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SOME TOPICS IN LINEAR ESTIMATION†

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INTRODUCTION

It must be with some trepidation that one ventures to speak about the problems of linear estimation to an audience already well familiar with the overwhelmingly more difficult nonlinear filtering problem. However, perhaps to compensate for this spectacle, the organizers have given me the opportunity to speak first, with considerable latitude in the choice of my topics.

For such an audience, there will be no need to present a tutorial on linear filtering, especially of the Kalman-Bucy type. I chose, therefore, to focus on some aspects generally less familiar to those with a 'modern' control theory background, i.e., largely a state-space background. In particular, we shall begin with a discussion of integral equations and of the important Wiener-Hopf technique. We shall specialize this to stationary processes over infinite intervals, and then describe some alternative, often computationally better, solution methods of Ambartzumian-Chandrasekhar and Krein-Levinson for finite-interval problems. For nonstationary processes, we start first with state-space models and build up to a brief description of the scattering theory framework for linear estimation. This will then lead us to nonstationary versions of the Ambartzumian-

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M. Hazewinkel and J. C. Willems (eds.), Stochastic Systems: The Mathematics of Filtering and Identification and Applications, 307–350.

Chandrasekhar and Krein-Levinson equations, which we shall only consider very briefly, providing detailed references for further reading. We shall conclude with a remark on a possible implication for nonlinear filtering.

Our presentation is confined to continuous-time processes; a recent survey of the discrete-time case can be found in Kailath (1980).

In writing this chapter, it was a great help to have a carefully prepared preliminary reduction of the actual lectures, contributed by B. Hanzon, B. Ursin and D. Ocone. It is a pleasure also to thank M. Hazewinkel for these and several other organizational touches that made for an outstanding symposium.

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1. The Integral Equations of Smoothing and Filtering

Our estimation problems will be discussed in the context of the following model for the observed random process $\{y_t\}$:

$$y_t = z_t + v_t \quad , 0 \le t \le T \tag{1}$$

where z_t and v_t are R^p-valued stochastic variables with mean zero, and such that †

$$E\, v_t v_s' := I_p \delta(t-s) \ ,$$

where δ is the Dirac-delta distribution, and

$$E(z_t z_s' + z_t v_s' + v_t z_s') := K(t,s) \ ,$$

a continuous function on $[0,T]\times[0,T]$. This model describes a situation in which a signal, z_t, can only be observed with additive white noise v_t. We note that $K(t,s)$ *does not have to be a covariance function; it is necessary only that*

$$R(t,s) := E\, y_t y_s' = I_p\, \delta(t-s) + K(t,s)$$

be a covariance function.

It will be useful to consider two special cases:

(i) $v \perp z_s$ for all s,t with $0 \le s \le T$, $0 \le t \le T$. In this case $K(t,s) = E\, z_t z_s'$ is a covariance function

(ii) $v_t \perp z_s$ for all $t > s$. This possibility allows *causal dependence* of z on y (feedback!).

1a. The Smoothing Problem: Fredholm Equations

The *smoothing problem* is as follows. Given the observations $\{y_s : 0 \le s \le T\}$, find

† No special notation will be used to indicate matrix on vector quantities. Primes will denote transposes and E denotes expectation.

$$\hat{z}_{t|T} = \int_0^T H(t,s) y_s \, ds \tag{2a}$$

such that $E(z_t - \hat{z}_{t|T})'(z_t - \hat{z}_{t|T})$ is minimum, where the minimization is taken over all matrix-valued functions $H(t,s)$ of s, with t fixed, in the Hilbert space $L^2[0,T]$. It is well known that the following holds.

Theorem: A necessary and sufficient condition for the solution of this smoothing problem is

$$E(z_t - \hat{z}_{t|T}) y_s' = 0 \text{ for } all \;\; s \; \varepsilon [0,T] \; . \tag{2b}$$

Put differently: every component of $z_t - \hat{z}_{t|T}$ must be orthogonal to every component y_s, for all $s \; \varepsilon [0,T]$, where the orthogonality is induced by the inner product $(a,b) := E\, ab$, a and b scalar stochastic variables.

Suppose now further that the special case (i) holds, namely that z_t is orthogonal to v_s for all s,t with $0 \le s \le T$, $0 \le t \le T$. Then the condition (2b) leads to the equation

$$H(t,s) + \int_0^T h(t,\tau) K(\tau,s) \, d\tau = K(t,\tau), \; 0 \le s,t \le T \; . \tag{3}$$

This is a Fredholm equation of the second kind (see, e.g., Courant and Hilbert, Vol. I, Ch. III) and the solution *H(t,s)* is called the ***Fredholm resolvent*** of ***K(t,s)*** on $[0,T] \times [0,T]$. Introduce the following operator notation:

$$HK \textit{ is defined as } (HK)(t,s) = \int_0^T H(t,\tau) K(\tau,s) \, d\tau \; ,$$

and I is the identity operator with kernel $I(t,s) = I_p \, \delta(t-s)$. In this notation the equation (3) can be written as

$$H + HK = K \tag{4a}$$

or in the equivalent forms

$$(I - H)(I + K) = I = (I + K)(I - H) . \tag{4b}$$

Clearly $I\text{-}H$ is the inverse, in the sense of the "operator multiplication" that we have just defined, of $I+K$. Note that in this case of complete orthogonality of z_t and v_s, the smoothing filter is precisely the resolvent of K.

How can the resolvent be computed? One answer is provided by the so-called Mercer expansion of $K(t,s)$ (see, e.g., Riesz and Nagy, p. 245; we use an extension to the vector case):

$$K(t,s) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(t)\varphi_i'(s) \tag{5a}$$

where the φ_i are vector-valued orthonormal eigenfunctions of the operator K with eigenvalue φ_i:

$$\int_0^T K(t,\tau)\varphi_i(\tau)\, d\tau = \lambda_i \varphi_i(t) \ , \quad i = 1,2,\ldots \ ,0 \le t \le T \tag{5b}$$

Then it can be seen easily that the Fredholm resolvent of K can be written as

$$H(t,s) = \sum_1^{\infty} (\lambda_i / 1 + \lambda_i)\varphi_i(t)\varphi_i'(s) \tag{6}$$

1b. The Filtering Problem: Wiener-Hopf Equations

In the special case that $T=t$, the smoothing problem becomes what is known as a ***filtering problem***. We shall assume further that we are in one of the special cases (i) and (ii), viz., that v_t is either orthogonal to z_s for all s,t or just for all $s<t$. The filtered estimate can be written

$$\hat{z}_{t|t} = \int_0^t h(t,s) y_s \, ds \tag{7}$$

where $h(t,s)$ satisfies

$$h(t,s) + \int_0^t h(t,\tau)K(\tau,s)\, d\tau = K(t,s) \ , \quad 0 \le s \le t \le T \ , \tag{8}$$

Note that for each fixed t, we have a smoothing problem. The point is now that we have a collection of Fredholm integral equations, one for each value of t, and unlike as in (3) is more than just an indexing parameter in the family of equations. The filtering integral equation is said to be of "Wiener-Hopf type" rather than of "Fredholm type" and the solution can not be as simply expressed in terms of Mercer expansions as in the smoothing problem.

In one sense then, smoothing appears to be "easier" than filtering, a statement counter to the intuition current in the Kalman-Bucy state-space theory (see, e.g., Meditch (1969) and also the discussion following (15b) below). However, the following facts give some justification to this claim:

1. In the Wiener theory of estimation of stationary processes over infinite (smoothing) or semi-infinite (filtering) intervals, the smoothing solution is readily determined by Fourier transformation, while the filtering solution requires the more difficult Wiener-Hopf technique (further elaborated below).
2. In estimation given a fixed time-interval, smoothing can be implemented with *time invariant* filters (convolutions or fast Fourier transforms, see Lévy, Kailath, Ljung and Morf (1979)), while this will never be true for filtering.

1c. The Generalized Wiener-Hopf Technique

Wiener-Hopf equations first appeared in astrophysics and radiative transfer theory around 1900. In 1931, Wiener and Hopf invented an ingenious method for solving the equation, and it has since borne their name. Their so-called Wiener-Hopf technique plays a central role in linear filtering theory, and we will present it briefly here, both as a framework for later discussion and as a service to our state-space friends who might conceivably have never seen it! The technique was originally developed for difference (or convolution) kernels R; here we describe a generalized form (for arbitrary kernels) that captures the main idea.

To focus on the main idea, the treatment will leave aside technical issues (hypotheses on kernel functions, specification of function spaces, etc.) that are needed to build a rigorous theory (see, for example, Devinatz and Shinbrot (1967) and Gohberg and Feldman (1974)). To describe the technique, we first develop an operator notation for (8) that expresses the constraint $s \le t$. Thus if L is an integral operator associated with the kernel $L(t,s)$, define

$$\{L\}_+ f(t) := \int \{L(t,s)\}_+ f(s)\, ds$$

with

$$\{L(t,s)\}_+ := \begin{cases} L(t,s) & \text{if} \quad s \le t \\ 0 & \text{if} \quad s > t \end{cases}$$

Accordingly, define $\{I\}_+ := I$. $\{L\}_+$ is called the *causal part* of L. With this notation the Wiener-Hopf equation (8) becomes

$$\{hR\}_+ = \{K\}_+ \,. \tag{9}$$

As only the values of $h(t,s)$ for $s \le t$ play a role in this problem, $h(t,s)$ can be (and will be) taken equal to its causal part: $h = \{h\}_+$. We assume that $R = I + K$, $R = R'$, R is positive definite as a kernel, and K does not contain I term (alternatively: $I_p\, \delta(t-s)$ does not appear in $K(t,s)$).

The key idea of the method of Wiener and Hopf (1931) is to assume that R can be suitably factored. In our case, as

$$R = R^{1/2} R^{*/2} \,,$$

where $R^{1/2}$ is a causal and causally invertible operator, that is, $R^{1/2} = \{R^{1/2}\}_+$ and $R^{-1/2} = [R^{1/2}]^{-1}$ exists and satisfies $R^{-1/2} = \{R^{1/2}\}_+$. Here $R^{*/2}$ denotes the adjoint of $R^{1/2}$, that is $R^{*/2}(t,s) = R^{1/2}(s,t)'$. Such an $R^{1/2}$ is called the

canonical factor of R, and when it exists it will be unique as a consequence of the causal and causally invertible requirement. [Observe that, despite the notation, $R^{1/2}$ is not the traditional operator-theoretic (symmetric) square root of a positive operator.]

Now make the simple but crucial observation that if h solves (9), there must exist some function g such that

$$\{g\}_+ = 0 \text{ and } hR = K + g \quad . \tag{10}$$

Here g is ***strictly*** anti-causal, i.e., it does not have any I component. Multiplying (10) on the right by $R^{-*/2}$, we have

$$hR^{1/2} = KR^{-*/2} + gR^{-*/2} \quad . \tag{11}$$

Now apply $\{\ \}_+$ to both sides of (11). Since h is causal and $R^{1/2}$ is causal and since the composition of two causal operators is again causal, $\{hR^{1/2}\}_+=hR^{1/2}$. Likewise the composition of a strictly anti-causal operator (g) with an anti-causal operator ($R^{-*/2}$) is strictly anti-causal; ($R^{-*/2}=(R^{-1/2})^*$ is anti-causal since $R^{-1/2}$ is causal). Thus $\{gR^{-*/2}\}_+=0$. The end result is $hR^{1/2}=\{KR^{-*/2}\}_+$, or

$$h = \{KR^{-*/2}\}_+ R^{-1/2} \quad . \tag{12}$$

This is the solution of the Wiener-Hopf equation.

However, we have not really so far used the assumption that R has the special form $R=I+K$. For this case further important results are available from canonical factorization. First observe that, since $K=R-I$,

$$\begin{aligned} h &= \{(R-I)R^{-*/2}\}_+ R^{-1/2} \\ &= \{R^{1/2} - R^{-*/2}\}_+ R^{-1/2} \\ &= I - \{R^{-*/2}\}_+ R^{-1/2} \quad . \end{aligned} \tag{13}$$

However, when $R=I+K$, $R^{1/2}$ must have the form $R^{1/2}=I+h$, where h is some strictly causal operator, and so also $R^{-1/2}$ must have the form $R^{-1/2}=I+l$, where l is strictly causal. Therefore, $\{R^{-*/2}\}_+=\{I+l^*\}_+=I$, since l^* is strictly anti-causal. Hence (13) reduces to

$$h = I - R^{-1/2} \tag{14}$$

This striking formula has several interesting implications.

First, and most important, it shows that for this problem canonical factorization and filtering are *equivalent* problems; h is *immediately* determined if $R^{-1/2}$ is known and vice versa.

Secondly, if $R^{-1/2}$ is applied as a filter to y, we have

$$R^{-1/2}y = (I - h)y = y - \hat{z} \ .$$

Since $y=z+v$, $\hat{z}$ is the estimate of y_t given $\{y_s, s<t\}$, so that it is reasonable to expect that $\{y_t-\hat{y}_t\}$, the *new information* or *innovation* process, is a white noise process, consistent with the calculation

$$<R^{-1/2}y, R^{-1/2}y> = R^{-1/2}<y,y>R^{-*/2} = R^{-1/2}RR^{-*/2} = I \ .$$

This result can be rigorously established under quite general conditions (see Kailath (1968), Kailath (1971), Meyer (1972)).

1d. A Resolvent Identity relating Smoothing and Filtering

Recall the Fredholm resolvent of K, defined by $I-H=(I+K)^{-1}=R^{-1}$. By virtue of (14), we can write

$$I - H = R^{-*/2}R^{-1/2} = (I - h^*)(I - h) \tag{15a}$$

which immediately yields the nice formula

$$H = h^* + h - h^*h \ . \tag{15b}$$

This is actually an old identity, known by the early 1950's when it was discovered, in a differential version, independently by Siegert, Krein, Bellman and others (see references in Kailath (1974)).

Now when the signal and noise are completely uncorrelated, we saw earlier (cf. (4b)) that the Fredholm resolvent H is just the smoothing filter, the identity (15) then shows that the causal filter h determines the smoothing filter. This seems to contradict the remarks we made in Section 2b about the relative difficulties of smoothing and filtering as they appeared from thinking of the Wiener-Hopf equation as an infinite family of Fredholm equations. In the approach via canonical factorization, it would appear that filtering comes first, and then smoothing. In Sec. 3d, which describes a scattering theory approach, this sequence will again be reversed.

We shall illustrate these different relationships between filtering and smoothing by considering several specific examples in the next two sections.

2. Some Examples - Stationary Processes

2a. Scalar Stationary Processes Over Infinite Intervals

These problems were studied by Wiener, Kolmogorov and Krein. For a concise exposition of Wiener's work, see a paper by N. Levinson (1947), reprinted as Appendix C of Wiener (1949). The papers of Kolmogorov and Krein are reprinted in Kailath (1977).

We suppose y_t, z_t, v_t to be scalar and stationary, then $R(t,s)=R(t-s)$. We assume the existence of

$$S_y(\omega) := F\{R(t)\}(\omega) = \int_{-\infty}^{\infty} R(t)e^{-j\omega t}\,dt \quad ,$$

the Fourier transform of $R(t)$, where j is the imaginary unit. $S_y(\omega)$ is nonnegative (for real ω) and is known as the power spectral density of the process y. We assume further that

$$\int_{-\infty}^{\infty} \frac{|ln\ S_y(\omega)|}{1+\omega^2} d\omega < \infty \tag{16}$$

If this is not the case, then Kolmogorov and Wiener showed that y_t can be predicted perfectly from its own past (see, e.g., Doob (1953), Ch. 12).

Under this assumption the canonical factorization of $R(t)$ over $(-\infty,\infty)$ corresponds to a factorization of $S_y(\omega)$ as

$$S_y(\omega) = S_y^+(\omega)S_y^-(\omega) \ , \tag{17}$$

where $S_y^+(\sigma+j\omega)$ is analytic and bounded in the right half plane $\sigma>0$, $S_y^-(\omega)$ is analytic in the left half plane and

$$S_y^-(\sigma + j\omega) = \bar{S}_y^+(\sigma - j\omega) \tag{17a}$$

It can be shown that (see, e.g., Solodovnikov (1960))

$$S_y^+(\omega) = \sqrt{S_y(\omega)}\, e^{j\vartheta(\omega)} \ , \tag{18}$$

where $\vartheta(\omega)$ is the Hilbert transform of $\ln\sqrt{S_y(\omega)}$.

In the case that $S_y(\omega+i\sigma)$ is rational, $S_y^+(\omega+i\sigma)$ can be found as follows:

$$S_y^+(\omega + j\sigma) = \text{constant x Monic polynomial of left half plane zeros}) \times \\ (\text{Monic polynomial of all left half plane poles})^{-1}. \tag{19}$$

This follows immediately from the fact that $S_y^+(\sigma+j\omega)$ and $1/S_y^+(\sigma+j\omega)$ must be analytic in the upper half plane $\sigma>0$, and from (17a). Note that (17a) implies $S_y^-(\omega)=\overline{S_y^+(\omega)}=S_y^+(-\omega)$, because $R(t)$ is real. Therefore,

Now the canonical causal factor (see the text between (9) and (10)) can be found as

$$R^{1/2} = F^{-1}\{S_y^+(\omega)\} \ , \tag{20}$$

the inverse Fourier transform of $S_y^+(\omega)$. The optimal filter (see (14)) is then equal to

$$h = F^{-1}\left\{1 - \frac{1}{S_y^+(\omega)}\right\} \tag{21}$$

How About Smoothing?

Consider the Fourier transform of the smoothing filter H (see (15)):

$$\begin{aligned} F\{H\} &= F\{h + h^* - h^*h\} = \\ &= \left\{1 - \frac{1}{S_y^+(\omega)}\right\} + \left\{1 - \frac{1}{S_y^+(-\omega)}\right\} - \left\{1 - \frac{1}{S_y^-(\omega)}\right\}\left\{1 - \frac{1}{S_y^+(\omega)}\right\} = \\ &= 1 - \frac{1}{S_y^+(\omega)S_y^+(-\omega)} = 1 - \frac{1}{S_y(\omega)} \end{aligned} \tag{22}$$

This is a well known formula easy to derive directly from the equality $H=I-R^{-1}$. Note that smoothing does not require special factorization, so that the smoothing solution is easier to find from the given data than the filtering solution.

Important Remark: When $S_y(\omega)$ is rational, $S_y^+(\omega)$ is rational and so is $F\{h\} = 1 - \frac{1}{S_y^+(\omega)}$. This can be readily implemented in (a variety of) state-space forms, so that z can be "recursively computed", as noted by Whittle ((1963), p. 35) and others, independently of the direct state-space formulation of Kalman and Bucy (1961).

2b. Finite Intervals - The Ambartzumian-Chandrasekhar Equations

We shall next talk about the more difficult case of filtering stationary, scalar processes defined on finite intervals. This may be considered the first natural extension of Wiener's work and it began to be studied in the engineering literature around 1950, shortly after Wiener's work became public (see, e.g., Zadeh and Ragazzini (1950)). The finite interval case presented a new challenge because spectral factorization, in its traditional sense, does not work, and thus researchers tried various other methods to find the solution (see, e.g., Solodovnikov (1962)).

However, the astrophysicists V. A. Ambartzumian (USSR) and S. Chandrasekhar (USA) had already studied such problems in the mid-1940's, independently of engineers, and had demonstrated that the Wiener-Hopf equation could be replaced by an equivalent Riccati equation. Their results greatly simplified the numerical computation of solutions, and since computation in those days meant calculation by hand, they were considered to be a great success. The Ambartzumian-Chandrasekhar theory (see Chandrasekhar (1950)) assumes that the kernel of the Wiener-Hopf equation has of the form

$$K(t-s) = \int_0^1 e^{\alpha|t-s|} w(\alpha)\, d\alpha \, , \qquad (23)$$

in which $w(\alpha)$ is some known function. This form arose from the physical situation they considered, in which light incident at an angle α is propagating through a medium. If we assume that the light is incident at a finite number of values α, so that $w(\alpha)$ is a sum of δ-functions, the process y wll have a rational spectral density

$$S_y(\omega) = 1 + \sum_{i=1}^{n} \frac{2w_i \alpha_i}{\alpha_i^2 + \omega^2} . \tag{24}$$

The first result of the Ambartzumian-Chandrasekhar theory is that to solve the Wiener-Hopf equation it suffices to find the solution $Q(t,\alpha,\beta)$ to a Riccati-type partial differential equation

$$\frac{\partial}{\partial t} Q(t,\alpha,\beta) = P + k\ Q(t,\alpha,\beta) + \int_0^1 Q(t,\alpha,\beta')w(\beta')\, d\beta' \tag{25}$$
$$+ \int_0^1 Q(t,\alpha',\beta)w(\alpha')\, d\alpha' + \int_0^1\int_0^1 Q(t,\alpha,\beta')Q(t,\alpha',\beta)w(\alpha')w(\beta')\, d\alpha' d\beta' .$$

Q can be computed by discretization of this equation to obtain a finite dimensional system of ordinary Ricatti differential equations This has great computational advantages, though since it may be required to compute $Q(t,\alpha,\beta)$ for t, α and β ranging over a large set of values, this is still burdensome.

However using certain physical *invariance* arguments, Ambartzumian (1943) was able to show that Q could actually be computed in terms of two functions $X(t,\gamma)$ and $Y(t,\gamma)$, of *two* rather than *three* variables. Then Chandrasekhar (1947) derived a pair of differential equations for X and Y, considerably simpler than the original Riccati equation (25).

$$\frac{\partial X(t,\gamma)}{\partial t} = -\ Y(t,\gamma) \int_0^1 Y(t,\beta)w(\beta)\, d\beta \tag{26a}$$

$$\frac{\partial Y(t,\gamma)}{\partial t} = -\ Y(t,\gamma) - X(t,\gamma) \int_0^1 Y(t,\beta)w(\beta)\, d\beta \tag{26b}$$

with

$$X(0,\gamma) = Y(0,\gamma) = 1 , \quad 0 < \gamma < 1$$

Astrophysicists were quick to recognize the value of the recursive solution of the equations (26) (see, e.g., Sobolev (1965), p. 79). The ideas of Ambartzumian and Chandrasekhar were brought to the attention of applied mathematicians by the extensive work of Bellman, Kalaba and their colleagues on what they called *invariant imbedding* (see, e.g., Bellman and Wing (1975)). The equations (26) were first introduced into the estimation literature by Casti, Kalaba and Murthy (1972). Their extension to nonstationary processes was made by Kailath (1973).

2c. Sobolev's Identity and The Krein-Levinson Equations

Fundamental work on the finite interval, stationary case did not end with Ambartzumian and Chandrasekhar. In particular Sobolev (1965) went on to address the problem of arbitrary $K(t-s)$, for which a representation such as (23) is not given, and he succeeded in developing a much more direct approach. His idea was to exploit the Toeplitz structure of $K(t,s)=K(t-s)$ more deeply than in the previous theory. By this approach, he established a very powerful constraint on the Fredholm resolvent (smoothing kernel) $H(t,s;T)$ of a Toeplitz kernel:

If $A(T;t)$ is defined via the equation

$$A(T;t) + \int_0^T A(T;u)K(u-t)\,du = K(T-t)\,, \quad 0 \le t \le T \tag{27}$$

and $B(T;t)$ via

$$B(T;t) + \int_0^T B(T;u)K(u-t)\,du = K(-t)\,, \quad 0 \le t \le T \tag{28}$$

then Sobolev showed that

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right] H(t,s;T) = A'(T,t)A(T;s) - B'(T;t)B(T;s) \tag{29}$$

with

$$A(T;t) = H(T,t;T) = H'(t,T;T) \tag{30a}$$

and

$$B(T;t) = H(0,t;T) = H'(t,0;T) \, . \tag{30b}$$

(Note that these equations are written for the general case of matrices K, H, A and B.) Sobolev's identity shows that the resolvent $H(s,t;T)$ is determined for $(t,s) \in [0,T]\times[0,T]$ by its values on the boundaries of $[0,T]\times[0,T]$, i.e., by two functions of one variable.

Sobolev's identity is even more striking in its integrated form, which, when translated into operator form, is

$$I - H = (I - \mathbf{a}^*)(I - \mathbf{a}) - \mathbf{b}^*\mathbf{b} \tag{31}$$

where $\mathbf{a}$ and $\mathbf{b}$ are not only causal, but also Toeplitz. This means, for example, that the filter determind by $\mathbf{a}$,

$$(\mathbf{a}y)_t = \int_0^t a(t-s)y_s \, ds$$

is ***time-invariant***. Equation (31) is a useful modification of the formula (15a): $(I-H)=(I-\mathbf{h}^*)(I-\mathbf{h})$, because it expresses H only in terms of time invariant (causal and anticausal) operators, whereas $\mathbf{h}$, even for Toeplitz K, is not generally time invariant. Since time invariant filters are much easier to implement than time-variant ones, it is reasonable to use $\mathbf{a}$ instead of $\mathbf{h}$, whenever $\mathbf{h}$ appears. Of course, an error is then incurred, but the remarkable implication of (31) is that if $\mathbf{a}$ replaces $\mathbf{h}$ in (15a), the correction can again be made with a time invariant filter $\mathbf{b}$. Explicit formulas for a and b are as follows.

$$a(t) = A(T;t) \tag{32a}$$

$$b(t) = B(T;T-t) \, . \tag{32b}$$

For more on the applications of this identity to smoothing and other problems, see Lévy et al. (1979) and Kailath et al. (1978).

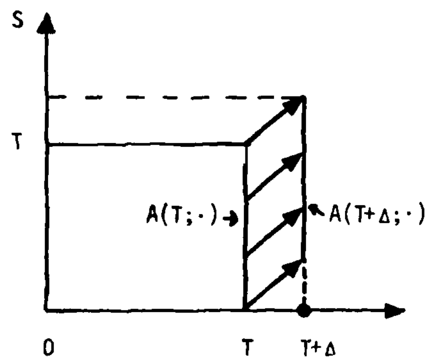
Sobolev's identity also has important implications for fast computation of the resolvent kernel $H(t,s;T)$ because we need only develop an efficient, recursive method for updating the boundary values $A(T;t)$ and $B(T;t)$ of $H(t,s;T)$. In fact, Krein (1955) had obtained such equations, which we shall present here in the special case of scalar processes:

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right\}A(T;s) = -A(T;T-s)A(T;0) \quad (36)$$

To see what this means, consider the naive discretization, $T=N\Delta$ and

$$A(T+\Delta;i\Delta+\Delta) = A(T;i\Delta) - A(T;T-i\Delta)A(T;0)\Delta \quad (36)$$

which propagates as illustrated in the figure



The one point not picked up by this scheme is $A(T+\Delta;0)$ and so it is computed by using the integral equation (27):

$$A(T+\Delta;0) = \int_0^{T+\Delta} A(T+\Delta;u)K(u)\,du + K(T+\Delta) \quad (37)$$
$$\approx \sum_{i=1}^{T+\Delta} A(T+\Delta;i\Delta)K(i\Delta)\Delta + K(T+\Delta) .$$

How fast is this algorithm? If $T=N\cdot\Delta$, it takes $N+1$ multiplications to go from T to $T+\Delta$. Therefore, to generate the boundary out to T, takes $1+2+\ldots+N=N(N+1)/2=0(N^2)$ multiplications. Without the Toeplitz structure, we would need $O(N^3)$ operations to compute $H(t,s;T)$.

The above method of solving (36) via discretization leads to recursions very similar to those introduced for the prediction of discrete-time processes by Levinson (1947) and since then widely studied in the signal processing literature. We therefore call (36) a *Krein-Levinson* equation. Kailath, Ljung and Morf (1978) have extended these techniques to nonstationary processes.

We should also mention that when a representation of $K(t-s)$ in the exponential form (23) is available, the Krein-Levinson equations can in fact be reduced to the Ambartzumian-Chandrasekhar equations (cf. Kailath, Ljung, Morf (1976)).

3. Some Examples – Nonstationary Processes

3a. State Space Process Models - Kalman-Bucy Filters

The most common approach to nonstationary process estimation is via state-space models. It is assumed that the signal process z_t can be described as

$$z_t = H_t x_t \tag{38a}$$

$$\dot{x}_t = F_t x_t + G_t u_t \, , \quad t \geq t_0 \, , \tag{38b}$$

where x_t is an $n \times 1$ "state" vector, z_t is a $p \times 1$ vector and u_t is an $m \times 1$ "white noise" vector. The observed process is

$$y_t = z_t + v_t \tag{38c}$$

where is the observation noise v_t such that†

$$E\left\{\begin{bmatrix} u_t \\ v_t \end{bmatrix} [u_s' v_s']\right\} = \begin{bmatrix} I & C_t \\ C_t' & I \end{bmatrix} \delta(t - s) \, . \tag{38d}$$

The initial state x_{t_0} is assumed to be random, with

$$E\, x_{t_0} = 0 \, , \quad E\, x_{t_0} x_{t_0}' = \Pi_0 \, . \tag{38e}$$

and furthermore, it is also assumed that

$$E\, u_t x_{t_0}' = 0 \, ; \quad E\, v_t x_{t_0}' = 0 \text{ for } all \quad t \geq t_0 \, . \tag{38f}$$

Finally, the matrices F_t, G_t, H_t, Π_0 are all assumed to be known. The above assumptions ensure that

$$E\, v_t z_s' = 0 \text{ if } s < t \, .$$

They also ensure that x_t is a Markov process, so that the signal z_t is modeled

† In the literature it is common to assume $E\, u_t u_s' = Q_t \delta(t - s)$ and $E\, v_t v_s' = R_t \delta(t - s)$, $R_t > 0$, but without loss of generality we can make the convenient normalizations of (38d).

as a so-called *projection* of the Markov state process x_t. Such descriptions had been widely used by physicists (see, e.g., the papers in Wax (1954)) and mathematicians (see, e.g., Doob (1944), (1948)) for stationary processes with rational power spectral densities. The extension to models with time-variant coefficients, as in (38) above, is at least in retrospect fairly natural and it began to be made in estimation theory in the late fifties by Laning and Battin (1958, p. 304), Stratonovich (1959), (1960), and most notably by Kalman (1960) and Kalman and Bucy (1961).

The celebrated Kalman-Bucy filter for the model (38) is

$$\hat{z}_t = H_t \hat{x}_t, \quad t \geq t_0 \tag{39a}$$

$$\dot{\hat{x}} = F_t \hat{x}_t + K_t \nu_t \tag{39b}$$

$$\hat{x}_{t_0} = 0 \tag{39c}$$

Here ν_t is the "innovation process",

$$\nu_t = y_t - H_t \hat{x}_t \tag{39d}$$

which is known to be white with

$$E\, \nu_t \nu_s' = I\, \delta(t - s) \tag{39e}$$

The $n \times p$ matrix K_t, which we shall call the "Kalman gain", can be computed as

$$K_t = P_t H_t' + G_t C_t \tag{40a}$$

where the $n \times n$ matrix P_t is the covariance matrix of the errors,

$$P_t = E\, \tilde{x}_t \tilde{x}_t', \quad \tilde{x}_t = x_t - \hat{x}_t . \tag{40b}$$

Kalman and Bucy showed that P_t could be recursively determined as the unique solution of the nonlinear Riccati-type differential equation

$$\dot{P}_t = F_t P_t + P_t F_t' + G_t G_t' - K_t K_t', \quad P_{t_0} = \Pi_0 \tag{40c}$$

This is all very well known to estimation theorists by now. It is perhaps not so widely known that the recognition of the importance of state-space models and Markov processes in signal estimation problems is due independently to Stratonovich, who actually studied the nonlinear filtering problem, and, using "Gaussian approximation" methods, derived what was later called the "extended Kalman filter". For the linear case, Stratonovich gave an exact solution which is exactly the Kalman-Bucy filter. However, no stability analysis was given and no intensive study of the Riccati equation was made; these were the vital contributions of Kalman and Bucy.

It is often said that the reason for the wide applicability of the Kalman-Bucy filter (in particular, to general time-variant models) is that giving a state-space model avoids the difficult problem of "spectral factorization". This is totally wrong! Canonical spectral, or in the nonstationary case "covariance", factorization, results in a model that is not only causal but also causally invertible. This is clearly not the case for the assumed model (38): knowledge of $y = \{y_s,\ t_0 \le s \le t\}$ does *not* allow us to reconstruct u, v and x_{t_0}. Therefore to find the filter, canonical factorization still has to be done in one way or another.

In fact, as we noted in Section 2c (cf. (14)), knowledge of the canonical spectral factor should immediately determine the filter *and vice versa*. Here we have the filter and could ask how to obtain the canonical factor. The answer is simple: just rewrite (39) in the form

$$\dot{\hat{x}}_t = F_t \hat{x}_t + K_t \nu_t \tag{41}$$

$$y_t = H_t \hat{x}_t + \nu_t\,, \quad \hat{x}_{t_0} = 0\ .$$

We see that (41) describes a causal and causally invertible relation between a white noise input ν and the desired process y. We call it the *innovations representation* (IR) of y. Thus the above equations determine the true canonical factor. But to find this factor, we have to do quite a bit of work, viz., solve the Riccati equation. In other words, assuming a state-space model does not in any sense allow us to avoid determining the canonical filter (unless, of course, we start with such a model).

A Remark on Derivations

By now numerous proofs of the Kalman-Bucy equations are available. In the present context, it may be interesting to note that if

$$\hat{x}_t = \int_{t_0}^{t} h_x(t,s) y_s \, ds$$

then $h_x(t,s)$ obeys the Wiener-Hopf equation

$$h_{xy}(t,s) + \int_0^t h_{xy}(t,\tau) K(\tau,s)\, d\tau = K(t,s), \quad t_0 \leq s < t$$

where $K(t,s)$ can be readily computed from the given model (38)--in fact, see (44) below. Some calculation will then show that

$$\frac{\partial}{\partial t} h_{xy}(t,s) = [F(t) - h_{xy}(t,t)] h_{xy}(t,s), \quad s < t$$

so that

$$\begin{aligned} \frac{d}{dt}\hat{x}_t &= \int_0^t [F(t) - h_{xy}(t,t)H(t)] h_{xy}(t,s) y_s \, ds + h_{xy}(t,t) y_{ty} \\ &= F(t)\hat{x}_t + h_{xy}(t,t)[y_t - H(t)\hat{x}_t], \end{aligned}$$

which will be the Kalman-Bucy equation (39a) when we show that $h_{xy}(t,t) = K_t$ as given by (40). We shall omit these calculations.

3b. State Space Covariance Models - Recursive Wiener Filters

While many different models $\{F_t, G_t, H_t, Q_t G_t, \Pi_0\}$ can yield the same covariance function

$$R(t,s) = I \cdot \delta(t-s) + K(t,s),$$

the impulse response of the canonical factor is known to be uniquely determined by $R(t,s)$. Therefore one would expect that it is possible to compute the Kalman gain K_t (which should not be confused with the term $K(t,s)$ in the covariance funtion) ***directly*** from the parameters of the covariance function $R(t,s)$. This can in fact be done (Kailath and Geesey, 1971), as can be seen by examining the special form of the covariance function associated with a state-space model of the form (38).

Let $\Phi(t,s)$ be the state transition matrix, which is the unique solution of the linear differential equation

$$\frac{d\Phi(t,s)}{dt} = F_t\Phi(t,s)\ ;\ \ \Phi(s;s) = I$$

Let Π_t be the $n \times n$ covariance matrix of x_t,

$$\Pi_t = E\, x_t x_t' \ . \tag{42}$$

It is easy to check that Π_t is the solution to the Lyapunov-type equation

$$\dot{\Pi}_t = F_t\Pi_t + \Pi_t F_t' + G_t G_t' \ , \tag{43}$$

with given initial value Π_0. By direct calculation we have

$$R(t,s) = \begin{cases} I \cdot \delta(t-s) + H_t\Phi(t,s)N_s & \text{if } t \geq s \\ I \cdot \delta(t-s) + N_t'\Phi'(s,t)H_s' & \text{if } t \leq s \end{cases} \tag{44}$$

where

$$N_t = \Pi_t H_t' + G_t C_t \ . \tag{45}$$

We will now assume that (only) H_t, F_t and N_t are known. Define

$$\Sigma_t := E\, \hat{x}_t \hat{x}_t' \ . \tag{46}$$

If we recall that $P_t := E\, \tilde{x}_t \tilde{x}_t'$, where $\tilde{x}_t = x_t - \tilde{x}_t$, then the orthogonality of $\hat{x}_t$ and $\tilde{x}_t$ immediately yields the identity

$$\Pi_t = P_t + \Sigma_t \tag{47}$$

Therefore we can now rewrite the Kalman gain (cf. (39f)) as

$$K_t := P_t H_t' + G_t C_t = \Pi_t H_t' + G_t C_t - \Sigma_t H_t = N_t - \Sigma_t H_t' \tag{48}$$

Moreover, note that

$$\dot{\Sigma}_t = \dot{\Pi}_t - \dot{P}_t = F_t \Sigma_t + \Sigma_t F_t' + K_t K_t' \, ; \; \Sigma_{t_0} = 0 \ . \tag{49}$$

The equations, (48) and (49), determine K_t completely in terms of the parameters $\{H_t, F_t, N_t\}$ of the covariance function $R(t,s)$.

We note that we have effectively also obtained a recursive form of the solution to the Wiener-Hopf integral equation (8) for estimating the signal z_t from y_t, provided the covariance function $R(t,s)$ is given in the (factored) form (44). This provides a nice answer to the efforts of several investigators in the mid-1950's attacking Wiener-Hopf equations with nonstationary kernels (see, e.g., Dolph and Woodbury (1952), Shinbrot (1956), Zadeh and Miller (1956), Laning and Battin (1958)). We may thus call our solution (39), (48)-(49) a ***recursive Wiener filter*** as compared to the Kalman-Bucy filter (39)-(40).

The close connection we noted in Sec. 1c between the solution of the filtering problem and of the canonical factorization problem means that the above results, especially (41) and (48)-(49), also give us an expression of the canonical factor (innovations representation) directly in terms of the parameters of the covariance function. The notation $\hat{x}$ for the state of the model (40) is, of course, just a reflection of the original noncanonical model (38) that we started with. x has no particular significance if we are only given a process y with covariance $R = I + K$, and therefore it is preferable to rewrite (40) in a different notation. Following Faurre (1969), we shall write the canonical innovations model as

$$\dot{x}_* = F_t x_* + K_* \nu_* \ , \quad x_{*0} = 0 \tag{50a}$$

$$y_t = H_t x_* + \nu_* \tag{50b}$$

where

$$K_{*t} = N_t - \Sigma_{*t} H_t' \tag{50c}$$

and Σ_* obeys the Riccati equation

$$\Sigma_{*t} = F_t \Sigma_{*t} + \Sigma_{*t} F_t' + K_{*t} K_{*t}' , \quad \Sigma_{*0} = 0 . \tag{50d}$$

The value of our particular derivation of the IR is that it shows that of all (causal) models of a given covariance triple $\{F_t, H_t, N_t\}$, this one has the "smallest" state variance matrix, because its state, x_{*t}, is the projection (on the space spanned by $\{y_s, s<t\}$) of the state, x_t, of any other model.

There has been some interest in studying the set of all causal models associated with a given covariance triple; in particular, one might ask by analogy with the above for a *maximum-variance* causal model. It turns out that this model can be defined via a certain *smoothing* problem, rather than a *filtering* problem as for the minimum-variance model. We refer to Kailath and Ljung (1981) for a discussion of this result, and its implications for the so-called *stochastic realization problem* of studying all causal models of a given process. There have been a number of recent papers on this, see especially, the book of Clerget, Faurre, Germain (1978), the thesis of Ruckebusch (1979), and Lindquist and Picci (1981), which contains references to several of their earlier works.

3c. Orthogonal Decomposition of the Space of Random Variables

Here we shall continue instead with another aspect of the interplay between smoothing and filtering theory, as brought out by a recent result of Weinert and Desai (1980), which we can formulate as follows.

Given the state-space model (38), it is easy to see that we cannot recover the "input" random variables $\{u_t, v_t, x_{t_0}\}$ just from knowledge of the "output" random variables $\{y_t\}$, unless we have additional information. The nice observation of Weinert and Desai is that this additional information can be provided by a model 'adjoint' to (38) in a certain sense. In particular, define $\{\eta_t, \Theta\}$ by the equations

$$\chi_t = -F_t' \chi_t - H_t' v_t \ , \ \chi_{t_f} = 0 \tag{51a}$$

$$\eta_t = -G_t' \chi_t + v_t \tag{51b}$$

and

$$\Theta = -\Pi_0 \chi_{t_0} + x_{t_0} \tag{51c}$$

Then we can check by direct calculation that

$$E\, \Theta y_t' = 0 \ , \ E\, \chi_t y_t' = 0 \tag{52}$$

and that the sets

$$\{u_t, v_t, x_{t_0}\} \text{ and } \{y_t, \eta_t, \Theta\} \tag{53}$$

can each be recovered from the other by causal linear operations. In fact, this latter equivalence can be seen by first combining the equations (38) and (51) into the set

$$\begin{bmatrix} \dot{x}_t \\ \dot{\chi}_t \end{bmatrix} = \begin{bmatrix} F_t & 0 \\ 0 & -F_t' \end{bmatrix} \begin{bmatrix} x_t \\ \chi_t \end{bmatrix} + \begin{bmatrix} G_t & 0 \\ 0 & -H_t' \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix} \tag{54a}$$

$$\begin{bmatrix} \eta_t \\ y_t \end{bmatrix} = \begin{bmatrix} F_t & -G_t' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \chi_t \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \end{bmatrix} \tag{54b}$$

Then, by elimination, we can write

$$\begin{bmatrix} \dot{x}_t \\ \dot{\chi}_t \end{bmatrix} = \begin{bmatrix} F_t & -G_t G_t' \\ -H_t' H_t & -F_t' \end{bmatrix} \begin{bmatrix} x_t \\ \chi_t \end{bmatrix} + \begin{bmatrix} G_t \eta_t \\ -H_t' y_t \end{bmatrix} \tag{55a}$$

with the "two-point" boundary value conditions

$$\chi_{t_f} = 0 \ , \ x_{t_0} - \Pi_0 \chi_{t_0} = \Theta \tag{55b}$$

Given $\{\Theta, y_t, \eta_t\}$, we can solve this linear two-point boundary value problem for $\{x_t, \chi_t\}$ and then calculate $\{u_t, v_t, x(0)\}$ from (51b,c).

We can summarize the above discussion by saying that the result of Weinert and Desai shows that the linear space of random variables spanned by $\{u_t, v_t, x_0\}$ can, at each t, be *orthogonally decomposed* into the space spanned by $\{y_t\}$ and $\{\eta_t, \Theta\}$.

[The following remarks might help explain the origin of the above equations. The point is that given $\{y_t, t_0 \le t \le t_f\}$, we cannot reconstruct $\{u_t, v_t, x_0\}$ but at best only their smoothed estimates $\{\hat{u}_t, \hat{v}_t, \hat{x}_0\}$. For a full reconstruction we must augment $\{y_t\}$ by the errors $\{\tilde{u}_t, \tilde{v}_t, \tilde{x}_0\}$. Now some simple calculation, which is facilitated and in fact illuminated by the use of operator notation, will show that the $\{\eta_t, \Theta\}$ as defined above span the same space as the $\{\tilde{u}_t, \tilde{v}_t, \tilde{x}_0\}$. Hence ...]

Of the several interesting implications of the above results, we shall here focus on just one.

3d. The Hamiltonian Equations and a Scattering Framework for Estimation Theory

Consider the projection of the random variables in the equations (55) onto the space spanned by $\{y_t, t_0 \le t \le t_f\}$.

Now, if we define

$$\lambda_{t|t_f} = \hat{\chi}_{t|t_f} \tag{56}$$

and note that the orthogonality properties (52) imply that $\hat{\eta}_{t|t_f} = 0$ and $\hat{\Theta}_{|t_f} = 0$, the equations (55) reduce to

$$\begin{bmatrix} \dot{\hat{x}}_{t|t_f} \\ -\dot{\lambda}_{t|t_f} \end{bmatrix} = \begin{bmatrix} F_t & -G_t G_t' \\ H_t' H_t & F_t' \end{bmatrix} + \begin{bmatrix} 0 \\ H_t' \end{bmatrix} y_t \tag{57a}$$

with boundary conditions

$$\lambda_{t_f|t_f} = 0 \, , \quad \hat{x}_{t_0|t_f} = \Pi_0 \lambda_{t_0|t_f} \, . \tag{57b}$$

These are the so-called *Hamiltonian equations* for linear estimation. They were first obtained by Bryson and Frazier (1963) in exploring a calculus of variations approach (maximizing a 'likelihood' function) to least-squares estimation. Verghese, Friedlander and Kailath (1980) used them to develop a so-called scattering theory approach to linear estimation. To introduce this, we shall for convenience change notation in (57):

$$t_0 \to \tau, \; t \to s, \; t_f \to t \; .$$

Then using Euler discretization, e.g.,

$$\dot{\lambda}_{s|t} = [\lambda_{s+\Delta|t} - \lambda_{s|t}]/\Delta + o(\Delta)$$

and the approximation

$$\int_s^{s+\Delta} y_\sigma \, d\sigma = y_s \Delta + o(\Delta) \, ,$$

we can obtain the following discrete approximation to (57):

$$\begin{bmatrix} \hat{x}_{s+\Delta|t} \\ \lambda_{s|t} \end{bmatrix} = \begin{bmatrix} I+F_s\Delta & G_s G_s'\Delta \\ -H_s'H_s\Delta & I+F_s'\Delta \end{bmatrix} \begin{bmatrix} \hat{x}_{s|t} \\ \lambda_{s+\Delta|t} \end{bmatrix} + \begin{bmatrix} 0 \\ H_s'y_s\Delta \end{bmatrix} \tag{58}$$

where we have, and shall in the future, consistently omit all $o(\Delta)$ terms. Note the arguments of the $\lambda_{\cdot|t}$, which are reversed from those of $\hat{x}_{\cdot|t}$. Because of this, we can depict (58) graphically as in Fig. 1, which suggests that we can regard $\hat{x}(\cdot|t)$ as a ***forward*** wave and $\lambda(\cdot|t)$ as a ***backward*** wave travelling through an incremental section at s of some *scattering* medium specified by the quantities

$I + F_s\Delta =$ the incremental ***forward transmission coefficient***

$I + F_s'\Delta =$ the incremental ***backward transmission coefficient***

$-H_s'H_s\Delta =$ the incremental ***left reflection coefficient***

$G_s G_s'\Delta =$ the incremental ***right reflection coefficient*** and

$H_s' y_s \Delta =$ the incremental *internal backward source excitation*.

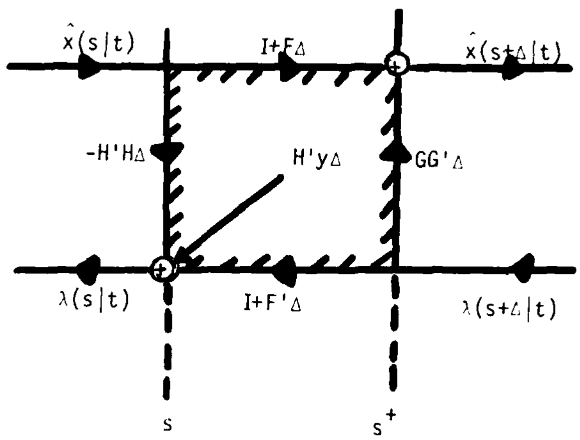


Figure 1. An incremental scattering layer corresponding to Eq. (58).

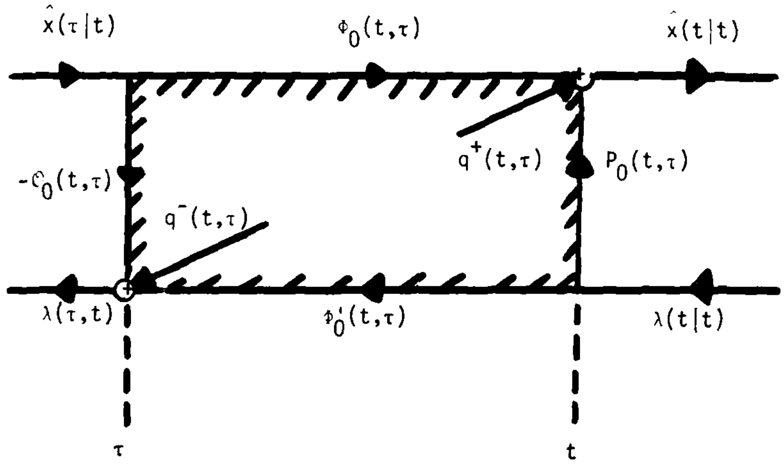


Figure 2. A macroscopic scattering section

We can put together such incremental sections to get a macroscopic section of the scattering medium from say $s=\tau$ to $s=t$. This is shown in Fig. 2, where, for reasons that will be clear very soon, we have denoted

$\Phi_0(t,\tau) =$ the *forward* transmission operator

$\Phi_0^{'}(t,\tau) =$ the *backward* transmission operator

$P_0(t,\tau) =$ the *right* reflection operator

$O_0(t,\tau) =$ the *left* reflection operator

$q_0^{+}(t,\tau) =$ the *forward* internal source (i.e., $y(.)$) contribution

$q_0^{-}(t,\tau) =$ the *backward* internal source contribution

The reasons for this notation can be seen by considering the effect of adding an incremental section from t to $t+\Delta$, as shown in Fig. 3. By tracing paths through the figure, we can write

$$\begin{aligned}\Phi_0(t+\Delta,\tau) &= (I+F\Delta)\Phi_0(t,\tau) - (I+F\Delta)P_0H'H\Delta\Phi_0(t,\tau)\\ &\quad + (I+F\Delta)P_0H'HP_0H'H\Delta^2\Phi_0(t,\tau) - \cdots\\ &= (I+F\Delta)(I - P_0H'H\Delta + o(\Delta)\Phi_0(t,\tau)\end{aligned} \tag{59}$$

where $o(\Delta)$ denotes terms that go to zero faster than Δ as $\Delta\to 0$. Then we see that

$$\lim \frac{\Phi_0(t+\Delta,t) - \Phi_0(t,\tau)}{\Delta} = (F - P_0H'H)\Phi_0(t,\tau) \tag{60}$$

which identifies $\Phi_0(t,\tau)$ as the state-transition matrix of $F-P_0H'H$.

Similarly, we can see by tracing paths through Figure 3 that

$$P_0(t+\Delta,\tau) = GG'\Delta + (I+F\Delta)(P_0 - P_0H'HP_0\Delta + o(\Delta))(I+F'\Delta) \tag{61}$$

so that

$$\lim \frac{P_0(t+\Delta,\tau) - P_0(t,\tau)}{\Delta} = GG' + FP_0 + P_0F' - P_0H'HP \tag{62}$$

which identifies $P_0(t,\tau)$ as the solution of a Riccati differential equation. A similar calculation will identify $\mathbf{O}_0(t,\tau)$ as an observability Gramian of the matrices $\{F-P_0H'H, H\}$

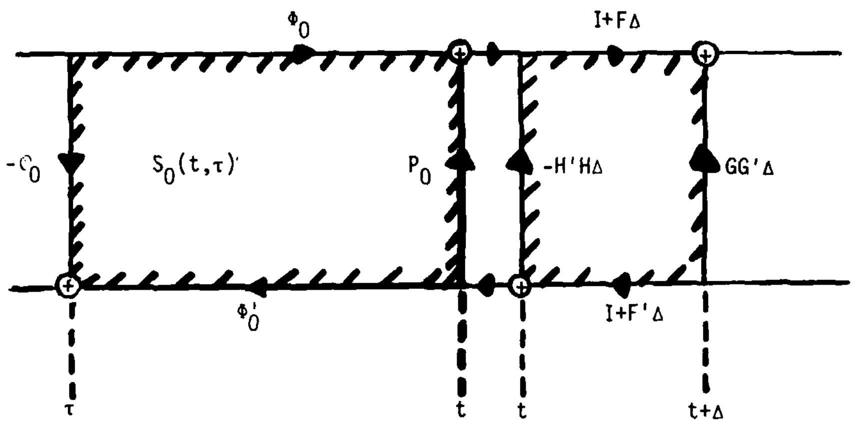


Figure 3. To determine the (forward) evolution equation of $S_0(t,\tau)$

We shall collect these operators in a so-called *scattering matrix*

$$S_0(t,\tau) = \begin{bmatrix} \Phi_0(t,\tau) & P_0(t,\tau) \\ -\mathbf{O}_0(t,\tau) & \Phi_0'(t,\tau) \end{bmatrix} \tag{63}$$

and our previous calculations show that

$$\frac{\partial}{\partial t} S_0(t,\tau) = \begin{bmatrix} (F-P_0H'H)\Phi_0 & FP_0+P_0F'+GG'-P_0H'HP_0 \\ \Phi_0'H'H\Phi_0 & \Phi_0'(F-P_0H'H)' \end{bmatrix} \tag{64a}$$

with initial conditions

$$S_0(\tau,\tau) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{64b}$$

These initial conditions help to explain the subscript '0' on the various quantities above, and we can get some more insight into this by going back to the boundary conditions of the original Hamiltonian equations and trying to incorporate them into our scattering picture. The $\lambda_{t|t}=0$ condition means we have no backwards incoming wave, while the condition

$$\hat{x}_{\tau|t} = \Pi_0 \lambda_{\tau|t}$$

can be incorporated, as shown in Fig. 4, by adding a 'boundary' layer to the left of the section $S_0(t,\tau)$ of Fig. 2.

One immediate result from this figure is that we can identify $\{q_0^+, q_0^-\}$ as the *emerging waves from the medium when* $\Pi_0=0$. We shall denote these as

$$q_0^-(t,\tau) = \lambda_0(\tau|t), \quad q_0^+(t,\tau) = \hat{x}_0(t|t) \tag{65}$$

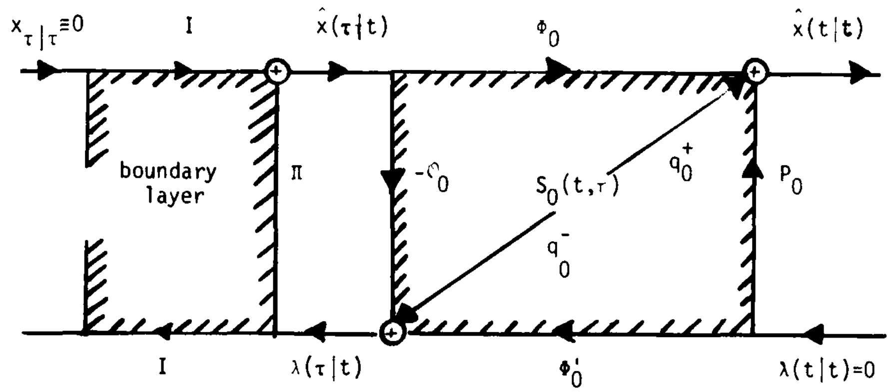


Figure 4. Incorporating the boundary conditions.

We shall now show how to derive forward differential equations for λ_0 and $\hat{x}_0$ and also backward equations as in (17)-(18). For forward equations, we start by adding an incremental section to the one in Fig. 4 (but with $\Pi=0=x_0$). Doing this gives the result shown in Fig. 5b.

Combining the signals in the two parts of Fig. 5a we obtain for the quantities in the combined section of Fig. 5b, the relations

$$\lambda_0(t\,|t+\Delta) = \lambda_0(\tau|t) + \Phi_0'H'(y - H\hat{x}_0(t\,|t+\Delta)\Delta + o(\Delta)$$

so that

$$\frac{\partial\lambda_0(\tau|t)}{\partial t} = \Phi_0'(t,\tau)H'(t)(y(t) - H(t)\hat{x}_0(t\,|t)) \tag{66}$$

So also, we can read off the relations

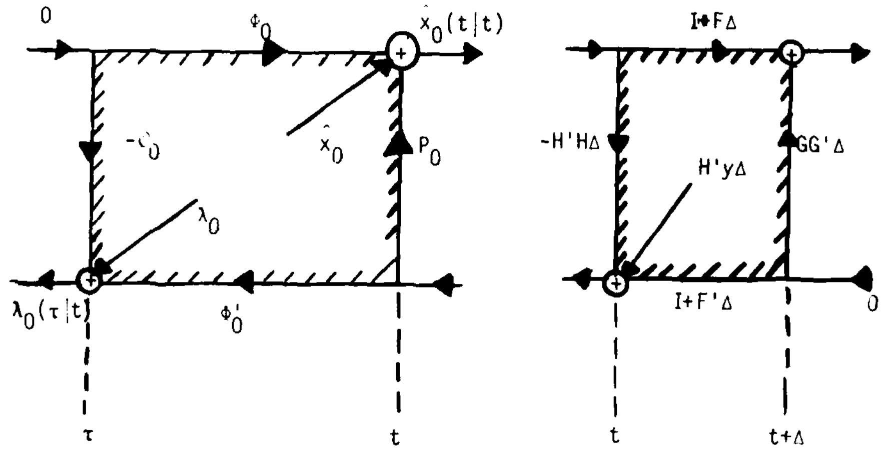
$$\hat{x}_0(t+\Delta|t+\Delta) = (I + F\Delta)\hat{x}_0(t\,|t+\Delta)$$
$$\hat{x}_0(t\,|t+\Delta) = \hat{x}_0(t\,|t) + P_0H'(y - H\hat{x}_0(t\,|t+\Delta)) + o(\Delta)$$

so that

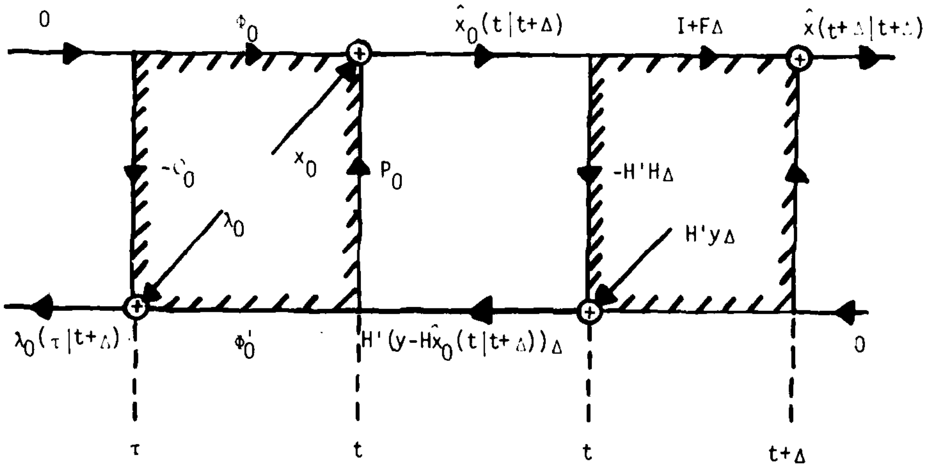
$$\dot{\hat{x}}_0(t\,|t) = F\,\hat{x}_0(t\,|t) + P_0H'(y(t) - H(t)\hat{x}_0(t\,|t)) \tag{67}$$

which can immediately be recognized as the Kalman-Bucy equation, thus explaining the notation in (65). What we wished to illustrate here is that the state-space filtering equations can be derived from the (scattering framework based on the) Hamiltononian equations for the smoothed estimates which, of course, is the opposite of the usual order.

There are many other illuminating consequences of the scattering picture, which not only helps to organize in a very efficient way many of the special results and identities associated with state-space estimation, and has led to new results on backwards Markovian models, smoothing formulas, asymptotic properties, fast algorithms, decentralized estimation, etc. However, we shall content ourselves here with mention of the papers Kailath (1975), Ljung, Kailath and Friedlander (1976), Friedlander, Kailath and Ljung (1976), Verghese, Friedlander and Kailath (1980), Lévy (1981).



(a) Adding an incremental layer



(b) The combined section showing the resulting signals at different points

Figure 5. Determining the forward evolution

3e. Nonstationary Processes Studied as Processes Close to Stationary

In studying the Ambartzumian-Chandrasekhar equations of Sec. 2b and especially their extension to nonstationary processes generated by constant coefficient state-space models (Kailath (1973)) in a scattering framework, Kailath and Ljung (1975) noticed that the Sobolev and Krein-Levinson equations of Sec. 2c could be extended to nonstationary processes by using the concept of an index of nonstationarity of a kernel or process. We shall briefly outline the main idea here.

We first define a so-called displacement operator $\lrcorner$ by

$$\lrcorner := \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \tag{68}$$

and we note that if $K(t,s) = K(t-s)$ (and is differentiable), then

$$\lrcorner K = 0$$

For a non-stationary process we may have

$$\lrcorner K(t,s) = \sum_{i=1}^{\tilde{\alpha}} \varphi_i(t)\varphi_i^{'}(s) \ , \quad \tilde{\alpha} < \infty \tag{69}$$

for some smallest $\tilde{\alpha}$, and a family of functions $\{\varphi_i(t)\}$. [For simplicity, we have assumed $p=1$, i.e., scalar processes and scalar kernels.]

Example 1.

For the Wiener process $K(t,s)=min(t,s)$ and $\tilde{\alpha}=1$.

Example 2.

For a state-space model (38) with constant parameters, it can be shown that $\tilde{\alpha} \leq n$, the dimension of the state space.

Example 3.

For a composition of kernels

$$(K_1K_2)(t,s) = \int_0^T K_1(t,u)K_2(u,s)du$$

we obtain the rule

$$\_|(K_1K_2) = (\_|K_1)K_2 + K_1(\_|K_2) + K_1\delta_0K_2 - K_1\delta_TK_2 \tag{70}$$

where

$$K_1\delta_gK_2 = \int_0^T K(t,u)\delta(u-g)K_2(u,s)\,du = K_1(t,g)K_2(g,s) \ .$$

The composition rule gives us an easy derivation of the Sobolev identity. For this, we apply the displacement operator to the equation $H+HK=K$ and use the rule (70) to obtain, after some rearrangement, the result

$$(\_|H)(I+K) = (I-H)\_|K + H(\delta_T - \delta_0)K$$

Noting that $(I+K)^{-1}=I-H$ and $K(I+K)^{-1}=H$ we then obtain

$$\_|H = (I-JH)(\_|K)(I-H) + H(\delta_T - \delta_0)H \ . \tag{71}$$

In the stationary case $\_|K=0$ and we have

$$\_|H = H\delta_TH - H\delta_0H$$

or written out

$$\_|H(t,s;T) = H(t,T;T)H(T,s;T) - H(t,0;T)H(0,s;T) \ . \tag{72}$$

which is just the Sobolev identity (29) of Sec. 2c.

The interesting fact is that this identity can be extended to nonstationary processes by using a slight modification of the representation (69). Let us rewrite (69) in the form

$$\_|K = K(t,0)K(0,s) + \sum_{i=1}^{\alpha} \lambda_j d_i(t)d_i'(s) = K\delta_0K + D_t'\Lambda D_s \ , \ say \tag{73}$$

Substituting into (71) we will have

$$\begin{aligned} \lrcorner H &= (I - H)(K\delta_0 K + D_t'\Lambda D_s)(I - H) + H\delta_T H - H\delta_0 H \\ &= H\delta_0 H + (I - H)D_t'\Lambda D_s(I - H) + H\delta_T H - H\delta_0 H \\ &= H\delta_T H + C_t'\Lambda C_s \text{ , } \textit{say} \end{aligned} \tag{74}$$

where $C := D(I-H)$, or equivalently, C satisfies the integral equation

$$C(1 + K) = D$$

Note that C will be a $1\times\alpha$ vector of functions, with $\alpha = 1$ in the stationary case. Thus α can serve as a measure of nonstationarity of the process, and (74) is the corresponding Sobolev identity for nonstationary processes. By simple further calculations we can also obtain generalized versions of the Krein-Levinson equations of Sec. 2c.

We refer to Kailath, Ljung, Morf (1976), (1978) for discussions of the computational aspects of these equations and of the role of the parameter α We regret that reasons of space and time do not permit us to describe here results on some efficient, so-called ladder-form, implementations of these equations. These results allow us to carry over to processes without (finite-dimensional) state-space models, the basic computational advantages of the state-space assumption. These were briefly mentioned in the conference lectures; for more details we refer to the theses of D. T. L. Lee (1980) and H. Lev-Ari (1981).

4. A Concluding Remark

In the nonlinear filtering problem, the state-space assumption has by no means been as useful as in the linear case, since it leads to difficult nonlinear stochastic partial differential equations. It may be that return to an input-output formulation, perhaps based on the Wiener-Volterra reprsentation, can be combined with analysis along the lines of Secs. 2c and 3c to make some computational progress in the nonlinear filtering problem.

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It must be with some trepidation that one ventures to speak about the problems of linear estimation to an audience already well familiar with the overwhelmingly more difficult nonlinear filtering problem. However, perhaps to compensate for this spectacle, the organizers have given me the opportunity to speak first, with considerable latitude in the choice of my topics.

For such an audience, there will be no need to present a tutorial on linear filtering, especially of the Kalman-Bucy type. I chose, therefore, (CONTINUED)

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ITEM #20, CONTINUED: to focus on some aspects generally less familiar to those with a 'modern' control theory background, i.e., largely a state-space background. In particular, we shall begin with a discussion of integral equations and of the important Wiener-Hopf technique. We shall specialize this to stationary processes over infinite intervals, and then describe some alternative, often computationally better, solution methods of Ambartzumian-Chandrasekhar and Krein-Levinson for finite-interval problems. For nonstationary problems, we start first with state-space models and build up to a brief description of the scattering theory framework for linear estimation. This will then lead us to nonstationary versions of the Ambartsumian-Chandrasekhar and Krein-Levinson equations, which we shall only consider very briefly, providing detailed references for further reading. We shall conclude with a remark on a possible implication for nonlinear filtering.

Our presentation is confined to continuous-time processes; a recent survey of the discrete-time case can be found in Kailath (1980).

In writing this chapter, it was a great help to have a carefully prepared preliminary reduction of the actual lectures, contributed by B. Hanzon, B. Ursin and D. Ocone. It is a pleasure also to thank M. Hazewinkel for these and several other organizational touches that made for an outstanding symposium.

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