RAY TRACING METHODS FOR CALCULATION
OF ATMOSPHERIC REFRACTION

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ABSTRACT

Equations for tracing rays through an atmospheric medium of continuously
variable refractive index are obtained in spherical coordinates from Fermat's
principle by applying the Euler equation. By introducing canonical variables
they are reduced to a set of first order differential equations in normal form,
suitable for stepwise numerical integration. Altitude and azimuth angles are
introduced and a transformation is derived for determining the refraction errors,
including lateral refraction, from the integrated results. The spherically
symmetrical case is considered in more detail and leads to an equation for the
error in altitude angle expressible as a quadrature over the radial coordinate.
A perturbation formula for obtaining the part of the refraction error due to
differences between an actual atmospheric profile and some standard atmospheric
profile is derived by taking the functional (or variational) derivative. The
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Equations for tracing rays through an atmospheric medium of continuously variable refractive index are obtained in spherical coordinates from Fermat's principle by applying the Euler equation. By introducing canonical variables they are reduced to a set of first order differential equations in normal form, suitable for stepwise numerical integration. Altitude and azimuth angles are introduced and a transformation is derived for determining the refraction errors, including lateral refraction, from the integrated results. The spherically symmetrical case is considered in more detail and leads to an equation for the error in

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RAY TRACING IN SPHERICAL COORDINATES

According to Fermat's principle[1] (also referred to as the principle of least time), the ray joining any two arbitrary points, P_1 and P_2, is determined by the condition that its optical length

$$S = \int_{P_1}^{P_2} n \, ds \tag{1}$$

be stationary as compared with the optical lengths of arbitrary neighboring curves joining P_1 and P_2. If the refractive index n is considered to be a given smooth continuous function of position and the location along the path is given in terms of a parameter t, then an actual ray path must furnish an extremum

$$\delta \int_{P_1}^{P_2} n(r, \Theta, \phi) S(r, \Theta, \dot{r}, \dot{\Theta}, \dot{\phi}) dt = 0 \tag{2}$$

where spherical coordinates are indicated with

$$\frac{ds}{dt} = S(r, \Theta, \dot{r}, \dot{\Theta}, \dot{\phi}) = \sqrt{\dot{r}^2 + r^2\dot{\Theta}^2 + r^2 \sin^2\Theta\dot{\phi}^2} \tag{3}$$

and where the dots indicate differentiation with respect to t. The partial derivatives

$$\frac{\partial S}{\partial r} = \frac{r(\dot{\Theta}^2 + \sin^2\Theta\,\dot{\phi}^2)}{S}, \quad \frac{\partial S}{\partial \dot{r}} = \frac{\dot{r}}{S}$$

$$\frac{\partial S}{\partial \Theta} = \frac{r^2 \sin\Theta \cos\Theta\,\dot{\phi}^2}{S}, \quad \frac{\partial S}{\partial \dot{\Theta}} = \frac{r^2\dot{\Theta}}{S} \tag{4}$$

$$\frac{\partial S}{\partial \phi} = 0 \qquad , \quad \frac{\partial S}{\partial \dot{\phi}} = \frac{r^2 \sin^2\Theta\,\dot{\phi}}{S}$$

will be useful in evaluating the Euler equations in the derivation that follows.

Taking

$$f(r, \Theta, \phi, \dot{r}, \dot{\Theta}, \dot{\phi}) = n(r, \Theta, \phi) S(r, \Theta, \dot{r}, \dot{\Theta}, \dot{\phi}) \tag{5}$$

in equation 2, the rays must lie along curves satisfying an Euler equation
for each coordinate

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{r}}\right) - \frac{\partial f}{\partial r} = 0$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\theta}}\right) - \frac{\partial f}{\partial \theta} = 0 \qquad (6)$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\phi}}\right) - \frac{\partial f}{\partial \phi} = 0$$

or by making use of the relations given in equations 4 and 5

$$\frac{d}{dt}\left(n\,\frac{\dot{r}}{S}\right) - S\frac{\partial n}{\partial r} - n\,\frac{r(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)}{S} = 0$$

$$\frac{d}{dt}\left(n\,\frac{r^2\dot{\theta}}{S}\right) - S\frac{\partial n}{\partial \theta} - n\frac{r^2\sin\theta\,\cos\theta\,\dot{\phi}^2}{S} = 0 \qquad (7)$$

$$\frac{d}{dt}\left(n\,\frac{r^2\sin^2\theta\,\dot{\phi}}{S}\right) - S\frac{\partial n}{\partial \phi} = 0$$

By taking the parameterization to be given in terms of arc length s along a ray

$$t = s$$

$$S = \frac{ds}{dt} = 1 \qquad (8)$$

the total differential system for the rays is simplified by eliminating the
radicals appearing in S above.

$$\frac{d}{ds}(n\dot{r}) - \frac{\partial n}{\partial r} - nr(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = 0$$

$$\frac{d}{ds}(nr^2\dot{\theta}) - \frac{\partial n}{\partial \theta} - nr^2\sin\theta\,\cos\theta\,\dot{\phi}^2 = 0 \qquad (9)$$

$$\frac{d}{ds}(nr^2\sin^2\theta\,\dot{\phi}) - \frac{\partial n}{\partial \phi} = 0$$

2

If a canonical system of variable is introduced where

$$p_r = n\dot{r}$$

$$p_\Theta = nr^2\dot{\Theta} \tag{10}$$

$$p_\phi = nr^2 \sin^2 \Theta \, \dot{\phi}$$

the corresponding first order differential system is easily put in normal form.

$$\dot{p}_r = \frac{1}{nr^3}\left(p_\Theta^2 + \frac{p_\phi^2}{\sin^2\Theta}\right) + \frac{\partial n}{\partial r}$$

$$\dot{p}_\Theta = \frac{\cos\Theta \, p_\phi^2}{nr^2 \sin^3\Theta} + \frac{\partial n}{\partial\Theta}$$

$$\dot{p}_\phi = \frac{\partial n}{\partial\phi} \tag{11}$$

$$\dot{r} = \frac{p_r}{n}$$

$$\dot{\Theta} = \frac{p_\Theta}{nr^2}$$

$$\dot{\phi} = \frac{p_\phi}{nr^2 \sin^2\Theta}$$

This system is suitable for numerical integration by many standard methods including the Runge-Kutta method. The equations are not completely independent but are inter-related by the implicit relationship from equations 3 and 8

$$\dot{r}^2 + r^2\dot{\Theta}^2 + r^2 \sin^2\Theta \, \dot{\phi}^2 = 1 \tag{12}$$

which requires that the sum of the squares of the local direction cosines of the tangent to the ray at any point be unity. This permits the integration

to be initiated from a knowledge of position coordinates and two angles sighted along a ray; for example altitude and azimuth angles. It can also facilitate the change of independent variable from s to one of the coordinates if desired; for example if it is desired to increment the radial distance r in fixed predetermined amounts. In such a case, the six equations given by 11 are reduced to five. For a general integration with s as the independent variable, the initial conditions consist of the coordinates r, Θ, ϕ and the direction cosines α_r, α_Θ, α_ϕ, related to the conjugate variables, as follows.

$$\alpha_r = \frac{dr}{ds} \qquad\qquad\qquad p_r = n\alpha_r$$

$$\alpha_\Theta = r\frac{d\Theta}{ds} \qquad\qquad\qquad p_\Theta = nr\alpha_\Theta \qquad\qquad (13)$$

$$\alpha_\phi = r\sin\Theta\frac{d\phi}{ds} \qquad\qquad p_\phi = nr\sin\Theta\alpha_\phi$$

The altitude angle a and azimuth angle A are given by

$$\sin a = \alpha_r$$

$$\tan A = \pm\frac{\alpha_\phi}{\alpha_\Theta} \qquad\qquad\qquad (14)$$

where the ambiguity of sign must be rectified to conform with the spherical coordinates, since various defining conventions are used for azimuth. By making use of the identity

$$\alpha_r^2 + \alpha_\Theta^2 + \alpha_\phi^2 = 1 \qquad\qquad\qquad (15)$$

it is easy to obtain the direction cosines in terms of altitude and azimuth.

$$\alpha_r = \sin a$$

$$\alpha_\Theta = \cos a \cos A \qquad\qquad\qquad (16)$$

$$\alpha_\phi = \pm\cos a \sin A$$

4

ATMOSPHERIC REFRACTION INCLUDING LATERAL REFRACTION

Assuming the quantities $\frac{\partial n}{\partial r}$, $\frac{\partial \Theta}{\partial r}$, $\frac{\partial \phi}{\partial r}$ are known functions of position, a
ray may now be traced up through the atmosphere by using the system of equa-
tions 11, for any starting location r_o, Θ_o, ϕ_o and direction α_{r_o}, α_{Θ_o}, α_{ϕ_o}.
Assuming the initial altitude angle is great enough that atmospheric ducting
and subsequent return of the ray does not occur, the ray eventually will
emerge from the atmosphere at some location r_f, Θ_f, ϕ_f with local direction coor-
dinates α_{r_f}, α_{Θ_f}, α_{ϕ_f}. In order to determine the amount of bending of the
ray, it is necessary to know the transformation of the final direction coor-
dinates back into the initial frame. This transformation will now be obtained.
For a general position vector $\vec{R}$ given in rectangular components but expressed
in spherical coordinates

$$\vec{R} = \hat{i} \, r \sin \Theta \cos \phi + \hat{j} \, r \sin \Theta \sin \phi + \hat{k} \, r \cos \Theta \, . \tag{17}$$

A local reference frame of unit vectors $\hat{r}$, $\hat{\Theta}$, $\hat{\phi}$ maybe defined by

$$\hat{r} = \frac{\partial \vec{R}}{\partial r} \Big/ \left| \frac{\partial \vec{R}}{\partial r} \right| = \hat{i} \sin \Theta \cos \phi + \hat{j} \sin \Theta \sin \phi - \hat{k} \cos \Theta$$

$$\hat{\Theta} = \frac{\partial \vec{R}}{\partial \Theta} \Big/ \left| \frac{\partial \vec{R}}{\partial \Theta} \right| = \hat{i} \cos \Theta \cos \phi + \hat{j} \cos \Theta \sin \phi + \hat{k} \sin \Theta \tag{18}$$

$$\hat{\phi} = \frac{\partial \vec{R}}{\partial \phi} \Big/ \left| \frac{\partial \vec{R}}{\partial \phi} \right| = -\hat{i} \sin \phi + \hat{j} \cos \phi$$

If the unit direction vector of the ray

$$\hat{\alpha} = \alpha_r \, \hat{r} + \alpha_\Theta \, \hat{\Theta} + \alpha_\phi \, \hat{\Theta} \tag{19}$$

is expressed in terms of the initial frame, the components are found to depend
on the cosines of angles between the initial and current frame vectors

$$\hat{\alpha}' = \hat{r}_o[(\hat{r}\cdot\hat{r}_o)\alpha_r + (\hat{\theta}\cdot\hat{r}_o)\alpha_\theta + (\hat{\phi}\cdot\hat{r}_o)\alpha_\phi]$$

$$+ \hat{\theta}_o[(\hat{r}\cdot\hat{\theta}_o)\alpha_r + (\hat{\theta}\cdot\hat{\theta}_o)\alpha_\theta + (\hat{\phi}\cdot\hat{\theta}_o)\alpha_\phi] \tag{20}$$

$$+ \hat{\phi}_o[(\hat{r}\cdot\hat{\phi}_o)\alpha_r + (\hat{\phi}\cdot\hat{\phi}_o)\alpha_\theta + (\hat{\phi}\cdot\hat{\phi}_o)\alpha_\phi]$$

where the direction cosines involved are readily obtained from equations 18 applied at the initial and current positions. (The prime added to $\hat{\alpha}'$ is to avoid confusion with the starting direction $\hat{\alpha}_o$).

$$\hat{r}\cdot\hat{r}_o = \sin\theta\,\sin\theta_o\,\cos(\phi-\phi_o) + \cos\theta\,\cos\theta_o$$

$$\hat{r}\cdot\hat{\theta}_o = \sin\theta\,\cos\theta_o\,\cos(\phi-\phi_o) - \cos\theta\,\sin\theta_o$$

$$\hat{r}\cdot\hat{\phi}_o = \sin\theta\,\sin(\phi-\phi_o)$$

$$\hat{\theta}\cdot\hat{r}_o = \cos\theta\,\sin\theta_o\,\cos(\phi-\phi_o) - \sin\theta\,\cos\theta_o \tag{21}$$

$$\hat{\theta}\cdot\hat{\theta}_o = \cos\theta\,\cos\theta_o\,\cos(\phi-\phi_o) + \sin\theta\,\sin\theta_o$$

$$\hat{\theta}\cdot\hat{\phi} = \cos\theta\,\sin(\phi-\phi_o)$$

$$\hat{\phi}\cdot\hat{r}_o = -\sin\theta_o\,\sin(\phi-\phi_o)$$

$$\hat{\phi}\cdot\hat{\theta}_o = -\cos\theta_o\,\sin(\phi-\phi_o)$$

$$\hat{\phi}\cdot\hat{\phi}_o = \cos(\phi-\phi_o)$$

By applying equations 20 and 21 to the emerging ray direction $\hat{\alpha}_f$, the components referred to the initial frame can be expressed in matrix form as given by equation 22.

$$\begin{bmatrix} \alpha'_{rf} \\ \alpha'_{\theta f} \\ \alpha'_{\phi f} \end{bmatrix} = \begin{bmatrix} \sin\theta_f\sin\theta_o\cos(\phi_f-\phi_o)+\cos\theta_f\cos\theta_o & \cos\theta_f\sin\theta_o\cos(\phi_f-\phi_o)-\sin\theta_f\cos\theta_o & -\sin\theta_o\sin(\phi_f-\phi_o) \\ \sin\theta_f\cos\theta_o\cos(\phi_f-\phi_o)-\cos\theta_f\sin\theta_o & \cos\theta_f\cos\theta_o\cos(\phi_f-\phi_o)+\sin\theta_f\sin\theta_o & -\cos\theta_o\sin(\phi_f-\phi_o) \\ \sin\theta_f\sin(\phi_f-\phi_o) & \cos\theta_f\sin(\phi_f-\phi_o) & \cos(\phi_f-\phi_o) \end{bmatrix} \begin{bmatrix} \alpha_{rf} \\ \alpha_{\theta f} \\ \alpha_{\phi f} \end{bmatrix} \tag{22}$$

6

It was found that the transformation matrix could be factored as

$$\begin{bmatrix} \alpha'_{rf} \\ \alpha'_{\Theta f} \\ \alpha'_{\phi f} \end{bmatrix} = \begin{bmatrix} \sin \Theta_o & \cos \Theta_o & 0 \\ \cos \Theta_o & -\sin \Theta_o & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \phi_o & \sin \phi_o & 0 \\ 0 & 0 & 1 \\ -\sin \phi_o & \cos \phi_o & 0 \end{bmatrix} \begin{bmatrix} \cos \phi_f & \sin \phi_f & 0 \\ \sin \phi_f & -\cos \phi_f & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot$$

$$\begin{bmatrix} \sin \Theta_f & \cos \Theta_f & 0 \\ 0 & 0 & -1 \\ \cos \Theta_f & -\sin \Theta_f & 0 \end{bmatrix} \begin{bmatrix} \alpha_{rf} \\ \alpha_{\Theta f} \\ \alpha_{\phi f} \end{bmatrix} \tag{23}$$

or alternatively in the form given by equation 24 as probably the most convenient for computations.

$$\begin{bmatrix} \alpha'_{rf} \\ \alpha'_{\Theta f} \\ \alpha'_{\phi f} \end{bmatrix} = \begin{bmatrix} \sin \Theta_o & \cos \Theta_o & 0 \\ \cos \Theta_o & -\sin \Theta_o & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi_f - \phi_o) & \sin(\phi_f - \phi_o) & 0 \\ 0 & 0 & 1 \\ \sin(\phi_f - \phi_o) & -\cos(\phi_f - \phi_o) & 0 \end{bmatrix} \cdot$$

$$\begin{bmatrix} \sin \Theta_f & \cos \Theta_f & 0 \\ 0 & 0 & -1 \\ \cos \Theta_f & -\sin \Theta_f & 0 \end{bmatrix} \begin{bmatrix} \alpha_{rf} \\ \alpha_{\Theta f} \\ \alpha_{\phi f} \end{bmatrix} \tag{24}$$

Returning to equation 14, the vertical refraction correction is given by

$$a'_f - a'_o = \arccos \alpha'_{rf} - \arccos \alpha_{ro} \tag{25}$$

and the lateral refraction error by

$$A'_f - A_o = \arctan (\pm \frac{\alpha'_{\phi f}}{\alpha'_{\Theta f}}) - \arctan (\pm \frac{\alpha_{\phi o}}{\alpha_{\Theta o}}) \tag{26}$$

$$= \pm \arctan (\frac{\alpha'_{\phi f} \alpha_{\Theta o} - \alpha_{\phi o} \alpha'_{\Theta f}}{\alpha'_{\Theta f} \alpha_{\Theta o} - \alpha_{\phi o} \alpha'_{\phi f}})$$

where the sign again depends on the convention used for azimuth

THE SPHERICALLY SYMMETRICAL CASE

For the case where the refractive index depends only upon r, equations 9 become

$$\frac{d}{ds}(n\dot{r}) - \frac{\partial n}{\partial r} - nr(\dot{\Theta}^2 + \sin^2\Theta \, \dot{\phi}^2) = 0$$

$$\frac{d}{ds}(n \, r^2\dot{\Theta}) - nr^2 \sin\Theta \cos\Theta \, \dot{\phi}^2 = 0 \qquad (27)$$

$$nr^2 \, \sin^2\Theta \, \dot{\phi} = C_1$$

where an integral has been found for the last equation. The coordinate system may be chosen so that initially $\frac{d\phi}{ds} = 0$. Then, $C_1 = 0$ and $\frac{d\phi}{ds}$ vanishes identically

$$\phi = \phi_0 = \text{constant} \qquad (28)$$

and the problem is reduced to two dimensions. Using the fact that $\dot{\phi} = 0$, the second equation of 27 becomes integrable.

$$nr^2 \, \frac{d\Theta}{ds} = C_2 \qquad (29)$$

Inserting the resultant value for $\frac{d\Theta}{ds}$ into the first equation of 27 (together with $\dot{\phi} = 0$) yields the following relationship.

$$\frac{d}{ds}(n \, \frac{dr}{ds}) - \frac{\partial n}{\partial r} - \frac{C_2^2}{nr^3} = 0 \qquad (30)$$

Multiplying by n and using the relationship $\frac{d}{ds} = \frac{dr}{ds} \, \frac{d}{dr}$,

$$n \frac{dr}{ds} \, \frac{d}{dr}(n \, \frac{dr}{ds}) - n\frac{dn}{dr} - \frac{C_2^2}{r^3} = 0 \qquad (31)$$

and integrating yields

$$(n\frac{dr}{ds})^2 - n^2 + \frac{C_2^2}{r^2} = C_3 \qquad (32)$$

8

If $\frac{C_2{}^2}{r^2}$ is replaced by $(nr\frac{d\Theta}{ds})^2$ from equation 29, it is found that

$$n^2[(\frac{dr}{ds})^2 + (r\frac{d\Theta}{ds})^2 - 1] = C_3.$$

(33)

The quantity in square brackets must vanish because arc length s is defined by $ds = \sqrt{dr^2 + r^2\ d\Theta^2}$ and hence $C_3 = 0$. It then also follows that equations 29 and 30 (in r and Θ) are not independent. As a matter of convenience, equation 29 will be used and the geometrical relations between dr, ds and dΘ will be exploited.

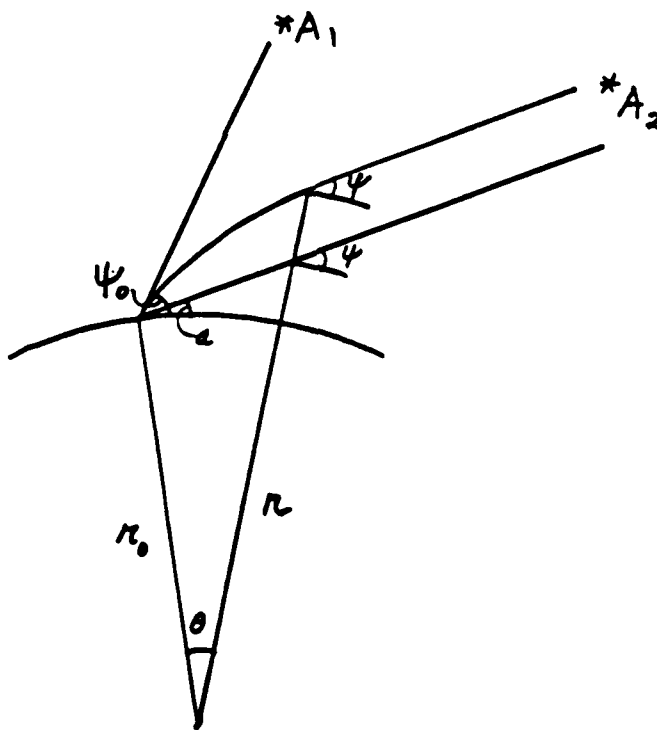As demonstrated in Figure 1, a star is observed at the apparent position A_1 given by angle ψ_o.

Figure 1 - Geometrical Parameters for the Atmospheric Ray Path

9

If no atmosphere were present, the actual position A_2 would coincide with A_1. For a ray travelling in the reverse direction and emanating at the surface at angle Ψ_o in a medium with variable refractive index, the ray path is curved and its inclination Ψ at r is given by

$$nr(r \frac{d\Theta}{ds}) = nr \cos \Psi = C \tag{34}$$

from equation 29 where the constant is determined from the initial values of n, r and Ψ.

$$C = n_o r_o \cos \Psi_o \tag{35}$$

After passing through the region of variable index, the ray will emerge at r_f, Θ_f in the direction Ψ_f toward A_2. By the optical Principle of Reversibility, an object at A_2 would be observed to have elevation Ψ_o, whereas, if the refractive index were constant (atmosphere removed) it would have its true elevation angle a. As layers of variable refractive index are added in the reversed ray system, a would change and so in this inbedded sense can be regarded as a function of r.

From Figure 1,

$$a = \Psi - \Theta \tag{36}$$

or

$$\frac{da}{dr} = \frac{d\Psi}{dr} - \frac{d\Theta}{dr} \tag{37}$$

and a may be determined by integrating equation 37. By rewriting equation 34 in the form

$$nr^2 \frac{dr}{ds} \frac{d\Theta}{dr} = C \tag{38}$$

10

and using the fact that $\sin \psi = \frac{dr}{ds}$, an expression for $\frac{d\Theta}{dr}$ is found

$$\sin \psi \frac{d\Theta}{dr} = \frac{C}{nr^2} \tag{39}$$

By differentiating equation 34 in the form

$$\cos \psi = \frac{C}{nr} \tag{40}$$

an expression containing $\frac{d\psi}{dr}$ is obtained.

$$\sin \psi \frac{d\psi}{dr} = \frac{C}{nr^2} + \frac{C}{n^2 r} \frac{dn}{dr} \tag{41}$$

Combining equations 39 and 41

$$\sin \psi \left(\frac{d\psi}{dr} - \frac{d\Theta}{dr} \right) = \frac{C}{n^2 r} \frac{dn}{dr} \tag{42}$$

and using the fact that

$$\sin \psi = \sqrt{1 - \cos^2 \psi} = \sqrt{1 - \left(\frac{C}{nr} \right)^2} \tag{43}$$

an expression for $\frac{da}{dr}$ is readily found.

$$\frac{da}{dr} = \frac{C \frac{dn}{dr}}{n^2 r \sqrt{1 - \left(\frac{C}{nr} \right)^2}} = \frac{\frac{r}{C} \frac{dn}{dr}}{\frac{nr}{C} \sqrt{\left(\frac{nr}{C} \right)^2 - 1}} \tag{44}$$

If this expression is integrated by parts from r_o to r_f, the value for the observational error is found

$$\delta a = a_f - a_o = \int_{r_o}^{r_f} \frac{\frac{r}{C} \frac{dn}{dr}}{\frac{nr}{C} \sqrt{\left(\frac{nr}{C} \right)^2 - 1}} \, dr \tag{45}$$

$$\delta a = \text{arcsec} \left(\frac{n_f r_f}{C} \right) - \text{arcsec} \left(\frac{n_o r_o}{C} \right) - \int_{r_o}^{r_f} \frac{dr}{r \sqrt{\left(\frac{nr}{n_o r_o} \sec \psi_o \right)^2 - 1}} \tag{46}$$

11

where $a_o = \Psi_o$, r_o, n_o, and a_f, r_f, n_f, are initial and final values and where C is given by equation 35.

$$\delta a = \text{arcsec}\ (\frac{n_f r_f}{n_o r_o} \sec \Psi_o) - \Psi_o - \int_{r_o}^{r_f} \frac{dr}{r\sqrt{(\frac{nr}{n_o r_o}\sec \Psi_o)^2 - 1}} \tag{47}$$

For determination of δa by numerical integration, equation 45 should be preferable to equation 47 by virtue of its simplicity and certainly a need to carry fewer significant figures. It can be easily evaluated with the trapazoidal rule, using a linear interpolation for $\frac{dn}{dr}$. For higher degree approximations, standard spline methods are suggested. Although it can be integrated by quadrature formulae (e.g. Newton-Cotes), equation 47 appears to offer no distinct advantage.

PERTURBATION OF THE SOLUTION

The refractive index function $n(r)$ is given a variation $\epsilon m(r)$ and the new error in altitude angle is obtained from equation 45

$$J = \int_{r_o}^{r_f} \frac{\frac{r}{C} \frac{d\tilde{n}}{dr}}{\frac{\tilde{n}r}{C} \sqrt{(\frac{\tilde{n}r}{C})^2 - 1}}\ dr \tag{48}$$

where

$$\tilde{n}(r) = n(r) + \epsilon m(r) \tag{49}$$

and

$$\frac{d\tilde{n}}{dr} = \frac{dn}{dr} + \epsilon \frac{dm}{dr} \tag{50}$$

12

By making use of equation 50 and the following Taylor expansion in ε,

$$\frac{\frac{r}{C}}{\frac{\tilde{n}r}{C}\sqrt{(\frac{\tilde{n}r}{C})^2 - 1}} = \frac{\frac{r}{C}}{\frac{nr}{C}\sqrt{(\frac{nr}{C})^2 - 1}} - \frac{\frac{r}{C}(\frac{mr}{C})\left[2(\frac{nr}{C})^2 - 1\right]}{(\frac{nr}{C})^2\left[(\frac{nr}{C})^2 - 1\right]^{3/2}}\varepsilon + O(\varepsilon^2) \tag{51}$$

the expression given below is obtained for the perturbed (or varied) integral

$$J = \int_{r_0}^{r_f} \frac{\frac{r}{C}\frac{dn}{dr}}{\frac{nr}{C}\sqrt{(\frac{nr}{C})^2 - 1}}\,dr + \varepsilon\int_{r_0}^{r_f}\left\{\frac{\frac{-r}{C}(\frac{mr}{C})\left[2(\frac{nr}{C})^2 - 1\right]}{(\frac{nr}{C})^2\left[(\frac{nr}{C})^2 - 1\right]^{3/2}}\frac{dn}{dr} + \frac{\frac{r}{C}}{\frac{nr}{C}\sqrt{(\frac{nr}{C})^2 - 1}}\frac{dm}{dr}\right\}dr \tag{52}$$

$$+ O(\varepsilon^2)$$

In equation 52, the full variation is obtained for $\varepsilon = 1$ and the conditions that the first order term give a good representation of the corresponding variation in J are

$$\varepsilon m(r) << n(r)$$

$$\varepsilon\frac{dm}{dr} << \frac{dn}{dr} \tag{53}$$

where ε has been carried in equation 52 mainly for purposes of identification.

By differentiation and a considerable amount of algebraic manipulation, the following identity may be obtained

13

$$\frac{\frac{r}{C}\frac{dm}{dr}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}} = \frac{d}{dr}\left[\frac{\frac{mr}{C}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}}\right] + \frac{\frac{m}{C}\left(\frac{nr}{C}\right)}{\left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}}$$

$$+ \frac{\frac{r}{C}\left(\frac{mr}{C}\right)\left[2\left(\frac{nr}{C}\right)^2 - 1\right]}{\left(\frac{nr}{C}\right)^2\left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}}\frac{dn}{dr}$$

(54)

and this is useful in further simplifying the form of the integral.

$$J = \int_{r_o}^{r_f}\frac{\frac{r}{C}\frac{dn}{dr}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}}\,dr + \varepsilon\int_{r_o}^{r_f}\frac{\frac{m}{C}\left(\frac{nr}{C}\right)}{\left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}}\,dr$$

(55)

$$+ \varepsilon\left[\frac{\frac{mr}{C}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}}\right]\Bigg|_{r_o}^{r_f} + O(\varepsilon^2)$$

By assuming the value of m to vanish at the endpoints,

$$m(r_o) = m(r_f) = 0$$

(56)

the quantity bracketed in equation 55 will also vanish. For the upper endpoint, this is a reasonable assumption since the refractive index should assume the value for vacuum and variation or perturbation is not reasonable. For the lower endpoint, it is necessary on practical grounds, since any variation of refractive index will disturb the value of C (initial condition) used throughout the entire range of integration.

The first term of equation 55 is the unperturbed error. The integral in the second term is known as the variational or functional derivative of J of first order. Taking $\varepsilon = 1$ and ignoring higher order terms yields the first order or linear perturbation of J.

$$\delta J \cong \int_{r_0}^{r_f} \frac{\frac{m}{C}\left(\frac{nr}{C}\right)}{\left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}} dr \tag{57}$$

It can be used for approximately determining refraction errors in the altitude angle due to differences between the actual refractive index profile and the profile of some standard atmospheric model.

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