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WEAKLY NONLINEAR

HIGH FREQUENCY WAVES

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ABSTRACT

In this paper we derive a method for finding small amplitude high frequency solutions to hyperbolic systems of quasilinear partial differential equations. Our solution is the sum of two parts: (i) a superposition of small amplitude high frequency waves; (ii) a slowly varying 'mean solution'. Each high frequency wave displays nonlinear distortion of the wave profile and shocks may form. Shock conditions are derived for conservative systems. Different high frequency waves do not interact provided the frequencies and wave numbers of two waves are not linearly related to those of a third. The mean solution is found by solving a linear partial differential equation. This method generalizes Whitham's nonlinearization technique [9] for single waves, to problems where many waves are present. We obtain these results by generalizing a scheme first proposed by Choquet-Bruhat [1] which employs the method of multiple scales.
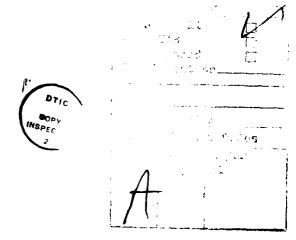
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SIGNIFICANCE AND EXPLANATION

Problems involving high frequency waves arise in many applications. For example they arise in optics (light waves propagating through a medium of slowly varying refractive index), acoustics (sound waves propagating through the atmosphere) and oceanography (water waves propagating over a gently sloping ocean bed). If the waves are sufficiently small in amplitude they satisfy linear differential equations and there is a powerful theory for calculating their behaviour called geometrical optics. Larger amplitude waves may satisfy nonlinear equations, and then they display qualitatively different effects from linear waves. The most significant effect of nonlinearity is that waves can form shocks, surfaces across which the wave amplitude changes sharply. The most familiar example of a shock is the sonic boom generated by an aircraft in supersonic flight. In this paper we develop a generalization of geometrical optics to small amplitude nonlinear waves which allows us to calculate how they propagate, and to predict effects such as the formation of shocks.

CONTENTS

WEAKLY NONLINEAR HIGH FREQUENCY WAVES

John Hunter

1. <u>Introduction</u>

In this paper we consider waves which satisfy nonlinear hyperbolic
partial differential equations. Work on such waves has been centered on one-
dimensional problems. The problems either involve only two independent
variables, usually time and one space variable, or involve only a single wave
and thus are essentially one-dimensional.

Here we develop a method for problems in any number of dimensions and
with any number of waves present. We call our method weakly nonlinear
geometrical optics, since it reduces to geometrical optics for linear
systems. It applies to small amplitude, high frequency waves which are
solutions of quasi-linear hyperbolic partial differential equations.

We obtain the first term in an asymptotic expansion for such a solution,
which is the sum of two parts: (i) a slowly varying mean solution; and (ii)
the superposition of a number of small amplitude, high frequency waves. The
profile of each high frequency wave in the superposition is distorted by the
nonlinear self-interaction of the wave. This can cause the solution for the
wave to become multivalued. To obtain a single-valued solution we must
introduce discontinuities, and for conservative systems we derive an equal
area rule that allows us to fit shocks into the wave solution.

We show that there is no interaction between different waves in the
superposition provided that (a) the mean solution is correctly chosen; and (b)
a certain resonance condition never holds. The mean solution must be chosen

so that an average of the wave profile is zero. Thus the mean solution is the average of the first term in the asymptotic expansion. It is found by solving a linear system of partial differential equations.

Resonance can occur when the frequencies and wave numbers of two waves are linearly related to those of a third. This threefold resonance is a consequence of the quadratically nonlinear interactions between the waves, and a more general form of solution than that considered here must be used if there is resonance.

The method by which we obtain these results is an extension of one proposed by Choquet-Bruhat [1], and employs the method of multiple scales. It gives exactly the same result for a single wave as the nonlinearization technique proposed by Landau [2] and Whitham [3], and later derived by Varley and Cumberbatch ([4], [5]) using the method of relatively undistorted waves.

The result that two small amplitude waves do not interact to first order in the wave amplitude, has been found previously in one dimension by Mortell and Varley [6] and Seymour and Mortell [7]. The present method generalizes this result for high frequency waves to any number of dimensions.

An outline of the contents of this paper are as follows. In section 2 we summarize the results obtained by the method of weakly nonlinear geometrical optics. The formal derivation of the equations stated in section 2 for the mean solution and the high frequency waves, is given in section 3. The equal area rule for fitting shocks into the high frequency waves is derived in section 4.

In the next two sections we compare our results with previous results. In section 5 we show that Whitham's nonlinearization technique is a consequence of the present method. This demonstrates the agreement between weakly nonlinear geometrical optics, when applied to a single wave, and the

method of relatively undistorted waves, when expanded for small amplitudes. Both methods reduce to the nonlinearization technique in the common case of a single small amplitude wave.

In section 6 we consider time-dependent plane and spherically symmetric waves in gas dynamics. The equations governing such waves involve only two independent variables and therefore can be treated using the method of characteristics. We verify the results of weakly nonlinear geometrical acoustics in this case by showing that they also follow from the method of characteristics. In particular we see that two waves do not interact if the mean solution is chosen as described above.

In section 7 we discuss conditions under which resonance can occur, giving necessary and sufficient conditions for it to happen. This gives us the conditions under which a solution consisting of the superposition of non-interacting waves is applicable.

In section 8 we consider systems with multiple characteristics and show how the results above are modified, assuming that the multiplicity of characteristics does not change.

Finally in the appendix we apply the method to the gas dynamics equations. The result is a theory of weakly nonlinear geometrical acoustics. Applications of the theory will be described in a future paper.

2. Summary of Weakly Nonlinear Geometrical Optics

We consider a strictly hyperbolic system of quasi-linear first order partial differential equations

$$(2.1) \qquad \sum_{i=1}^{n} A^{(i)}(x,u)u_{x_i} + B(x,u) = 0 \quad .$$

In (2.1) $x = (x_1,\ldots,x_n)$ is an n-vector, B and $u = (u_1,\ldots,u_N)$ are N-vectors and each $A^{(i)}$ is an $N \times N$ matrix. Usually one of the independent variables x_i is time. We assume that $A^{(i)}$ and B are continuously differentiable with respect to u.

The weakly nonlinear geometrical optics solution to (2.1) is

$$(2.2) \quad u = \bar{u}(x,\epsilon) + \epsilon \sum_{j=1}^{m} a^{(j)}(x,\phi^{(j)}(x)/\epsilon)R^{(j)}(x) + 0(\epsilon^2) \text{ (as } \epsilon \to 0) \quad .$$

The derivation of (2.2) is given in section 3. Here we explain what the various functions appearing in (2.2) are and how they are determined.

Equation (2.2) represents u as the sume of a mean solution $\bar{u}$ and a superposition of m small amplitude high frequency waves $a^{(j)}R^{(j)}$. The small parameter ϵ is introduced into the problem through initial and boundary conditions. It is the ratio of a typical dimensioned wave amplitude relative to a parameter with the same dimensions appearing in the problem. It is also the ratio of a typical wavelength relative to the length scale over which a wave is modulated. We assume that dimensionless variables can be chosen in the original problem leading to (2.1) so that both the wave amplitude and the wavelength are $0(\epsilon)$.

Now we describe how to calculate the functions in (2.2). The mean solution $\bar{u}$ satisfies (2.1) to $0(\epsilon)$. For $\bar{u}$ regular in ϵ we expand it as

$$(2.3) \qquad \bar{u} = u^{(0)}(x) + \epsilon\bar{v}(x) + 0(\epsilon^2) \quad .$$

We use (2.3) in (2.1), expand $A^{(i)}$ and B in Taylor series about $u = u^{(0)}$,

and formally equate to zero the coefficients of ε^0 and ε^1. It follows that $u^{(0)}$ is a solution of (2.1), and $\bar{v}(x)$ satisfies the system of equations that results from linearizing (2.1) about $u = u^{(0)}$.

$$(2.4) \quad \sum_{i=1}^{n} A^{(i)}(x,u^{(0)})\bar{v}_{x_i} + \sum_{i=1}^{n} A_u^{(i)}(x,u^{(0)})\bar{v}u_{x_i}^{(0)} + B_u(x,u^{(0)})\bar{v} = 0 \quad .$$

The notation $A_u v$ means $\sum_{j=1}^{N} A_{u_j} v_j$. We suppose $u^{(0)}(x)$ is a known function, and then $\bar{v}(x)$ is found from (2.4).

The sum of terms $a^{(j)} R^{(j)}$ in (2.2) is a superposition of high frequency waves. The jth wave depends on the rapidly varying phase $\phi^{(j)}(x)/\varepsilon$, and the wave is proportional to the vector $R^{(j)}(x)$ determined by the phase function $\phi^{(j)}$. The scalar $a^{(j)}(x,\phi^{(j)}(x)/\varepsilon)$ is the amplitude of the wave (if $R^{(j)}$ is normalized to unit length).

The mean of the wave amplitude over the rapidly varying phase is required to be zero. That is

$$(2.5) \quad \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T a^{(j)}(x,\theta)d\theta\right) = 0 \quad .$$

In (2.5) the mean is taken over $[0,\infty)$, but it could equally well be taken over any other semi-infinite or infinite interval.

The functions $\phi^{(j)}$, $R^{(j)}$ and $a^{(j)}$ are determined as follows. Each phase function satisfies an eiconal equation

$$(2.6) \quad \det\left(\sum_{i=1}^{n} \phi_{x_i}^{(j)} A^{(i)}(x,u^{(0)})\right) = 0 \quad .$$

Notice that (2.6) really is an equation for $\phi^{(j)}$ because $A^{(i)}(x,u)$ is evaluated at $u = u^{(0)}(x)$.

Equation (2.6) is a first order partial differential equation for a scalar. Therefore it can be written as a system of ordinary differential equations along a set of curves in $\mathbb{R}^n$. These curves are the rays or bicharacteristics of (2.1) which correspond to the solution $u = u^{(0)}$ and the

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phase function $\phi^{(j)}$. We introduce ray coordinates $(s_j, \beta_{j,1}, \ldots, \beta_{j,n-1})$, where $s_j(x)$ is a function of arclength along a ray and $\beta_j(x) = (\beta_{j,1}, \ldots, \beta_{j,n-1})$ is constant on each ray.

Then [8] the rays are solutions of

$$(2.7) \qquad \frac{dx_i}{ds_j}\bigg|_{\beta_j} = L^{(j)}(x) A^{(i)}(x, u^{(0)}) R^{(j)}(x) \qquad (i = 1, \ldots, n) \quad .$$

In (2.7) $L^{(j)}$ and $R^{(j)}$ (the vector appearing in (2.2)) are the left and right null vectors of the matrix

$$(2.8) \qquad \sum_{i=1}^{n} \phi_{x_i}^{(j)} A^{(i)} \quad .$$

In (2.8), and in the future, we omit showing explicitly that $A^{(i)}$ and B are evaluated at $u = u^{(0)}$. Since (2.1) is strictly hyperbolic $L^{(j)}$ and $R^{(j)}$ are uniquely determined up to a scalar multiple.

The amplitude of the jth wave is given by

$$(2.9) \qquad a^{(j)} = F_j(\beta_j, \zeta_j/\epsilon) E_j(s_j, \beta_j) \quad .$$

In (2.9) F_j is an arbitrary function with zero mean, which describes the wave profile by its dependence on ζ_j/ϵ. The factor E_j in (2.9) describes the modulation of the wave amplitude due to inhomogeneities and changes in the ray geometry. It is

$$(2.10) \qquad E_j = \exp\{-\int_{s_0}^{s_j} [\sum_{i=1}^{n} L^{(j)} A^{(i)} R_{x_i}^{(j)} u_{x_i}^{(0)} + L^{(j)} B_u R^{(j)}] ds_j\} \quad .$$

The integral in (2.10) is taken with respect to s_j along a ray $\beta_j = $ constant. The integral is taken from an arbitrary point $s_j = s_0(\beta_j)$ on the ray.

The wavefronts of (2.9) are given by $\zeta_j = $ constant. The modified phase function $\zeta_j(x, \epsilon)$ is defined implicitly by

$$(2.11) \qquad \zeta_j = \phi^{(j)} - \epsilon F_j(\beta_j, \zeta_j/\epsilon) I_j - \epsilon K_j \quad .$$

In (2.11) $I_j(s_j, \beta_j)$ and $K_j(s_j, \beta_j)$ are given by

$$I_j = \sum_{i=1}^{n} \int_{s_0}^{s_j} L^{(j)} \phi_{x_i}^{(j)} A_u^{(i)} R^{(j)} R^{(j)} E_j ds_j$$

(2.12)

$$K_j = \sum_{i=1}^{n} \int_{s_0}^{s_j} L^{(j)} \phi_{x_i}^{(j)} A_u^{(i)} \bar{v} R^{(j)} ds_j \quad .$$

The integrals in (2.12) are taken along a ray as in (2.10).

The expression (2.11) for ζ_j involves only F_j and does not depend on F_k for any $k \neq j$. The different high frequency waves are completely uncoupled.

If (2.1) is linear then $\zeta_j = \phi^{(j)}$ and the wavefronts are given by $\phi^{(j)}$ = constant. Equation (2.11) expresses the fact that the wavefronts ζ_j = constant differ from level surfaces of $\phi^{(j)}$ because (2.1) is nonlinear. The term $\epsilon F_j I_j$ in (2.11) gives the correction to the wavefronts because of the waves nonlinear self-interaction. The term ϵK_j gives the effect of the departure of the mean solution $\bar{u}(x, \epsilon)$ from $u^{(0)}(x)$.

Although both the correction terms in (2.11) are small in magnitude the nonlinear case is qualitatively different from the linear one. In the nonlinear case (2.11) defines $\zeta_j(x, \epsilon)$ implicitly, and because of this ζ_j can become a multivalued function of x. Then a discontinuity or shock must be introduced into the solution for $a^{(j)}$ in order to make $a^{(j)}$ a single-valued function of x.

For a conservative system such a shock satisfies an equal area rule. Suppose that the ray β_j = constant meets a shock in the jth wave at $s_j = f(\beta_j)$. Denote the two wavefronts meeting the shock at this point by $\zeta_j = z_1(\beta_j, \epsilon)$ and $\zeta_j = z_2(\beta_j, \epsilon)$. Then z_1 and z_2 are related by the equal area rule

(2.13) $\int_{z_1}^{z_2} F_j(\beta_j, \zeta/\epsilon)d\zeta = \frac{1}{2}(z_2 - z_1)\{F_j(\beta_j, z_1/\epsilon) + F_j(\beta_j, z_2/\epsilon)\}$.

We also have from (2.11) and the definition of z_1 and z_2 that

(2.14) $z_k = \phi^{(j)} - \epsilon F_j(\beta_j, z_k/\epsilon)I_j - \epsilon K_j$ $(k = 1,2)$.

All functions of x in (2.14) are evaluated at $s_j = f(\beta_j)$. Equations (2.13) and (2.14) provide three equations for the unknown functions z_1, z_2 and f. These equations are essentially the usual equations for weak one-dimensional shocks as described in [9].

Now we briefly describe how to use these results to solve an initial-boundary value problem. Suppose we are given boundary values for a small amplitude high frequency perturbation about a solution $u^{(0)}(x)$ of (2.1) which is assumed known. For example in gas dynamics $u^{(0)}$ might correspond to gas at rest with constant density.

We choose dimensionless variables in which the perturbation amplitude is $0(\epsilon)$ and the frequency is $0(1/\epsilon)$. Then $\bar{v}(x)$ is found by solving (2.4), a linear system of partial differential equations, subject to boundary values obtained by taking the mean of the perturbation's boundary values over the fast phase variables. The infinite or semi-infinite interval over which each fast variable is averaged may be choosen so as to make the boundary values for $\bar{v}$ as simple as possible.

Next the m phase functions $\phi^{(j)}(x)$ are found by solving (2.6), which can be done by integrating a system of ordinary differential equations. Initial conditions for $\phi^{(j)}$ are chosen so that $\phi^{(j)}$ takes the same value on the boundary as the appropriate rapidly varying function on which the boundary values depend. Once $\phi^{(j)}$ is known it is an algebraic problem to calculate the corresponding null vector $R^{(j)}$.

The amplitude $a^{(j)}$ is given by (2.9) and the function F_j is chosen to make (2.2) agree with the boundary conditions. The function E_j is given in (2.10) and the modified phase variable ζ_j is found from (2.11). If shocks form they are fitted into the solution using (2.13) and (2.14). This completes the solution of the boundary value problem. As an example of this procedure, a general initial value problem for the unsteady one dimensional gas dynamics equations is solved in section 6.

There are two limitations on the use of (2.2). Firstly the results above apply to the nonresonant case. Resonant interaction between the jth wave and the kth wave ($j \neq k$) can occur if there are nonzero scalars $c_j(x)$ and $c_k(x)$ such that the vector $p(x)$ defined by

$$(2.15) \qquad\qquad p = c_j \nabla\phi^{(j)} + c_k \nabla\phi^{(k)} \quad,$$

satisfies the eiconal equation

$$(2.16) \qquad\qquad \det\{ \sum_{i=1}^{n} p_i A^{(i)} \} = 0 \quad.$$

The results above apply provided (2.15) and (2.16) do not hold for any distinct j and k at any x. Included in this condition is the case $p = 0$, when $\nabla\phi^{(j)}$ and $\nabla\phi^{(k)}$ are parallel. This condition is a sufficient condition for resonance not to occur. A necessary and sufficient condition is derived in section 7.

A second limitation on the use of (2.2) is that in general the expansion will break down for large times of $O(1/\epsilon)$. One source of this nonuniformity is the expansion (2.3) of $\bar{u}$. Another is the cumulative effect of lower order (cubic) nonlinearities which we expect to become significant after times of $O(1/\epsilon)$. If necessary one could extend the validity of (2.2) to larger times by introducing further multiple scales $X = \epsilon x$.

When (2.1) is linear the equations summarized above reduce to those of geometrical optics. We finish this section by stating the major effects introduced by the nonlinearity of (2.1). There are three. Firstly for there to be no interaction between different waves the mean wave profile (2.5) must be zero. Secondly the wavefronts z_j = constant are distorted by the nonlinearity according to (2.11), and this can cause shocks to form. Lastly there is the possibility of resonant interactions between waves when (2.14) holds.

3. Derivation of the Equations

In this section we derive the equations for $\bar{v}(x)$ and $a^{(j)}$ stated in section 2. Following Choquet-Bruhat [1] we shall seek small amplitude, high frequency solutions to (2.1) of the form

$$(3.1) \qquad u = u^{(0)}(x) + \varepsilon v(x,\theta,\varepsilon) \ .$$

In (3.1) $u^{(0)}$ is a solution of (2.1), as before, and $\theta = (\theta_1,\ldots,\theta_m)$ where $\theta_j = \phi^{(j)}(x)/\varepsilon$. The v and $\phi^{(j)}$ are to be determined. The difference between (3.1) and the form of v used by Choquet-Bruhat is that in (3.1) v depends on m fast variables θ_j instead of just one. This allows the presence of many waves, permitting us to deal with the interactions between them.

We shall use the method of multiple scales, in which x and θ are treated as independent variables, to obtain an asymptotic expansion for v as $\varepsilon \to 0$. We shall assume that v, v_{x_i} and v_{θ_i} are bounded functions of x and θ and that $v(x,\theta,\varepsilon)$ has an asymptotic expansion as $\varepsilon \to 0$ of the form

$$(3.2) \qquad v(x,\theta;\varepsilon) = v^{(0)}(x,\theta) + \varepsilon v^{(1)}(x,\theta) + 0(\varepsilon^2) \ .$$

We now use (3.2) in (3.1) and substitute the result into (2.1). Then we expand $A^{(i)}(x,u)$ and $B(x,u)$ in Taylor series in powers of ε about $u = u^{(0)}$, and replace the partial derivative $\dfrac{\partial}{\partial x_i}$ by

$$\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \sum_{j=1}^{m} \phi^{(j)}_{x_i} \frac{\partial}{\partial \theta_j} \ .$$

Equating to zero the coefficients of ε^0 and ε^1 in the resulting formula gives the following equations:

$$(3.3) \qquad \sum_{j=1}^{m} \sum_{i=1}^{n} \phi^{(j)}_{x_i} A^{(i)} v^{(0)}_{\theta_j} = 0 \ ,$$

$$(3.4) \quad \sum_{j=1}^{m} \sum_{i=1}^{n} \phi_{x_i}^{(j)} A^{(i)} v_{\theta_j}^{(1)} = - \left[\sum_{i=1}^{n} (A^{(i)} v_{x_i}^{(0)} + A_u^{(i)} v^{(0)} u_{x_i}^{(0)}) \right.$$

$$\left. + \sum_{j=1}^{m} \sum_{i=1}^{n} \phi_{x_i}^{(j)} A_u^{(i)} v^{(0)} v_{\theta_j}^{(0)} + B_u v^{(0)} \right] .$$

In (3.3) and (3.4) $A^{(i)}$, $A_u^{(i)}$ and B are all evaluated at $u = u^{(0)}$.

For a single wave propagating along the j-th characteristic only one term in the sum over j in (3.3) appears, and therefore it is zero. Motivated by the idea of extending the superposition of single waves from the linear to the weakly nonlinear case, we shall look for solutions such that each term in the sum over j in (3.3) vanishes separately. Therefore we suppose that

$$(3.5) \quad \sum_{i=1}^{n} \phi_{x_i}^{(j)} A^{(i)}(x,u^{(0)}) v_{\theta_j}^{(0)} = 0 \quad \text{for} \quad j = 1,\ldots,m .$$

In order for $v_{\theta_j}^{(0)}$ to be non-zero the matrix in (3.5) must be singular. This gives the eiconal equation (2.6) for $\phi^{(j)}(x)$. Also $v_{\theta_j}^{(0)}$ must be parallel to the right null vector $R^{(j)}$ of (2.8). Therefore we seek a solution for $v^{(0)}$ of the form

$$(3.6) \quad v^{(0)} = \bar{v}(x) + \sum_{j=1}^{n} a^{(j)}(x,\theta_j) R^{(j)}(x) .$$

Clearly (3.6) satisfies (3.5) for an arbitrary vector function $\bar{v}(x)$ and arbitrary scalar functions $a^{(j)}(x,\theta_j)$. It is not the most general form of the solution to (3.5), but we shall see that a solution of this form can be found in the absence of resonance.

Without any loss of generality we take $\bar{v}(x)$ in (3.6) equal to the mean of $v^{(0)}(x,\theta)$ with respect to θ. The mean of $v^{(0)}$ is defined because $v^{(0)}$ is a bounded function of θ at each x. Then the mean of each $a^{(j)}(x,\theta_j)$ with respect to θ_j is zero. Thus:

$$\bar{v}(x) = \lim_{T \to \infty} \left\{ \frac{1}{T^m} \int_0^T \cdots \int_0^T v^{(0)}(x,\theta) d\theta_1 \cdots d\theta_m \right\} ,$$

(3.7)

$$\lim_{T \to \infty} \left\{ \frac{1}{T} \int_0^T a^{(j)}(x,\theta_j) d\theta_j \right\} = 0 .$$

The functions $\bar{v}$ and $a^{(j)}$ will be found from solvability conditions for (3.4).

The average of the derivative of a bounded function is zero. If the expansion (3.2) is uniformly valid for all x and θ, $v^{(0)}$ and $v^{(1)}$ must be bounded functions of x and θ. We use (3.6) in (3.4) and average the resulting equation over $\theta_1, \ldots, \theta_m$. The average of the left hand side is zero and we obtain equation (2.4) for the mean $\bar{v}(x)$.

We obtain equations for the m scalars $a^{(j)}(x,\theta_j)$ by using (3.6) in (3.4), averaging the result over all θ_k for $k \neq j$, and then taking the scalar product with $L^{(j)}$, the left null vector of (2.8). The left hand side of (3.4) is annihilated and we obtain an equation for $a^{(j)}$. In this equation we recognize the sum of partial derivatives with respect to the x_i as a directional derivative along the jth set of rays (2.7). We rewrite the sum of these derivatives as a derivative with respect to s_j and find that $a^{(j)}$ satisfies

(3.8)
$$a^{(j)}_{s_j} + \sum_{i=1}^{n} \{ L^{(j)} \phi^{(j)}_{x_i} A^{(i)}_u R^{(j)} R^{(j)} a^{(j)} + L^{(j)} \phi^{(j)}_{x_i} A^{(i)}_u \bar{v} R^{(j)} \} a^{(j)}_{\theta_j}$$
$$+ L^{(j)} Q^{(j)} a^{(j)} = 0 .$$

In (3.8)

(3.9)
$$Q^{(j)} = \sum_{i=1}^{n} (A^{(i)} R^{(j)}_{x_i} + A^{(i)}_u R^{(j)} u^{(0)}_{x_i}) + B_u R^{(j)} .$$

The coefficient of $a^{(j)}_{\theta_j}$ in (3.8) is the propagation speed of disturbances in the (s_j, θ_j) plane.

The equations for the scalars $a^{(j)}$ are uncoupled. No $a^{(k)}$ with $k \neq j$ appears in (3.8). Equation (3.8) is a nonlinear equation for $a^{(j)}$ in two independent variables. It reduces to the transport equation of (linear) geometrical optics if equation (2.1) is linear, when the coefficient of $a^{(j)}_{\theta_j}$ in (3.8) is zero.

If the resonance condition (2.4) holds then there is a stronger solvability condition than (3.8) for (3.4) and it is not always possible to find a solution of the form (3.6) for $v^{(0)}$. This is discussed further in section 7. Here we assume that there is no resonance.

Then $a^{(j)}$ satisfies (3.8) which we can integrate by introducing characteristics. Let

$$M_j(s_j,\beta_j) = \sum_{i=1}^{n} L^{(j)}\phi^{(j)}_{x_i} A^{(i)}_u R^{(j)} R^{(j)} \ ,$$

(3.10)
$$N_j(s_j,\beta_j) = \sum_{i=1}^{n} L^{(j)}\phi^{(j)}_{x_i} A^{(i)}_u \bar{v} R^{(j)} \ ,$$

$$P_j(s_j,\beta_j) = L^{(j)} Q^{(j)} \ .$$

Then (3.8) is

(3.11)
$$a^{(j)}_{s_j} + (M_j a^{(j)} + N_j) a^{(j)}_{\theta_j} + P_j a^{(j)} = 0 \ .$$

We may write (3.11) as

(3.12)
$$\frac{d}{ds_j} a^{(j)} = -P_j a^{(j)} \ ,$$

on the characteristic curves

(3.13)
$$\frac{d}{ds_j} \theta_j = M_j a^{(j)} + N_j \ .$$

The solution to (3.12) is

(3.14)
$$a^{(j)} = F_j(\beta_j,\xi) E_j(s_j,\beta_j) \ .$$

In (3.14) F_j is an arbitrary function depending on β_j and a scalar ξ which parametrizes the set of characteristic curves (3.13). The function E_j is defined by (2.10).

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We use (3.14) in (3.13) and integrate keeping ξ constant. The result is

$$(3.15) \qquad \theta_j = F_j(\beta_j,\xi) \int_0^{s_j} M_j(s,\beta_j) E_j(s,\beta_j) ds + \int_0^{s_j} N_j(s,\beta_j) ds$$
$$+ \Xi_j(\beta_j,\xi) \ .$$

In (3.15) K_j is an arbitrary function which determines how ξ parametrizes the curves (3.13). Usually we take $\Xi_j(\beta_j,\xi) \equiv \xi$.

The final step in the method of multiple scales is to let $\theta_j = \phi^{(j)}(x)/\epsilon$. We use this in (3.15) and also let $\xi = \zeta_j/\epsilon$ in (3.14) and (3.15). Then we obtain (2.9) from (3.14) and (2.11) from (3.15) after multiplying through by ϵ.

The only equation that we still have to derive is the equal area rule (2.13) and we shall do this in the next section.

4. Derivation of the Shock Conditions

In this section we suppose that (2.1) comes from a conservation law and derive the equal area rule (2.13) for $a^{(j)}$ from the Rankine-Hugoniot shock conditions for (2.1).

Therefore suppose that (2.1) can be written in conservation form

$$(4.1) \qquad \sum_{i=1}^{n} \{F^{(i)}(x,u)\}_{x_i} + H(x,u) = 0 \quad .$$

In (4.1) $F^{(i)}$ and H are N-vectors such that

$$F_u^{(i)}(x,u) = A^{(i)}(x,u) \quad ,$$

$$(4.2)$$

$$H(x,u) + \sum_{i=1}^{n} F_{x_i}^{(i)}(x,u) = B(x,u) \quad .$$

The $F^{(i)}$ correspond to conserved quantities, and H is the source density of these quantities.

The generalized Rakine-Hugoniot shock conditions for (4.1) which hold across a shock front, are

$$(4.3) \qquad \sum_{i=1}^{n} [F^{(i)}]n_i = 0 \quad .$$

In (4.3) $[F^{(i)}]$ denotes the change in $F^{(i)}$ across the shock front, and n is the normal to the shock.

We consider the jth wave with amplitude $a^{(j)}$. Linear theory predicts that a shock in this wave travels along the linear characteristics $\phi^{(j)}(x) =$ constant. This should be the zeroth order result for weakly nonlinear waves. We suppose that the position of a shock in the j-th wave is given by

$$(4.4) \qquad h(x,\theta;\epsilon) = 0 \quad .$$

From the linear theory $h(x,\theta;0)$ should be a function of x and θ_j alone, since then the shock position is

$$\phi^{(j)}(x) = \text{constant} + 0(\varepsilon) \quad .$$

Therefore we seek a power series expansion of (4.4) in the form

$$(4.5) \qquad h(x,\theta;\varepsilon) = h^{(0)}(x,\theta_j) + \varepsilon h^{(1)}(x,\theta) + 0(\varepsilon^2) \quad .$$

We assume that $h^{(0)}$, $h^{(1)}$ and their derivatives are bounded functions of x and θ.

The normal n to the shock front is proportional to ∇h and its component n_i is proportional to

$$(4.6) \qquad h_{x_i} = \frac{1}{\varepsilon} \phi^{(j)}_{x_i} h^{(0)}_{\theta_j} + h^{(0)}_{x_i} + \sum_{k=1}^{m} \phi^{(k)}_{x_i} h^{(1)}_{\theta_k} + 0(\varepsilon) \quad .$$

We use (3.1) and expand $[F^{(i)}]$ in a Taylor series in powers of ε about $u = u^{(0)}$. We assume $u^{(0)}$ is continuous across the shock. Using (4.2) this gives

$$(4.7) \qquad [F^{(i)}] = \varepsilon A^{(i)}(x,u^{(0)})[v] + \frac{1}{2} \varepsilon^2 A_u^{(i)}(x,u^{(0)}) + 0(\varepsilon^3) \quad .$$

We use (4.6) and (4.7) in (4.3) and equate the coefficients of ε^0 and ε^1 to zero. This gives

$$(4.8) \qquad h^{(0)}_{\theta_j} \sum_{i=1}^{n} \phi^{(j)}_{x_i} A^{(i)}[v^{(0)}] = 0 \quad ,$$

$$(4.9) \qquad h^{(0)}_{\theta_j} \sum_{i=1}^{n} \phi^{(j)}_{x_i} A^{(i)}[v^{(1)}] = -\left\{ \frac{1}{2} h^{(0)}_{\theta_j} \sum_{i=1}^{n} \phi^{(j)}_{x_i} A_u^{(i)}[v^{(0)}v^{(0)}] \right.$$

$$+ \sum_{i=1}^{n} h^{(0)}_{x_i} A^{(i)}[v^{(0)}]$$

$$\left. + \sum_{i=1}^{n} \sum_{k=1}^{m} h^{(1)}_{\theta_k} \phi^{(k)}_{x_i} A^{(i)}[v^{(0)}] \right\} \quad .$$

Equation (4.8) is satisfied if $[v^{(0)}]$ is proportional to $R^{(j)}$. Therefore we suppose that $\bar{v}$ and $a^{(k)}(k \neq j)$ are continuous across the shock. Then from (3.6)

$$(4.10) \qquad [v^{(0)}] = [a^{(j)}]R^{(j)} \quad .$$

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Now we use (3.6) and (4.10) in (4.9). We average the resulting equation over all θ_k, $k \neq j$, and take the scalar product with $L^{(j)}$. The result is

$$(4.11) \quad \sum_{i=1}^{n} \left\{ [a^{(j)}] L^{(j)} A^{(i)} R^{(j)} h_{x_i}^{(0)} + \frac{1}{2} ([a^{(j)2}] L^{(j)} \phi_{x_i}^{(j)} A_u^{(i)} R^{(j)} R^{(j)} \right.$$

$$\left. + [a^{(j)}] L^{(j)} \phi_{x_i}^{(j)} A_u^{(j)} (\bar{v} R^{(j)} + R^{(j)} \bar{v}) h_{\theta_j}^{(0)} \right\} = 0 \quad .$$

In (4.11) the sum of partial derivatives with respect to x_i can be written, as before, as a derivative with respect to s_j. Then we divide (4.11) through by $[a^{(j)}]$ to obtain:

$$(4.12) \quad h_{s_j}^{(0)} + \sum_{i=1}^{n} \left\{ L^{(j)} \phi_{x_i}^{(j)} A_u^{(i)} R^{(j)} R^{(j)} \langle a^{(j)} \rangle \right.$$

$$\left. + \frac{1}{2} L^{(j)} \phi_{x_i}^{(j)} A_u^{(i)} (\bar{v} R^{(j)} + R^{(j)} \bar{v}) h_{\theta_j}^{(0)} \right\} = 0 \quad .$$

In (4.12) we have written $[a^{(j)2}]/2[a^{(j)}]$ as $\langle a^{(j)} \rangle$ where

$$(4.13) \qquad \langle a^{(j)} \rangle \equiv \frac{1}{2} \{a_+^{(j)} + a_-^{(j)}\} \quad .$$

In (4.13) $a_+^{(j)}$ and $a_-^{(j)}$ are the limiting values of $a^{(j)}$ on the two sides of the shock, and $\langle a^{(j)} \rangle$ is their average.

We claim that for systems derived from a conservation law, if S and T are any N-vectors,

$$(4.14) \qquad\qquad A_u^{(i)} ST = A_u^{(i)} TS \quad .$$

<u>Proof.</u> From (4.2)

$$A_{jk}^{(i)} = \frac{\partial F_j^{(i)}}{\partial u_k} \quad .$$

Therefore

$$A_u^{(i)}{}_{ST} = \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{\partial^2 F_j^{(i)}}{\partial u_k \partial u_\ell} S_\ell T_k$$

$$= \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{\partial^2 F_j^{(i)}}{\partial u_\ell \partial u_k} T_k S_\ell$$

$$= A_u^{(i)}{}_{TS} \quad . \quad \blacksquare$$

We use (4.14), with $S = R^{(j)}$ and $T = \bar{v}$, in (4.12) and divide through by $h_{\theta_j}^{(0)}$ to obtain the following expression for $-h_{s_j}^{(0)}/h_{\theta_j}^{(0)}$, which is the speed of the shock in the (s_j, θ_j) plane:

$$
\text{(4.15)} \qquad
\begin{aligned}
-h_{s_j}^{(0)}/h_{\theta_j}^{(0)} = \sum_{i=1}^{n} \{ & L^{(j)} \phi_{x_i}^{(j)} A_u^{(i)} R^{(j)} R^{(j)} \langle a^{(j)} \rangle \\
& + L^{(j)} \phi_{x_i}^{(j)} A_u^{(i)} \bar{v} R^{(j)} \} \quad .
\end{aligned}
$$

Comparing equation (4.15) with (3.8), we see (4.15) expresses the well known fact that the speed of a weak shock is the average of the propagation speeds on the two sides of the shock. The shock problem (3.11) and (4.15) always involves only two independent variables, s_j and θ_j, however many independent variables there are in (2.1), and the equal area rule (2.13) then follows from (4.15) exactly as described in [9].

5. Comparison with the Nonlinearization Technique

Landau [2] and Whitham [3] independently proposed a technique for nonlinearizing geometrical optics. In the technique, the deviation of the rays from their linear position is neglected, but in treating the propagation of disturbances along the rays the first order nonlinear correction to the propagation speed is taken into account.

We consider a single wave propagating on one family of characteristics with amplitude α. Whitham describes the nonlinearization technique, as follows [9].

Geometrical optics provides a ray geometry and gives along each ray

$$(5.1) \qquad\qquad \alpha = f(\phi)E(s) \; .$$

In (5.1), s is arclength measured along the ray. The function $E(s)$ gives the wave amplitude, $f(\phi)$ gives the wave profile, and $\phi(s)$ is the phase defined by

$$(5.2) \qquad\qquad \phi(s) = t - \int_0^s \frac{ds'}{c_0(s')} \; .$$

In (5.2), $c_0(s)$ is the velocity at which disturbances are propagated along the ray according to linear theory.

The nonlinear velocity on the ray $c(s,\alpha)$ is expanded for small α as

$$(5.3) \qquad\qquad c(s,\alpha) = c_0(s) + c_1(s)\alpha + 0(\alpha^2) \; .$$

Then the result of nonlinearizing (5.1) is

$$(5.4) \qquad\qquad \alpha = f(\tau)E(s) \; .$$

In (5.4), the nonlinearized characteristic variable τ is defined implicitly by

$$(5.5) \qquad t - \int_0^s \frac{ds'}{c_0(s')} = \tau - f(\tau) \int_0^s c_1(s')c_0^{-2}(s')E(s')ds' \; .$$

We show that (5.4) and (5.5) follow from weakly nonlinear geometrical optics. Let the wave amplitude be εa. Then a satisfies equation (3.11).

That is

(5.6)
$$a_s + \{M(s)a + N(s)\}a_\theta + P(s)a = 0 \quad .$$

In (5.6), $M(s)$, $N(s)$ and $P(s)$ are given by (3.10). Also $\theta = \phi/\varepsilon$, where ϕ is a solution to the eiconal equation (2.6). We may take ϕ to be given by (5.2), provided the matrices $A^{(i)}$ are independent of t.

Suppose we linearize (2.1) about $u = u^{(0)}$ and use linear geometrical optics. The transport equation is obtained by dropping the terms in (5.6) proportional to the square of the wave amplitude. That is we drop $M(s)aa_\theta$, and $N(s)a_\theta$ since $N(s)$ is proportional to the mean $\bar{v}(x)$.

Thus the equation for a obtained by linear theory is

(5.7)
$$a_s + P(s)a = 0 \quad .$$

The solution to (5.7) is

(5.8)
$$a = F(\theta)E(s) \quad .$$

In (5.8), $F(\theta)$ is an arbitrary function and $E(s)$ is given by (2.10).

Let us compare (5.8) with the solution according to the weakly nonlinear theory. We write (5.6) as

(5.9)
$$\frac{da}{ds} + P(s)a = 0 \quad ,$$

on

(5.10)
$$\frac{d\theta}{ds} = M(s)a + N(s) \quad .$$

The solution to (5.9) is

(5.11)
$$a = F(\xi)E(s) \quad .$$

We use (5.11) in (5.10) and integrate taking our constant of integration to be ξ. Then we let $\theta = \phi/\varepsilon$, use (5.2) and multiply through by ε. In this way we find that

(5.12)
$$t - \int_0^s \frac{ds'}{c_0(s')} = \varepsilon\xi + \varepsilon F(\xi) \int_0^s M(s')E(s')ds'$$
$$+ \varepsilon \int_0^s N(s')ds' \quad .$$

We see from the weakly nonlinear solution (5.11) and (5.12) that the amplitude a has the same form as in the linear theory, but the characteristic variable ξ is constant along characteristics (5.12) determined using nonlinear theory. This is just what is supposed in the nonlinearization technique.

When $N(s) = 0$, (5.4) and (5.5) agree exactly with (5.11) and (5.12) if we let $\tau = \varepsilon\xi$ and $f(\tau) = \varepsilon F(\tau/\varepsilon)$, provided

$$(5.13) \qquad M(s) = -c_1(s)c_0^{-2}(s) \quad .$$

We show that (5.13) is the case.

Suppose u satisfies

$$(5.14) \qquad u_t + \sum_{i=1}^{n} A^{(i)}(x,u)u_{x_i} + B(x,u) = 0 \quad .$$

The eiconal equation is

$$(5.15) \qquad \det\{\Phi_t I + \sum_{i=1}^{n} \Phi_{x_i} A^{(i)}(x,u)\} = 0 \quad .$$

The phase speed $c(x,u)$ equals $-\Phi_t/|\nabla\Phi|$, and from (5.15) satisfies

$$(5.16) \qquad \det\{c(u)I - \sum_{i=1}^{n} \eta_i A^{(i)}(u)\} = 0 \quad .$$

The speed c also depends on the direction of the normal to the characteristic surface, $\eta = \dfrac{\nabla\Phi}{|\nabla\Phi|}$. We omit showing the dependence of c and $A^{(i)}$ on x in (5.16).

Let $L(u)$ and $R(u)$ be the left and right null vectors of $c(u)I - \sum_{i=1}^{n} \eta_i A^{(i)}(u)$. Then

$$(5.17) \qquad \sum_{i=1}^{n} \eta_i A^{(i)}(u)R(u) = c(u)R(u) \quad .$$

We expand about $u = 0$ (there is no loss in generality in taking $u^{(0)} = 0$):

$$c(u) = c_0 + c_u u + 0(u^2) \quad ,$$

$$R(u) = R + R_u u + 0(u^2) \quad ,$$

(5.18)

$$L(u) = L + L_u u + 0(u^2) \quad ,$$

$$\sum_{i=1}^{n} \eta_i A^{(i)}(u) = \sum_{i=1}^{n} \eta_i A^{(i)} + \sum_{i=1}^{n} \eta_i A_u^{(i)} u + 0(u^2) \quad .$$

On the right hand side R, L, $A^{(i)}$ and their derivatives are evaluated at $u = 0$.

We use (5.18) in (5.17) and equate coefficients of powers of u. Then

(5.19)
$$\sum_{i=1}^{n} \eta_i A^{(i)} R = c_0 R \quad ,$$

(5.20)
$$\left(\sum_{i=1}^{n} \eta_i A_u^{(i)} u - c_u u I \right) R = - \left(\sum_{i=1}^{n} \eta_i A^{(i)} - c_0 I \right) R_u u \quad .$$

From (5.19), R is the right null vector of $\sum_{i=1}^{n} \eta_i A^{(i)} - c_0 I$. Similarly L is the left null vector. Then we take the scalar produce to (5.20) with L and solve the resulting equation for $c_u u$. We find

(5.21)
$$c_u u = \frac{\sum_{i=1}^{n} \eta_i L A_u^{(i)} u R}{LR} \quad .$$

To find the velocity along the linear rays, we take $\eta_i = \phi_{x_i} / |\nabla \phi|$ in (5.21) where ϕ is given by (5.2). Then since $c_0 = -\phi_t / |\nabla \phi|$ and $\phi_t = 1$.

(5.22)
$$\eta_i = -c_0 \phi_{x_i} \quad .$$

Next the derivative with respect to s in (5.6) is taken along the rays

(5.23)
$$\frac{dt}{ds} = LR \quad , \quad \frac{d}{ds} x_i = L A^{(i)} R \quad .$$

We differentiate (5.2) with respect to s, and use $\frac{d\phi}{ds} = 0$, since ϕ is constant on the rays. The result is

(5.24)
$$\frac{dt}{ds} = c_0^{-1} \quad .$$

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Therefore from (5.23) and (5.24)

$$(5.25) \qquad LR = c_0^{-1} \ .$$

(This is the normalization of L and R required so that the parameter s is arclength.)

We use (5.22) and (5.25) in (5.2). Then

$$(5.26) \qquad c_u u = -c_0^2 \sum_{i=1}^{n} \phi_{x_i} L A_u^{(i)} u R \ .$$

Finally we let $u = \alpha R$ in (5.26), and use the result in (5.18). Then

$$(5.27) \qquad c(u) = c_0 - \alpha c_0^2 M(s) + 0(\alpha^2) \ .$$

In (5.27), we have used (3.10) to write $\sum_{i=1}^{n} \phi_{x_i} L A_u^{(i)} R R = M(s)$. Now, c was expanded in the nonlinearization technique as (5.3). We see that (5.13) follows from (5.3) and (5.27). This shows that for weak waves the nonlinearization technique follows from weakly nonlinear geometrical optics, when we apply it to single waves $(N = 0)$.

6. Comparison with the Method of Characteristics

Systems of hyperbolic equations in two independent and two dependent variables can be solved by introducing characteristic coordinates [10]. In this section we show that the results calculated using characteristic coordinates for the one dimensional gas dynamics equations agree to $O(\epsilon)$ with those calculated using weakly nonlinear geometrical acoustics.

The gas dynamics equations for plane $(N = 0)$ axisymmetric $(N = 1)$, and spherically symmetric $(N = 2)$ flows are

$$c_t + uc_x + \frac{\gamma-1}{2} c(u_x + N \frac{u}{x}) = 0 \quad,$$

(6.1)

$$u_t + uu_x + \frac{2}{\gamma-1} cc_x = 0 \quad.$$

In (6.1) u is the gas velocity and c is the local sound speed. To be definite we shall consider an initial value problem for (6.1), and take as initial values

$$c(x,t=0) = c_0 + \epsilon c_0 g(x,x/\epsilon) + O(\epsilon^2)$$

(6.2)

$$u(x,t=0) = \epsilon c_0 f(x,x/\epsilon) + O(\epsilon^2) \quad.$$

In (6.2) $f(x,\theta)$ and $g(x,\theta)$ are arbitrary continuously differentiable, bounded functions.

Here we shall solve (6.1) and (6.2) for plane and spherical flows. The results for axisymmetric flows are entirely analogous to those for spherical flows, although the formulae are more complicated. The details are given in [11].

We shall solve (6.1) and (6.2), first by weakly nonlinear geometrical acoustics and second using characteristic coordinates. Then we shall show that the two solutions agree.

To solve (6.1) and (6.2) by weakly nonlinear geometrical acoustics we seek a solution of the form (A.16) with $c_0(x,t) = c_0$ and $U(x,t) = 0$. In this one-dimensional problem there are only sound waves. The phase functions $\phi^{(j)}(x,t)$ satisfy (A.7) which is

$$(6.3) \qquad (\phi_t^{(j)})^2 - c_0^2(\phi_x^{(j)})^2 = 0 \quad .$$

We want the fast variables $\phi^{(j)}/\varepsilon$ in (A.16) to equal the fast variable x/ε appearing in (6.2) at $t = 0$. Therefore

$$(6.4) \qquad \phi^{(j)}(x,t=0) = x \quad .$$

There are two solutions, $\phi^{(j)} = \phi^{(\pm)}$, to (6.3) and (6.4):

$$(6.5) \qquad \phi^{(\pm)}(x,t) = x \mp c_0 t \quad .$$

From (A.18) the corresponding null vectors are

$$(6.6) \qquad R^{(\pm)} = c_0 \begin{bmatrix} 1 \\ \pm 2/(\gamma-1) \end{bmatrix} \quad ,$$

where we have chosen the arbitrary scalar $\lambda_{\pm}$ in (A.18) to be

$$(6.7) \qquad \lambda_{\pm} = \pm 2c_0/(\gamma-1) \quad .$$

Using (6.5) and (6.6) in (A.16) we have

$$(6.8) \qquad \begin{bmatrix} c \\ u \end{bmatrix} = \begin{bmatrix} c_0 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{c}(x,t) \\ \bar{u}(x,t) \end{bmatrix} + \varepsilon c_0 a^{(+)}\left(x,t,\frac{x-c_0 t}{\varepsilon}\right) \begin{bmatrix} 1 \\ 2/(\gamma-1) \end{bmatrix}$$
$$+ \varepsilon c_0 a^{(-)}\left(x,t,\frac{x+c_0 t}{\varepsilon}\right) \begin{bmatrix} 1 \\ -2/(\gamma-1) \end{bmatrix} + 0(\varepsilon^2) \quad .$$

The means $\bar{c}$ and $\bar{u}$ satisfy (A.17) with $U = 0$

$$(6.9) \qquad \bar{c}_t + \frac{\gamma-1}{2} c_0(\bar{u}_x + N\frac{\bar{u}}{x}) = 0 \quad ,$$
$$\bar{u}_t + \frac{2}{\gamma-1} c_0 \bar{c}_x = 0 \quad .$$

Initial conditions for (6.9) are found by averaging the initial conditions (6.2) for (6.1):

$$(6.10) \quad \begin{aligned} \bar{c}(x, t=0) &= \bar{g}(x) \quad , \\ \bar{u}(x, t=0) &= \bar{f}(x) \quad . \end{aligned}$$

In (6.10) we have defined

$$(6.11) \quad \bar{f}(x) = \lim_{T \to \infty} \{ \frac{1}{T} \int_0^T f(x, \theta) d\theta \}, \quad \bar{g}(x) = \lim_{T \to 0} \{ \frac{1}{T} \int_0^T g(x, \theta) d\theta \} \quad .$$

For $N = 0$ the solution to (6.9) and (6.10) is

$$(6.12) \quad \begin{aligned} \bar{c} &= \bar{p}_0(x - c_0 t) + \bar{q}_0(x + c_0 t) \quad , \\ \bar{u} &= \frac{2}{\gamma - 1} \{ \bar{p}_0(x - c_0 t) - \bar{q}_0(x + c_0 t) \} \quad , \end{aligned}$$

where

$$(6.13) \quad \begin{aligned} \bar{p}_0(x) &= \frac{c_0}{2} \{ \frac{\gamma - 1}{2} \bar{f}(x) + \bar{g}(x) \} \quad , \\ \bar{q}_0(x) &= \frac{c_0}{2} \{ -(\frac{\gamma - 1}{2}) \bar{f}(x) + \bar{g}(x) \} \quad . \end{aligned}$$

For $N = 2$ the solution to (6.9) and (6.10) is [9]

$$(6.14) \quad \begin{aligned} \bar{c} &= \frac{\bar{p}_2(x - c_0 t) + \bar{q}_2(x + c_0 t)}{x} \quad , \\ \bar{u} &= \frac{2}{\gamma - 1} \{ \frac{\bar{p}_2(x - c_0 t) - \bar{q}_2(x + c_0 t)}{x} - \frac{\bar{P}_2(x - c_0 t) - \bar{Q}_2(x + c_0 t)}{x^2} \} \quad , \end{aligned}$$

where

$$(6.15) \quad \begin{aligned} \bar{p}_2(x) &= \frac{d}{dx} \bar{P}_2(x) = \frac{c_0}{2} \{ \frac{\gamma - 1}{2} x \bar{f}(x) + \frac{\gamma - 1}{2} \int^x \bar{f}(x') dx' + x \bar{g}(x) \} \quad , \\ \bar{q}_2(x) &= \frac{d}{dx} \bar{Q}_2(x) = \frac{c_0}{2} \{ -(\frac{\gamma - 1}{2}) x \bar{f}(x) - (\frac{\gamma - 1}{2}) \int^x \bar{f}(x') dx' + x \bar{g}(x) \} \quad . \end{aligned}$$

Now we calculate $a^{(\pm)}$. Ray coordinates $(s_\pm, \beta_\pm)$ corresponding to $\phi^{(\pm)}$ are

$$(6.16) \quad s_\pm = x \pm c_0 t \quad , \quad \beta_\pm = x \mp c_0 t \quad .$$

With ray coordinates (6.16) we must take μ_j in (A.8) to be

(6.17)
$$\mu_j = -c_0^{-2}/2 \ .$$

From (6.5) we have

(6.18)
$$\phi_{tt}^{(\pm)} - c_0^2 \Delta\phi^{(\pm)} = -Nc_0^2/x \ .$$

We use (6.17) and (6.18) in (A.10) then integrate with respect to

$s_\pm = x \pm c_0 t$ along the ray $\beta_\pm = x \mp c_0 t = $ constant. This gives

(6.19)
$$E^{(\pm)} = x^{-N/2} \ .$$

Equation (6.19) is the usual result of linear geometrical acoustics: the

cross-sectional area of a ray tube increases like x^N, so the wave amplitude

decays like $x^{-N/2}$ in order that energy (which is proportional to the

amplitude squared) is conserved along the ray tube.

We use (6.19) in (2.9) and obtain

(6.20)
$$a^{(\pm)} = F_\pm^{(N)}(x \mp c_0 t, \ \zeta_\pm/\epsilon)x^{-N/2} \ .$$

The functions $F_\pm^{(N)}$ are found from the initial conditions. The fast variable

$\zeta_\pm/\epsilon$ in (6.20) should equal the fast variable x/ϵ in (6.2) at $t = 0$.

Therefore

(6.21)
$$\zeta_\pm(x,t=0,\epsilon) = x \ .$$

Then we use (6.20) in (6.8), substitute the result into (6.2), use (6.21), and

solve for $F_\pm^{(N)}$. This gives

$$F_+^{(N)}(x,\zeta/\epsilon) = x^{N/2} \frac{\gamma-1}{4} \{f(x,\zeta/\epsilon) - \bar{f}(x)\} + \frac{x^{N/2}}{2} \{g(x,\zeta/\epsilon) - \bar{g}(x)\} \ ,$$
(6.22)
$$F_-^{(N)}(x,\zeta/\epsilon) = -x^{N/2} \frac{\gamma-1}{4} \{f(x,\zeta/\epsilon) - \bar{f}(x)\} + \frac{x^{N/2}}{2} \{g(x,\zeta/\epsilon) - \bar{g}(x)\} \ .$$

We use (6.5), (6.7), (6.16), (6.17) and (6.19) in (A.11) and integrate.

To satisfy (6.21) we take $s_0(\beta_\pm) = \beta_\pm$ so that $\zeta_\pm = \phi^{(\pm)} = x$ at $t = 0$,

when $s_\pm = \beta_\pm$. Thus we find

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(6.23) $\quad \zeta_{\pm} = \phi^{(\pm)} - \epsilon \frac{\gamma+1}{\gamma-1} F_{\pm}^{(N)}(x \mp c_0 t, \zeta_{\pm}/\epsilon) h_{\pm}^{(N)}(x,t) \mp \frac{\epsilon}{2c_0} \int_{x-c_0 t}^{x+c_0 t} (\bar{c} \pm \bar{u}) ds_{\pm}$.

In (6.23)

$$h_{\pm}^{(0)} = \pm c_0 t \quad ,$$

(6.24) $\qquad\qquad\qquad\qquad h_{\pm}^{(2)} = \log \left| \frac{x}{x \mp c_0 t} \right| \quad .$

The integral of $\bar{c}$ and $\bar{u}$ in (6.23) is taken with respect to $s_{\pm}$ keeping $\beta_{\pm}$ constant.

This completes the solution of (6.1) and (6.2) by the method of weakly nonlinear geometrical acoustics. The solution is given by (6.8). In (6.8) $\bar{c}$ and $\bar{u}$ are given by (6.12) and (6.13) (N=0) or (6.14) and (6.15) (N=2). The amplitudes $a^{(\pm)}$ are given by (6.20) and (6.22) - (6.24).

Now we solve the same problem using characteristic coordinates. When written with respect to characteristic coordinates A and B (6.1) become [8]

(6.25)
$$\frac{2}{\gamma-1} c_A - u_A + \frac{Nuc}{x} t_A = 0 \quad , \quad x_A = (u-c)t_A \quad ,$$

$$\frac{2}{\gamma-1} c_B + u_B + \frac{Nuc}{x} t_B = 0 \quad , \quad x_B = (u+c)t_B \quad .$$

Without any loss of generality we can impose that $A = B = x$ at $t = 0$. Then from (6.2) the initial conditions for (6.25) are

$$c(A,B=A) = c_0 + \epsilon c_0 g(A,A/\epsilon) + O(\epsilon^2) \quad , \quad x(A,B=A) = A \quad ,$$

(6.26)

$$u(A,B=A) = \epsilon c_0 f(A,A/\epsilon) + O(\epsilon^2) \qquad , \quad t(A,B=A) = 0 \quad .$$

We shall solve (6.25) and (6.26) by seeking a generalized asymptotic expansion for c, u, x and t of the form

$$c = c_0 + \epsilon c^{(1)}(A,B,\epsilon) + 0(\epsilon^2) \quad ,$$

$$u = \epsilon u^{(1)}(A,B,\epsilon) + 0(\epsilon^2) \quad ,$$

(6.27)

$$x = x^{(0)}(A,B,\epsilon) + \epsilon x^{(1)}(A,B,\epsilon) + 0(\epsilon^2) \quad ,$$

$$t = t^{(0)}(A,B,\epsilon) + \epsilon t^{(1)}(A,B,\epsilon) + 0(\epsilon^2) \quad .$$

We use (6.27) in (6.25) and (6.26), and equate explicit powers of ϵ. From the coefficients of ϵ^0 we find that

$$x_A^{(0)} + c_0 t_A^{(0)} = 0 \quad , \quad x^{(0)}(A,B=A) = A \quad ,$$

(6.28)

$$x_B^{(0)} - c_0 t_B^{(0)} = 0 \quad , \quad t^{(0)}(A,B=A) = 0 \quad .$$

The solution to (6.28) is

$$x^{(0)} = \frac{1}{2}(A+B) \quad ,$$

(6.29)

$$c_0 t^{(0)} = \frac{1}{2}(B-A) \quad .$$

Equating the coefficients of ϵ^1 and using (6.29) we find that $u^{(1)}$ and $c^{(1)}$ satisfy

$$\frac{2}{\gamma-1} c_A^{(1)} - u_A^{(1)} - \frac{N u^{(1)}}{A+B} = 0 \quad , \quad c^{(1)}(A,B=A,\epsilon) = c_0(g(A,A/\epsilon) \quad ,$$

(6.30)

$$\frac{2}{\gamma-1} c_B^{(1)} + u_B^{(1)} + \frac{N u^{(1)}}{A+B} = 0 \quad , \quad u^{(1)}(A,B=A,\epsilon) = c_0 f(A,A/\epsilon) \quad .$$

Also $x^{(1)}$ and $t^{(1)}$ satisfy

$$x_A^{(1)} + c_0 t_A^{(1)} = \{-c^{(1)} + u^{(1)}\}/2c_0 \quad , \quad x^{(1)}(A,B=A,\epsilon) = 0 \quad ,$$

(6.31)

$$x_B^{(1)} - c_0 t_B^{(1)} = \{c^{(1)} + u^{(1)}\}/2c_0 \quad , \quad t^{(1)}(A,B=A,\epsilon) = 0 \quad .$$

Equations (6.30) have the same form as the linearized gas dynamics equations, so we can write down their solution. For plane flows (N=0) the solution to (6.30) is

$$c^{(1)}(A,B,\varepsilon) = p_0(A,A/\varepsilon) + q_0(B,B/\varepsilon) \quad,$$

(6.32)

$$u^{(1)}(A,B,\varepsilon) = \frac{2}{\gamma-1} \{p_0(A,A/\varepsilon) - q_0(B,B/\varepsilon)\} \quad,$$

where

$$p_0(A,A/\varepsilon) = \frac{c_0}{2} \{\frac{\gamma-1}{2} f(A,A/\varepsilon) + g(A,A/\varepsilon)\} \quad,$$

(6.33)

$$q_0(B,B/\varepsilon) = \frac{c_0}{2} \{-(\frac{\gamma-1}{2})f(B,B/\varepsilon) + g(B,B/\varepsilon)\} \quad.$$

For spherically symmetric flows (N=2) the solution to (6.30) is

$$c^{(1)}(A,B,\varepsilon) = \frac{2}{A+B} \{p_2(A,A/\varepsilon,\varepsilon) + q_2(B,B/\varepsilon,\varepsilon)\}$$

(6.34)

$$u^{(1)}(A,B,\varepsilon) = \frac{2}{\gamma-1} \{\frac{2}{A+B}(p_2(A,A/\varepsilon,\varepsilon) - q_2(B,B/\varepsilon,\varepsilon)) -$$
$$- \frac{4}{(A+B)^2} (P_2(A,\varepsilon) - Q_2(B,\varepsilon))\} \quad.$$

In (6.34)

$$p_2(A,A/\varepsilon,\varepsilon) = \frac{d}{dA} P_2(A,\varepsilon) = \frac{c_0}{2} \{\frac{\gamma-1}{2} Af(A,A/\varepsilon)$$

$$+ \frac{\gamma-1}{2} \int^A f(A',A'/\varepsilon)dA' + Ag(A,A/\varepsilon)\} \quad,$$

(6.35)

$$q_2(B,B/\varepsilon,\varepsilon) = \frac{d}{dB} Q_2(B,\varepsilon) = \frac{c_0}{2} \{-(\frac{\gamma-1}{2})Bf(B,B/\varepsilon)$$

$$- \frac{\gamma-1}{2} \int^B f(B',B'/\varepsilon)dB' + Bg(B,B/\varepsilon)\} \quad.$$

We integrate (6.31) to find $x^{(1)}$ and $t^{(1)}$ in terms of $c^{(1)}$ and $u^{(1)}$, and use the result and (6.29) in (6.27). After rearrangement, this gives the following expressions for A and B

$$A = x - c_0 t - \frac{\epsilon}{2c_0} \int_A^B \{c^{(1)}(A,B',\epsilon) + u^{(1)}(A,B',\epsilon)\} dB' + 0(\epsilon^2) \quad ,$$

(6.36)
$$B = x + c_0 t + \frac{\epsilon}{2c_0} \int_A^B \{c^{(1)}(A',B,\epsilon) - u^{(1)}(A',B,\epsilon)\} dA' + 0(\epsilon^2) \quad .$$

In (6.36) $c^{(1)}$ and $u^{(1)}$ are given by (6.32) when $N = 0$, and by (6.34) when $N = 2$.

This completes the solution of (6.1) and (6.2) using characteristic coordinates. The functions c and u are given by (6.27) and (6.32) or (6.34), while x and t are given by (6.36). Now we show that this solution equals the solution found using weakly nonlinear geometrical acoustics to first order in ϵ.

We shall need the following lemma.

Lemma 6.1. Let $f(x,\theta)$ be a differentiable function of x and integrable with respect to θ on $[0,\infty)$. Define

(6.37)
$$\bar{f}(x) = \lim_{T \to \infty} \{\frac{1}{T} \int_0^T f(x,\theta) d\theta\} \quad ,$$

and suppose that $\bar{f}(x)$ is differentiable and

(6.38)
$$\bar{f}_x(x) = \lim_{T \to \infty} \{\frac{1}{T} \int_0^T f_x(x,\theta) d\theta\}$$

uniformly in x. Then

(6.39)
$$\lim_{\epsilon \to 0} \int_a^b f(x,x/\epsilon) dx = \int_a^b \bar{f}(x) dx \quad .$$

Proof: Let

(6.40)
$$F(x,\theta) = f(x,\theta) - \bar{f}(x) \quad ,$$

and

(6.41)
$$u(x,\epsilon) = \int_0^{x/\epsilon} F(x,\theta) d\theta \quad .$$

Then

(6.42)
$$u_x = \frac{1}{\epsilon} F(x,x/\epsilon) + \int_0^{x/\epsilon} F_x(x,\theta) d\theta \quad .$$

We multiply (6.42) by ε and integrate with respect to x over $[a,b]$. This gives

$$(6.43) \qquad \int_a^b F(x,x/\varepsilon)dx = \varepsilon \int_0^{b/\varepsilon} F(b,\theta)d\theta - \varepsilon \int_0^{a/\varepsilon} F(a,\theta)d\theta -$$

$$- \varepsilon \int_a^b dx \int_0^{x/\varepsilon} F_x(x,\theta)d\theta \quad .$$

Taking the limit of (6.43) as $\varepsilon \to 0$ and using (6.37), (6.38) and (6.40), it follows that

$$(6.44) \qquad \lim_{\varepsilon \to 0} \int_a^b F(x,x/\varepsilon)dx = 0 \quad .$$

Then (6.39) follows from (6.40) and (6.44). ∎

In fact, with certain additional assumptions on $f(x,\theta)$ it is shown in [11] that as $\varepsilon \to 0$

$$\int_a^b f(x,x/\varepsilon)dx = \int_a^b \bar{f}(x)dx + O(\varepsilon) \quad .$$

We shall also need the following three equations. From (6.16) and (6.36)

$$(6.45) \qquad \begin{aligned} x - c_0 t &= s_- = \beta_+ = A + O(\varepsilon) \quad , \\ x + c_0 t &= s_+ = \beta_- = B + O(\varepsilon) \quad . \end{aligned}$$

Then if $h(x,y,\theta,\sigma)$ is any continuously differentiable function of x and y, using (6.45) and expanding in a Taylor series about $A = x - c_0 t$, $B = x + c_0 t$ we have

$$(6.46) \qquad h(A,B,A/\varepsilon,B/\varepsilon) = h(x-c_0 t,x+c_0 t,A/\varepsilon,B/\varepsilon) + O(\varepsilon) \quad .$$

Finally suppose two continuously differentiable functions $\tilde{w}(A,B)$ and $\bar{w}(x,t)$ are related by

$$\tilde{w}(A,B) = \bar{w}(x,t) + O(\varepsilon) \quad .$$

Then using (6.45) and expanding in Taylor series we have

$$\int_A^B \tilde{w}(A,B')dB' = \int_{x-c_0t}^{x+c_0t} \bar{w}ds_+ + 0(\epsilon) \quad ,$$

(6.47)

$$\int_A^B \tilde{w}(A',B)dA' = \int_{x-c_0t}^{x+c_0t} \bar{w}ds_- + 0(\epsilon) \quad .$$

In (6.47) the integral with respect to s_+ is taken keeping β_+ constant and the integral with respect to s_- is taken keeping β_- constant.

Now we compare the two solutions. We take the plane and spherical cases separately and consider plane flows first. Using (6.13), (6.22) and (6.33) we can rewrite the solution (6.32) for $c^{(1)}$ and $u^{(1)}$ as

$$c^{(1)}(A,B) = \tilde{c}(A,B) + c_0F_+^{(0)}(A,A/\epsilon) + c_0F_-^{(0)}(B,B/\epsilon) \quad ,$$

(6.48)

$$u^{(1)}(A,B) = \tilde{u}(A,B) + \frac{2c_0}{\gamma-1} F_+^{(0)}(A,A/\epsilon) - \frac{2c_0}{\gamma-1} F_-^{(0)}(B,B/\epsilon) \quad .$$

In (6.48)

$$\tilde{c}(A,B) = \bar{p}_0(A) + \bar{q}_0(B) \quad ,$$

(6.49)

$$\tilde{u}(A,B) = \frac{2}{\gamma-1} \{\bar{p}_0(A) - \bar{q}_0(B)\} \quad .$$

Now using (6.12) and (6.46) in (6.49) we have

$$\tilde{c}(A,B) = \bar{c}(x,t) + 0(\epsilon) \quad ,$$

(6.50)

$$\tilde{u}(A,B) = \bar{u}(x,t) + 0(\epsilon) \quad .$$

We use (6.46) and (6.50) in (6.48) to obtain

$$c^{(1)}(A,B) = \bar{c}(x,t) + c_0F_+^{(0)}(x-c_0t,A/\epsilon) + c_0F_-^{(0)}(x+c_0t,B/\epsilon) + 0(\epsilon)$$

(6.51)

$$u^{(1)}(A,B) = \bar{u}(x,t) + \frac{2c_0}{\gamma-1} F_+^{(0)}(x-c_0t,A/\epsilon) - \frac{2c_0}{\gamma-1} F_-^{(0)}(x+c_0t,B/\epsilon) + 0(\epsilon) \quad .$$

Using (6.51) in (6.27) and comparing the result with (6.8) and (6.20) we see that the two solutions for c and u agree to $0(\epsilon)$ provided that

(6.52) $$A/\epsilon = \zeta_+/\epsilon + 0(\epsilon), \quad B/\epsilon = \zeta_-/\epsilon + 0(\epsilon) \quad .$$

To show that (6.52) does hold, we use (6.48) in (6.36), and integrate. This yields

$$A = x - c_0 t - \varepsilon \frac{\gamma+1}{\gamma-1} F_+^{(0)}(A, A/\varepsilon) \frac{B-A}{2} + \frac{\varepsilon}{2} \frac{3-\gamma}{\gamma-1} \int_A^B F_-^{(0)}(B', B'/\varepsilon) dB'$$

$$- \frac{\varepsilon}{2c_0} \int_A^B \{\tilde{c}(A,B') + \tilde{u}(A,B')\} dB' + O(\varepsilon^2) \quad ,$$

(6.53)

$$B = x + c_0 t + \varepsilon \frac{\gamma+1}{\gamma-1} F_-^{(0)}(B, B/\varepsilon) \frac{B-A}{2} - \frac{\varepsilon}{2} \frac{3-\gamma}{\gamma-1} \int_A^B F_+^{(0)}(A', A'/\varepsilon) dA'$$

$$+ \frac{\varepsilon}{2c_0} \int_A^B \{\tilde{c}(A',B) - \tilde{u}(A',B)\} dB' + O(\varepsilon^2) \quad .$$

Now from (6.22) $F_\pm^{(N)}(x,\theta)$ has zero mean over θ, and therefore using lemma 6.1

$$\int_A^B F_+^{(N)}(A', A'/\varepsilon) dA' = O(\varepsilon) \quad ,$$

(6.54)

$$\int_A^B F_-^{(N)}(B', B'/\varepsilon) dB' = O(\varepsilon) \quad .$$

Also using (6.50) and (6.47) with $\tilde{w} = \tilde{c} \pm \tilde{u}$ we have

$$\int_A^B \{\tilde{c}(A,B') + \tilde{u}(A,B')\} dB' = \int_{x-c_0 t}^{x+c_0 t} \{\bar{c} + \bar{u}\} ds_+ + O(\varepsilon) \quad ,$$

(6.55)

$$\int_A^B \{\tilde{c}(A',B) - \tilde{u}(A',B)\} dA' = \int_{x-c_0 t}^{x+c_0 t} \{\bar{c} - \bar{u}\} ds_- + O(\varepsilon) \quad .$$

We use (6.46), (6.54) and (6.55) in (6.53). This gives

$$A = x - c_0 t - \varepsilon \frac{\gamma+1}{\gamma-1} F_+^{(0)}(x - c_0 t, A/\varepsilon) c_0 t - \frac{\varepsilon}{2c_0} \int_{x-c_0 t}^{x+c_0 t} \{\bar{c} + \bar{u}\} ds_+ + O(\varepsilon^2)$$

(6.56)

$$B = x + c_0 t + \varepsilon \frac{\gamma+1}{\gamma-1} F_-^{(0)}(x + c_0 t, B/\varepsilon) c_0 t + \frac{\varepsilon}{2c_0} \int_{x-c_0 t}^{x+c_0 t} \{\bar{c} - \bar{u}\} ds_- + O(\varepsilon^2) \quad .$$

Comparing (6.56) with (6.23), and using (6.5) and (6.24), shows that (6.52) is true for plane flows. This completes the confimation that the two methods agree for plane flows.

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Next consider the spherical case. Applying lemma 6.1 to (6.35) we have

$$p_2(A,A/\epsilon,\epsilon) = \frac{d}{dA} P_2(A,\epsilon) = \frac{c_0}{2} \{\frac{\gamma-1}{2} Af(A,A/\epsilon) +$$

$$+ \frac{\gamma-1}{2} \int^A \bar{f}(A')dA' + Ag(A,A/\epsilon)\} + 0(\epsilon) \quad,$$

(6.57)

$$q_2(B,B/\epsilon,\epsilon) = \frac{d}{dB} Q_2(B,\epsilon) = \frac{c_0}{2} \{-(\frac{\gamma-1}{2})Bf(B,B/\epsilon) -$$

$$- (\frac{\gamma-1}{2}) \int^B \bar{f}(B')dB' + Bg(B,B/\epsilon)\} + 0(\epsilon) \quad.$$

Integrating (6.57) to obtain P_2 and Q_2 and applying Lemma 6.1 again gives

$$P_2(A,\epsilon) = \frac{c_0}{2} \{\frac{\gamma-1}{2} \int^A A'\bar{f}(A')dA' + \frac{\gamma-1}{2} \int^A \bar{f}(A'')dA'' +$$

$$+ \int^A A'\bar{g}(A')dA'\} + 0(\epsilon) \quad,$$

(6.58) $$Q_2(A,\epsilon) = \frac{c_0}{2} \{-(\frac{\gamma-1}{2}) \int^B B'\bar{f}(B')dB' - (\frac{\gamma-1}{2}) \int^B \bar{f}(B'')dB'' +$$

$$+ \int^B B'\bar{g}(B')dB'\} + 0(\epsilon) \quad.$$

Thus from (6.58)

$$P_2(A,\epsilon) = \bar{P}_2(A) + 0(\epsilon) \quad,$$

(6.59)

$$Q_2(A,\epsilon) = \bar{Q}_2(A) + 0(\epsilon) \quad,$$

where from (6.15)

(6.60) $$\frac{d}{dA} \bar{P}_2(A) = \bar{p}_2(A), \frac{d}{dB} \bar{Q}_2(B) = \bar{q}_2(B) \quad.$$

Now using (6.15) and (6.22) we can rewrite (6.57) as

$$p_2(A,A/\epsilon,\epsilon) = \bar{p}_2(A) + c_0 F_+^{(2)}(A,A/\epsilon) + 0(\epsilon) \quad,$$

(6.61)

$$q_2(B,B/\epsilon,\epsilon) = \bar{q}_2(B) + c_0 F_-^{(2)}(B,B/\epsilon) + 0(\epsilon) \quad.$$

Then we use (6.59) and (6.61) in (6.34) which gives

$$c^{(1)} = \tilde{c}(A,B) + \frac{2c_0}{A+B} \{F_+^{(2)}(A,A/\epsilon) + F_-^{(2)}(B,B/\epsilon)\} + 0(\epsilon) \quad,$$

(6.62)

$$u^{(1)} = \tilde{u}(A,B) + \frac{2}{\gamma-1} \frac{2c_0}{A+B} \{F_+^{(2)}(A,A/\epsilon) - F_-^{(2)}(B,B/\epsilon)\} + 0(\epsilon) \quad.$$

In (6.62)

$$\tilde{c}(A,B) = \frac{2}{A+B} \{\bar{p}_2(A) + \bar{q}_2(B)\} \quad,$$

(6.63)

$$\tilde{u}(A,B) = \frac{2}{\gamma-1} \frac{2}{A+B} \{\bar{p}_2(A) - \bar{q}_2(B)\} - \frac{2}{\gamma-1} \frac{4}{(A+B)^2} \{\bar{P}_2(A) - \bar{Q}_2(B)\} \quad.$$

We use (6.46) and (6.14) in (6.63) which leads to (6.50). Then we use (6.46) and (6.50) in (6.62) and find

$$c^{(1)} = \bar{c}(x,t) + \frac{c_0}{x} \{F_+^{(2)}(x-c_0t,A/\epsilon) + F_-^{(2)}(x+c_0t,B/\epsilon)\} + 0(\epsilon) \quad,$$

(6.64)

$$u^{(1)} = \bar{u}(x,t) + \frac{2}{\gamma-1} \frac{c_0}{x} \{F_+^{(2)}(x-c_0t,A/\epsilon) - F_-^{(2)}(x+c_0t,B/\epsilon)\} + 0(\epsilon) \quad.$$

Using (6.64) in (6.27) and comparing the result with (6.8) and (6.20) we see that the two solutions agree provided (6.52) holds.

To show that (6.52) holds for spherical waves, we use (6.62) in (6.36) which gives

$$A = x-c_0t - \epsilon \frac{\gamma+1}{\gamma-1} F_+^{(2)}(A,A/\epsilon)\log\left(\frac{A+B}{2A}\right) + \epsilon \frac{3-\gamma}{\gamma-1} \int_A^B \frac{F_-^{(2)}(B',B'/\epsilon)}{A+B'} dB'$$

$$- \frac{\epsilon}{2c_0} \int_A^B \{\tilde{c}(A,B') + \tilde{u}(A,B')\}dB' + 0(\epsilon) \quad,$$

(6.65)

$$B = x+c_0t - \epsilon \frac{\gamma+1}{\gamma-1} F_-^{(2)}(B,B/\epsilon)\log\left(\frac{A+B}{2B}\right) - \epsilon \left(\frac{3-\gamma}{\gamma-1}\right) \int_A^B \frac{F_+^{(2)}(A',A'/\epsilon)}{A'+B} dA'$$

$$+ \frac{\epsilon}{2c_0} \int_A^B \{\tilde{c}(A',B) - \tilde{u}(A',B)\}dA' + 0(\epsilon) \quad.$$

Now we use (6.46), (6.54) and (6.47) with $\tilde{w} = \tilde{c} \pm \tilde{u}$ in (6.65). The result is

$$A = x - c_0 t - \varepsilon \frac{\gamma+1}{\gamma-1} F_+^{(2)}(x - c_0 t, A/\varepsilon) \log\left(\frac{x}{x - c_0 t}\right) - \frac{\varepsilon}{2c_0} \int_{x-c_0 t}^{x+c_0 t} \{\bar{c} + \bar{u}\} ds_+ + 0(\varepsilon) \quad ,$$

(6.66)

$$B = x + c_0 t - \varepsilon \frac{\gamma+1}{\gamma-1} F_-^{(2)}(x + c_0 t, B/\varepsilon) \log\left(\frac{x}{x + c_0 t}\right) + \frac{\varepsilon}{2c_0} \int_{x-c_0 t}^{x+c_0 t} \{\bar{c} - \bar{u}\} ds_- + 0(\varepsilon) \quad .$$

Comparing (6.66) with (6.23), and using (6.5) and (6.24), shows that (6.52) holds for spherical flows. Thus, weakly nonlinear geometrical acoustics and the method of characteristics agree to first order in the wave amplitude ε.

7. Resonance Conditions

Equations for $\bar{v}(x)$ and $a^{(j)}(x,\theta)$ were found in section 3 as solvability conditions for (3.4). In this section we show that there can be solvability conditions additional to those considered in section 3, and that the solution derived in section 3 satisfies all these solvability conditions whenever the resonance condition (2.15) does not hold for distinct j and k at any x. Even if (2.15) does hold, and there is the possibility of resonance, the solution derived in section 3 may still be valid and we give necessary and sufficient conditions for this to be the case.

We abbreviate (3.4) to

$$(7.1) \qquad \sum_{j=1}^{m} \sum_{i=1}^{n} \phi_{x_i}^{(j)} A^{(i)} v_{\theta_j} = f(\theta) \quad .$$

We have dropped the superscript on $v^{(1)}$ and omit showing any x-dependence explicitly, because x is effectively constant in (7.1). We shall derive conditions which $f(\theta)$ must satisfy if (7.1) has a bounded solution for $v(\theta)$.

In fact all we require on $v(\theta)$ is that

$$(7.2) \qquad v(\theta) = o(|\theta|) \quad \text{as} \quad |\theta| \to \infty \quad .$$

Then using (7.2)

$$\epsilon v^{(1)}(\phi^{(1)}/\epsilon,\ldots,\phi^{(m)}/\epsilon) = o(1) \quad \text{as} \quad \epsilon \to 0 \quad ,$$

and the asymptotic expansion (3.2) remains valid when θ_j is evaluated at $\phi^{(j)}/\epsilon$.

If $v(\theta)$ satisfies (7.2) then

$$(7.3) \qquad \lim_{T\to\infty} \{\frac{1}{T} \int_0^T v_{\theta_j} d\theta_j\} = 0 \quad .$$

That is any derivative of $v(\theta)$ has zero mean.

Therefore averaging (7.1) with respect to θ we find as in section 3 that $f(\theta)$ has zero mean, which gives (2.4) for the mean $\bar{v}$ of $v^{(0)}(\theta)$. Next suppose that there is a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that

$$(7.4) \qquad \sum_{j=1}^{m} \lambda_j \nabla \phi^{(j)} = p$$

where $p = (p_1, \ldots, p_n)$ satisfies (2.15)

Denote by L the left null vector of $\sum_{i=1}^{n} p_i A^{(i)}$. We assume throughout this section that (2.1) is strictly hyperbolic, so L is uniquely defined (up to a scalar factor). Also let

$$(7.5) \qquad \sigma = \sum_{j=1}^{m} \lambda_j \theta_j \quad ,$$

and define the hyperplane $S(\sigma)$ by

$$(7.6) \qquad S(\sigma) = \{\theta \in \mathbb{R}^m : \sigma = \text{constant}\} \quad .$$

Then we average (7.1) with respect to θ over $S(\sigma)$. To do this introduce coordinates $(\sigma, \eta_1, \ldots, \eta_{m-1})$ in $\mathbb{R}^m$, where the η_j depend linearly on θ. Then using (7.5), (7.1) becomes

$$(7.7) \qquad \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \phi_{x_i}^{(j)} A^{(i)} v_\sigma + \sum_{k=1}^{m-1} \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial \eta_k}{\partial \theta_j} \phi_{x_i}^{(j)} A^{(i)} v_{\eta_k} = f(\theta) \quad .$$

Now we average (7.7) over $S(\sigma)$, integrating with respect to $\eta_1, \ldots, \eta_{m-1}$, when the mean of the terms proportional to v_{η_k} is zero from (7.2). Then we take the scalar product of the result with L, which makes the coefficient of v_σ zero. Therefore

$$(7.8) \qquad L(f)_{avS} = 0 \quad .$$

In (7.8) $(f)_{avS}$ is the average of f over $S(\sigma)$.

$$(7.9) \qquad (f)_{avS} = \lim_{T \to \infty} \left\{ \frac{1}{T^{m-1}} \int_0^T \cdots \int_0^T f \, d\eta_1 \ldots d\eta_{m-1} \right\} \quad .$$

Thus $(f)_{avS}$ is a function of σ.

In general it is possible for there to be many vectors λ such that $\sum_{i,j} \lambda_j \phi_{x_i}^{(j)} A^{(i)}$ has a given left null vector L. The set of all such λ forms a vector subspace of $\mathbb{R}^m$. Pick a basis $\{\lambda_1, \ldots, \lambda_\nu\}$ of this space, where $\lambda_\ell = (\lambda_{\ell 1}, \ldots, \lambda_{\ell m})$, and define σ_ℓ and $S(\sigma)$ analagously to (7.5) and (7.6):

$$\sigma_\ell = \sum_{j=1}^{m} \lambda_{\ell j} \theta_j \quad (\ell = 1, \ldots, \nu) \quad,$$

(7.10)

$$S(\sigma) = \{\theta \in \mathbb{R}^m : (\sigma_1, \ldots, \sigma_\nu) \equiv \sigma = \text{constant}\} \quad.$$

Then exactly as before we obtain the solvability condition (7.8). We obtain one such condition for each subspace S of the kind defined above.

Next we show that the solution (3.6) for $v^{(0)}$ always satisfies these solvability conditions if (2.15) never holds. Using (3.6) in (3.4) and comparing with (7.1) we obtain

$$(7.11) \quad -f(x, \theta) = \sum_{i=1}^{n} \{A^{(i)} \bar{v}_{x_i} + A_u^{(i)} \bar{v} u_{x_i}^{(0)}\} + B_u \bar{v}$$

$$+ \sum_{k=1}^{m} \sum_{i=1}^{n} \{A^{(i)} R^{(k)} a_{x_i}^{(k)} + \phi_{x_i}^{(k)} A_u^{(i)} \bar{v} R^{(k)} a_{\theta_k}^{(k)} + Q^{(k)} a^{(k)}\}$$

$$+ \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{i=1}^{n} \phi_{x_i}^{(j)} A_u^{(i)} R^{(k)} R^{(\ell)} a^{(k)} a_{\theta_j}^{(j)} \quad.$$

In (7.11) $Q^{(k)}$ is defined by (3.9).

Now we claim that the conditions (7.8) are satisfied for f given by (7.11), with $\bar{v}$ satisfying (2.4) and $a^{(j)}$ satisfying (3.8), if and only if the nonlinear terms

$$(7.12) \quad \sum_{j=1}^{m} \sum_{k=1}^{j-1} \sum_{i=1}^{m} \{L \phi_{x_i}^{(j)} A_u^{(i)} R^{(k)} R^{(j)} a^{(k)} a_{\theta_j}^{(j)} + L \phi_{x_i}^{(k)} A_u^{(i)} R^{(j)} R^{(k)} a^{(j)} a_{\theta_j}^{(k)}\}$$

have zero mean over $S(\sigma)$. The reason for this is as follows. The first three terms in (7.11) proportional to $\bar{v}$ sum to zero by (2.4). The terms

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linear in $a^{(k)}$ and the nonlinear term proportional to

$a^{(k)} a^{(k)}_{\theta_k} = \frac{1}{2} \frac{\partial}{\partial \theta_k} [a^{(k)}]^2$ have zero average over $S(\sigma)$ unless θ_k is

constant on $S(\sigma)$. If θ_k is constant on $S(\sigma)$ then we must have $L = L^{(k)}$,

and then the scalar produce of the sum of these terms with L is zero by

(3.8).

Thus a necessary and sufficient condition for the solution derived in

section 3 to satisfy the solvability conditions (7.8) is that the terms (7.12)

have zero mean over $S(\sigma)$. However, suppose that θ_j and θ_k are not

constant or functionally dependent on each other on $S(\sigma)$. Then we can use

$\eta_1 = \theta_j$ and $\eta_2 = \theta_k$ as two coordinates on $S(\sigma)$, and the mean of (7.12)

over $S(\sigma)$ is clearly zero.

If θ_j and θ_k are constant or functionally dependent on each other

then we must have for some $c_j(x)$, $c_k(x)$ and $(\mu_1(x),\ldots,\mu_\nu(x))$

(7.13)
$$c_j \theta_j + c_k \theta_k = \sum_{\ell=1}^{\nu} \mu_\ell \sigma_\ell \ .$$

Using (7.10)

(7.14)
$$c_j \theta_j + c_k \theta_k = \sum_{r=1}^{m} \sum_{\ell=1}^{\nu} \mu_\ell \lambda_{\ell r} \theta_r \ .$$

Since (7.14) holds for all θ, in particular it follows that

(7.15)
$$c_j = \sum_{\ell=1}^{\nu} \mu_\ell \lambda_{\ell j} \ ,$$

$$c_k = \sum_{\ell=1}^{\nu} \mu_\ell \lambda_{\ell k} \ .$$

But by the definition of $\lambda_{\ell r}$, $\sum_{r=1}^{m} \lambda_{\ell r} \phi_{x_i}^{(r)} A^{(i)}$ has left null vector L for

$\ell = 1,\ldots,\nu$. Therefore if p is defined by (2.15) using (7.15) the matrix

$\sum_{i=1}^{n} p_i A^{(i)}$ has left null vector L, and (2.16) must hold.

This proves that if (2.15) and (2.16) do not hold at any x for distinct j and k, then the mean of all terms (7.12) over $S(\sigma)$ is zero, and $\bar{v}$ and $a^{(j)}$ satisfy the solvability conditions (7.8). On the other hand, suppose (2.15) and (2.16) do hold for some pairs $\{j,k\}$. Then the solution described in section 2 is valid if and only if for each pair $\{j,k\}$ the average of the term (7.12) is zero over the space $S(\sigma)$ corresponding to the left null vector of L of $\sum\limits_{i=1}^{n} p_i A^{(i)}$.

In section 3 we took a particular form of solution for ϕ and $v^{(0)}$ to (3.3). We have shown that this is sufficient provided that re⸱ ⸱⸱⸱e does not occur. To treat problems involving resonant interactions a more general form of solution to (3.3) for ϕ and $v^{(0)}$ must be used.

8. Multiple Characteristics

In this section we generalize the results of section 2 to the case of multiple characteristics for a large class of systems. This class includes systems derived from a set of conservation laws.

Let us regard the eiconal equation (2.6) as an equation for ϕ_{x_1}, where we can suppose that x_1 is a timelike coordinate. Equation (2.6) is an m-th degree polynomial in ϕ_{x_1}. We denote its m real roots by $g_j(x, \phi_{x_2}, \ldots, \phi_{x_n})$. Suppose that (2.1) has multiple characteristics. Then ν of the g_j's will be the same. We suppose $g_j \equiv g_1$ for $j = 1, \ldots, \nu$. Therefore we shall take $\phi^{(j)}(x) = \phi^{(1)}(x)$ and $\theta_j \equiv \theta_1$ for $j = 1, \ldots, \nu$. We assume that the multiplicity ν does not change, and we also assume that there is no resonance i.e. that (2.15) never holds.

We use (3.2) in (3.1) and use the result in (2.1). Equating to zero the coefficient of ϵ^0 we obtain (3.3). Setting each term in the sum over m equal to zero gives (3.5). We take as solutions to the eiconal equation (2.6), $\phi^{(1)}(x)$, $\phi^{(\nu+1)}(x), \ldots, \phi^{(m)}(x)$. To simplify notation we suppose that the other families of characteristics are simple. This does not affect our final results which are the same whether or not there are other multiple characteristics. In fact, several of the $\phi^{(j)}$'s can correspond to the same multiple characteristic.

Once the $\phi^{(j)}$ are determined $v^{(0)}$ satisfies (3.5). The matrix $\sum_{i=1}^{n} \phi_{x_i}^{(1)} A_u^{(i)}$ has a null space of dimension ν. We denote by $\{R^{(1)}, \ldots, R^{(\nu)}\}$ a basis of the space of right null vectors, and by $\{L^{(1)}, \ldots, L^{(\nu)}\}$ a basis of the space of left null vectors. We write $v^{(0)}$ as

$$(8.1) \quad v^{(0)}(x,\theta) = \bar{v}(x) + \sum_{j=1}^{\nu} a^{(j)}(x, \theta_1) R^{(j)} + \sum_{j=\nu+1}^{m} a^{(j)}(x, \theta_j) R^{(j)} \ .$$

Then $v^{(0)}$ clearly satisfies (3.5).

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The only difference between (8.1) and (3.6) is that in (8.1) the first $\nu\, a^{(j)}$ all depend on the same fast variable θ_1.

Equating the coefficient of ε to zero we obtain

$$- \sum_{k=1}^{m} \sum_{i=1}^{n} \phi_{x_i}^{(k)} A^{(i)} v_{\theta_k}^{(1)} = \sum_{i=1}^{n} \{A^{(i)} \bar{v}_{x_i} + A_u^{(i)} \overline{vu}_{x_i}^{(0)}\} + B_u \bar{v}$$

(8.2)
$$+ \sum_{k=1}^{m} \sum_{k=1}^{n} \{(a_{x_i}^{(k)} A^{(i)} R^{(k)} + a_{\theta_k}^{(k)} \phi_{x_i}^{(k)} A_u^{(i)} \overline{v R}^{(k)}) + a^{(k)} Q^{(k)}\}$$

$$+ \sum_{k=1}^{m} \sum_{\ell=1}^{m} a^{(k)} a_{\theta_\ell}^{(\ell)} \sum_{i=1}^{n} \phi_{x_i}^{(\ell)} A_u^{(i)} R^{(k)} R^{(\ell)} \quad .$$

In (8.2) θ_j is taken equal to θ_1 for $j = 1,\ldots,\nu$. Now equation (2.4) for $\bar{v}(x)$ follows from (8.2) by averaging it over θ provided that the mean of the nonlinear terms proportional to $a^{(j)} a_{\theta_j}^{(k)}$ is zero.

In fact this is not always the case when there are multiple characteristics. The mean of $a^{(j)}(x,\theta_1) a_{\theta_1}^{(k)}(x,\theta_1)$ is not necessarily zero when $j \neq k$, and $1 \leq j, k \leq \nu$. In general the terms $[a^{(j)} a_{\theta_1}^{(k)}]_{av\theta}$ couple together the equations for the mean $\bar{v}(x)$ and the amplitudes $a^{(j)}(x,\theta_1)$.

However we shall show that the mean of the nonlinear terms is zero for systems which satisfy the following condition.

Condition 8.1. For all $1 \leq j, k \leq \nu$:

(8.3)
$$\sum_{i=1}^{n} \phi_{x_i}^{(1)} A_u^{(i)} R^{(j)} R^{(k)} = \sum_{i=1}^{n} \phi_{x_i}^{(1)} A_u^{(i)} R^{(k)} R^{(j)} \quad .$$

If there are other multiple characteristics they also satisfy the analagous equation to (8.3).

From (4.14) condition 8.1 is always satisfied by any system which is derived from a set of conservation laws. We now show that the mean of the terms $a^{(j)} a_{\theta_k}^{(k)}$ is zero. Trouble can only arise when $1 \leq j, k \leq \nu$ so that both $a^{(j)}$ and $a^{(k)}$ depend on θ_1.

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If $j = k$ then

$$a^{(j)}a_{\theta_1}^{(j)} = \frac{1}{2}\frac{\partial}{\partial\theta_1}[a^{(j)}]^2 .$$

Thus $a^{(j)}a_{\theta_1}^{(j)}$ is an exact derivative and has zero mean, since $a^{(j)}$ is a bounded function of θ_1. If $j \neq k$ we consider the terms proportional to $a^{(j)}a_{\theta_1}^{(k)}$ and $a^{(k)}a_{\theta_1}^{(j)}$. From (8.2) these are

$$(8.4) \qquad \sum_{i=1}^{n} \phi_{x_i}^{(1)}A_u^{(i)}R^{(j)}R^{(k)}a^{(j)}a_{\theta_1}^{(k)} + \sum_{i=1}^{n} \phi_{x_i}^{(1)}A_u^{(i)}R^{(k)}R^{(j)}a^{(k)}a_{\theta_1}^{(j)} .$$

Using (8.3) we rewrite (8.4) as an exact derivative

$$(8.5) \qquad \sum_{i=1}^{n} \phi_{x_i}^{(1)}A_u^{(i)}R^{(j)}R^{(k)}\frac{\partial}{\partial\theta_1}(a^{(j)}a^{(k)}) .$$

Therefore (8.4) averages to zero. This shows that $\bar{v}(x)$ satisfies (2.4).

To obtain equations satisfied by $a^{(j)}$ for $j = 1,\ldots,\nu$ we average (8.2) over $(\theta_{\nu+1},\ldots,\theta_m)$ and take the scalar product of the resulting equation with $L^{(p)}(p = 1,\ldots,\nu)$. Then we find the same equations as those obtained by Choquet-Bruhat [1]:

$$
\begin{aligned}
(8.6) \qquad & \sum_{j=1}^{\nu}\sum_{i=1}^{n} L^{(p)}A^{(i)}R^{(j)}a_{x_i}^{(j)} + \sum_{j=1}^{\nu}\sum_{k=1}^{\nu}\sum_{i=1}^{n} L^{(p)}\phi_{x_i}^{(1)}A_u^{(i)}R^{(k)}R^{(j)}a^{(k)}a_{\theta_1}^{(j)} \\
& + \sum_{j=1}^{\nu}\sum_{i=1}^{n} L^{(p)}\phi_{x_i}^{(1)}A_u^{(i)}\bar{v}R^{(j)}a_{\theta_1}^{(j)} + \sum_{j=1}^{\nu} L^{(p)}Q^{(j)}a^{(j)} = 0 .
\end{aligned}
$$

We have the following lemma:

Lemma 8.1. There are scalar functions $c_i(x)$, for $i = 1,\ldots,n$, such that for all $j = 1,\ldots,\nu$ and $p = 1,\ldots,\nu$:

$$L^{(p)}A^{(i)}R^{(j)} = c_i L^{(p)}R^{(j)} .$$

The proof of lemma 8.1 follows that of Lewis [12] almost exactly.

Using lemma 8.1 in (8.6) we find

$$(8.7) \quad \sum_{j=1}^{\nu} \sum_{i=1}^{n} L^{(p)} R^{(j)} a_\sigma^{(j)} + \sum_{j=1}^{\nu} \sum_{k=1}^{\nu} \sum_{i=1}^{n} L^{(p)} \phi_{x_i}^{(1)} A_u^{(i)} R^{(k)} R^{(j)} a^{(k)} a_{\theta_1}^{(j)}$$

$$+ \sum_{j=1}^{\nu} \sum_{i=1}^{n} L^{(p)} \phi_{x_i}^{(1)} A_u^{(i)} \bar{v} R^{(j)} a_{\theta_1}^{(j)} + \sum_{j=1}^{\nu} L^{(p)} Q^{(j)} a^{(j)} = 0 \quad .$$

$$(p = 1,\ldots,\nu) \quad .$$

In equation (8.7), σ is defined by

$$(8.8) \quad \frac{dx_i}{d\sigma} = c_i(x) \quad .$$

Equation (8.7) is a system of quasi-linear equations for the ν scalars $a^{(j)}$ in two independent variables σ and θ_1.

Equation (8.7) reduces to a system of ordinary differential equations in the special case that there are $\lambda(x) \in R$ and $\mu(x) \in R$ such that

$$(8.9) \quad \sum_{i=1}^{n} L^{(p)} \phi_{x_i}^{(1)} A_u^{(i)} R^{(k)} R^{(j)} = \lambda L^{(p)} R^{(j)} \delta_{jk} \quad ,$$

$$\sum_{i=1}^{n} L^{(p)} \phi_{x_i}^{(1)} A_u^{(i)} \bar{v} R^{(j)} = \mu L^{(p)} R^{(j)} \quad .$$

Appendix

In this appendix we specialize the results of section 2 to the gas dynamics equations

$$\rho_t + \text{div}(\rho u) = 0$$

(A.1)
$$\rho u_t + \rho u \cdot \nabla u + \nabla p = 0$$

$$p = \kappa \rho^\gamma \quad .$$

Equations (A.1) are the equations of motion for the isentropic flow of a compressible, inviscid, ideal gas. In (A.1) ρ is the gas density, p the pressure and u the velocity of the gas. The quantities κ and γ are constants.

The weakly nonlinear geometrical acoustics solution (2.2) to (A.1) for x and u in $\mathbb{R}^3$ is

$$(A.2) \quad \begin{bmatrix} \rho \\ u \end{bmatrix} = \begin{bmatrix} \rho(x,t) \\ U(x,t) \end{bmatrix} + \epsilon \begin{bmatrix} \bar{\rho}(x,t) \\ \bar{u}(x,t) \end{bmatrix} + \epsilon \sum_{j=1}^{m} a^{(j)}(x,t,\phi^{(j)}(\epsilon)R^{(j)}(x,t)$$

$$+ \epsilon \sum_{k=1}^{m'} \{a^{(k,1)}(x,t,\psi^{(k)}/\epsilon)R^{(k,1)}(x,t) + a^{(k,2)}(x,t,\psi^{(k)}/\epsilon)R^{(k,2)}(x,t)\}$$

$$+ 0(\epsilon^2) \quad .$$

In (A.2) (ρ_0,U) is any exact solution to (A.1). The mean value of the perturbation in (ρ,u) about (ρ_0,U) is $(\epsilon\bar{\rho},\epsilon\bar{u})$. The sum over j represents a superposition of high frequency sound waves, and the sum over k is a superposition of high frequency vorticity waves.

The means $(\bar{\rho},\bar{u})$ satisfy (2.4) which gives the acoustics equations

$$\bar{\rho}_t + \text{div}(\rho_0\bar{u} + \bar{\rho}U) = 0 \quad ,$$

(A.3)
$$\rho_0\bar{u}_T + \nabla(c_0^2\bar{\rho}) + \bar{\rho}U_T + \rho_0\bar{u} \cdot \nabla U = 0 \quad .$$

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In (A.3) c_0 is the linearized sound speed

(A.4)
$$c_0 = \left(\kappa\gamma\rho_0^{\gamma-1}\right)^{1/2} \quad,$$

and $\frac{\partial}{\partial T}$ is the derivative taken along the streamlines of U

(A.5)
$$\frac{\partial}{\partial T} \equiv \frac{\partial}{\partial t} + U\cdot\nabla \quad.$$

The eiconal equation (2.6) for (A.1) when $x \in R^n$ is

(A.6)
$$(\phi_T^2 - c_0^2|\nabla\phi|^2)\phi_T^{n-1} = 0 \quad.$$

The phase functions $\phi^{(j)}$ and $\psi^{(j)}$ in (A.2) satisfy

(A.7)
$$(\phi_T^{(j)})^2 - c_0^2|\nabla\phi^{(j)}|^2 = 0 \quad, \quad \psi_T^{(k)} = 0 \quad.$$

We consider the sound waves (which depend on $\phi^{(j)}$) and the vorticity waves (which depend on $\psi^{(k)}$) separately.

For the sound waves the rays (2.7) corresponding to $\phi^{(j)}$ are given by

(A.8)
$$\frac{dt}{ds_j}\bigg|_{\beta_j} = \mu_j\phi_T^{(j)} \quad, \quad \frac{dx}{ds_j}\bigg|_{\beta_j} = \mu_j\phi_T^{(j)}U - \mu_j c_0^2\nabla\phi^{(j)} \quad,$$

and the vector $R^{(j)}$ is given in terms of $\phi^{(j)}$ by

(A.9)
$$R^{(j)} = \lambda_j \begin{bmatrix} -\rho_0 c_0^{-2}\phi_T^{(j)} \\ \\ \nabla\phi^{(j)} \end{bmatrix} \quad.$$

The scalars $\mu_j(x,t)$ and $\lambda_j(x,t)$ in (A.8) and (A.9) are arbitrary functions which may be chosen in whatever way is convenient.

The amplitude $a^{(j)}(x,t,\phi^{(j)}/\epsilon)$ of the jth sound wave is given by (2.9). For (A.1) $E_j(s_j,\beta_j)$ is

(A.10)
$$E^{(j)} = \exp\left[-\int_{s_0}^{s_j}\left\{\frac{\phi_{TT}^{(j)}-\Delta\phi^{(j)}}{2} - \frac{\gamma-1}{2}\phi_T^{(j)}\operatorname{div} U - \frac{\gamma c_0^2}{2\rho_0}\nabla\rho_0\cdot\nabla\phi^{(j)}\right\}\mu_j ds_j\right] \quad.$$

The integral in (A.10) is taken along the rays (A.8). The first term in the integrand gives the change in the wave amplitude due to changes in the ray geometry. The second two terms in the integrand give the change in the wave amplitude due to inhomogeneities in the zeroth order flow $\rho = \rho_0$, $u = U$.

The modified phase function ζ_j in (2.9) is defined implicitly by (2.11) which becomes

$$\zeta_j = \phi^{(j)} - \varepsilon \frac{\gamma-1}{2} F_j(\beta_j, \zeta_j/\varepsilon) \int_{s_0}^{s_j} \lambda_j (\phi_T^{(j)})^3 c_0^{-2} E_j \mu_j ds_j'$$

(A.11)

$$- \varepsilon \int_{s_0}^{s_j} \phi_T^{(j)} \overline{u} \cdot \nabla \phi^{(j)} \mu_j ds_j' + \varepsilon \frac{\gamma-1}{2} \int_{s_0}^{s_j} (\phi_T^{(j)})^2 \overline{\rho} \rho_0^{-1} \phi_j ds_j' \quad .$$

If shocks form they are fitted into the solution using (2.13).

The phase function $\psi^{(k)}$ for the vorticity waves corresponds to a characteristic of multiplicity $n - 1$ when $x \in R^n$. For three-dimensional flows the characteristic has multiplicity two. Therefore as explained in section 8 we obtain two null vectors $R^{(k,1)}$ and $R^{(k,2)}$ for $\psi^{(k)}$ and two amplitudes $a^{(k,1)}$ and $a^{(k,2)}$ which are coupled together. The equations satisfied by $a^{(k,1)}$ and $a^{(k,2)}$ happen to be linear for (A.1).

The null vectors $R^{(k,1)}$ and $R^{(k,2)}$ are given in terms of $\psi^{(k)}$ by

(A.12)
$$R^{(k,1)} = \lambda_k \begin{bmatrix} 0 \\ \psi_{x_2} \\ -\psi_{x_1}^{(k)} \\ 0 \end{bmatrix} \quad , \quad R^{(k,2)} = \lambda_k \begin{bmatrix} 0 \\ 0 \\ \psi_{x_3}^{(k)} \\ -\psi_{x_2}^{(k)} \end{bmatrix} \quad .$$

In (A.12) $\lambda_k(x,t)$ is an arbitrary scalar. To put the equations for $a^{(k,1)}$ and $a^{(k,2)}$ in their simplest form it is convenient to let

(A.13)
$$\begin{bmatrix} w_1^{(k)} \\ w_2^{(k)} \\ w_3^{(k)} \end{bmatrix} = \lambda_k a^{(k,1)} \begin{bmatrix} \psi_{x_2}^{(k)} \\ -\psi_{x_1}^{(k)} \\ 0 \end{bmatrix} + \lambda_k a^{(k,2)} \begin{bmatrix} 0 \\ \psi_{x_3}^{(k)} \\ -\psi_{x_2}^{(k)} \end{bmatrix} \quad .$$

Then $w^{(k)}(x,t,\psi^{(k)}/\varepsilon)$ is a velocity vector orthogonal to $\nabla\psi^{(k)}$ and satisfies

$$(A.14) \qquad w^{(k)}_T + \bar{u}\cdot\nabla\psi^{(k)} w^{(k)}_\sigma + w^{(k)}\cdot\nabla U - \frac{\nabla\psi^{(k)}\cdot(w^{(k)}\cdot\nabla U)}{|\nabla\psi^{(k)}|^2}\nabla\psi^{(k)} = 0 \quad .$$

We solve (A.14) for $w^{(k)}(x,t,\sigma)$ and evaluate σ at $\psi^{(k)}/\varepsilon$. When U is independent of x (A.14) gives three uncoupled equations for $w^{(k)}_1$, $w^{(k)}_2$ and $w^{(k)}_3$.

We may want to use the local sound speed $c(x,t)$ as a dependent variable instead of $\rho(x,t)$. They are related by

$$(A.15) \qquad c^2 = \kappa\gamma\rho^{\gamma-1} \quad .$$

Then we seek a solution

$$
\begin{bmatrix} c \\ \\ u \end{bmatrix}
=
\begin{bmatrix} c_0(x,t) \\ \\ U(x,t) \end{bmatrix}
+ \varepsilon
\begin{bmatrix} \bar{c}(x,t) \\ \\ \bar{u}(x,t) \end{bmatrix}
+ \varepsilon \sum_{j=1}^{m} a^{(j)}(x,t,\phi^{(j)}/\varepsilon)R^{(j)}(x,t)
$$

$$(A.16)$$

$$
+ \varepsilon \sum_{k=1}^{m} \{a^{(k,1)}(x,t,\psi^{(k)}/\varepsilon)R^{(k,1)} + a^{(k,2)}(x,t,\psi^{(k)}/\varepsilon)R^{(k,2)}\} + 0(\varepsilon^2) \quad .
$$

In (A.16) $c = c_0$, $u = U$ is an exact solution of the gas dynamics equations. The means $\bar{c}$ and $\bar{u}$ satisfy

$$
\bar{c}_T + \frac{\gamma-1}{2} c_0 \operatorname{div} \bar{u} + \frac{\gamma-1}{2} \bar{c} \operatorname{div} U + \bar{u}\cdot\nabla c_0 = 0 \quad ,
$$

$$(A.17)$$

$$
\bar{u}_T + \frac{2}{\gamma-1}\nabla(c_0\bar{c}) + \bar{u}\cdot\nabla U = 0 \quad .
$$

The $R^{(j)}$ in (A.16) are given by

$$(A.18) \qquad R^{(j)} = \lambda_j \begin{bmatrix} -(\gamma-1)\phi^{(j)}_T/2c_0 \\ \\ \nabla\phi^{(j)} \end{bmatrix} \quad ,$$

where $\lambda_j(x,t)$ is an arbitrary scalar. The remaining equations are exactly as before: $\phi^{(j)}$ satisfies (A.7), $a^{(j)}$ is found from (2.9), (A.10) and

(A.11) with $\frac{\gamma-1}{2} \bar{\rho}/\rho_0 = \bar{c}/c_0$, $\psi^{(k)}$ satisfies (A.7), $R^{(k,1)}$ and $R^{(k,2)}$ are given by (A.12) and $w^{(k)}$ satisfies (A.14).

The equations for plane flow $(x \in \mathbb{R}^2)$ are obtained from those above by suppressing the x_3-dependence and setting $a^{(k,2)} = 0$. The equations for one dimensional flow follow by suppressing the x_2 and x_3-dependence and setting $a^{(k,1)} = a^{(k,2)} = 0$.

References

1. V. Choquet-Bruhat, Ondes asymptotique et approchées pour systèmes d'équations aux dérivées partielles nonlinéaires, J. Math. Pures et Appl., $\underline{48}$, (1969), 117-158.

2. L. D. Landau, On shock waves at large distances from the place of their origin, Soviet Journal of Physics, $\underline{9}$, (1945), 496-500.

3. G. B. Whitham, The flow pattern of a supersonic projectile, Comm. Pure Appl. Math., $\underline{5}$, (1952), 301-348.

4. E. Varley and E. Cumberbatch, Nonlinear high frequency sound waves, J. Inst. Math. Appl., $\underline{2}$, (1966), 133-143.

5. E. Varley and E. Cumberbatch, Large amplitude waves in stratified media: acoustic pulses, J. Fluid Mech., $\underline{43}$, (1970), 513-537.

6. M. P. Mortell and E. Varley, Finite amplitude waves in bounded media: nonlinear free vibrations of an elastic panel, Proc. Roy. Soc. London, $\underline{A318}$, (1970), 169-196.

7. B. R. Seymour and M. P. Mortell, Nonlinear geometrical acoustics, in Mechanics Today, vol. 2, S. Nemat-Nasser, editor, Pergamon Press (1975).

8. R. Courant, Methods of Mathematical Physics, vol. II, John Wiley & Sons, New York (1962).

9. G. B. Whitham, Linear and Nonlinear Waves, John Wiley & Sons, New York (1974).

10. P. A. Fox, Perturbation theory of wave propagation based on the method of characteristics, J. Math. and Phys., $\underline{34}$, (1955), 133-151.

11. J. K. Hunter, Weakley Nonlinear Wave Propagation, Ph.D. thesis, Stanford University (1981).

12. R. M. Lewis, Discontinuous initial value problems for symmetric

hyperbolic linear differential equations, J. Math. Mech. $\underline{7}$, (1958), 571-

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In this paper we derive a method for finding small amplitude high frequency solutions to hyperbolic systems of quasilinear partial differential equations. Our solution is the sum of two parts: (i) a superposition of small amplitude high frequency waves; (ii) a slowly varying 'mean solution'. Each high frequency wave displays nonlinear distortion of the wave profile and shocks may form. Shock conditions are derived for conservative systems. Different high frequency waves do not interact provided the frequencies and wave numbers of two waves are not linearly related to those of a third. The mean solution is found by solving a linear partial

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ABSTRACT (continued)

differential equation. This method generalizes Whitham's nonlinearization technique [9] for single waves, to problems where many waves are present. We obtain these results by generalizing a scheme first proposed by Choquet-Bruhat [1] which employs the method of multiple scales.