapted to F_t, iii) a bounded progressively measurable process $i(t,\omega)$ with values in $\mathbf{N}^*$, iv) a family $(y_x(t))_{x\in\overline{\mathcal{O}}}$ of solutions of the equation:

(25) $$\begin{cases} dy_x(t) = \sigma^{i(t)}(y_x(t))dB_t - b^{i(t)}(y_x(t))dt,\ t \geq 0 \\ y_x = x \ ; \end{cases}$$

where $\sigma^i(x) = \sqrt{2}\,(a^i(x))^{1/2}$ (for example). For each admissible system A, we define a cost function $J(x,A)$

$$J(x,A) = E\int_0^{\tau_x} f_{i(t)}(y_x(t))\exp\{-\int_0^t c^{i(s)}(y_x(s))ds\}dt$$

where τ_x is the first exit time from $\bar{\mathcal{O}}$ of the process $(y_x(t))$. Finally, we minimize $J(x,A)$ overall admissible systems A:

$$u(x) = \inf_A J(x,A) \quad . \tag{26}$$

Let us recall briefly a few known results: 1) If $c^i(x) \geq 0$ ($\forall x \in \bar{\mathcal{O}}$, $\forall i \geq 1$) and if there exists $u \in W^{2,\infty}(\mathcal{O})$ solution of (23) then u is given by (26) (and in addition one can define ε-optimal Markovian controls); 2) If $c^i(x) \geq 0$ and u given by (26) belongs to $W^{2,\infty}(\mathcal{O})$ then u solves (23) - for the proofs of these two facts, see N. V. Krylov [26], M. Nisio [40], A. Bensoussan and J. L. Lions [5]; 3) <u>If</u> $c^i(x) \geq 0$, <u>then there exists</u> $u \in W^{2,\infty}(\mathcal{O})$ <u>solution of</u> (23) <u>and thus</u> u <u>is given by</u> (26) - see P. L. Lions [31], L. C. Evans and P. L. Lions [17] for the proof of this result; in N. V. Krylov [25], H. Brezis and L. C. Evans [9], P. L. Lions [29], L. C. Evans and A. Friedman [16], P. L. Lions and J. L. Menaldi [38] some previous results concerning the solution of (23) were obtained. Finally let us mention that the most general results concerning the solution of (23) are given in P. L. Lions [30], [33], [34] - including the case when the operators A_i degenerate.

<u>Remark II.1</u>: In these references, sometimes, instead of "control" processes $i(t,\omega)$ with values in $\mathbb{N}^*$, are taken controls $v(t,\omega)$ with values in a closed set of $\mathbb{R}^m$ (for example). In this case the only additional assumption is that $a(x,v)$, $b(x,v)$, $c(x,v)$, $f(x,v)$ are continuous with respect to v and everything we say below remains valid in this case (remark that by taking a dense family $(v_i)_{i \geq 1}$ in V, one can reduce this case to the preceding one).

Before concluding this section, we want to mention the method of proof used in [31], [17] in order to solve (23): one considers the following penalized system (for each $m \geq 1$ fixed):

$$
(25)\quad \begin{cases}
A_1 u_\varepsilon^1 + \beta_2(u_\varepsilon^1 - u_\varepsilon^2) = f_1 \quad \text{in} \quad \mathcal{O},\ u_\varepsilon^1 \in C^2(\overline{\mathcal{O}}),\ u_\varepsilon^1 = 0 \ \text{on} \ \partial\mathcal{O}\ , \\
A_2 u_\varepsilon^2 + \beta_\varepsilon(u_\varepsilon^2 - u_\varepsilon^3) = f_2 \quad \text{in} \quad \mathcal{O},\ u_\varepsilon^2 \in C^2(\overline{\mathcal{O}}),\ u_\varepsilon^2 = 0 \ \text{on} \ \partial\mathcal{O}\ , \\
\cdot \\
\cdot \\
\cdot \\
A_m u_\varepsilon^m + \beta_\varepsilon(u_\varepsilon^m - u_\varepsilon^1) = f_m \quad \text{in} \quad \mathcal{O},\ u_\varepsilon^m \in C^2(\overline{\mathcal{O}}),\ u_\varepsilon^m = 0 \ \text{on} \ \partial\mathcal{O}\ ;
\end{cases}
$$

where $\beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta(t)$, $\beta \in C^\infty(\mathbb{R})$, $\beta(t) = 0$ if $t \leq 0$, $\beta'(t) > 0$ if $t > 0$, $\beta''(t) > 0$ if $t > 0$.

In [31], [17]; it is proved that if $c^i > 0$ ($\forall i \geq 1$) and if we assume (21), (22), (24) then: $\|u_\varepsilon^i\|_{W^{2,\infty}(O)} \leq C$ (indep. of m and of ε); and, as ε goes to 0, $u_\varepsilon^i \underset{C^1(\overline{\mathcal{O}})}{\to} u_m \in W^{2,\infty}(\mathcal{O})$ which is the solution of

(23-m) $\quad \sup\limits_{1\leq i\leq m} (A_i u_m - f_i) = 0$ a.e. in $\mathcal{O}$, $u_m \in W^{2,\infty}(\mathcal{O})$, $u_m = 0$ on $\partial\mathcal{O}$.

Since, u_m is bounded in $W^{2,\infty}(\mathcal{O})$, taking $m \to \infty$, one obtains the solution $u = \lim\limits_m u_m$ of (23).

<u>Remark II.2</u>: But, an easy examination of the proof of [31], [17] shows that if for ε small enough there exist $(u_\varepsilon^i)_{1\leq i\leq m}$ solution of (25), if $\|u_\varepsilon^i\|_{L^\infty(O)} \leq C$ (indep. of ε, i, m) and if we assume (21), (22) and

$$
(24')\quad \begin{cases}
\sup\limits_{i\geq 1} \|f_i\|_{W^{1,\infty}(\mathcal{O})} < \infty \\
\exists C > 0,\ \forall \chi \in \mathbb{R}^N\ |\chi| = 1\ ,\ \dfrac{\partial^\varepsilon f_i}{\partial \chi^2} \leq C \ \text{in} \ \mathcal{D}'(\mathcal{O})\ (\forall i \geq 1)\ ,
\end{cases}
$$

then we have: $\|u_\varepsilon^1\|_{W^{2,\infty}(\mathcal{O})} \leq C$ (indep. of ε, i, m) and we obtain in this case, as in [31], [17], a solution of (23-m) for all m and a solution of (23). We will use this remark many times in what follows.

Remark II.3: In L. C. Evans [14], [15], it is proved that if (21), (22) and (24) are satisfied then any solution u of (23) satisfies: $u \in C^{2,\alpha}(\mathcal{O})$ for some $0 < \alpha < 1$ (depending only on and the bounds in (21), (22) and (24)).

In what follows, we show in section II.2 that roughly speaking, if (23) possesses a pair of ordered sub and supersolutions, then there exists a solution of (23) between these two functions. This auxiliary result will be one of the key ingredient which enables us to prove in section *II.3 the existence of demi-eigenvalues such as indicated in the* Introduction. Finally in section II.4, we present various applications and comments on these demi-eigenvalues.

II.2 An auxiliary result:

Let $f_i(x;t)$ be given functions on $\bar{\mathcal{O}} \times \mathbb{R}$ satisfying:

$$\sup_{i \geq 1} \|f_i(x,t)\|_{W^{2,\infty}(\mathcal{O}\times B_R)} < +\infty \ , \quad \forall R < \infty \ ; \tag{26}$$

where $B_R = \{\xi \in \mathbb{R}, \ |\xi| < R\}$.

We will study in this section the equation:

$$\left\{\begin{array}{l} \sup_{i \geq 1} \{A_i u(x) - f_i(x,u(x))\} = 0 \quad \text{a.e. in } \mathcal{O} \\ \\ \qquad u \in W^{2,\infty}(\mathcal{O}), \ u = 0 \ \text{ on } \partial\mathcal{O} \ . \end{array}\right. \tag{23'}$$

We will assume that there exist $\underline{u}, \bar{u}$ respectively subsolution and supersolution of (23') that is satisfying:

$$\left\{\begin{array}{l} \sup_{i \geq 1} \{A_i \underline{u}(x) - f_i(x,\underline{u}(x))\} \leq 0 \quad \text{a.e. in } \mathcal{O} \\ \\ \sup_{i \geq 1} \{A_i \bar{u}(x) - f_i(x,\bar{u}(x))\} \geq 0 \quad \text{a.e. in } \mathcal{O} \\ \\ \underline{u}, \bar{u} \in W^{2,\infty}(\mathcal{O}), \ \underline{u} \leq 0 \leq \bar{u} \ \text{ on } \partial\mathcal{O}, \ \underline{u} \leq \bar{u} \ \text{ in } \bar{\mathcal{O}} \ . \end{array}\right. \tag{27}$$

Then we have the

Theorem II.1: Under assuptions (21), (22), (26) and if there exist $\underline{u}, \bar{u}$ satisfying (27); then there exists u solution of (23') which satisfies in addition $\underline{u} \leq u \leq \bar{u}$ in $\bar{\mathcal{O}}$.

This result will be very useful in the following sections and of course is an extension of the well-known result corresponding to the case $A_i = A (\forall i \geq 1)$, $f_i = f (\forall i \geq 1)$ (see H. Amann [1], [2] for this special case).

<u>Proof of Theorem II.1</u>: We first make some preliminary substitutions: there exists $K > 0$ such that: $c^i(x) + K \geq 1$ in $\bar{\mathcal{O}}(\forall i)$ and $f_i(x,t) + Kt$ is increasing for $t \in [-C_0, C_0]$ (where C_0 is some fixed constant larger than $\max(\|\underline{u}\|_{L^\infty(\mathcal{O})}, \|\bar{u}\|_{L^\infty(\mathcal{O})})$. We will denote by $\tilde{f}_i(x,t)$ the functions defined by:

$$\tilde{f}_i(x,t) = f_i(x, t \wedge \bar{u}(x)) + K(t \wedge \bar{u}(x)) \quad ;$$

and we denote by $\tilde{A}_i$ the operator:

$$\tilde{A}_i = -a^i_{k\ell}(x)\partial^2_{k\ell} + b^i_k(x)\partial_k + (c^i(x) + K) \quad .$$

We first claim that it is enough to show the existence of u satisfying:

$$(23'')\quad \begin{cases} \sup\limits_{i \geq 1}\{\tilde{A}_i u(x) - \tilde{f}_i(x, u(x))\} = 0 \quad \text{a.e. in } \mathcal{O} \\ u \in W^{2,\infty}(\mathcal{O}),\ u = 0 \ \text{ on } \partial\mathcal{O},\ u \geq \underline{u} \ \text{ in } \bar{\mathcal{O}} \quad . \end{cases}$$

Indeed if this is the case, remarking that we have:

$$\tilde{f}_i(x, u(x)) \leq f_i(x, \bar{u}(x)) + K\bar{u}(x) \quad \text{in } \bar{\mathcal{O}}$$

we deduce from the definition of $\bar{u}$:

$$\begin{cases} \sup\limits_{i \geq 1}\{\tilde{A}_i \bar{u}(x) - \tilde{f}_i(x, \bar{u}(x))\} \geq 0 \quad \text{a.e. in } \mathcal{O} \\ \bar{u} \in W^{2,\infty}(\mathcal{O}),\ \bar{u} \geq 0 \ \text{ on } \partial\mathcal{O} \quad . \end{cases}$$

and we conclude: $u \leq \bar{u}$ in $\bar{\mathcal{O}}$, from the following lemma:

<u>Lemma II.1</u>: <u>We assume</u> (21), (22) <u>and let</u> $(f_i)_{i \geq 1}(g_i)_{i \geq 1}$ <u>be two sequences of functions satisfying</u>:

$$\sup_{i \geq 1}\{\|f_i\|_{L^\infty(\mathcal{O})} + \|g_i\|_{L^\infty(\mathcal{O})}\} < \infty;\ f_i \geq g_i \ \underline{\text{a.e. in}}\ \mathcal{O},\ \forall i \geq 1 \quad .$$

<u>We assume there exist</u> $u, v \in W^{2,\infty}(\mathcal{O})$ <u>satisfying</u>

$$\sup_{i \geq 1}(A_i u - f_i) \geq \ \underline{\text{a.e. in}}\ \mathcal{O},\ \sup_{i \geq 1}(A_i v - g_i) \leq 0 \ \underline{\text{a.e. in}}\ \mathcal{O}$$

$$u \geq v \ \underline{\text{on}}\ \partial\mathcal{O} \quad .$$

i) <u>If</u> $u \geq v$ <u>in</u> $\bar{\mathcal{O}}$, <u>then either</u> $u \equiv v$ <u>in</u> $\mathcal{O}$ <u>or</u> $u(x) > v(x)$ <u>in</u> $\mathcal{O}$ <u>and</u> $\frac{\partial u}{\partial n} < \frac{\partial v}{\partial n}$ <u>on</u> $\partial\mathcal{O}$ (n <u>denotes the unit outward normal</u>).

ii) <u>If</u> $c^i(x) \geq 0$ ($\forall i \geq 1$, $\forall x \in \bar{O}$) <u>then</u>: $u \geq v$ <u>in</u> $\bar{O}$.

(The proof of this Lemma will be given later on.)

Next, to prove the existence of u satisfying (23"), we argue as follows: for each m fixed, we consider the penalized system:

$$(25')\quad \begin{cases} \tilde{A}_1 u_\varepsilon^1 + \beta_\varepsilon(u_\varepsilon^1 - u_\varepsilon^2) = \tilde{f}_1(x,u_\varepsilon^1) & \text{in } \mathcal{O},\ u_\varepsilon^1 \in C^2(\bar{\mathcal{O}}),\ u_\varepsilon^1 = 0 \text{ on } \partial\mathcal{O} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{A}_m u_\varepsilon^m + \beta_\varepsilon(u_\varepsilon^m - u_\varepsilon^1) = \tilde{f}_m(x,u_\varepsilon^m) & \text{in } \mathcal{O},\ u_\varepsilon^m \in C^2(\bar{\mathcal{O}}),\ u_\varepsilon^m = 0 \text{ on } \partial\mathcal{O} \end{cases}.$$

Obviously $\underline{u}$ is a subsolution of (25') since:

$$\forall i \geq 1,\ \tilde{A}_i \underline{u} \leq \tilde{f}_i(x,\underline{u}(x)) \quad \text{in } \mathcal{O},\ \underline{u} \in W^{2,\infty}(\mathcal{O}),\ \underline{u} \leq 0 \quad \text{on } \partial\mathcal{O} .$$

Next, recall that in view of the results of L. C. Evans and A. Friedman [16] if $g_1,\dots,g_m \in C(\bar{\mathcal{O}})$, there exists a unique solution $U = KF$ of:

$$(25")\quad \begin{cases} \tilde{A}_1 u^m + \beta_\varepsilon(u^1 - u^2) = f_1 & \text{in } \mathcal{O},\ u^1 \in W^{2,p}(\mathcal{O})(p < \infty),\ u^1 = 0 \text{ on } \partial\mathcal{O} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{A}_m u^m + \beta_\varepsilon(u^m - u^1) = f_m & \text{in } \mathcal{O},\ u^m \in W^{2,p}(\mathcal{O})(p < \infty),\ u^m = 0 \text{ on } \partial\mathcal{O} \end{cases},$$

where $U = (u^1,\dots,u^m)$, $F = (f^1,\dots,f^m)$. In addition K is a compact mapping from $C(\bar{\mathcal{O}})$ into $C(\bar{\mathcal{O}})$ and: $KF^1 \geq KF^2$ if $F^1 \geq F^2$ (where $F^1 \geq F^2$ means $f_i^1 \geq f_i^2$, for $1 \leq i \leq m$). Now if $v \in C(\bar{\mathcal{O}})$, $v \geq \underline{u}$ in $\bar{\mathcal{O}}$ then $\|\tilde{f}_i(x,v)\|_{L^\infty(\mathcal{O})} \leq C$ (indep. of ε, i, m) and thus (see for example [16], [38])

$$\sup_{1 \leq i \leq m} \|K_i(v)\|_{L^\infty(\mathcal{O})} \leq C_1 (\text{indep. of } \varepsilon, i, m)$$

where $K_i(v)$ is the solution of (25") corresponding to $g_i = \tilde{f}_i(x,v)$. Then if $C = \{v \in C(\bar{\mathcal{O}}),\ v \geq \underline{u} \text{ in } \bar{\mathcal{O}},\ \|v\|_{L^\infty(\mathcal{O})} \leq C_1\}$, the map $(K_1(v),\dots,K_m(v))$ is a compact continuous map from the convex set $\mathcal{C}$ into $\mathcal{C}$ and thus by Schauder fixed point theorem, there exist $(u^1_\varepsilon,\dots,u^m_\varepsilon)$ solution of (25') and in addition we have:

$$u^i_\varepsilon \geq \underline{u} \quad \text{in } \overline{\mathcal{O}}, \quad \|u^i_\varepsilon\|_{L^\infty(\mathcal{O})} \leq C_1 \quad .$$

But this last bound enables us to obtain the following estimate by the same method as mentioned in Remark II.2:

$$\|u^i_\varepsilon\|_{W^{2,\infty}(\mathcal{O})} \leq C \quad (\text{indep. of } \varepsilon, i, m) \quad .$$

And passing to the limit $(\varepsilon \to 0, m \to \infty)$ exactly as in [16], [31] we prove the existence of u satisfying (23"). And this completes the proof of Theorem II.1.

Proof of Lemma II.1: We will only prove part i) of Lemma II.1 since part ii) is obtained by exactly the same argument as in the proof of uniqueness in P. L. Lions [31].

Now, to prove i), we first claim that we may assume without loss of generality that $c^i(x) \geq \alpha > 0$ (for some $\alpha > 0$). Indeed we have for all $\mu > 0$:

$$\sup_{i\geq 1}(A_i u + \mu u - (f_i + \mu u)) \geq 0 \quad \text{a.e. in } \mathcal{O}$$

$$\sup_{i\geq 1}(A_i v + \mu v - (g_i + \mu v)) \leq 0 \quad \text{a.e. in } \mathcal{O}$$

and choosing μ large enough, we are done. Thus we assume: $c^i(x) \geq \alpha > 0$ $(\forall i \geq 1, \forall x \in \overline{\mathcal{O}})$.

Next, we remark that we have:

$$\sup_{i\geq 1}(A_i(u - v)) \geq 0 \quad \text{a.e. in } \mathcal{O}, \; u-v \in W^{2,\infty}(\mathcal{O}), \; u-v \geq 0 \quad \text{in } \overline{\mathcal{O}} \quad .$$

And thus there exist $\alpha_{k\ell}, \beta_k, \gamma \in L^\infty(\mathcal{O})$ satisfying:

$$\begin{cases} -\alpha_{k\ell}\partial^2_{k\ell}(u-v) + \beta_k\partial_k(u-v) + \gamma(u-v) \geq 0 \quad \text{a.e. in } \mathcal{O} \\ \alpha_{k\ell}(x)\xi_k\xi_\ell \geq \nu|\xi|^2 \;\forall \xi \in \mathbb{R}^N \quad \text{a.e. } x \in \mathcal{O} \\ \gamma(x) \geq \alpha \quad \text{a.e. in } \mathcal{O} \quad . \end{cases}$$

We may now apply Bony's maximum principle [8] to show that either $u \equiv v$ or $u > v$ in $\mathcal{O}$. And if $u > v$ in $\mathcal{O}$, following the proof of M. H. Protter and H. F. Weinberger [41] concerning the Hopf maximum principle, one obtains easily: $\frac{\partial u}{\partial n} < \frac{\partial v}{\partial n}$ on $\partial\mathcal{O}$.

II.3 Existence and properties of demi-eigenvalues:

We will keep the notations of the preceding sections (II.1 and II.2) and we introduce the nonlinear operator A:

$$A\varphi = \sup_{i\geq 1}(A_i\varphi) \in L^\infty(\mathcal{O}), \quad \text{if} \quad \varphi \in W^{2,\infty}(\mathcal{O}) \quad .$$

Let us recall that we denote by $\lambda_1(A_i)$ the first (or lowest) eigenvalue of the operator A_i (with Dirichlet boundary values).

Our main result is the following:

Theorem II.2: Under assumptions (21), (22); there exist two constants $\underline{\lambda}_1$, $\overline{\lambda}_1$ such that:

i) The following inequalities are satisfied:

(28) $$\underline{\lambda}_1 \leq \inf_{i\geq 1} \lambda_1(A_i) \leq \sup_{i\geq 1} \lambda_1(A_i) \leq \overline{\lambda}_1 \quad ;$$

(29) $$\text{If } c^i(x) \geq 0 \text{ in } \overline{\mathcal{O}} \text{ for all } i \geq 1; \text{ then: } \underline{\lambda}_1 > 0 \quad .$$

ii) If $\lambda < \underline{\lambda}_1$ and if $(f_i(x))_{i\geq 1}$ is a sequence of functions satisfying (24) then there exists a unique solution $u \in W^{2,\infty}(\mathcal{O})$ of

(4) $$\sup_{i\geq 1}(A_i u - f_i) = \lambda u \text{ a.e. in } \mathcal{O},\ u = 0 \text{ on } \partial\mathcal{O} \quad .$$

iii) If $\lambda < \overline{\lambda}_1$ and if $(f_i(x))$ is a sequence of nonnegative functions satisfying (24) then there exists a unique nonnegative solution $u \in W^{2,\infty}(\mathcal{O})$ of (4).

iv) There exist $\varphi_1, \psi_1 \in W^{2,\infty}(\mathcal{O})$ satisfying:

(5) $$A\varphi_1 = \sup_{i\geq 1}(A_i\varphi_1) = \underline{\lambda}_1\varphi_1 \text{ a.e. in } \mathcal{O},\ \varphi_1 < 0 \text{ in } \mathcal{O},\ \varphi_1 = 0 \text{ on } \partial\mathcal{O}$$

(6) $$A\psi_1 = \sup_{i\geq 1}(A_i\psi_1) = \overline{\lambda}_1\psi_1 \text{ a.e. in } \mathcal{O},\ \psi_1 > 0 \text{ in } \mathcal{O},\ \psi_1 = 0 \text{ on } \partial\mathcal{O} \ .$$

v) Let $(\varphi,\lambda) \in W^{2,\infty}(\mathcal{O}) \times \mathbb{R}$ satisfy:

$$A\varphi = \lambda\varphi \text{ a.e. in } \quad , \varphi = 0 \text{ on } \partial\mathcal{O} \quad .$$

If $\varphi \leq 0$ in $\overline{\mathcal{O}}$, then $\lambda = \underline{\lambda}_1$ and $\varphi = \theta\varphi_1$ for some $\theta > 0$. And if $\varphi \geq 0$ in $\overline{\mathcal{O}}$, then $\lambda = \overline{\lambda}_1$ and $\varphi = \theta\psi_1$ for some $\theta > 0$.

Remark II.4: In view of this result, it is clear that $\underline{\lambda}_1$, $\overline{\lambda}_1$ play the role of the first eigenvalue for the nonlinear operator $\mathcal{A}$. And we see that the nonlinearity [illegible] the [illegible] form and thus typically Lipschitz roughly speaking) creates a pair of what we call demi-eigenvalues. This phenomenon was first observed by H. Berestycki [7] in a totally different setting namely bifurcation theory with non-differentiable mappings: in [7] was considered the case:

$$-u'' = \lambda u^+ - \mu u^- \quad \text{in} \quad (0,1),\ u \in C^2([0,1]),\ u(0) = u(1) = 0$$

where $\lambda, \mu > 0$. But this is actually a particular case of the above case: indeed set $\mu = \lambda\alpha$ and remark that the above equation is equivalent to, if $\alpha \geq 1$,

$$\max(-u'', -\frac{1}{\alpha}u'') = \lambda u \quad \text{in} \quad (0,1),\ u \in C^2([0,1]),\ u(0) = u(1) = 0$$

and to, if $\alpha \leq 1$,

$$\min(-u'', -\frac{1}{\alpha}u'') = \lambda u \quad \text{in} \quad (0,1),\ u \in C^2([0,1]),\ u(0) = u(1) = 0 \ .$$

Since in this specific context it is possible to show the existence of infinitely many demi-eigenvalues, it would be interesting to see if this remains true in the general context of Hamilton-Jacobi-Bellman equations.

Remark II.5: Let us mention that, using the results of L. C. Evans [14], [15] one can prove $\varphi_1, \psi_1 \in C^{2,\alpha}(\mathcal{O})$ (for some $\alpha \in (0,1)$). In addition from Lemma II.1 one deduces: $\frac{\partial}{\partial n}(\varphi_1) > 0$ on $\partial\mathcal{O}$, $\frac{\partial}{\partial n}(\psi_1) < 0$ on ∂. Finally let us mention that part ii) of Lemma II.1 remains valid if $\underline{\lambda}_1 > 0$ while it remains valid if $\overline{\lambda}_1 > 0$ and if f_i, g_i, u, v are nonnegative.

It is possible to give a purely analytical proof of Theorem II.2 without any help from Probability theory but we prefer to make a simpler proof which uses both Partial Differential Equations and Probabilistic techniques. This will enable us to prove at the same time (we keep the notations of section II.1):

Theorem II.3: Under assumptions (21), (22); the two constants $\underline{\lambda}_1$, $\overline{\lambda}_1$ of Theorem II.2 satisfy:

i) <u>we have</u>

$$(30)\quad \begin{cases} \underline{\lambda}_1 = \sup(\lambda \in \mathbf{R},\ \sup\limits_{x\in\bar{\mathcal{O}}} \sup\limits_{A} E[\int_0^{\tau_x} \exp(\lambda t - \int_0^t c^{i(s)}(y_x(s))ds)dt] < \infty) \\ \quad = \sup(\lambda \in \mathbf{R},\ \sup\limits_{x\in\bar{\mathcal{O}}} \sup\limits_{A} E[\exp(\lambda\tau_x - \int_0^{\tau_x} c^{i(t)}(y_x(t))dt)] < \infty) \end{cases}$$

$$(31)\quad \begin{cases} \bar{\lambda}_1 = \sup(\lambda \in \mathbf{R},\ \sup\limits_{x\in\bar{\mathcal{O}}} \inf\limits_{A} E[\int_0^{\tau_x} \exp(\lambda t - \int_0^t c^{i(s)}(y_x(s))ds)dt] < \infty) \\ \quad = \sup(\lambda \in \mathbf{R},\ \sup\limits_{x\in\bar{\mathcal{O}}} \inf\limits_{A} E[\exp(\lambda\tau_x - \int_0^{\tau_x} c^{i(t)}(y_x(t))dt)] < \infty) \end{cases} ;$$

ii) <u>If</u> $\lambda < \underline{\lambda}_1$ <u>and if</u> $(f_i(x))_{i\geq 1}$ <u>is a sequence of functions satisfying</u> (24) <u>then the unique solution</u> u <u>in</u> $W^{2,\infty}(\mathcal{O})$ <u>of</u> (4) <u>is given by</u>:

$$(32)\qquad u(x) = \inf E[\int_0^{\tau_x} f_{i(t)}(y_x(t)) \exp\{\lambda t - \int_0^t c^{i(s)}(y_x(s))ds\}dt] \quad ;$$

iii) <u>If</u> $\lambda < \bar{\lambda}_1$ <u>and if</u> $(f_i(x))_{i\geq 1}$ <u>is a sequence of nonnegative functions satisfying</u> (24) <u>then the unique nonnegative solution</u> u <u>of</u> (4) <u>in</u> $W^{2,\infty}(\mathcal{O})$ <u>is given by</u> (32).

We may now turn to the proof of Theorems II.2 - II.3 which is divided in several steps: we need first to introduce a few notations. We will denote by $\mathcal{B}$ the nonlinear operator defined by:

$$\mathcal{B}\varphi = \inf_{i\geq 1} (A_i\varphi) \in L^\infty(\mathcal{O}), \quad \text{for} \quad \varphi \in W^{2,\infty}(\mathcal{O}) \quad .$$

And we introduce the sets I *and* J:

$I = \{\lambda \in \mathbf{R},\ \exists u_\lambda \in W^{2,\infty}(\mathcal{O}):\ Au_\lambda = \lambda u_\lambda + 1$ a.e. in $\mathcal{O}$, $u_\lambda > 0$ in $\mathcal{O}$, $u_\lambda = 0$ on $\partial\mathcal{O}\}$

$J = \{\lambda \in \mathbf{R},\ \exists v_\lambda \in W^{2,\infty}(\mathcal{O}):\ \mathcal{B}v_\lambda = \lambda v_\lambda + 1$ a.e. in $\mathcal{O}$, $v_\lambda > 0$ in $\mathcal{O}$, $v_\lambda = 0$ on $\partial\mathcal{O}\}$

- in view of (17), if $\lambda \leq -\sup\limits_{i\geq 1} \|c_i\|_{L^\infty(\mathcal{O})}$, $\lambda \in I \cap J$ - .

Our proof consists of six steps: <u>Step 1</u>: There exist $\underline{\lambda}_1$, $\bar{\lambda}_1 \in \mathbf{R} \cup \{+\infty\}$ such that $I =]-\infty, \bar{\lambda}_1[$, $J =]-\infty, \underline{\lambda}_1[$ and $\underline{\lambda}_1 \leq \bar{\lambda}_1$; <u>Step 2</u>: $\underline{\lambda}_1$ (resp. $\bar{\lambda}_1$) is less than the constant defined in (30) (resp. (31)) and they are finite; <u>Step 3</u>: Proof of parts ii),

iii) of Theorem II.2 and of parts ii), iii) of Theorem II.3; Step 4: $\|u_\lambda\|_{L^\infty}$ (resp. $\|v_\lambda\|_{L^\infty}$) $\to +\infty$ as $\lambda \to \overline{\lambda}_1$ (resp. $\lambda \to \underline{\lambda}_1$) and proof of (30) - (31); Step 5: Proof of parts iv), v) of Theorem II.2; Step 6: Proof of part i) of Theorem II.2.

Step 1: There exist $\underline{\lambda}_1, \overline{\lambda}_1 \in \mathbb{R} \cup \{+\infty\}$ such that $I =]-\infty, \overline{\lambda}_1[$ and $J =]-\infty, \underline{\lambda}_1[$ and $\underline{\lambda}_1 \leq \overline{\lambda}_1$:

We first prove that if $\lambda \in I$ (resp. $\lambda \in J$) then $\mu \in I$ (resp. $\mu \in J$) for all $\mu \leq \lambda$. Indeed, let us take for example the case of I, if $\lambda \in I$ than for $\mu \leq \lambda$:

$$Au_\lambda \geq \mu u_\lambda + 1 \text{ a.e. in } \mathcal{O},\ u_\lambda \geq 0 \text{ in } \mathcal{O},\ u_\lambda = 0 \text{ on } \partial\mathcal{O}$$

while obviously: $A0 = 0$. Thus by Theorem II.1, there exists $u_\mu \in W^{2,\infty}(\mathcal{O})$ satisfying:

$$Au_\mu = \mu u_\mu + f \text{ a.e. in } \mathcal{O},\ u_\lambda \geq u_\mu \geq 0 \text{ in } \mathcal{O},\ u_\mu = 0 \text{ on } \partial\mathcal{O}\ ;$$

and $\mu \in I$.

Next, to prove that, for example, I is open, we argue as follows: let $\lambda \in I$, we need to prove there exists $\varepsilon > 0$ such that $\lambda + \varepsilon \in I$. But, if $k \geq 1$, we have:

$$A(ku_\lambda) = \lambda(k\, u_\lambda) + k = (\lambda+\varepsilon)k\, u_\lambda + 1 + \{k - 1 - \varepsilon k\, u_\lambda\} \text{ a. e. in } \mathcal{O}\ .$$

Thus choosing ε small enough such that $\varepsilon\|u_\lambda\|_{L^\infty(\mathcal{O})} < \frac{1}{2}$ and $k \geq 2$, we obtain:

$$A(ku_\lambda) \geq (\lambda+\varepsilon)(ku_\lambda) + 1 \text{ a.e. in } \mathcal{O},\ ku_\lambda = 0 \text{ on } \partial\mathcal{O}\ .$$

Since $A0 = 0$, we deduce from Theorem II.1 the existence of $u(=u_{\lambda+\varepsilon})$ such that: $Au = (\lambda+\varepsilon)\, u + 1$ a.e. in $\mathcal{O}$, $u \geq 0$ in $\mathcal{O}$, $u = 0$ on $\partial\mathcal{O}$.

Finally, if $\lambda \in J$, we have:

$$Av_\lambda \geq Bv_\lambda = \lambda v_\lambda + 1 \text{ a.e. in } \mathcal{O},\ v_\lambda \geq 0 \text{ in } \mathcal{O},\ v_\lambda = 0 \text{ on } \partial\mathcal{O}$$

and since $0 = 0$, we conclude that $\lambda \in J$ from Theorem II.1.

Step 2: $\underline{\lambda}_1$ (resp. $\overline{\lambda}_1$) is less than the constant defined in (30) (resp. (31)) and they are finite.

More precisely, we denote by $\underline{\mu}_1, \overline{\mu}_1$:

$$\underline{\mu}_1 = \sup(\lambda \in \mathbb{R},\ \sup_{x\in\overline{\mathcal{O}}}\ \sup_{A}\ E[\int_0^{\tau_x} \exp(\lambda t - \int_0^t c^{i(s)}(y_x(s))ds)dt] < \infty)$$

$$\overline{\mu}_1 = \sup(\lambda \in \mathbb{R},\ \sup_{x\in\overline{\mathcal{O}}}\ \inf_{A}\ E[\int_0^{\tau_x} \exp(\lambda t - \int_0^t c^{i(s)}(y_x(s))ds)dt] < \infty)\ .$$

It is a simple exercise that we skip to check that $\underline{\mu}_1, \overline{\mu}_1$ are also given by:

$$\underline{\mu}_1 = \sup(\lambda \in \mathbf{R},\ \sup_{x\in\mathcal{O}} \sup_{A} E[\exp\{\lambda\tau_x - \int_0^{\tau_x} c^{i(s)}(y_x(s))ds\}] < \infty)$$

$$\overline{\mu}_1 = \sup(\lambda \in \mathbf{R},\ \sup_{x\in\overline{\ }} \inf_{A} E[\exp\{\lambda\tau_x - \int_0^{\tau_x} c^{i(s)}(y_x(s))ds\}] < \infty) \quad .$$

Next, let $\lambda \in J$ (for example) and let us prove that $\lambda \leq \underline{\mu}_1$. Let v_λ be such that: $\mathcal{B}v_\lambda = \lambda v_\lambda + 1$ a.e. in $\mathcal{O}$, $v_\lambda > 0$ in $\mathcal{O}$, $v_\lambda = 0$ on $\partial\mathcal{O}$. Let A be an admissible system, we apply Ito's formula to $v_\lambda(y_x(t))\exp\{\lambda t - \int_0^t c^{i(s)}(y_x(s))ds\}$ between 0 and $T \wedge \tau_x$, and we find:

$$v_\lambda(x) = E[v_\lambda(y_x(T\wedge\tau_x))\exp[\lambda T\wedge\tau_x - \int_0^{T\wedge\tau_x} c^{i(t)}(y_x(t))dt\}] +$$

$$+ E\int_0^{T\wedge\tau_x}\{A_{i(t)}(y_x(t)) - \lambda v_\lambda(y_x(t))\}\{\exp(\lambda t - \int_0^t c^{i(s)}(y_x(s))ds)dt\}$$

and this yields:

$$v_\lambda(x) \geq E\int_0^{T\wedge\tau_x} \exp(\lambda t - \int_0^t c^{i(s)}(y_x(s))ds)dt$$

for all $T < \infty$. Taking $T \to \infty$, we conclude.

It just remains to prove that $\overline{\mu}_1$ is finite (i.e. $\overline{\mu}_1 < +\infty$). Without loss of generality, we my assume that for some $\delta > 0$, $Q =]0,\delta[^N \subset \mathcal{O}$. And clearly $\overline{\mu}_1 \leq \overline{\mu}$ where $\overline{\mu}$ is given by:

$$\overline{\mu} = \sup(\lambda \in \mathbf{R},\ \sup_{x\in\mathcal{O}} \inf_{A} E[\int_0^{\overline{\tau}_x} \exp(\lambda t - \int_0^t c^{i(s)}(y_x(s))ds)dt] < \infty$$

where $\overline{\tau}_x$ is the first exit time from $\overline{Q}$ of the process $y_x(t)$.

Now consider the one dimensional control problem:

$$dy_x(t) = \sum_{j=1}^N \sigma_j(t,\omega)dW_j(t) + b(t,\omega)dt,\ y_x(0) = x \in [0,\delta]$$

where σ_j, b_j are any progressively measurable processes such that:

$$\nu \leq \frac{1}{2}\sum_{j=1}^N |\sigma_j|^2 \leq \nu C \quad , \quad |b| \leq C$$

where C is such that: $\|a^1_{11}\|_{L^\infty(\mathcal{O})} \leq \nu C$, $\|b^1_1\|_{L^\infty(\mathcal{O})} \leq C$. Therefore, it is clear that

$\bar{\mu} \leq \mu_0$ given by:

$$\mu_0 = \sup(\lambda \in R,\ \sup_{x\in[0,\delta]} \inf_{\tilde{A}} E[\int_0^{\sigma_x} \exp(\lambda t - \lambda_0 t)dt] < \infty)$$

where $\tilde{A}$ is any admissible system corresponding to the one-dimensional problem, where σ_x is the first exit time from $[0,\delta]$ of $y_x(t)$ and where $\lambda_0 = \sup_1 \|c^1\|_{L^\infty(0)}$. We claim now that μ_0 is also given by:

$$\mu_0 = \sup(\lambda \in R,\ \exists u_\lambda \in C^2([0,\delta]):\ -\nu u'' + |u'| = \frac{\lambda-\lambda_0}{C} u_\lambda + 1 \text{ in }]0,\delta[,$$
$$u_\lambda > 0,\ u_\lambda(0) = u_\lambda(\delta) = 0) .$$

Indeed if we denote by $\bar{\mu}_0$ this last constant, remarking that the following equations are equivalent if $\lambda > \lambda_0$

$$-\nu u_\lambda'' + |u_\lambda'| = \frac{\lambda-\lambda_0}{C} u_\lambda + \frac{1}{C} \text{ in }]0,\delta[,\ u_\lambda > 0$$

or

$$\max_{\substack{\nu\leq a\leq C\nu\\ |b|\leq C}} [-a u_\lambda'' + b u_\lambda'] = (\lambda-\lambda_0)u_\lambda + 1 \text{ in }]0,\delta[,\ u_\lambda > 0$$

we already know that if $\bar{\mu}_0 > \lambda_0$ then $\bar{\mu}_0 \leq \mu_0$. But it is very easy to show that $\bar{\mu}_0 > \lambda_0$ and thus we know that $\bar{\mu}_0 \leq \mu_0$. Next, we show that if $\bar{\mu}_0 < \infty$ then $|u_\lambda|_{L^\infty} \to +\infty$ as $\lambda \to \bar{\mu}_0$. Indeed if it is not the case, there exists $u_{\bar{\mu}_0}$ solution of:

$$-\nu u_{\bar{\mu}_0}'' + |u_{\bar{\mu}_0}'| = \frac{\bar{\mu}_0 - \lambda_0}{C} u_{\bar{\mu}_0} + 1 \text{ in }]0,\delta[,\ u_{\bar{\mu}_0} \in C^2([0,\delta]) ,$$
$$u_{\bar{\mu}_0}(0) = u_{\bar{\mu}_0}(\delta) = 0 ,\quad u_{\bar{\mu}_0} > 0 \text{ in } [0,\delta] .$$

But since the set of λ such that there exists u_λ as above is open (see Step 1) we have a contradiction with the definition of $\bar{\mu}_0$. Thus, $|u_\lambda|_{L^\infty} \to +\infty$ as $\lambda \to \bar{\mu}_0$, if $\bar{\mu}_0$ is finite. But, by the same argument as above (applying Ito's formula), we see that for $\lambda < \bar{\mu}_0$

$$u_\lambda(x) = C \inf_{\tilde{A}} E[\int_0^{\sigma_x} \exp[(\lambda - \lambda_0)t]dt]$$

(since $\lambda < \mu_0$, this is a verification result totally similar to those introduced in N. V. Krylov [25], [26]).

Since $\max\limits_{[0,\delta]} u_\lambda(x) \to +\infty$, as $\lambda \to \bar{\mu}_0$, one sees immediately that

$\sup\limits_{[0,\delta]} \inf\limits_{A} E[\int_0^{\sigma_x} \exp\{(\bar{\mu}_0 - \lambda_0)t\}dt] = +\infty$; thus $\mu_0 = \bar{\mu}_0$.

We now conclude by proving that $\mu_0 < \infty$. Indeed let $\lambda < \mu_0$ and let $u_\lambda \in C^2([0,\delta])$ satisfy:

$$- \nu u_\lambda'' + |u_\lambda'| = \frac{\lambda-\lambda_0}{C} u_\lambda + 1 \quad \text{in} \quad]0,\delta[\;,\; u_\lambda > 0,\; u_\lambda(0) = u_\lambda(\delta) = 0 \;.$$

As indicated before: $u_\lambda(x) = C \inf\limits_{A} E[\int_0^{\sigma_x} \exp\{(\lambda - \lambda_0)t\}dt]$ and thus is unique, therefore $u_\lambda(x) = u_\lambda(\delta-x)$ for $x \in [0,\delta]$. In particular $u_\lambda'(\frac{\delta}{2}) = 0$. In addition it is easy to prove that: $u_\lambda' > 0$ on $[0, \frac{\delta}{2}]$ (one may use for example the general results of Gidas-Ni-Nirenberg [22]). Now from the equation one sees that $u_\lambda'' \in W^{1,\infty}(0,\delta)$ and $v = u_\lambda'$ satisfies:

$$\begin{cases} - \nu v'' + v' = \dfrac{\lambda-\lambda_0}{C} v \quad \text{in} \quad]0,\frac{\delta}{2}[\quad , v(0) \leq 0,\; v(\frac{\lambda}{2}) = 0 \\ v \in C^2(0,\frac{\delta}{2}) \cap C^1([0,\frac{\delta}{2}]) \end{cases}$$

and thus $\frac{\lambda-\lambda_0}{C}$ is less than the lowest eigenvalue λ_1 of the operator $- \nu \frac{d^2}{dx^2} + \frac{d}{dx}$ on the domain $]0,\frac{\delta}{2}[$ with Dirichlet boundary conditions at 0. Therefore $\mu_0 \leq \lambda_0 + C\lambda_1$ and we conclude.

Step 3: Proof of parts ii), iii) of Theorems II.2, II.3:

Let $\lambda \in J$ (i.e. $\lambda < \underline{\lambda}_1$), from the definition of J, there exists $u_\lambda, v_\lambda \in W^{2,\infty}(\mathcal{O})$ such that:

$$\begin{cases} Au_\lambda = \lambda u_\lambda + 1 \quad \text{a.e. in} \quad \mathcal{O},\; u_\lambda > 0 \quad \text{in} \quad \mathcal{O},\; u_\lambda = 0 \quad \text{on} \quad \partial\mathcal{O} \\ A(-v_\lambda) = \lambda(-v_\lambda) - 1 \quad \text{a.e. in} \quad \mathcal{O},\; -v_\lambda < 0 \quad \text{in} \quad ,\; -v_\lambda = 0 \quad \text{on} \quad \partial\mathcal{O} \;. \end{cases}$$

And thus if $(f_i(x))_{i\geq 1}$ is a sequence of functions satisfying (24) then for k large enough we have:

$$\sup_{i\geq 1} \{A_i(ku_\lambda) - f_i\} \geq \lambda(ku_\lambda) \quad \text{a.e. in} \quad \mathcal{O},\quad ku_\lambda > 0 \quad \text{in} \quad \bar{\mathcal{O}}$$

$$\sup_{i\geq 1} [A_i(-\kappa v_\lambda) - f_i] \leq \lambda(-\kappa v_\lambda) \text{ a.e. in } \mathcal{O},\ -\kappa v_\lambda \leq 0 \text{ in } \bar{\mathcal{O}}$$

and we deduce from Theorem II.2, the existence of $u \in W^{2,\infty}(\mathcal{O})$ solution of:

$$\sup_{i\geq 1} [A_i u - f_i] = \lambda u \text{ a.e. in } \mathcal{O},\ u = 0 \text{ on } \partial\mathcal{O} .$$

In the same way, if $\lambda < \bar{\lambda}_1$ and if $(f_i(x))_{i\geq 1}$ is a sequence of nonnegative functions satisfying (24), we prove the existence of a nonnegative solution u of the same problem (in this case take 0 as a subsolution since $f_i \geq 0,\ \forall i \geq 1$).

Finally to prove uniqueness of such solutions u, it is enough to show the stochastic representation (32). But this is a simple remake of the arguments introduced by N. V. Krylov [25], [26], using the fact that if $\lambda < \underline{\lambda}_1$ (resp. $\lambda < \bar{\lambda}_1$) then $\lambda < \underline{\mu}_1$ (resp. $\lambda < \bar{\mu}_1$).

Step 4: $\|u_\lambda\|_{L^\infty} \to +\infty$ as $\lambda \to \bar{\lambda}_1$; $\|v_\lambda\|_{L^\infty} \to +\infty$ as $\lambda \to \underline{\lambda}_1$. Remark first that in view of the representation proved above:

$$u_\lambda(x) = \inf_A E[\int_0^{\tau_x} \exp\{\lambda t - \int_0^t c^{i(s)}(y_x(s))ds\}dt]$$

$$v_\lambda(x) = \sup_A E[\int_0^{\tau_x} \exp\{\lambda t - \int_0^t c^{i(s)}(y_x(s))ds\}dt] \quad ;$$

this implies immediately: $\underline{\lambda}_1 = \underline{\mu}_1$, $\bar{\lambda}_1 = \bar{\mu}_1$; that is part i) of Theorem II.3. Let us prove now that, for example, $\|u_\lambda\|_{L^\infty(\mathcal{O})} \to +\infty$ as $\lambda \to \bar{\lambda}_1$. If this were not the case, there would exist $\lambda_n \underset{n}{\to} \bar{\lambda}_1$ such that $\|u_{\lambda_n}\|_{L^\infty} \leq C$ (indep. of n). We are going to prove that this would imply: $\|u_{\lambda_n}\|_{W^{2,\infty}(\mathcal{O})} \leq C$ (indep. of n). But this would show that there exists $u_{\bar{\lambda}_1} \in W^{2,\infty}(\mathcal{O})$ solution of

$$Au_{\bar{\lambda}_1} = \bar{\lambda}_1 u_{\bar{\lambda}_1} + 1 \text{ a.e. in } \mathcal{O},\ u_{\bar{\lambda}_1} \geq 0 \text{ in } \mathcal{O},\ u_{\bar{\lambda}_1} = 0 \text{ on } \partial\mathcal{O}$$

(pass to the limit, as $n \to \infty$, as in [13], [16]). And this would contradict the definition of $\bar{\lambda}_1$ since I is open (Step 1).

Therefore, we need to prove: $\|u_{\lambda_n}\|_{W^{2,\infty}(\mathcal{O})} \leq C$ (indep. of n) as soon as $\|u_{\lambda_n}\|_{L^\infty}$ is bounded. To simplify the notations, we denote by $u^n = u_{\lambda_n}$. Without loss of generality we may assume that $c^i \geq 0,\ \forall i \geq 1$ (if this is not the case, add a constant to both c^1 and λ_n). As in the proof of Theorem II.1, we know there exist $u_\varepsilon^{n,i}$ solution of: ($m \geq 1$ is fixed)

$$\begin{cases} A_1 u_\varepsilon^{n,1} + \beta_\varepsilon(u_\varepsilon^{n,1} - u_\varepsilon^{n,2}) = \lambda_n(u_\varepsilon^{n,1} \wedge C_0) + 1 \quad \text{in } \mathcal{O} \\ \cdot \\ \cdot \\ \cdot \\ A_m u_\varepsilon^{u,m} + \beta_\varepsilon(u_\varepsilon^{n,m} - u_\varepsilon^{n,1}) = \lambda_n(u_\varepsilon^{n,m} \wedge C_0) + 1 \quad \text{in } \mathcal{O} \end{cases}$$

with $u_\varepsilon^{n,i} \in C^2(\overline{\mathcal{O}})$, $u_\varepsilon^{n,i} \geq 0$ in $\overline{\mathcal{O}}$, $u_\varepsilon^{n,i} = 0$ on $\partial\mathcal{O}$ and where $C_0 > \|u^n\|_{L^\infty(\mathcal{O})}$ ($\forall n \geq 1$). In addition $\|u_\varepsilon^{n,i}\|_{L^\infty(\mathcal{O})} \leq C_1$ for some constant C_1 (indep. of m, n, i, ε). But this implies as in Remark II.2 and in the proof of Theorem II.1:

$$\|u_\varepsilon^{n,i}\|_{W^{2,\infty}(\mathcal{O})} \leq C_2 \quad \text{(indep. of } m, n, i, \varepsilon) \ .$$

Now, taking $\varepsilon \to 0$, $m \to \infty$, we obtain the existence of $\tilde{u}^n$ solution of:

$$\begin{cases} A\tilde{u}^n = \lambda_n(\tilde{u}^n \wedge C_0) + 1 \quad \text{in } \mathcal{O},\ \tilde{u}^n \in W^{2,\infty}(\mathcal{O}),\ \|\tilde{u}^n\|_{W^{2,\infty}(\mathcal{O})} \leq C_2 \\ \tilde{u}^n \geq 0 \quad \text{in } \mathcal{O},\ \tilde{u}^n = 0 \quad \text{on } \partial\mathcal{O} \ . \end{cases}$$

To conclude, we are going to prove that $\tilde{u}^n = u^n$. But, rewriting the above equation as

$$\begin{cases} A\tilde{u}^n + \lambda_n(\tilde{u}^n - C_0)^+ = \lambda_n\tilde{u}^n + 1 \quad \text{in } \mathcal{O},\ \tilde{u}^n \in W^{2,\infty}(\mathcal{O}) \\ \tilde{u}^n \geq 0 \quad \text{in } \mathcal{O},\ \tilde{u}^n = 0 \quad \text{on } \partial\mathcal{O} \ ; \end{cases}$$

we prove as above that any solution of the preceding equation is given by

$$\tilde{u}^n(x) = \inf_{A,\delta} E[\int_0^{\tau_x}\{1 + \delta(t)\lambda_n C_0\}\exp\{(1 - \delta(t)\lambda_n t - \int_0^t c^{i(s)}(g_x(s))ds\}dt$$

where $\delta(t,\omega)$ is any progressively measurable process with values in $\{0,1\}$. Now since $\|u^n\|_{L^\infty(\mathcal{O})} < C_0$, u^n is also a solution of the preceding equations and $u^n = \tilde{u}^n$.

<u>Step 5</u>: <u>Proofs of parts</u> iv) - v) <u>of Theorem</u> II.2.

From Step 4, we know that $\|u_\lambda\|_{L^\infty(\mathcal{O})} \to +\infty$ as $\lambda \to \overline{\lambda}_1$ and $\|v_\lambda\|_{L^\infty(\mathcal{O})} \to +\infty$ as $\lambda \to \underline{\lambda}_1$. We then define $\varphi_\lambda = -\dfrac{v_\lambda}{\|v_\lambda\|_{L^\infty(\mathcal{O})}}$ and $\psi_\lambda = \dfrac{u_\lambda}{\|u_\lambda\|_{L^\infty(\mathcal{O})}}$; obviously we have: φ_λ, $\psi_\lambda \in W^{2,\infty}(\mathcal{O})$

$$A\varphi_\lambda = \lambda\varphi_\lambda + \frac{1}{\|u_\lambda\|_{L^\infty(\mathcal{O})}} \quad \text{a.e. in } \mathcal{O},\ \varphi_\lambda > 0 \text{ in } \mathcal{O},\ \varphi_\lambda = 0 \text{ on } \partial\mathcal{O}$$

$$A\psi_\lambda = \lambda\psi_\lambda - \frac{1}{\|v_\lambda\|_{L^\infty(\mathcal{O})}} \quad \text{a.e. in } \mathcal{O},\ \psi_\lambda < 0 \text{ in } \mathcal{O},\ \psi_\lambda = 0 \text{ on } \partial\mathcal{O}$$

and $\|\varphi_\lambda\|_{L^\infty(\mathcal{O})} = \|\psi_\lambda\|_{L^\infty(\mathcal{O})} = 1$.

Now exactly as in the proof of Step 4, we obtain:

$$\|\varphi_\lambda\|_{W^{2,\infty}(\mathcal{O})} \le C_0 \quad , \quad \|\psi_\lambda\|_{W^{2,\infty}(\mathcal{O})} \le C_0$$

where C_0 does not depend on λ (for $\lambda \ge \underline{\lambda}_1 - 1$). And passing to the limit as $\lambda \to \underline{\lambda}_1$ or $\lambda \to \overline{\lambda}_1$ (in the same way as in [13], [16], [31]) we obtain part iv) of Theorem II.2.

We next prove part v) of Theorem II.2. For example let $(\psi,\lambda) \in W^{2,\infty}(\mathcal{O}) \times \mathbf{R}$ be such that:

$$A\psi = \lambda\psi \quad \text{a.e. in } \mathcal{O},\ \psi > 0 \text{ in } \mathcal{O},\ \psi = 0 \text{ on } \partial\mathcal{O} .$$

We first show that $\lambda = \overline{\lambda}_1$; indeed if $\lambda < \overline{\lambda}_1$, it is then trivial to deduce from the stochastic representation (or the uniqueness) that $\psi \equiv 0$ in $\overline{\mathcal{O}}$. On the other hand if $\lambda > \overline{\lambda}_1$, we argue as follows: first, we remark that Lemma II.1 implies that if $\psi \not\equiv 0$, there exists $\alpha, \beta = 0$ such that: $\beta\psi_1 \ge \psi \ge \alpha\psi_1 > 0$ in $\mathcal{O}$. Now by the same verification method as the one introduced by N. V. Krylov [25], [26] one obtains easily:

$$\psi_1(x) = \inf_A E[\psi_1(y_x(T\wedge\tau_x))\exp\{\overline{\lambda}_1 T\wedge\tau_x - \int_0^{T\wedge\tau_x} c^{i(t)}(y_x(t))dt\}]$$

$$\psi(x) = \inf_A E[\psi(y_x(T\wedge\tau_x))\exp\{\lambda T\wedge\tau_x - \int_0^{T\wedge\tau_x} c^{i(t)}(y_x(t))dt\}]$$

for all $T < \infty$. Therefore:

$$\psi_1(x) \le \frac{1}{\alpha} \inf_A E[\psi(y_x(T\wedge\tau_x))\exp\{\lambda T\wedge\tau_x - \int_0^{T\wedge\tau_x} c^{i(t)}(y_x(t))dt\} \cdot 1_{(T<\tau_x)}]$$

$$\le e^{-(\lambda-\overline{\lambda}_1)T} \frac{\beta}{\alpha} \psi_1(x) \quad \text{in } \mathcal{O} ,$$

and choosing T large enough, we obtain a contradiction that proves: $\lambda = \overline{\lambda}_1$.

We now show that, necessarily, $\psi = \theta\psi_1$ for some $\theta > 0$. Indeed, let us first remark that by a simple application of Lemma II.1 we have:

$$\psi(x),\ \psi_1(x) > 0 \quad \text{for} \quad x \in \mathcal{O},\ \frac{\partial\psi}{\partial n},\ \frac{\partial\psi_1}{\partial n} < 0 \quad \text{on} \quad \partial\mathcal{O} \quad .$$

Thus if $\psi \neq \theta\psi_1$ (for any $\theta > 0$), and if we denote by

$$\theta = \sup(\mu > 0,\ \mu\psi_1 \leq \psi \quad \text{in} \quad \overline{\mathcal{O}})$$

then necessarily: $\psi \geq \theta\psi_1$ in $\overline{\mathcal{O}}$ and $\psi \not\equiv \theta\psi_1$. But let $\lambda > 0$ be such that $\lambda + \overline{\lambda}_1 > 0$, we have:

$$A\psi + \lambda\psi = (\lambda + \overline{\lambda}_1)\psi \quad \text{in} \quad \mathcal{O}$$

$$\geq (\lambda + \overline{\lambda}_1)\ (\theta\psi_1) = A(\theta\psi_1 + \lambda(\theta\psi_1) \quad \text{in} \quad \mathcal{O} \quad .$$

And applying Lemma II.1, we deduce: $\psi - \theta\psi_1 > 0$ in $\mathcal{O}$ and $\frac{\partial}{\partial n}(\psi - \theta\psi_1) < 0$ on $\partial\mathcal{O}$. Thus there exists $\varepsilon > 0$ such that:

$$(\theta+\varepsilon)\psi_1 \leq \psi \quad \text{in} \quad \overline{\mathcal{O}} \quad ,$$

and this contradicts the definition of θ. This proves our claim. (Let us mention that the above argument is an adaptation of a device due to T. Laetsch [24].)

Step 6: Proof of part i) of Theorem II.2.

We first prove (29): indeed in view of [17], if $c^i \geq 0$ ($\forall i \geq 1$) then there exists a solution $v_0 \in W^{2,\infty}(\mathcal{O})$ of

$$\mathcal{B} v_0 = 1 \quad \text{in} \quad \mathcal{O},\ v_0 = 0 \quad \text{on} \quad \partial\mathcal{O}$$

and thus $0 \in J$; since J is open, this yields (29).

From (5), we deduce:

$$A_i\varphi_1 \leq \underline{\lambda}_1\varphi_1 \quad \text{a.e. in} \quad \mathcal{O},\ \varphi_1 < 0 \quad \text{in} \quad \mathcal{O},\ \varphi_1 = 0 \quad \text{on} \quad \partial\mathcal{O} \quad ;$$

but this implies: $\lambda_1(A_i) \geq \underline{\lambda}_1$; and we obtain the first part of (28).

Finally to prove the second part: let $\lambda < \sup_i \lambda_1(A_i)$, there exists i such that $\lambda_1(A_i) > \lambda$. Thus, it is well-known that there exists $\overline{u}_\lambda \in C^2(\overline{\mathcal{O}})$ satisfying:

$$A_i\overline{u}_\lambda = 1 + \lambda\overline{u}_\lambda \quad \text{in} \quad \mathcal{O},\ \overline{u}_\lambda > 0 \quad \text{in} \quad \mathcal{O},\ \overline{u}_\lambda = 0 \quad \text{on} \quad \partial\mathcal{O} \quad ;$$

and therefore: $A\overline{u}_\lambda \geq 1 + \lambda\overline{u}_\lambda$ in $\mathcal{O}$, $\overline{u}_\lambda \geq 0$ in $\overline{\mathcal{O}}$. since we have: $A0 = 0 \leq 1$; applying

Theorem II.1 we obtain the existence of u_λ (between 0 and $\bar{u}_\lambda$) satisfying:

$$Au_\lambda = 1 + \lambda u_\lambda \text{ a.e. in } \mathcal{O},\ u_\lambda > 0 \text{ in } \bar{\mathcal{O}},\ u_\lambda = 0 \text{ on } \partial\mathcal{O} .$$

Thus $\lambda \in I$ and $\lambda < \bar{\lambda}_1$. This proves (28) and completes the proof of Theorems II.2 - II.3.

Remark II.6: As we will see in the next section, $\underline{\lambda}_1$ and $\bar{\lambda}_1$ possess many of the properties of the lowest eigenvalue for a second-order uniformly elliptic operator. For the moment let us just mention that, obviously, $\underline{\lambda}_1 = \lambda_1(A_i)$ (for some $i \geq 1$) if and only if, denoting by v_1 the eigenfunction corresponding to $\lambda_1(A_i)$, we have:

$$A_j v_1 - \underline{\lambda}_1 v_1 \geq 0 \text{ in } \mathcal{O}, \text{ for all } j \neq i .$$

In this case we have in addition: $\varphi_1 = -\theta v_1$, for some $\theta > 0$. Indeed if $\underline{\lambda}_1 = \lambda_1(A_i)$, we have:

$$A_i\varphi_1 - \lambda_1(A_i)\varphi_1 \leq 0 \text{ in } \mathcal{O},\ \varphi_1 < 0 \text{ in } \mathcal{O},\ \varphi_1 = 0 \text{ on } \partial\mathcal{O}$$

and this implies: $\varphi_1 = -\theta v_1$ for some $\theta > 0$; and we conclude.

In the same way, $\bar{\lambda}_1 = \lambda_1(A_i)$ if and only if we have:

$$A_j v_1 - \bar{\lambda}_1 v_1 \leq 0 \text{ in } \mathcal{O}, \text{ for all } j \neq i .$$

In this case we have in addition: $\psi_1 = \theta v_1$, for some $\theta > 0$.

II.4 Applications and properties of demi-eigenvalues.

We first give a very simple bifurcation result which has only the value of an example. We keep the notations of the preceding sections and we consider the equation:

(33) $$Au + \lambda|u|^{p-1}u = \lambda u \text{ a.e. in } \mathcal{O},\ u \in W^{2,\infty}(\mathcal{O}),\ u = 0 \text{ on } \partial\mathcal{O}$$

and we take $\lambda > 0$; we assume (21), (22) and $c^i \geq 0$ ($\forall i \geq 1$) for simplicity. Finally let $p > 1$. We will consider here only the existence of solutions with constant sign.

We then have:

Theorem II.4: Under assumptions (21), (22) and if $c^i \geq 0$ ($\forall i \geq 1$); then we have:

i) If $\lambda < \underline{\lambda}_1$, the only solution of (33) is: $u \equiv 0$.

ii) If $\lambda \leq \bar{\lambda}_1$, the only nonnegative solution of (33) is: $u \equiv 0$.

iii) If $\lambda > \underline{\lambda}_1$, there exists a unique negative solution $\underline{u}_\lambda$ of (33).

iv) If $\lambda > \bar{\lambda}_1$, there exists a unique positive solution $\bar{u}_\lambda$ of (33).

In other words, we have the following bifurcation diagram for the equation [illegible]: for solutions of constant sign):

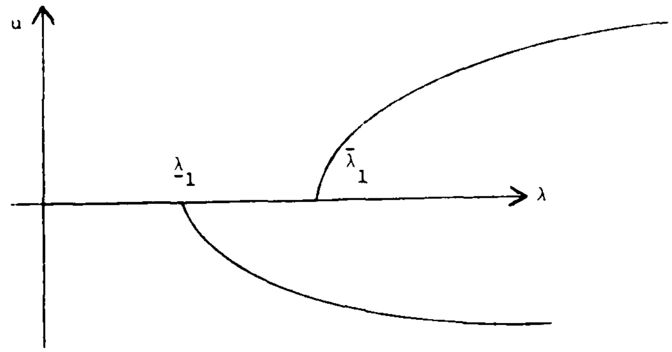

Figure 1

<u>Remark II.7</u>: It is possible to give, as in part I, the stochastic interpretation of $\underline{u}_\lambda$, $\overline{u}_\lambda$: we will not do it here. It is also possible to show that $\underline{u}_\lambda$, $\overline{u}_\lambda$ are continuous with respect to λ (in the space $C^{1,\alpha}(\overline{O})$ for any $\alpha < 1$) and that $\underline{u}_\lambda \to 0$ as $\lambda \to \underline{\lambda}_1$ and $\overline{u}_\lambda \to 0$ as $\lambda \to \overline{\lambda}_1$.

<u>Remark II.8</u>: On some simple examples, it is possible to show that there may be between $\underline{\lambda}_1$ and $\overline{\lambda}_1$ bifurcation of continua of solutions with no constant sign.

<u>Remark II.9</u>: This type of split bifurcation diagram (and of the existence of demi-eigenvalues) is intimately connected with the Lipschitzian character of the nonlinearity arising in A: we will give below a striking example explaining this claim (see also [7]).

<u>Proof of Theorem II.4</u>: We first show parts i) and ii). Now if $\lambda < \underline{\lambda}_1$, using Theorem II.3 (and the fact that $d(x) = \lambda|u|^{p-1} \geq 0$ in $\overline{O}$), the stochastic interpretation of:

$$Au + du = \lambda u \quad \text{a.e. in } O, \; u \in W^{2,\infty}(O), \; u = 0 \text{ on } \partial O \;,$$

immediately yields: $u \equiv 0$.

Next, if $\lambda \leq \overline{\lambda}_1$, the claim follows obviously from the following Lemma, proved below:

Lemma II.2: Under assumptions (21), (22); and let $(d^i)_i$ be a bounded sequence of nonnegative functions in $L^\infty(\mathcal{O})$. We assume there exists $(\psi,\lambda) \in W^{2,\infty}(\mathcal{O}) \times R$ such that:

$$\sup_{i \geq 1} (A_i \psi + d^i \psi) = \lambda\psi \quad \text{a.e. in } \mathcal{O}, \quad \psi = 0 \text{ on } \partial\mathcal{O} . \tag{34}$$

i) *If* $\psi > 0$ *in* $\mathcal{O}$, *then* $\lambda \geq \overline{\lambda}_1$ *and* $\lambda = \overline{\lambda}_1$ *if and only if* $\psi = \theta\psi_1$ *(for some* $\theta > 0$*): in particular we have then* $\inf_i d^i = 0$ *a.e.*

ii) *If* $\psi < 0$ *in* $\mathcal{O}$, *then* $\lambda \geq \underline{\lambda}_1$ *and* $\lambda = \underline{\lambda}_1$ *if and only if* $\psi = \theta\varphi_1$ *(for some* $\theta < 0$*): in particular we have then* $\sup_i d^i = 0$ *a.e.*

Proof of Lemma II.2: We will only prove part i) since i) and ii) are totally similar. We first show: $\lambda \geq \overline{\lambda}_1$. Indeed if $\lambda < \overline{\lambda}_1$, from Theorem II.3 and using the stochastic representations, it is easy to conclude: $\psi \equiv 0$. Now, if $\lambda = \overline{\lambda}_1$, let θ be defined by:

$$\theta = \sup\,(\mu > 0,\ \mu\psi_1 \leq \psi) \quad .$$

If $\psi \equiv \theta\psi_1$, we are done since we have then:

$$\sup_{i\geq 1}(A_i\psi_1) = \sup_{i\geq 1}(A_i\psi_1 + d^i\psi_1) \geq \sup_{i\geq 1}(A_i\psi_1) + (\inf_{i\geq 1} d^i)\psi_1 \quad .$$

If $\psi \not\equiv \theta\psi_1$, we argue as follows: we first observe that

$$\sup_{i\geq 1}(A_i\psi + d_i\psi) - \sup_{i\geq 1}(A_i\theta\psi_1) = \overline{\lambda}_1\,(\psi - \theta\psi_1) \quad \text{a.e. in} \quad \mathcal{O} .$$

On the other hand there exist $\alpha_{k\ell}, \beta_k, \gamma \in L^\infty(\mathcal{O})$ satisfying:

$$\alpha_{k\ell}(x)\xi_k\xi_\ell \geq \nu|\xi|^2, \ \forall\xi \in R^N, \quad \text{a.e.} \quad \text{in} \quad \mathcal{O}$$

and such that:

$$\sup_{i\geq 1}(A_i\psi) - \sup_{i\geq 1}(A_i\theta\psi_1) = -\alpha_{k\ell}\partial_{k\ell}(\psi - \theta\psi_1) + \beta_k\partial_k(\psi - \theta\psi_1) + \gamma(\psi - \theta\psi_1) \quad \text{a.e. in} \quad \mathcal{O} \quad .$$

And this yields:

$$\begin{aligned} -\alpha_{k\ell}\partial_{k\ell}(\psi - \theta\psi_1) + \beta_k\partial_k(\psi - \theta\psi_1) + (\gamma - \overline{\lambda}_1)(\psi - \theta\psi_1) &\geq \\ \geq (\inf_{i\geq 1} d^i)\psi \quad \text{a.e.} \quad \text{in} \quad \mathcal{O}& \end{aligned}$$

$$\psi - \theta\psi_1 \in W^{2,\infty}(\mathcal{O}), \ \psi - \theta\psi_1 \geq 0 \quad \text{in} \quad \mathcal{O} \quad .$$

And using Bony's maximum principle [8] as in the proof of Lemma II.1 we deduce:

$\psi - \theta\psi_1 > 0$ in $\mathcal{O}$, $\frac{\partial}{\partial n}(\psi - \theta\psi_1) < 0$ on $\partial\mathcal{O}$, and this contradicts the definition of θ. This completes the proof of the Lemma.

We now turn to the proof of iv) in Theorem II.4 (since the proof of iii) is identical, we will skip it). Let $\lambda > \overline{\lambda}_1$ and let us show the existence of $\overline{u}_\lambda$. We first remark that for ε small enough, we have:

$$\begin{cases} A(\varepsilon\psi_1) + \lambda\,\varepsilon^p\psi_1^p = \varepsilon\,\overline{\lambda}_1\psi_1 + \lambda\,\varepsilon^p\psi_1^p \\ \qquad\qquad \leq \varepsilon\,\lambda\,\psi_1 \quad \text{a.e. in } \mathcal{O} \\ \varepsilon\,\psi_1 \in W^{2,\infty}(\mathcal{O}), \quad \varepsilon\,\psi_1 = 0 \text{ on } \partial\mathcal{O}\,. \end{cases}$$

On the other hand there exists K large enough such that $K \geq \varepsilon\|\psi_1\|_{L^\infty}$ and $K \geq 1$: thus

$$A(K) + \lambda K^p \geq \lambda K \text{ in } \overline{\mathcal{O}}\ ,\quad K \geq \varepsilon\psi_1 \text{ in } \overline{\mathcal{O}}\ .$$

Then, applying Theorem II.1, the existence of $\overline{u}_\lambda$ is proved.

Next, let u_λ, v_λ be two positive solutions of (33); we may assume without loss of generality that $u_\lambda \not\leq v_\lambda$ and we are going to use an argument due to H. Amann and T. Laestch [3]. Let $k = \sup(\mu \in (0,1),\ \mu u_\lambda \leq v_\lambda$, necessarily $k < 1$ and we have: $v_\lambda \geq ku_\lambda$ in $\overline{}$. Now we have:

$$A(ku_\lambda) + \lambda k\ u_\lambda^p = \lambda ku_\lambda \quad\text{and thus}\quad A(ku_\lambda) + \lambda k^p u_\lambda^p \leq \lambda ku_\lambda$$

therefore:

$$Av_\lambda - A(ku_\lambda) + \{p\ v_\lambda^{p-1} - \lambda\}(v_\lambda - ku_\lambda) \geq 0 \text{ in } \mathcal{O}\quad ,$$

and from Lemma II.1 we deduce: $v_\lambda - ku_\lambda > 0$ in $\mathcal{O}$, $\frac{\partial}{\partial n}(v_\lambda - ku_\lambda) < 0$; on $\partial\mathcal{O}$; which contradicts the definition of k. This proves the uniqueness and we conclude.

Remark II.10: We only used the fact that $f(t) = \lambda t - \lambda t^p$ satisfies (for $\lambda > \lambda_1$): $t \in W^{2,\infty}_{\ell oc}(\mathbb{R})$, $f(t)t^{-1}$ is strictly decreasing on $\mathbb{R}_+$, $f'(0) > \overline{\lambda}_1$ and $\lim_{t\to+\infty} f(t)t^{-1} < \bar{\lambda}_1$.

We now conclude by a result, announced in Remark II.9, showing that the existence of demi-eigenvalues and of split bifurcation diagrams is mainly a consequence of the Lipschitz character (and non-differentiability) of the nonlinearity arising in the operator A. The example that we give below can be interpreted in terms of optimal stochastic control but we will not do it here.

We first remark that if we take for A_1:

$$A_1 = -\Delta + \partial_{\xi_1}$$

where ξ_1 is a dense family of S^{N-1} ($|\xi_1| = 1$), then:

$$A\varphi = -\Delta\varphi + |\nabla\varphi| \quad , \quad \forall \varphi \in C^2(\bar{\mathcal{O}}) \quad .$$

Thus Theorem II.2 yields in particular: $\bar{\lambda}_1 > \lambda_1$

i) If $\lambda < \bar{\lambda}_1$ there exists a unique solution $u_\lambda \in C^2(\bar{\mathcal{O}})$ of

$$-\Delta u_\lambda + |\nabla u_\lambda| = 1 + \lambda u_\lambda \quad \text{in } \mathcal{O}, \quad u_\lambda > 0 \quad \text{in } \mathcal{O}, \quad u_\lambda = 0 \quad \text{on } \partial\mathcal{O} \quad .$$

ii) As $\lambda \to \bar{\lambda}_1$, $\dfrac{u_\lambda}{\|u_\lambda\|_{L^\infty}}$ converges in $C^1(\bar{\mathcal{O}})$ (and thus in $C^2(\bar{\mathcal{O}})$) to the unique solution ψ_1 of:

$$-\Delta\psi_1 + |\nabla\psi_1| = \bar{\lambda}_1\psi_1 \quad \text{in} \quad , \psi_1 > 0 \quad \text{in } \mathcal{O}, \quad \|\psi_1\|_{L^\infty} = 1, \quad \psi_1 = 0 \quad \text{on } \partial\mathcal{O} \quad .$$

iii) Finally, if $\psi \in C^2(\bar{\mathcal{O}})$ satisfies:

$$-\Delta\psi + |\nabla\psi| = \lambda\psi \quad \text{in } \mathcal{O}, \quad \psi > 0 \quad \text{in } \mathcal{O}, \quad \psi = 0 \quad \text{on } \partial\mathcal{O}$$

then $\lambda = \bar{\lambda}_1$ and $\psi = \theta\psi_1$ for some $\theta > 0$.

We now consider a somewhat related problem, namely:

(35) $$-\Delta u_\lambda + |\nabla u_\lambda|^\beta = 1 + \lambda u_\lambda, \quad u_\lambda > 0 \quad \text{in } \mathcal{O}, \quad u_\lambda = 0 \quad \text{on } \partial\mathcal{O} \quad ;$$

(36) $$-\Delta\tilde{u}_\lambda + |\nabla\tilde{u}_\lambda|^\beta = \lambda\tilde{u}_\lambda, \quad \tilde{u}_\lambda > 0 \quad \text{in } \mathcal{O}, \quad \tilde{u}_\lambda = 0 \quad \text{on } \partial\mathcal{O} \quad ;$$

where $\beta > 1$.

Proposition II.1:

i) For all $\lambda > 0$, there exists a unique solution u_λ of (35) and u_λ is continuous with respect to λ (for example in the space $C^2(\bar{\mathcal{O}})$).

ii) If $\lambda \leq \lambda_1$, there is no solution of (36); while if $\lambda > \lambda_1$ there exists a unique positive solution $\tilde{u}_\lambda$ of (36) and $\tilde{u}_\lambda$ is continuous with respect to λ (for example in the space $C^2(\bar{\mathcal{O}})$).

Proof of Proposition II.1: Since this proposition is not essential for our concern here, we will indicate only the main lines of its proof. We first show that if λ is bounded, then solutions of (35), (36) are a priori bounded in $W^{1,\infty}(\mathcal{O})$ (and thus in $C^2(\bar{\mathcal{O}})$): the existence can then be obtained by the techniques of P. L. Lions [28]. We will then prove the uniqueness of u_λ, $\tilde{u}_\lambda$.

The proof of the a priori bounds for u_λ, $\tilde{u}_\lambda$ are totally similar and we will do it only for u_λ. We argue as follows: first remark that we have:

$$\int_{\mathcal{O}} |\nabla u_\lambda|^2 dx + \int_{\mathcal{O}} |\nabla u_\lambda|^\beta u_\lambda dx \leq \int_{\mathcal{O}} u_\lambda dx + \lambda \int_{\mathcal{O}} u_\lambda^2 dx$$

but $\int_{\mathcal{O}} |\nabla u_\lambda|^\beta u_\lambda dx = \left(\frac{\beta}{\beta+1}\right)^\beta \int_{\mathcal{O}} |\nabla u_\lambda^{1+1/\beta}|^\beta \geq C \int_{\mathcal{O}} u_\lambda^\delta dx$ for some constant $C(= C(\beta)) > 0$ and where $\delta = \frac{N}{N-1}(\beta+1) > 2$ and this shows: $\|u_\lambda\|_{H^1} \leq C$. Therefore u_λ is bounded in $L^{2N/(N-2)}(\mathcal{O})$ (in $L^p(\mathcal{O})$, $p < \infty$ if $N \leq 2$) and remarking that we have: $-\Delta u_\lambda \leq 1 + \lambda u_\lambda$ in $\mathcal{O}$, $u_\lambda \geq 0$ we then deduce by a straightforward bootstrap argument:

$$\|u_\lambda\|_{L^\infty(\mathcal{O})} \leq C .$$

Using this bound it is easy to show that, on a convenient neighborhood of the boundary $\Gamma_\varepsilon = \{x \in \mathcal{O},\ \mathrm{dist}(x,\partial\mathcal{O}) < \varepsilon\}$, we have:

$$-\Delta(\mu\delta) + |\nabla(\mu\delta)|^\beta \geq 1 + \lambda u_\lambda \quad \text{in } \Gamma_\varepsilon$$

$$\mu\delta \geq u_\lambda \quad \text{on } \partial\Gamma_\varepsilon$$

for some large enough constant $\mu > 0$ and where $\delta(x) = \mathrm{dist}(x,\partial\mathcal{O})$. This implies: $\left\|\frac{\partial u_\lambda}{\partial n}\right\|_{L^\infty(\partial\mathcal{O})} \leq C$. And using the results of P. L. Lions [28], one then obtains:

$$\|u_\lambda\|_{W^{1,\infty}(\mathcal{O})} \leq C .$$

We conclude by giving the proof of the uniqueness of u_λ (the same argument works for $\tilde{u}_\lambda$): again we will use the device of Laetsch [27]. If v_λ, u_λ are two solutions and if $u_\lambda \not\leq v_\lambda$, we denote by: $k = \sup(\mu \in (0,1),\ \mu u_\lambda \leq v_\lambda \text{ in } \mathcal{O})$. Then $k < 1$ and $ku_\lambda \leq v_\lambda$. Next, we have:

$$-\Delta(ku_\lambda) + |\nabla(ku_\lambda)|^\beta \leq k(1 + \lambda u_\lambda) < 1 + \gamma(ku_\lambda) \leq -\Delta v_\lambda + |\nabla v_\lambda|^\beta$$

and from the maximum principle, this yields:

$$ku_\lambda < v_\lambda \quad \text{in } \mathcal{O} \quad , \quad \frac{\partial}{\partial n}(ku_\lambda - v_\lambda) > 0 \quad \text{on } \partial\mathcal{O}$$

which contradicts the definition of k.

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Two questions concerning bifurcation theory and optimal stochastic control are considered. First, in a few examples, we give the interpretation of a bifurcation in terms of optimal stochastic control. Next, we introduce the analogue of the lowest eigenvalue for the nonlinear operator associated with the Hamilton-Jacobi-Bellman equations of Optimal Stochastic Control.

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