AD-A116 814 PITTSBURGH UNIV PA DEPT OF MATHEMATICS AND STATISTICS F/G 12/1
INEQUALITIES ON DISTRIBUTIONS: BIVARIATE AND MULTIVARIATE.(U)
MAY 82 H W BLOCK, A R SAMPSON N00014-76-C-0839
UNCLASSIFIED 82-11 NL

1 OF 1
AD A
116814

END
DATE
FILMED
08-82
DTIC

(12)

AD A116814

INEQUALITIES ON DISTRIBUTIONS: BIVARIATE AND MULTIVARIATE

Henry W. Block[1]

and

Allan R. Sampson[2]

University of Pittsburgh

May 1982

Technical Report No. 82-11

Center for Multivariate Analysis
University of Pittsburgh
Ninth Floor, Schenley Hall
Pittsburgh, PA 15260

[1]The work of the author is supported by ONR Contract N00014-76-C-0839.

[2]The work of this author is sponsored by the Air Force Office of Scientific Research under Contract F49629-82-K-001. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Accession For	
NTIS GRA&I	☑
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A	

INEQUALITIES ON DISTRIBUTIONS: BIVARIATE AND MULTIVARIATE

by

Henry W. Block

and

Allan R. Sampson

ABSTRACT

Inequalities concerning bivariate and multivariate distributions in statistics are surveyed, as well as historical background. Subjects treated include inequalities arising through positive and negative dependence; Boole, Bonferroni and Fréchét inequalities; convex symmetric set inequalities; stochastic ordering; stochastic majorization and inequalities obtained by majorization; Chebyshev and Kolmogorov-type inequalities; multivariate moment inequalities; and applications to simultaneous inference, unbiased testing and reliability theory.

Key Words: Positive and negative dependence; association; TP_2; elliptically symmetric density; Boole, Bonferroni and Fréchét inequalities; stochastic ordering; (stochastic) majorization; Chebyshev inequality; Kolmogorov inequality; simultaneous inference; reliability theory; unbiased testing.

1. Introduction

Inequalities have played a central and lasting role in probability theory as well as in mathematics in general. In mathemaics, the subject of inequalities has centuries old roots and many prominent researchers have contributed to its growth. The classic book by Hardy, Littlewood and Polya (1952), first published in 1934, provides a remarkable compendium of mathematical inequalities. While the development of probability inequalities is intertwined with the mathematical development, there have been separate major influences as a result of the special needs of probability theory. For instance one inequality common to both mathematics and probability is the famous Cauchy-Schwarz inequality which in its probabilistic form states that $\mathrm{Cov}(X,Y) \leq (\mathrm{Var}\ X)^{\frac{1}{2}}(\mathrm{Var}\ Y)^{\frac{1}{2}}$. On the other hand, *Chebyshev's* important inequality germinated in the context of probability theory, being developed in order to approximate probabilities of complex events.

The origins of probability inequalities for multivariate distributions are not new (for instance, Boole's inequality and also the Cauchy-Schwarz inequality). However, the dramatic growth of this subject area has taken place in the last half of this century. This growth parallels the major growth of multivariate analysis itself during this period. Recently there have been some attempts to impose structure on the area of multivariate probability inequalities. Two notable efforts are the monographs of Marshall and Olkin (1979) and Tong (1980). Also see the review paper of Eaton (1982). The multivariate inequalities presented here are divided roughly into three groupings, namely, inequalities among random variables, stochastic comparisons between random vectors, and moment inequalities.

While multivariate probability inequalities are important in their own right, equally important is the application of these inequalities to problems in statistics. The context in which they are most useful is dealing with probabilities for random vectors having complex dependencies. In many such situations, the evaluation of the probabilities is technically virtually impossible, so that bounds for these probabilities become highly useful. Three specific areas of extensive application, which are discussed in this paper, are reliability theory, simultaneous inference and unbiased testing.

2. Positive and Negative Dependence

Various inequalities arise when the entries of a random vector $\underline{X} = (X_1, \ldots, X_p)$ are positively dependent. (See Barlow and Proschan (1975) for a discussion.) For example, there are many circumstances where for all $x_1, \ldots, x_p$,

$$\Pr[X_1 > x_1, \ldots, X_p > x_p] \geq \prod_{i=1}^{p} \Pr[X_i > x_i]. \tag{1.1}$$

This is a type of positive dependence called positive upper orthant dependence (PUOD). A variant, positive lower orthant dependence (PLOD), is obtained by replacing all ">" by "$\leq$" in (1.1). In the bivariate case, considered by Lehmann (1966), PUOD and PLOD are equivalent; a discussion of the more general case is given by Tong (1980). A different positive dependence concept which implies both PUOD and PLOD is association. The entries of X are said to be associated if $\mathrm{Cov}(f(\underline{X})g(\underline{X})) \geq 0$ for all binary valued functions f and g which are nondecreasing in each argument (see Barlow and Proschan (1975)). A simple checkable condition which implies association is called TP_2 in pairs. See Block and Ting (1981) for a review of this and other positive dependence concepts, their relations, and references.

A concept of negative dependence, negative upper orthant dependent (NUOD), is obtained if "$\geq$" is replaced by "$\leq$" in (1.1). For example, the multinomial distribution is NUOD. Various other distributions which have the same structure as a multinomial (i.e., essentially $\sum_{i=1}^{p} X_i$ being constant) also are NUOD (see Block, Savits and Shaked (1982)). Other concepts of negative dependence are discussed by Karlin and Rinott (1980) and Ebrahimi and Ghosh (1981).

Concepts derived from (1.1) can be used to partially order, according to degree of positive dependence, random vectors whose one-dimensional marginal distributions agree. The random vector $\underline{X}$ is more PUOD than $\underline{Y}$ if $\Pr[X_1 > t_1, \ldots, X_p > t_p] \geq \Pr[Y_1 > t_1, \ldots, Y_p > t_p]$ for all $t_1, \ldots, t_p$. If ">" is replaced by "$\leq$" in the above, then $\underline{X}$ is said to be more PLOD than $\underline{Y}$. The bivariate version of the more PLOD ordering was first described by Tchen (1980).

When $\underline{X} \sim N(\underline{0}, \underline{\Sigma}_X)$, then (a) the density is TP_2 in pairs if and only if $\underline{\Lambda} = \underline{\Sigma}_X^{-1}$ satisfies the condition $\lambda_{ij} \leq 0$ for all $i \neq j$

(Barlow and Proschan (1975)); (b) $\underline{X}$ is associated if and only if $\underline{\Sigma}_X$ is a nonnegative matrix (Pitt (1982)); (c) $\underline{X}$ is PUOD and PLOD if and only if $\underline{\Sigma}_X$ is a nonnegative matrix (this follows from (b)); and (d) $\underline{X}$ is NUOD if and only if the off-diagonal elements of $\underline{\Sigma}_X$ are nonpositive (Block et al (1981)). If $\underline{Y} \sim N(\underline{0}, \underline{\Sigma}_Y)$, where the diagonal elements of $\underline{\Sigma}_X$ and $\underline{\Sigma}_Y$ are the same and $\underline{\Sigma}_X - \underline{\Sigma}_Y$ is a nonnegative matrix, then $\underline{X}$ is more PUOD (Das Gupta et al (1972, Remark 5.1)) and also more PLOD (Slepian (1962)) than $\underline{Y}$. The density of $|X_1|, \ldots, |X_p|$ is TP_2 in pairs if and only if there exists a diagonal matrix $\underline{D}$ with elements ± 1, such that the off-diagonal elements of $\underline{D}\,\underline{\Sigma}_X^{-1}\,\underline{D}$ are all nonpositive (Karlin and Rinott (1981), and Abdel-Hameed and Sampson (1978)). For certain structural conditions on $\underline{\Sigma}_X$, $|X_1|, \ldots, |X_p|$ are associated (Ahmed, León and Proschan (1981)) and also PUOD (Khatri (1967)); however, $|X_1|, \ldots, |X_p|$ are PLOD (Šidák (1967)) for all $\underline{\Sigma}_X$. Under certain conditions on $\underline{\Sigma}_X$ and $\underline{\Sigma}_Y$ it can be shown that $|X_1|, \ldots, |X_p|$ are more PLOD (Šidák (1968)) and also more PUOD (Šidák (1975)) than $|Y_1|, \ldots, |Y_p|$.

Let $\underline{X} \sim N(\underline{0}, \underline{\Sigma}_X)$, $\underline{Y} \sim N(\underline{0}, \underline{\Sigma}_Y)$, $s^2 \sim \chi_k^2$, $u^2 \sim \chi_k^2$ and assume $\underline{X}$, $\underline{Y}$, s^2 and u^2 are all independent. If $\underline{\Sigma}_X$ is a nonnegative matrix, then the scaled t-vector, $X_1/s, \ldots, X_p/s$ is both PLOD and PUOD. This result follows from Theorem 3.2.1 of Ahmed, Langberg, León and Proschan (1979), and the PLOD and PUOD result for $\underline{X}$. If $\underline{\Sigma}_X - \underline{\Sigma}_Y$ is a nonnegative matrix and $\underline{\Sigma}_X$ and $\underline{\Sigma}_Y$ have the same diagonal elements, then $X_1/s, \ldots, X_p/s$ is both more PLOD and more PUOD than $Y_1/u, \ldots, Y_p/u$ (see Das Gupta et al (1972, Theorem 5.1 and

Remark 5.1)). Under any conditions that allow $|X_1|,\ldots,|X_p|$ to be associated, $|X_1|/s,\ldots,|X_p|/s$ will be associated (see Abdel-Hameed and Sampson, (1978, Lemmas 4.1 and 4.2)). Šidák (1971) showed that $|X_1|/s,\ldots,|X_p|/s$ is PLOD for arbitrary $\underline{\Sigma}_X$. The analogous PUOD result has been established for certain special cases of $\underline{\Sigma}_X$ by among others Abdel-Hameed and Sampson (1978, Theorem 4.2) and Ahmed, Langberg, León and Proschan (1979, Sec. 5.6). If $\underline{\Sigma}_X - \underline{\Sigma}_Y$ is a positive semi-definite matrix then $|X_1|/s,\ldots,|X_p|/s$ is more PLOD than $|Y_1|/u,\ldots,|Y_p|/u$ (this follows from Das Gupta et al (1972, Theorem 3.3)). Many of these results also hold when s^2 and u^2 are arbitrary positive random variables, or when the denominators of the t-vectors are not all the same random variable. These scaled multivariate t-distributions and their generalizations arise naturally in regression problems, when the sample regression coefficients are studied.

Let $(X_{i1},\ldots,X_{ip})'$, $i=1,\ldots,n$, be i.i.d. according to $N(\underline{0},\underline{\Sigma})$, where $\underline{\Sigma}$ is any covariance matrix such that $|X_{i1}|,\ldots,|X_{ip}|$ are associated. Then $\Sigma X_{i1}^2,\ldots,\Sigma X_{ip}^2$, which can be considered (up to scaling) to be a p-dimensional multivariate χ^2, are associated. (The proof is a direct p-variate extension of Theorem 4.1 of Abdel-Hameed and Sampson (1978)). Similar results hold for multivariate F-distributions.

For distributions with an elliptically symmetric density, Das Gupta et al (1972) give a number of results concerning the random variables being more PLOD and more PUOD. Sampson (1982) gives necessary and sufficient conditions in the bivariate elliptically symmetric case for the density to be TP_2. For a random vector $\underline{X}$ having a distribution with a covariance scale parameter $\underline{\Sigma}$, Sampson (1980) gives sufficient conditions for the association of $\underline{X}$.

3. Boole, Bonferroni and Fréchét

For the special case when the random vector has entries which are indicator functions of sets, there are specialized probability inequalities. Let $I_{A_i}(X_i) = 1$, or 0, respectively, as $X_i \in A_i$ or $X_i \notin A_i$. The simplest such inequality is the well-known Boole's inequality which states $\max_{1 \le i \le p} \Pr[I_{A_i}(X_i) = 1] \le \Pr[\bigcup_{i=1}^{p} (I_{A_i}(X_i) = 1)] \le \sum_{i=1}^{p} \Pr[I_{A_i}(X_i) = 1]$. There are a string of more generalized inequalities called Bonferroni inequalities (see Feller (1968)), the best known of which is $\Pr[\bigcup_{i=1}^{p} (I_{A_i}(X_i) = 1)] \ge 1 - \sum_{i=1}^{p} \Pr[I_{A_i}(X_i) = 0]$. This can be extended by noting that $\Pr[\bigcap_{i=1}^{p} (I_{A_i}(X_i) = 1)] \le \min_{1 \le i \le p} \Pr[I_{A_i}(X_i) = 1]$. In the special case when $A_i = (-\infty, x_i]$, $i = 1, \ldots, p$, these latter two inequalities can be combined to give the multivariate Fréchét bounds which are written as

$$\max\left(0, \sum_{i=1}^{p} F_i(x_i) - (p-1)\right) \le F_{X_1, \ldots, X_p}(x_1, \ldots, x_p) \le \min_{1 \le i \le p} F_i(x_i).$$

This inequality provides bounds on the joint cumulative distribution function in terms of the one-dimensional marginal cumulative distribution.

4. Convex Symmetric Set Inequalities

There are a number of probability inequalities involving convex symmetric sets which may be viewed in some sense as generalizations of positive dependence and stochastic ordering. Anderson (1955) showed that if $\underline{X}$ has a density $g(\underline{x})$ symmetric around the origin and satisfying $\{(\underline{x}:g(\underline{x}) \geq t)\}$ is a convex set for all t, $0 \leq t \leq \infty$, then for any convex set C symmetric around the origin, $\Pr[\underline{X} + s\underline{y} \in C]$ is a monotone decreasing function of s, $0 \leq s \leq 1$, for every constant vector $\underline{y}$. Khatri (1967) showed for multivariate normal vector $\underline{X} \equiv (\underline{X}_1 : \underline{X}_2)$ with $\text{cov}(\underline{X}_1, \underline{X}_2)$ having rank one that $\Pr[\underline{X}_1 \in C_1, \underline{X}_2 \in C_2] \geq \Pr[\underline{X}_1 \in C_1] \Pr[\underline{X}_2 \in C_2]$ for any convex symmetric sets C_1 and C_2. Under certain conditions on the $\text{cov}(\underline{X}_1, \underline{X}_2)$, Khatri (1976) obtained the reverse version of this inequality with "$\geq$" replaced by "$\leq$", and C_1 and C_2 being complements of convex symmetric sets. Pitt (1977) has shown that if $\underline{X} = (X_1, X_2)'$, where $\underline{X} \sim N(\underline{0}, \underline{I})$ and C_1, C_2 are convex symmetric sets, then $\Pr[\underline{X} \in C_1 \cap C_2] \geq \Pr[\underline{X} \in C_1] \Pr[\underline{X} \in C_2]$. For certain types of convex sets involving quadratic forms, Dykstra (1980) has obtained inequalities for the multivariate normal. For instance, if $\underline{\Sigma}_{X_2} = \underline{I}$ and Cov $(\underline{X}_1, \underline{X}_2)$ arbitrary, then $\Pr(\underline{X}_1 \in C_1, \underline{X}_2' \underline{A}\, \underline{X}_2 \leq c_2) \geq \Pr(\underline{X}_1 \in C_1)$ $\Pr(\underline{X}_2' \underline{A}\, \underline{X}_2 \leq c_2)$ for all convex symmetric sets C_1 and real numbers $c_2 \geq 0$, where $\underline{A}$ is any matrix satisfying $\underline{A}^2 = \underline{A}$.

5. Stochastic Ordering

Stochastic ordering is a way of comparing the relative sizes of random variables (vectors). For example, if X and Y are univariate random variables on the same probability space one possible definition of the concept of X being less than or equal to Y is to require $\Pr[X \leq Y] = 1$. Because there is a problem when X and Y are not defined on the same space, the usual definition of stochastic ordering is $P\{X > t\} \leq P\{Y > t\}$ for all t. This is written $X \overset{st}{\leq} Y$, i.e., X is stochastically less than Y. Moreover, if $X \overset{st}{\leq} Y$, it can be shown that there exist random variables $\tilde{X}$ and $\tilde{Y}$ defined on the same probability space with the same marginal distributions as X and Y, respectively, such that $Pr[\tilde{X} \leq \tilde{Y}] = 1$. It also can be shown that $X \overset{st}{\leq} Y$ if and only if $E(\phi(X)) \leq E(\phi(Y))$ for all nondecreasing functions ϕ (see Marshall and Olkin (1979, p. 483)).

In the multivariate case, $\underline{X}$ is stochastically less than $\underline{Y}$, denoted by $\underline{X} \overset{st}{\leq} \underline{Y}$, if $E(\phi(\underline{X})) \leq E(\phi(\underline{Y}))$ for all nondecreasing functions ϕ, where $\phi\colon \mathbb{R}^p \to \mathbb{R}^1$. This condition implies, but is not equivalent to, $\underline{Y}$ being more PUOD than $\underline{X}$ (Marshall and Olkin (1979, p. 486)). (See the next xection for an example of multivariate stochastic ordering.) An existence theorem similar to the one in the one dimensional case holds, that is, $\underline{X} \overset{st}{\leq} \underline{Y}$ implies the existence of component-wise ordered random vectors on the same space with the same marginal distributions as $\underline{X}$ and $\underline{Y}$, respectively. See Arjas and Lehtonen (1978), and Marshall and Olkin (1979) for the proofs and discussions of these results in both the univariate and multivariate cases.

6. Stochastic Majorization and Inequalities Obtained by Majorization

One simple way to define stochastic majorization between random vectors $\underline{X}$ and $\underline{Y}$ is to require $Pr[\underline{X} \prec \underline{Y}] = 1$ where $\prec$ denotes ordinary majorization. This definition involves the joint distribution of $\underline{X}$ and $\underline{Y}$, and, hence, other definitions are preferable. An alternate definition is to say that $\underline{X}$ is stochastically majorized by $\underline{Y}$ if $E(\phi(\underline{X})) \leq E(\phi(\underline{Y}))$ for all Schur convex functions ϕ. See Marshall and Olkin (1979, pp. 281-5 and 311-7) for other possible definitions of stochastic majorization and their interrelationships. With the use of these concepts, various functions of random vectors corresponding to standard families can be shown to be Schur convex and useful inequalities can be obtained (see Marshall and Olkin (1979, Chapter 11)). Majorization techniques can also be used to show that $E(\phi(\underline{X})) \leq E(\phi(\underline{Y}))$ for other families of functions ϕ (see Marshall and Olkin (1979, Chapter 12)). For example, let $Y_1,\ldots,Y_p$, $Y_1',\ldots,Y_p'$ be 2p independent exponential random variables with means $\lambda_1^{-1},\ldots,\lambda_p^{-1}$, $(\lambda_1')^{-1},\ldots,(\lambda_p')^{-1}$, respectively. Proschan and Sethuraman (1976) show that if $\underline{\lambda} \prec \underline{\lambda}'$, then $E(\phi(Y_1,\ldots,Y_p)) \leq E(\phi(Y_1',\ldots,Y_p'))$ for all nondecreasing ϕ, i.e., $(Y_1,\ldots,Y_p) \overset{st}{\leq} (Y_1',\ldots,Y_p')$. Thus, if $\underline{Y}' = (Y_1',\ldots,Y_p')$ comes from a heterogeneous random sample with means $(\lambda_i')^{-1}$, $i = 1,\ldots,p$, and $\underline{Y} = (Y_1,\ldots,Y_p)$ comes from a homogeneous random sample with common mean λ^{-1}, where $\lambda = (\sum_{i=1}^{p} \lambda_i')/p$, then $\underline{Y} \overset{st}{\leq} \underline{Y}'$, since necessarily $(\lambda,\ldots,\lambda) \prec (\lambda_1',\ldots,\lambda_p')$. This implies that all of the order statistics of the homogeneous sample are stochastically smaller than the corresponding order statistics of the heterogeneous sample.

7. Chebyshev and Kolmogorov-Type Inequalities

A standard univariate version of the Chebyshev inequality is $\Pr[|X-\mu| \leq a\sigma] \geq 1 - a^{-2}$ where X has mean μ and finite variance σ^2. If $X_1,\ldots,X_p$ are independent with means μ_i and finite variance σ_i^2, $i=1,\ldots,p$, then $\Pr[\bigcap_{i=1}^{p}\{|X_i-\mu_i| \leq a_i\sigma_i\}] \geq \prod_{i=1}^{p}(1-a_i^{-2})$. If the X_i are dependent, various authors have obtained more general inequalities of which the previous inequality is a special case. One of the first of these was obtained by Berge (1937) in the bivariate case. Let X_1 and X_2 have means μ_1 and μ_2, variances σ_1^2 and σ_2^2 and correlation ρ. Then $\Pr[|X_1-\mu_1| \leq a\sigma_1, |X_2-\mu_2| \leq a\sigma_2] \geq 1-[1+(1-\rho^2)^{\frac{1}{2}}]a^2$. Various multivariate inequalities including the previous one can be obtained from the following general result. Let $\underline{X}=(X_1,\ldots,X_p)$ have mean vector $\underline{\mu}$ and covariance matrix $\underline{\Sigma}=(\rho_{ij}\sigma_i\sigma_j)$. For $a_i > 0$ define the matrix $T=(\tau_{ij})$, where $\tau_{ij}=\rho_{ij}/a_ia_j$. Then $\Pr[\bigcap_{i=1}^{p}\{|X_i-\mu_i| \leq a_i\sigma_i\}] \geq 1-\sum_{i=1}^{p}a_i^{-2}$. For other examples, see Tong (1980, pp. 153-154) or Karlin and Studden (1966, pp. 517-519). Both of these references also give bounds for one-sided probabilities; for example, lower bounds can be obtained on probabilities of the form $\Pr[\bigcap_{i=1}^{p}\{X_i \leq \mu_i + a\sigma_i\}]$, where $\underline{X}=(X_1,\ldots,X_p)$ has mean vector $\underline{\mu}$ and variances σ_i^2, $i=1,\ldots,p$, and for certain $a > 0$. For background and historical references pertaining to the Chebyshev inequality, see Karlin and Studden (1966, pp. 467-468). One of the earliest books to contain the material on multivariate Chebyshev inequalities was Godwin (1964).

A Kolmogorov-type inequality is similar to the above, except that the maximum of partial sums is employed. For example, if $X_1,\ldots,X_n$ are independent and have mean 0 and $S_n=X_1+\ldots+X_n$ with $\sigma(S_n) = \sqrt{Var(S_n)}$, then $Pr[\max_{1\le j\le n} |S_j|(\sigma(S_n))^{-1} \le a] \ge 1 - a^{-2}$. A multivariate version of the Kolmogorov inequality has been obtained by Sen (1971). For multivariate applications of the univariate (independent) result and one-sided analogs, see Tong (1980, Section 7.3).

8. Multivariate Moment Inequalities

For the moments and expectations of other functions of multivariate distributions, there are a number of inequalities. The most well-known states that $\underline{\Sigma}$ is the population covariance matrix of any random vector if and only if $\underline{\Sigma}$ is nonnegative definite. Moreover, if $\underline{\Sigma}$ is positive definite and the sample size large enough, the corresponding sample covariance matrix is positive definite with probability one (see Eaton and Perlman (1973)). For suitable bivariate distributions, there exists a canonical expansion (see Lancaster (1969)) and a sequence $\{\rho_i\}$ of canonical correlations. This sequence $\{\rho_i\}$ can be shown to satisfy certain inequalities, e.g., Griffiths (1970) or Thomas and Tyan (1975).

Chebyshev has given the following covariance inequality for similarly ordered univariate functions ϕ_1,ϕ_2 of a random vector $\underline{X}$ (see Hardy, Littlewood and Polya, (1952, Sec. 2.17) or Tong, (1980, Lemma 2.2.1)). If ϕ_1,ϕ_2 satisfy the condition that $(\phi_1(\underline{x}) - \phi_1(\underline{y}))(\phi_2(\underline{x}) - \phi_2(\underline{y})) \ge 0$ for all suitable $\underline{x},\underline{y}$ then $Cov(\phi_1(\underline{X}), \phi_2(\underline{X})) \ge 0$. A number of moment inequalities can be obtained from the result that for any nonnegative random vector $\underline{X}$, whose distribution is invariant under permutations, it follows that $E \Pi X_i^{\lambda_i}$ is a Schur-convex function of $(\lambda_1,\ldots,\lambda_p)$ (see Tong (1980, Lemma 6.2.4)). For example, if $\underline{X} \sim N(\mu \underline{e}, \sigma^2(1-\rho)\underline{I} + \rho \underline{e}\,\underline{e}')$, where $\underline{e} = (1,\ldots,1)'$, then $E X_j^{\Sigma \lambda_i} \geq E \prod_{i=1}^{p} X_i^{\lambda_i} \geq E(\prod_{i=1}^{p} X_i)^{\bar{\lambda}}$ where $\lambda_i \geq 0$, $i = 1,\ldots,p$ and $\bar{\lambda} = p^{-1}\Sigma \lambda_i$.

There are several results relating the more PLOD ordering to certain moment inequalities. If (X_1,X_2) is more PLOD than (Y_1,Y_2), then any of the following measures of association: Pearson's correlation, Kendall's τ, Spearman's ρ or Blomquist's q computed based on (X_1,X_2) are greater than or equal to the corresponding measure based on (Y_1,Y_2) (see Tchen (1980)).

Dykstra and Hewett (1978) have examined the positive dependence properties of the characteristic roots of the sample covariance matrix. If $\underline{S}$ is the sample covariance matrix based upon a random sample from $N(\underline{\mu},\underline{I})$, they show that the ordered characteristic roots are associated random variables.

9. Applications

Multivariate probability inequalities are very important for simultaneous confidence bounds, where lower bounds are sought on probabilities of events such as $\bigcap_{i=1}^{p} \{|\hat{\Theta}_i - \Theta_i| \leq c_i\}$, where the estimators $\hat{\Theta}_1,\ldots,\hat{\Theta}_p$ have some multivariate distribution possibly depending on nuisance parameters. The basic concept is to bound this probability by probabilities of marginal events, where no parameters are involved. For instance, if $\underline{X} \sim N(\underline{\mu},\underline{\Sigma})$, the fact that $|X_1 - \mu_1|,\ldots,|X_p - \mu_p|$ are PLOD provides conservative simultaneous confidence intervals for $\mu_1,\ldots,\mu_p$, when X_i, $i = 1,\ldots,p$ are known. General discussions of applications of probability inequalities to simultaneous inference can be found in Miller (1981), Krishnaiah (1979), and Tong (1980). Also found in the latter are applications of these techniques for establishing unbiasedness for certain multivariate tests of hypothesis.

Many of the dependence inequalities are useful in applications ot reliability theory. Consider a nonrepairable binary system consisting of p binary components with lifetimes $T_1,\ldots,T_p$ and having system lifetime T. The system lifetime T is generally a function of the component lifetimes such as $T = \max_{1\leq r\leq k} \min_{i\in P_r} T_i$ where the P_r are min path sets (see Barlow and Proschan (1975, Chapters 1 and 2, and p. 150)). In general the T_i are not independent and it is desired to approximate $\Pr[T\in B]$ by the $\Pr[T_i \in B_i]$, $i = 1,\ldots,p$, where B and the B_i's are usually intervals. To do this, various univariate and multivariate inequalities are used. As a simple example, if the T_i are PUOD, then $\Pr[T > t] =$
$\Pr[\bigcup_{r=1}^{k} \bigcap_{i\in P_r} (T_i > t)] \geq \max_{1\leq r\leq k} \Pr[\bigcap_{i\in P_r} (T_i > t)] \geq \max_{1\leq r\leq k} \prod_{i\in P_r} \Pr[T_i > t]$,
where the first inequality follows from Boole's inequality and the second follows from PUOD. If the distributions of the T_i are not known but the T_i lie in a class of wearout distributions, e.g., T_i has increasing failure rate, lower bounds on $\Pr[T>t]$ can be found in terms of the bounds on $\Pr[T_i > t]$. For example, Theorem 6.7 of Barlow and Proschan (1975, Chapter 4) can be em-

ployed in such an application. Many other applications of this type are contained in Barlow and Proschan (1975, Section 4.6). Generalizations of this type of result to multistate systems have been given by Block and Savits (1982).

REFERENCES

[1] Abdel-Hameed, M. and Sampson, A.R. (1978), Ann. Statist., 6, 1360-1368.

[2] Ahmed, A., Langberg, N., León, R. and Proschan, F. (1979), "Partial ordering of positive quadrant dependence with applications", unpublished report.

[3] Ahmed, A., Leon, R. and Proschan, F. (1981), Ann. Statist., 9, 168-176.

[4] Arjas, E. and Lehtonen, T. (1978), Math of O.R., 3, 205-223.

[5] Anderson, T.W. (1955), Proc. Amer. Math. Soc., 6, 170-176.

[6] Barlow, R.E. and Proschan, F. (1975), Statistical Theory of Reliability and Life Testing, Holt, Rinehart and Winston, New York. Many excellent applications of inequalities on distributions to reliability theory are discussed. Also this book contains a comprehensive treatment of dependence concepts in the bivariate case.

[7] Berge, P.O. (1937), Biometrika 29, 405-406.

[8] Block, H.W. and Savits, T.H. (1982), J. Appl. Prob., 19.

[9] Block, H.W., Savits, T.H. and Shaked, M. (1982), Ann. Probab., 10.

[10] Block, H.W., Savits, T.H. and Shaked, M. (1981), "A concept of negative dependence using stochastic ordering", unpublished report. This paper and the entry above provide a comprehensive view on the topic of negative dependence.

[11] Block, H.W. and Ting, M.-L. (1981), Comm. Statist., A10, 742-762.

[12] Das Gupta, S., Eaton, M., Olkin, I., Perlman, M., Savage, L. and Sobel, M. (1972), Proceedings of the Sixth Berkeley Symposium, Volume II, University of California Press, Berkeley, Cal., 241-265. One of the first papers extending normal theory inequalities to the broader class of elliptically symmetric distributions.

[13] Dykstra, R.L. (1980), J. Amer. Statist. Assoc., 75, 646-650.

[14] Dykstra, R.L. and Hewett, J.E. (1978), Ann. Statist., 6, 235-238.

[15] Eaton, M.L. (1982), Ann. Statist., 10, 11-43.

[16] Eaton, M.L. and Perlman, M. (1973), Ann. Statist., 1, 710-717.

[17] Ebrahimi, N. and Ghosh, M. (1981), Commun. Statist. Theor. Meth. A, 10, 307-337.

[18] Feller, W. (1968), An Introduction to Probability Theory and Its Applications, Volume 1, Third Edition, John Wiley and Sons, New York.

[19] Godwin, H.J. (1964), Inequalities on Distribution Functions, Hafner, New York. One of the earlier works on multivariate inequalities.

[20] Griffiths, R. (1970), Aust. J. Statist., 12, 162-165.

[21] Hardy, G.H., Littlewood, J.E. and Ploya, G. (1952), Inequalities, Cambridge University Press, Cambridge.

[22] Karlin, S. and Rinott, Y. (1980), J. Mult. Anal., 10, 400-516.

[23] Karlin, S. and Rinott, Y. (1981), Ann. Statist., 9, 1035-1049.

[24] Karlin, S. and Studden, W.J. (1966), Tchebycheff Systems, Interscience, New York.

[25] Khatri, C. (1967), Ann. Math. Statist., 38, 1853-1867.

[26] Khatri, C. (1976), Gujarat Statistical Review, 3, 1-12.

[27] Krishnaiah, P.R. (1979), Development in Statistics, Vol. 2, ed. by P.R. Krishnaiah, Academic Press, N.Y., 157-201.

[28] Lancaster, H. (1969), The Chi-Squared Distribution, John Wiley and Sons, New York.

[29] Lehmann, E.L. (1966), Ann. Math. Statist., 43, 1137-1153.

[30] Marshall, A.W. and Olkin, I. (1979), Inequalities: Theory of Majorization and Its Applications, Academic Press, N.Y. This is one of the two best references for inequalities on distributions.

[31] Miller, R. (1981), Simultaneous Statistical Inference, Springer-Verlag, New York. A very complete and readable presentation of simultaneous inference procedures.

[32] Pitt, L. (1977), Ann. Probab., 5, 470-474.

[33] Pitt, L. (1982), Ann. Probab., 10, 496-499.

[34]Proschan, F. and Sethurman, J. (1976), J. Mult. Anal., **6**, 608-616.

[35]Sampson, A.R. (1980), SIAM J. Alg. Disc. Meth., **1**, 284-291.

[36]Sampson, A.R. (1982), J. Mult. Anal., **12**.

[37]Sen, P.K. (1971), Ann. Math. Statist., **42**, 1132-1134.

[38]Šidák, Z. (1967), J. Amer. Statist. Assoc., **62**, 626-633.

[39]Šidák, Z. (1968), Ann. Math. Statist., **39**, 1425-1434.

[40]Šidák, Z. (1971), Ann. Math. Statist., **42**, 169-175.

[41]Šidák, Z. (1975), Ann. Inst. of Math. Statist., **27**, 181-184.

[42]Slepian, D. (1962), Bell System Technical Journal, 41, 463-501. One of the first modern papers on multivariate normal inequalities.

[43]Tchen, A. (1980), Ann. Probab., **8**, 814-827.

[44]Tong, Y.L. (1980), Probability Inequalities in Multivariate Distributions, Academic Press, New York. This is one of the two best references for inequalities on distributions.

[45]Thomas, J. and Tyan, S. (1975), J. Mult. Anal., **5**, 227-235.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 82- 11	2. GOVT ACCESSION NO. AD-A116 814	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Inequalities on Distributions: Bivariate and Multivariate		5. TYPE OF REPORT & PERIOD COVERED Technical-May 1982
		6. PERFORMING ORG. REPORT NUMBER 82-11
7. AUTHOR(s) Henry W. Block and Allan R. Sampson		8. CONTRACT OR GRANT NUMBER(s) F49629-82-K-001
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Multivariate Analysis University of Pittsburgh Pittsburgh, PA 15260		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS May 1982
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research Dept. of the Air Force Bolling Air Force Base, DC 20332		12. REPORT DATE May 1982
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Positive and negativedependence; association; TP_2; elliptically symmetric density; Boole, Bonferroni and Frechet inequalities; stochastic ordering; (stochastic) majorization; Chebyshev inequality; Kolmogorov inequality; simultaneous inference; reliability theory; unbiased testing.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Inequalities concerning vivariate and multivariate distributions in statistics are surveyed, as well as historical background. Subjects treated include inequalities arising through positive and negative dependence; Boole, Bonferroni and Frechet inequalities; convex symmetric set inequalities; stochastic ordering; stochastic majorization and inequalities obtained by majorization; Chebyshev and Kolmogorov-type inequalities; multivariate moment inequalities; and applications to simultaneous inference, unbiased testing and

DD FORM 1 JAN 73 1473

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

reliability theory.

ATE

LME