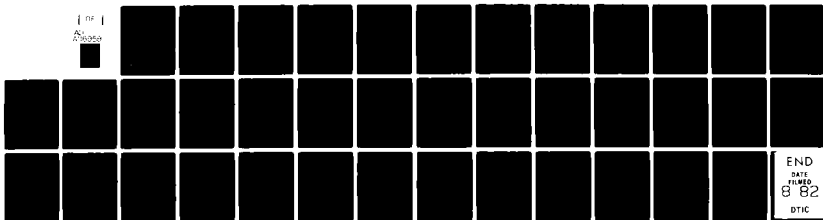


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SOME VIBRATING MEMBRANE EQUATIONS FOR THE
LINEAR ESTIMATION OF TWO-DIMENSIONAL ISOTROPIC
RANDOM FIELDS

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ABSTRACT

This paper considers the problem of estimating a two-dimensional isotropic random field given some noisy observations of this field over a disk of finite radius. By expanding the field and observations in Fourier series, and exploiting the covariance structure of the resulting Fourier coefficient processes, some vibrating membrane equations are obtained for estimating the random field. These equations provide a set of recursions for constructing the field estimates as the radius of the observation disk increases. In the spectral domain, these recursions take the form of Schrödinger equations which can be viewed as being associated to an inverse scattering problem.

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I. Introduction

In [1] - [5] some vibrating string equations were obtained to estimate a stationary stochastic process given some observations of this process over a finite interval. It was also shown in [6] - [8] (see also [5]) that the estimation problem over a finite interval can be formulated as an inverse scattering problem similar to those appearing in the study of transmission lines, or in quantum mechanics. In this paper, we will consider the problem of estimating a two-dimensional isotropic random field given some observations of this field over a disk of finite radius. By expanding the field and observations in Fourier series and by exploiting the covariance structure of the resulting coefficient processes, some vibrating membrane equations will be derived for solving a set of filtering problems associated to the coefficient processes. These equations will be used to estimate the random field and to obtain some recursions for the optimum filter as the radius of the observation disk increases.

The main feature of the estimation procedure proposed here is that it is recursive, i.e. as the radius of the observation disk increases, the optimum estimation filter can be updated. Earlier methods [9], [10] for estimating isotropic random fields were nonrecursive. An advantage of this procedure is that it is very efficient: it requires considerably fewer operations (i.e. multiplications and additions) than methods which solve directly the integral equation associated to the estimation problem over a finite disk. This property is not surprising if we note that the vibrating membrane equations discussed here generalize the Levinson recursions [11], [12] and the vibrating

string equations of [1] - [5] which are known to provide efficient solutions of the one-dimensional estimation problem over a finite interval.

The vibrating membrane equations that we consider are identical to equations that appear in the Gelfand-Levitan formulation of the inverse scattering problem of quantum mechanics [13], [14]. By transforming the estimation problem over a finite disk into an equivalent problem in the spectral domain, we obtain some two-dimensional Schrödinger equations. These equations are those satisfied by partial waves of nonzero angular momentum for a radially symmetric potential. By using this observation, we show that only one vibrating membrane equation needs to be solved to obtain a complete solution of the two-dimensional estimation problem.

This paper is organized as follows. In Section II we expand the observed random field in Fourier series and introduce some one-dimensional filtering problems for the Fourier coefficient processes. These filtering problems are solved by exploiting the structure of the covariance kernels of the coefficient processes, and in Section III this solution is used to estimate an arbitrary random variable. In Section III, we also transform the two dimensional estimation problem to the spectral domain, and we derive some Schrödinger equations. In Section IV a set of special transformations is introduced to relate the solutions of these Schrödinger equations and to show that only one of them needs to be computed. The construction of random field estimates is considered in Section V and the special case when we want to estimate the random field at the origin of the observation disk is examined. In this case, the optimum filter constitutes an approximation of the smoothing filter for observations over the whole plane. The Section VI presents some conclusions and discusses some possible extensions of these results.

II. Fourier series expansion of the observed field

In this paper, we will consider the estimation problem where we are given some observations

$$dy(r,\theta) = z(r,\theta) dA + dv(r,\theta) \quad (2.1)$$

of a two-dimensional isotropic zero-mean Gaussian Random field $z(\cdot, \cdot)$ over a disk D_R centered at the origin and of radius R , i.e. for

$$0 \leq \theta < 2\pi, \quad 0 \leq r < R. \quad (2.2)$$

Here $dA = r dr d\theta$ is an arbitrary surface element and $v(\cdot, \cdot)$ is a two-dimensional Wiener process which is uncorrelated with $z(\cdot, \cdot)$ and whose increments are such that

$$E[dv(r,\theta) dv(\rho,\phi)] = \begin{cases} dA & \text{for } r=\rho, \theta = \phi \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

Since the field $z(\cdot, \cdot)$ is isotropic, its covariance is given by

$$E[z(r,\theta) z(\rho,\phi)] = k(d) \quad (2.4)$$

where $d = (r^2 + \rho^2 - 2r\rho \cos(\theta - \phi))^{1/2}$ is the Euclidean distance between the points with polar coordinates (r, θ) and (ρ, ϕ) . If $z(\cdot, \cdot)$ is mean-square continuous, or equivalently if $k(\cdot)$ is continuous, $k(\cdot)$ admits a spectral representation of the form (see [15], [16])

$$k(d) = \int_0^\infty J_0(\lambda d) dM(\lambda) \quad (2.5)$$

where $J_0(\cdot)$ is the zero-th order Bessel function and where $M(\cdot)$ is bounded and nondecreasing.

The Hilbert spaces spanned by the observations and by the field $z(\cdot, \cdot)$ will be denoted respectively as $Y^R = H(dy(r, \theta); 0 \leq r < R, 0 \leq \theta < 2\pi)$ and $Z = H(z(r, \theta); 0 < r < \infty, 0 \leq \theta < 2\pi)$. If η and ξ are some arbitrary elements of Y^R and Z , they can be represented as

$$\eta = \int_{D_R} \eta(r, \theta) dy(r, \theta) \quad (2.6)$$

$$\xi = \int_{D_R} \xi(r, \theta) z(r, \theta) dA \quad (2.7)$$

where $\eta(\cdot, \cdot)$ and $\xi(\cdot, \cdot)$ are respectively square-integrable over D_R and D_∞ with respect to the measure dA . D_∞ denotes here the whole plane, or equivalently a disk of infinite radius centered at the origin. Note that while the observations are given over D_R , the field $z(\cdot, \cdot)$ is defined over D_∞ .

The estimation problem that will be studied here consists of finding the conditional mean of a random variable $\xi \in Z$ given Y^R . A special case that will be of interest is when $\xi = z(0)$ is the field at the origin. The optimum smoothing filter that is obtained in this case can be viewed as an approximation for the optimum smoothing filter based on observations over the whole plane.

A. Fourier coefficient processes

Since the field $z(r, \theta)$ is a periodic function of θ , it can be expanded in Fourier series as

$$z(r, \theta) = \sum_{n=-\infty}^{\infty} z_n(r) \exp jn\theta \quad (2.8)$$

where the Fourier coefficients

$$z_n(r) = \frac{1}{2\pi} \int_0^{2\pi} z(r, \theta) \exp -jn\theta d\theta \quad (2.9)$$

define some one-dimensional stochastic processes. It is shown in [16] that we have

$$E[z_n(r) z_m^*(s)] = \phi_n(r,s) \delta_{nm} \quad (2.10)$$

with

$$\phi_n(r,s) = \int_0^\infty J_n(\lambda r) J_n(\lambda s) dM(\lambda) \quad , \quad (2.11)$$

where $J_n(\cdot)$ is the n^{th} -order Bessel function and where $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ otherwise. Thus, the processes $z_n(\cdot)$ are uncorrelated, and if we denote by $Z_n = H(z_n(r); 0 \leq r < \infty)$ the Hilbert space spanned by the process $z_n(\cdot)$, Z can be decomposed orthogonally as

$$Z = \bigoplus_{n=-\infty}^{\infty} Z_n \quad . \quad (2.12)$$

Since the covariance of the processes $z_n(\cdot)$ takes the form (2.11) it is clear that $z_n(\cdot)$ is not stationary. However, the covariance ϕ_n has the following property.

Lemma 1. Displacement property of ϕ_n . If $k(\cdot)$ is twice differentiable, or equivalently if the field $z(\cdot, \cdot)$ is mean-square differentiable, we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) - \left(\frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} - \frac{n^2}{s^2} \right) \phi_n(r,s) \equiv 0 \quad (2.13)$$

with the boundary conditions

$$\frac{\partial}{\partial r} \phi_0(r,s) \Big|_{r=0} = 0 \quad (2.14a)$$

and

$$\phi_n(0, s) = 0 \text{ for } n \neq 0. \quad (2.14b)$$

Proof: Since $k(\cdot)$ is twice differentiable, $\phi_n(\cdot, \cdot)$ is also twice differentiable.

Then, since $J_n(\lambda r)$ obeys the differential equation

$$\ddot{J}_n + \frac{1}{r} \dot{J}_n + \left(\lambda^2 - \frac{n^2}{r^2}\right) J_n = 0$$

we have

$$\begin{aligned} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) \phi_n(r, s) &= - \int_0^\infty \lambda^2 J_n(\lambda r) J_n(\lambda s) dM(\lambda) \\ &= \left(\frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} - \frac{n^2}{s^2}\right) \phi_n(r, s) \end{aligned}$$

so that (2.13) is satisfied. The boundary conditions (2.14) can be obtained by noting that $\dot{J}_0(0) = 0$ and that $J_n(0) = 0$ for $n \neq 0$. ■

Thus, although $z_n(\cdot)$ is not stationary, it has as much structure as a stationary process. In fact, the displacement operator

$$\delta_{-n} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2}\right) - \left(\frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} - \frac{n^2}{s^2}\right) \quad (2.15)$$

associated to (2.13) can be viewed as a generalization of the displacement operator

$$\mathbb{I} = \frac{\partial}{\partial r} + \frac{\partial}{\partial s}$$

which was used in [11], [12] to characterize Toeplitz kernels, i.e. covariances of stationary processes. The operator δ_{-n} is also similar to the one-dimensional wave operator introduced in [5].

The noise and observation processes can also be expanded in Fourier series. Let $L_2(rdr; [0,R])$ denote the space of functions which are square-integrable over $[0,R]$ with respect to the measure rdr . Then the Fourier coefficient processes $v_n(\cdot)$ and $y_n(\cdot)$ associated to the noise and observations are such that if $a(\cdot)$ is an arbitrary function of $L_2(rdr; [0,R])$, one has

$$\int_0^R a(r) dv_n(r) \triangleq \frac{1}{2\pi} \int_{D_R} a(r) \exp-jn\theta dv(r,\theta) \quad (2.16)$$

$$\int_0^R a(r) dy_n(r) \triangleq \frac{1}{2\pi} \int_{D_R} a(r) \exp-jn\theta dy(r,\theta) \quad (2.17)$$

Formally, the increments of these processes can be denoted as

$$dv_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \exp-jn\theta dv(r,\theta)$$

$$dy_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \exp-jn\theta dy(r,\theta)$$

and from (2.1) one gets

$$dy_n(r) = z_n(r) rdr + dv_n(r), \quad 0 \leq r < R. \quad (2.18)$$

This shows that by expanding the signal and observations in Fourier series, the original two-dimensional estimation problem has been decomposed into a countable set of one-dimensional problems. By using the definition (2.16) for $v_n(\cdot)$ one finds also that

$$E[dv_n(r) dv_m^*(s)] = \begin{cases} \frac{r}{2\pi} dr & \text{for } r=s \text{ and } n=m \\ 0 & \text{otherwise} \end{cases} \quad (2.19)$$

so that the processes $v_n(\cdot)$ are uncorrelated Wiener processes of intensity $r/2\pi$.

A consequence of this property is that the estimation problems (2.18) can be solved independently of each other. Equivalently, if $Y_n^R = H(dy_n(r); 0 \leq r < R)$ is the Hilbert space spanned by the n^{th} observation process, by combining (2.10) and (2.19) one has $Y_n^R \perp Y_m^R$ for $n \neq m$. To show that

$$Y^R = \bigoplus_{n=-\infty}^{\infty} Y_n^R \quad (2.20)$$

one needs to prove that if

$$\eta = \int_{D_R} \eta(r, \theta) dy(r, \theta) \quad (2.21)$$

is an arbitrary element of Y^R , then η can be expressed as the sum of elements in Y_n^R for $n = 0, \pm 1, \pm 2 \dots$. But $\eta(r, \theta)$ can be expanded in Fourier series as

$$\eta(r, \theta) = \sum_{n=-\infty}^{\infty} \eta_n(r) \exp jn\theta$$

and if $\eta(\cdot, \cdot)$ is real $\eta_n^*(r) = \eta_{-n}(r)$. By substituting this expression inside (2.21) and by using the definition (2.17) of $y_n(\cdot)$ one finds that

$$\eta = 2\pi \sum_{n=-\infty}^{\infty} \int_0^R \eta_n^*(r) dy_n(r) \quad (2.22)$$

so that η is the sum of elements of Y_n^R with $n = 0, \pm 1, \pm 2 \dots$. This proves (2.20).

B. Filtering problem for the Fourier coefficients

The first step towards the solution of the two-dimensional estimation problem described above is to find the filtered estimate of the field $z(R, \theta)$

on the edge of the disk D_R given the observations Y^R . By using the decomposition (2.20) of Y^R , the conditional mean of $z(R, \theta)$ given Y^R can be expressed as

$$E[z(R, \theta) | Y^R] = \sum_{n=-\infty}^{\infty} E[z_n(R) | Y_n^R] \exp jn\theta \quad (2.23)$$

where

$$E[z_n(R) | Y_n^R] = \hat{z}_n(R|R) = \int_0^R g_n(R, s) dy_n(s) \quad (2.24)$$

denotes the filtered estimate of $z_n(R)$ given Y_n^R .

The two-dimensional filtering problem has therefore been reduced to a set of one-dimensional filtering problems for the Fourier coefficients. If the filtering error associated to (2.24) is denoted as

$$\tilde{z}_n(R|R) = z_n(R) - \hat{z}_n(R|R) \quad , \quad (2.25)$$

by using the orthogonality property $\tilde{z}_n(R|R) \perp Y_n^R$ of linear least-squares estimates, we find that $g_n(\cdot, \cdot)$ satisfies the integral equation

$$\phi_n(R, r) = \int_0^R g_n(R, s) \phi_n(s, r) s ds + \frac{1}{2\pi} g_n(R, r) \quad (2.26)$$

for $0 \leq r \leq R$.

Note that since we have assumed that $z(\cdot, \cdot)$ is mean-square continuous, the Fourier coefficients $z_n(\cdot)$ are also mean-square continuous, and therefore $\phi_n(\cdot, \cdot)$ is continuous. Thus, the operator

$$\underline{\Phi}_n: a(r) \rightarrow b(r) = \int_0^R \phi_n(r, s) a(s) s ds$$

is defined over $L_2(rdr; [0, R])$, and since $\phi_n(\cdot, \cdot)$ is a covariance kernel, the operator $\underline{\Phi}_n$ is self-adjoint and non-negative definite, so that $\underline{\Phi}_n + I/2\pi$ is

invertible. This guarantees the existence and unicity of a solution in $L_2(rdr; [0,R])$ to the integral equation (2.26).

To compute $g_n(\cdot, \cdot)$, instead of solving directly the integral equation (2.26), we can exploit the displacement property of ϕ_n . This gives the following result.

Theorem 1. Vibrating membrane equation for g_n . If the kernel $k(\cdot)$ is twice differentiable, or equivalently if $z(\cdot, \cdot)$ is mean-square differentiable, $g_n(\cdot, \cdot)$ satisfies the partial differential equation

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} - \frac{n^2}{R^2} \right) g_n(R, r) - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) g_n(R, r) = a_n(R) g_n(R, r) \quad (2.27)$$

with the boundary conditions

$$a_n(R) = -2 \frac{d}{dR} (Rg_n(R, R)) \quad (2.28)$$

and

$$\frac{\partial}{\partial R} g_n(R, r) \Big|_{r=0} = 0, \quad g_n(R, 0) = 0 \text{ for } n \neq 0 \quad (2.29)$$

The proof of Theorem 1 is given in Appendix A. The equation (2.27) can be viewed as being satisfied by a vibrating membrane submitted to a circularly symmetric excitation. For such a membrane R and r denote respectively the radial position and time. Since (2.27) takes nearly the same form as the displacement relation satisfied by ϕ_n , it can be viewed as a generalization of the Levinson recursions [11], [12] and of the vibrating string equations introduced in [5] for the one-dimensional estimation problem. Note indeed that these recursions were obtained by exploiting the displacement properties

of Toeplitz, and of Toeplitz plus Hankel kernels.

By performing the transformation

$$\ell_n(R, r) = (Rr)^{1/2} g_n(R, r) \quad (2.30)$$

the recursions (2.27) can be expressed as

$$\left(\frac{\partial^2}{\partial R^2} - \frac{1}{R^2} \left(n^2 - \frac{1}{4} \right) \right) - \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \left(n^2 - \frac{1}{4} \right) \right) \ell_n(R, r) = \alpha_n(R) \ell_n(R, r) \quad (2.31)$$

with the boundary condition

$$\alpha_n(R) = -2 \frac{d}{dR} \ell_n(R, R) \quad .$$

This equation appears in the study of inverse scattering problems of quantum mechanics [13], [14] and the relation between these problems and the linear estimation results discussed here will be explored in [17]. The filter $\ell_n(\cdot, \cdot)$ can be interpreted by considering the normalized estimation problem

$$d\bar{y}_n(r) = \bar{z}_n(r) dr + d\bar{v}_n(r) \quad , \quad 0 \leq r < R \quad (2.32)$$

where $d\bar{y}_n(r) = r^{-1/2} dy_n(r)$, $d\bar{v}_n(r) = r^{-1/2} dv_n(r)$ and $\bar{z}_n(r) = r^{1/2} z_n(r)$.

In this case, the noise process $\bar{v}_n(\cdot)$ has stationary increments, and $\ell_n(R, \cdot)$ is the optimal filter for estimating $\bar{z}_n(R)$ given $\bar{Y}_n^R = H(d\bar{y}_n(r), 0 \leq r < R)$.

The mean-square error associated to the filtering problem (2.18) can be denoted as

$$e_n^2(R) = E\{\tilde{z}_n^2(R|R)\}$$

and by using the orthogonality of linear least-squares estimates we find that

$$e_n^2(R) = g_n(R, R) \quad . \quad (2.33)$$

This relation provides an interpretation of the potential $q_n(R)$ given by (2.28).

C. Innovations processes for the Fourier coefficients

Given the observations (2.18), we have shown how to compute the filtered estimates $\hat{z}_n(r|r)$. These estimates can be used to generate an innovations process

$$dv_n(r) = dy_n(r) - \hat{z}_n(r|r) dr = \tilde{z}_n(r|r) dr + dv_n(r) \quad (2.34)$$

which is a Wiener process of intensity $r/2\pi$, i.e.

$$E[dv_n(r) dv_n^*(s)] = \begin{cases} \frac{r}{2\pi} dr & \text{for } r=s \\ 0 & \text{otherwise} \end{cases} \quad (2.35)$$

Furthermore, the Hilbert space $V_n^R = H(dv_n(r), 0 \leq r < R)$ generated by this process is identical to the Hilbert space Y_n^R associated to the n^{th} Fourier coefficient of the observations, so that the Hilbert space decomposition (2.20) reduces to

$$Y^R = \bigoplus_{n=-\infty}^{\infty} V_n^R \quad (2.36)$$

III. The general estimation problem

We have shown above how to solve the filtering problem associated to the Fourier coefficients of the observed field $z(\cdot, \cdot)$. This result will now be used to estimate an arbitrary random variable.

Theorem 2. Let ξ be a zero-mean random variable whose joint statistics with

variables in Y^R are known and are Gaussian. A special case is of course when $\xi \in \mathcal{Z}$. Then, the conditional mean and error variance of ξ given Y^R are given by

$$E[\xi | Y^R] = 2\pi \sum_{n=-\infty}^{\infty} \int_0^R \xi_n(r) dv_n(r) \quad (3.1)$$

and

$$e^2 = E[(\xi - E[\xi | Y^R])^2] = E[\xi^2] - 2\pi \sum_{n=-\infty}^{\infty} \int_0^R |\xi_n(r)|^2 r dr \quad (3.2)$$

where $\xi_n(\cdot)$ is such that

$$E[\xi dv_n^*(r)] = \xi_n(r) r dr . \quad (3.3)$$

Proof: Use the decomposition (2.36) and note that $v_n(\cdot)$ is a Wiener process of intensity $r/2\pi$.

The previous results can be expressed entirely in the spectral domain. To do so, we denote by $Y = H(dy(r, \theta); 0 \leq r < \infty, 0 \leq \theta < 2\pi)$ the Hilbert space spanned by the observations over the whole plane, and we use the Kolmogorov isometry

$$a = \int_{D_\infty} a(r, \theta) dy(r, \theta) \leftrightarrow \hat{a}(\lambda, \phi) = \int_{D_\infty} a(r, \theta) \exp(j\lambda r \cos(\theta - \phi)) r dr d\theta \quad (3.4)$$

between Y and $L_2(dF)$ where $F(\lambda, \phi)$ is the spectral measure associated to the observations $y(\cdot, \cdot)$. This measure is given by

$$k(r) + \frac{\delta(r)}{\pi r} = \int_{D_\infty} \exp(j\lambda r \cos(\theta - \phi)) dF(\lambda, \phi) \quad (3.5)$$

where $\delta(r)/\pi r$ is a two-dimensional impulse function written in polar coordinates. Note that in (3.4) and (3.5) the term $\lambda r \cos(\theta - \phi)$ corresponds to the inner product

between the vectors $\underline{x} = (r \cos \theta, r \sin \theta)$ and $\underline{u} = (\lambda \cos \phi, \lambda \sin \phi)$.

Since the left hand side of (3.5) depends on r only, $dF(\cdot, \cdot)$ is a function of λ only, i.e.

$$dF(\lambda, \phi) = dF_1(\lambda) \frac{d\phi}{2\pi} \quad (3.6)$$

so that (3.5) reduces to

$$k(r) + \frac{\delta(r)}{\pi r} = \int_0^{\infty} J_0(\lambda r) dF_1(\lambda) \quad (3.7)$$

where we have used the fact that

$$J_0(\lambda r) = \frac{1}{2\pi} \int_0^{2\pi} \exp(j\lambda r \cos(\theta - \phi)) d\phi$$

for θ arbitrary. Furthermore, $F_1(\cdot)$ can be expressed in function of the measure $M(\cdot)$ introduced in (2.5) as

$$dF_1(\lambda) = dM(\lambda) + \frac{\lambda d\lambda}{2\pi} \quad (3.8)$$

The isometry (3.4) is such that if a and b are two random variables of Y corresponding to some functions $\hat{a}(\lambda, \phi)$ and $\hat{b}(\lambda, \phi)$ of $L_2(dF)$, one has

$$E[ab] = \langle \hat{a}, \hat{b} \rangle_F \triangleq \int_{D_\infty} \hat{a}(\lambda, \phi) \hat{b}^*(\lambda, \phi) dF(\lambda, \phi) \quad (3.9)$$

In this framework, the subspace Y^R of Y generated by the observations over D_R is isomorphic to the subspace S^R of $L_2(dF)$ generated by the functions

$$\hat{a}(\lambda, \phi) = \int_{D_R} a(r, \theta) \exp(j\lambda r \cos(\theta - \phi)) r dr d\theta \quad (3.10)$$

where $a(\cdot, \cdot)$ is square-integrable over D_R with respect to $r dr d\theta$.

A. Decomposition of S^R

To obtain a decomposition of S^R equivalent to the decomposition (2.20) for Y^R , we note that if

$$n = \frac{1}{2\pi} \int_{D_R} n(r) \exp -jn\theta \, dy(r, \theta) \quad (3.11)$$

is an arbitrary random variable of Y_n^R , n is mapped into the function

$$\hat{n}(\lambda, \phi) = \frac{1}{2\pi} \int_{D_R} n(r) \exp j(\lambda r \cos(\theta - \phi) - n\theta) \, r dr d\theta. \quad (3.12)$$

By taking into account the Fourier series expansion

$$\exp(jx \cos \theta) = \sum_{n=-\infty}^{\infty} j^n J_n(x) \exp jn\theta$$

$\hat{n}(\cdot, \cdot)$ can be expressed as

$$\hat{n}(\lambda, \phi) = \hat{n}_1(\lambda) \exp -jn\phi \quad (3.13a)$$

where

$$\hat{n}_1(\lambda) = j^n \int_0^R n(r) J_n(\lambda r) \, r dr \quad (3.13b)$$

so that Y_n^R is isomorphic to the subspace S_n^R of $L_2(dF)$ generated by functions of the type (3.13). From (3.13a) it is clear that $S_n^R \perp S_m^R$ for $n \neq m$. Moreover, if $\hat{n}(\cdot, \cdot)$ and $\hat{\xi}(\cdot, \cdot)$ are two functions of S_n^R , we have

$$\langle \hat{n}, \hat{\xi} \rangle_F = \langle \hat{n}_1, \hat{\xi}_1 \rangle_{F_1} \stackrel{\Delta}{=} \int_0^{\infty} \hat{n}_1(\lambda) \hat{\xi}_1^*(\lambda) \, dF_1(\lambda), \quad (3.14)$$

and the mapping

$$\hat{r}(\lambda, \beta) = \hat{r}_1(\lambda) \exp - jn\phi \mapsto \hat{\eta}_1(\lambda) \quad (3.15)$$

is an isometry between S_n^R and the subspace L_n^R of $L_2(dF_1)$ generated by functions of the form (3.13b).

By transforming the decomposition (2.20) under the isometry (3.4), we find that

$$S^R = \bigoplus_{n=-\infty}^{\infty} S_n^R \quad (3.16)$$

In the special case when $R = \infty$, the decomposition (3.16) becomes

$$L_2(dF) = \bigoplus_{n=-\infty}^{\infty} S_n \quad (3.17)$$

where S_n is the subspace of $L_2(dF)$ spanned by the functions $\hat{\eta}_1(\lambda) \exp - jn\phi$, where $\hat{\eta}_1(\cdot)$ belongs to $L_2(dF_1)$. Note that if $\eta_1(\cdot)$ is arbitrary function of $L_2(dF_1)$, it can be expressed as

$$\hat{r}_1(\lambda) = j^n \int_0^{\infty} \eta(r) J_n(\lambda r) r dr$$

where $\eta(\cdot)$ is the n^{th} -order Hankel transform [18] of $\hat{\eta}_1(\cdot)$.

B. Orthogonalization of the functions $J_n(\lambda r)$

The relation (3.13b) shows that if $\hat{\eta}_1(\cdot)$ belongs to L_n^R , it can be expressed as linear combination of the functions $\{J_n(\lambda r), 0 \leq r \leq R\}$.

However, since we have

$$\langle J_n(\lambda r), J_n(\lambda s) \rangle_{F_1} = \frac{1}{2\pi r} \delta(r-s) + \phi_n(r,s) \quad (3.18)$$

these functions are not orthogonal in $L_2(dF_1)$. To orthogonalize them, we will use the correspondence

$$\int_0^R a(r) dv_n(r) \leftrightarrow j^n \int_0^r a(r) \gamma_n(r,\lambda) r dr \quad (3.19)$$

between Y_n^R and L_n^R , where

$$\gamma_n(r,\lambda) \triangleq J_n(\lambda r) - \int_0^r g_n(r,s) J_n(\lambda s) s ds \quad (3.20)$$

and where $a(\cdot)$ is an arbitrary function of $L_2(rdr; [0,R])$. Then, since the innovations process $v_n(\cdot)$ is a Wiener process of intensity $r/2\pi$, the functions $\gamma_n(\cdot,\lambda)$ are orthogonal and

$$\langle \gamma_n(r,\lambda), \gamma_n(s,\lambda) \rangle_{F_1} = \frac{1}{2\pi r} \delta(r-s) . \quad (3.21)$$

To characterize the functions $\gamma_n(\cdot,\cdot)$, instead of using the functions $g_n(\cdot,\cdot)$, we can use the following result.

Theorem 3. For $n \geq 0$, $\gamma_n(\cdot,\lambda)$ satisfies the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \left(\lambda^2 - g_n(r) - \frac{n^2}{r^2} \right) \right) \gamma_n(r,\lambda) = 0 \quad (3.22)$$

with the boundary condition

$$\lim_{r \rightarrow 0} 2^n n! (\lambda r)^{-n} \gamma_n(r,\lambda) = 1; \quad (3.23)$$

and for $n \leq 0$, we have

$$\gamma_n(r, \lambda) = (-1)^n \gamma_{-n}(r, \lambda) \quad (3.24)$$

Proof: By operating with

$$\Delta_n(r) = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \quad (3.25)$$

on (3.20), one gets

$$\begin{aligned} \Delta_n(r) \gamma_n(r, \lambda) = & -(\lambda^2 - q_n(r)) J_n(\lambda r) + r \frac{\partial}{\partial s} q_n(r, s) \Big|_{s=r} J_n(\lambda r) \\ & - \lambda r q_n(r, r) \frac{dJ_n}{dr}(\lambda r) - \int_0^r (\Delta_n(r) q_n(r, s)) J_n(\lambda s) s ds \end{aligned}$$

Then, since the displacement operator δ_n can be expressed as $\delta_n = \Delta_n(r) - \Delta_n(s)$, by using the equation (2.27) for $q_n(r, s)$ and integrating by parts, we obtain (3.22). The boundary conditions (3.23) and (3.24) are identical to those satisfied by $J_n(\lambda r)$ and can be derived from (3.20). ■

The differential equation (3.22) takes a familiar form if we perform the transformation

$$\psi_n(r, \lambda) = (r\lambda)^{1/2} \gamma_n(r, \lambda) \quad (3.26)$$

In this case (3.22) becomes

$$\ddot{\psi}_n(r, \lambda) + (\lambda^2 - q_n(r) - \frac{1}{r^2} (n^2 - \frac{1}{4})) \psi_n(r, \lambda) = 0 \quad (3.27)$$

with the boundary condition

$$\lim_{r \rightarrow 0} 2^n n! (\lambda r)^{-(n+1/2)} \psi_n(r, \lambda) = 1$$

for $n \geq 0$, and $\psi_n(r, \lambda) = (-1)^n \psi_{-n}(r, \lambda)$ for $n \leq 0$. For a space of dimension two, this is the radial Schrödinger equation satisfied by the partial wave of angular momentum n associated to a particle moving in the circularly symmetric potential $q_n(\cdot)$ (see [17] for more details).

C. Projection on S^R

Under the isomorphism (3.4), if ξ is an arbitrary random variable of Y corresponding to the function $\hat{\xi}(\lambda, \phi)$ of $L_2(dF)$, the problem of finding the conditional mean of ξ given Y^R is equivalent to the one of projecting $\hat{\xi}(\lambda, \phi)$ on S^R . Thus, we have the correspondence

$$E[\xi | Y^R] \leftrightarrow P_{S^R} \hat{\xi}(\lambda, \phi) \quad (3.28)$$

where P_{S^R} denotes the projection operator on S^R . Then, if

$$\hat{\xi}(\lambda, \phi) = \sum_{n=-\infty}^{\infty} \hat{\xi}_n(\lambda) \exp jn\phi \quad (3.29)$$

denotes the Fourier series expansion of $\hat{\xi}(\cdot, \cdot)$, Theorem 2 can be reformulated as follows.

Theorem 2': The projection of $\hat{\xi}(\lambda, \phi)$ on S^R is given by

$$P_{S^R} \hat{\xi}(\lambda, \phi) = 2\pi \sum_{n=-\infty}^{\infty} j^n \left(\int_0^R \xi_n(r) \gamma_n(r, \lambda) r dr \right) \exp -jn\phi \quad (3.30)$$

where

$$\xi_n(r) = \int_0^{\infty} \hat{\xi}_{-n}(\lambda) \gamma_n^*(r, \lambda) dF_1(\lambda) \quad (3.31)$$

satisfies (3.3).

Proof: By replacing n by $-n$ in (3.29), the expansion (3.29) can be viewed as decomposition of $\hat{\xi}(\cdot, \cdot)$ in terms of elements of S_n with $n = 0, \pm 1, \pm 2, \dots$. The decomposition (3.16) for S^R can then be used to show that

$$P_{S^R} \hat{\xi}(\lambda, \phi) = \sum_{n=-\infty}^{\infty} P_{L_n^R} \hat{\xi}_{-n}(\lambda) \exp -jn\phi$$

where $P_{L_n^R}$ denotes the projection operator on the subspace L_n^R of $L_2(dF_1)$. Furthermore, by using Theorem 2 and the isometry (3.19) between Y_n^P and L_n^R , we find that

$$P_{L_n^R} \hat{\xi}_{-n}(\lambda) = 2\pi j^n \int_0^R \xi_n(r) \gamma_n(r, \lambda) r dr$$

where $\xi_n(\cdot)$ is given by (3.3). The relation (3.31) is equivalent to (3.3) under the isometry (3.19). ■

Note that the solution of the general estimation problem that we have obtained in Theorems 2 and 2' depends on the functions $\{g_n, n \in \mathbb{Z}\}$ or $\{\gamma_n, n \in \mathbb{Z}\}$. It will now be shown that it is not necessary to compute all these functions, and that for $|n| \geq 1$, g_n and γ_n can be expressed in function of g_0 and γ_0 .

IV. Special transformations

The first step in order to relate g_{n+1} and g_n (respectively γ_{n+1} and γ_n) is to note that the covariance kernels $\phi_n(\cdot, \cdot)$ satisfy the following property.

Lemma 2. If $k(\cdot)$ is differentiable, for all n we have

$$\left(\frac{\partial}{\partial r} - \frac{n}{r}\right) \phi_n(r, s) + \left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) \phi_{n+1}(r, s) \equiv 0 \tag{4.1a}$$

$$\left(\frac{\partial}{\partial s} - \frac{n}{s}\right) \phi_n(r, s) + \left(\frac{\partial}{\partial r} + \frac{(n+1)}{r}\right) \phi_{n+1}(r, s) \equiv 0 \tag{4.1b}$$

Proof: The Bessel functions $\{J_n(r), n \in \mathbb{Z}\}$ satisfy the recursions

$$J_n'(r) + \frac{n}{r} J_n(r) = J_{n-1}(r)$$

$$J_n'(r) - \frac{n}{r} J_n(r) = J_{n+1}(r)$$

and $\phi_n(\cdot, \cdot)$ is given by (2.11). This implies that

$$\left(\frac{\partial}{\partial r} - \frac{n}{r}\right) \phi_n(r, s) = - \int_0^\infty \lambda J_{n+1}(\lambda r) J_n(\lambda s) dM(\lambda)$$

and

$$\left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) \phi_{n+1}(r, s) = \int_0^\infty \lambda J_{n+1}(\lambda r) J_n(\lambda s) dM(\lambda)$$

so that (4.1a) is satisfied. The relation (4.16) can be obtained by symmetry. ■

Note that if $\underline{\Delta}_n(r)$ is given by (3.25), we have

$$\underline{\Delta}_n(r) = \left(\frac{\partial}{\partial r} + \frac{(n+1)}{r}\right) \left(\frac{\partial}{\partial r} - \frac{n}{r}\right) \quad (4.2a)$$

$$\underline{\Delta}_{n+1}(r) = \left(\frac{\partial}{\partial r} - \frac{n}{r}\right) \left(\frac{\partial}{\partial r} + \frac{(n+1)}{r}\right) \quad (4.2b)$$

and since $\delta_{-n} = \underline{\Delta}_n(r) - \underline{\Delta}_n(s)$, the displacement property (2.13) for ϕ_n can be obtained by combining (4.1a) and (4.1b). Then, by using the integral equation (2.26) for g_n , we obtain the following result.

Theorem 4. If $k(\cdot)$ is differentiable, the functions $g_n(\cdot, \cdot)$ are such that

$$\left(\frac{\partial}{\partial r} - \frac{n}{r}\right) g_n(r, s) + \left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) g_{n+1}(r, s) = - a_n(r) g_n(r, s) \quad (4.3a)$$

$$\left(\frac{\partial}{\partial s} - \frac{n}{s}\right) g_n(r,s) + \left(\frac{\partial}{\partial r} + \frac{(n+1)}{r}\right) g_{n+1}(r,s) = a_n(r) g_{n+1}(r,s) \quad (4.3b)$$

where

$$a_n(r) = r (g_n(r,r) - g_{n+1}(r,r)). \quad (4.4)$$

The proof of Theorem 4 is given in Appendix A. This result can be used to derive the vibrating membrane equation (2.27), since by combining (4.3a) and (4.3b) and using the identities (4.2), we obtain (2.27) with

$$q_n(r) = a_n^2(r) - \dot{a}_n(r) - \frac{(2n+1)}{r} a_n(r) \quad (4.5a)$$

$$q_{n+1}(r) = a_n^2(r) + \dot{a}_n(r) - \frac{(2n+1)}{r} a_n(r) . \quad (4.5b)$$

The relations (4.5) show that the potentials $q_n(\cdot)$ and $q_{n+1}(\cdot)$ are related, since they can be expressed in terms of the same function $a_n(\cdot)$. Conversely, given $q_n(\cdot)$ or $q_{n+1}(\cdot)$ we can obtain $a_n(\cdot)$ by solving the Riccati equation (4.5a) or (4.5b) with the boundary condition

$$a_n(0) = 0 \quad \text{for all } n.$$

The functions $a_n(\cdot)$ play here a role similar to the reflection coefficient function of [5].

An alternative method of computing $a_n(\cdot)$ given either $q_n(\cdot)$ or $q_{n+1}(\cdot)$ is to substitute

$$a_n(r) = \frac{n}{r} - \frac{\dot{w}_n(r)}{w_n(r)} \quad (4.6a)$$

or

$$a_n(r) = \frac{(n+1)}{r} + \frac{\dot{u}_n(r)}{u_n(r)} \quad (4.6b)$$

with the boundary conditions

$$\lim_{r \rightarrow 0} 2^n n! r^{-n} w_n(r) = 1 \quad (4.7a)$$

$$\lim_{r \rightarrow \infty} r^{n+1} u_n(r) = 1 \quad (4.7b)$$

inside (4.5a) or (4.5b). Then, the equations (4.5a) and (4.5b) are linearized and take the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left(q_n(r) + \frac{n^2}{r^2} \right) \right) w_n(r) = 0 \quad (4.8a)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left(q_{n+1}(r) + \frac{(n+1)^2}{r^2} \right) \right) u_n(r) = 0 \quad (4.8b)$$

This shows that $w_n(r)$ and $u_{n-1}(r)$ are two solutions of the differential equation (4.8a). However $w_n(r)$ is regular at the origin, whereas by subtracting (4.6a) from (4.5b) and integrating, we find that

$$u_n(r) = \frac{C_n}{r w_n(r)} \quad (4.9)$$

where C_n is a nonzero constant. By taking into account (4.7a) we see therefore that $u_n(r)$ has a singularity of order $n+1$ at the origin. The identity (4.9) also shows that the singular solution associated to the potential $q_{n+1}(\cdot)$ can be obtained from the regular solution associated to $q_n(\cdot)$. We can identify

$$w_n(r) = \theta_n(r, 0) \quad (4.10)$$

where $\theta_n(r, \lambda) = \gamma_n(r, \lambda) / \lambda^n$, and by integrating (4.6a) we find that

$$w_n(r) = \frac{r^n}{2^n n!} \exp - \int_0^r a_n(u) du \quad (4.11)$$

The identities (4.3) of Theorem 4 can be rewritten as

$$r^n \frac{\partial}{\partial r} (r^{-n} g_n(r,s)) + s^{-(n+1)} \frac{\partial}{\partial s} (s^{n+1} g_{n+1}(r,s)) = -a_n(r) g_n(r,s)$$

$$s^n \frac{\partial}{\partial s} (s^{-n} g_n(r,s)) + r^{-(n+1)} \frac{\partial}{\partial r} (r^{n+1} g_{n+1}(r,s)) = a_n(r) g_{n+1}(r,s)$$

and therefore, by integration, g_{n+1} can be expressed in function of g_n and vice-versa. However, a simpler set of identities can be obtained by considering the functions $\gamma_n(\cdot, \cdot)$.

Theorem 4'. For all n , we have

$$\lambda \gamma_{n+1}(r, \lambda) = \left(\frac{d}{dr} - \frac{n}{r} \right) \gamma_n(r, \lambda) + a_n(r) \gamma_n(r, \lambda) \quad (4.12a)$$

$$\lambda \gamma_n(r, \lambda) = \left(\frac{d}{dr} + \frac{(n+1)}{r} \right) \gamma_{n+1}(r, \lambda) - a_n(r) \gamma_{n+1}(r, \lambda). \quad (4.12b)$$

Proof: By operating with $\frac{d}{dr} - \frac{n}{r}$ on (3.20), one gets

$$\begin{aligned} \left(\frac{d}{dr} - \frac{n}{r} \right) \gamma_n(r, \lambda) &= \lambda J_{n+1}(\lambda r) - r g_n(r, r) J_n(\lambda r) \\ &\quad - \int_0^r \left(\frac{\partial}{\partial r} - \frac{n}{r} \right) g_n(r, s) J_n(\lambda s) s ds. \end{aligned} \quad (4.13)$$

Then, if we use (4.3a) and integrate by parts, we find that

$$\begin{aligned} \int_0^r \left(\frac{\partial}{\partial r} - \frac{n}{r} \right) g_n(r, s) J_n(\lambda s) s ds &= - g_{n+1}(r, s) J_n(\lambda s) s \Big|_{s=0}^{s=r} \\ &\quad + \lambda \int_0^r g_{n+1}(r, s) J_{n+1}(\lambda s) s ds - a_n(r) \int_0^r g_n(r, s) J_n(\lambda s) s ds. \end{aligned}$$

By substituting this expression inside (4.13), and taking into account (4.4) and (3.20), we obtain (4.12a). The identity (4.12b) can be proved similarly. ■

When the relations (4.6) for $w_n(\cdot)$ and $u_n(\cdot)$ are taken into account, the identities (4.12) can be expressed as

$$\lambda \gamma_{n+1}(r, \lambda) = W[\gamma_n(r, \lambda), w_n(r)] / w_n(r) \quad (4.14a)$$

$$\lambda \gamma_n(r, \lambda) = W[\gamma_{n+1}(r, \lambda), u_n(r)] / u_n(r) \quad (4.14b)$$

where

$$W[f(r), g(r)] \triangleq \dot{f}(r) g(r) - f(r) \dot{g}(r)$$

denotes the Wronskian of $f(\cdot)$ and $g(\cdot)$. This class of transformations was introduced by Crum [19] and was later used by Krein [20] (see also [13], [14]) to study the inverse scattering problem of quantum mechanics for waves of nonzero angular momentum. Under such transformations, the solutions of the Schrödinger equation for a potential $q(\cdot)$ and angular momentum n can be transformed into solutions for a potential $q'(\cdot)$ and angular momentum n' , where $q'(\cdot)$ behaves like $q(\cdot)$ near $r = 0$ and as $r \rightarrow \infty$. For the problem consider here, we can use these transformations to express $\gamma_n(\cdot, \cdot)$ in function of $\gamma_0(\cdot, \cdot)$ for all n .

However, the relation (4.14a) alone is not sufficient to show that $\gamma_n(\cdot, \cdot)$ can be expressed in function of $\gamma_0(\cdot, \cdot)$ for all n . Since (4.14a) depends also on $w_n(\cdot)$, we need to prove that $w_{n+1}(\cdot)$ can be computed in function of $w_n(\cdot)$. To do so, note that the Wronskian

$$W(r, \lambda) = W[\gamma_n(r, \lambda), w_n(r)]$$

satisfies the differential equation

$$\frac{d}{dr} W + \frac{1}{r} W = -\lambda^2 \gamma_n(r, \lambda) w_n(r) \quad (4.15)$$

with $W(0, \lambda) = 0$ for all n . By integration, one obtains

$$W(r, \lambda) = -\frac{\lambda^2}{r} \int_0^r \gamma_n(s, \lambda) w_n(s) ds$$

and by using (4.14a), we find that

$$w_{n+1}(r) = \lim_{\lambda \rightarrow 0} \gamma_{n+1}(r, \lambda) \lambda^{n+1} = \frac{1}{r w_n(r)} \int_0^r w_n^2(s) ds \quad (4.16)$$

This shows that $\gamma_n(\cdot, \cdot)$ is a function of $\gamma_0(\cdot, \cdot)$, so that we need to solve only one of the Fourier coefficient filtering problems associated to our original two-dimensional estimation problem.

V. Random field estimates

In Section III, it was shown how to estimate an arbitrary random variable ξ whose joint statistics with $y(\cdot, \cdot)$ are known. We will now consider the case when $\xi = z(r, \theta)$. In this case, instead of using the estimation method described in Section III, we can decompose the conditional mean of $z(r, \theta)$ given Y^R as

$$E[z(r, \theta) | Y^R] = \sum_{n=-\infty}^{\infty} E[z_n(r) | Y_n^R] \exp jn\theta \quad (5.1)$$

and then denote

$$\hat{z}_n(r|R) = E[z_n(r) | Y_n^R] = \int_0^R H_n(r, s; R) dy_n(s) \quad (5.2)$$

The problem of estimating $z(r, \theta)$ given Y^R is now reduced to the one of computing $H_n(\cdot, \cdot; R)$ for all n .

By using the orthogonality property of linear least-squares estimates, we find that $H_n(\cdot, \cdot; R)$ satisfies the integral equation

$$\phi_n(r, s) = \int_0^R H_n(r, u; R) \phi_n(u, s) u du + \frac{1}{2\pi} H_n(r, s; R) \quad (5.3)$$

where $0 \leq s \leq R$ and $0 \leq r < \infty$. In the special case when $0 \leq r \leq R$, i.e. when the field $z(\cdot, \cdot)$ is estimated at a point located inside the disk D_R , if \underline{H}_n is the operator on $L_2(rdr; [0, R])$ defined by

$$\underline{H}_n: a(r) \rightarrow b(r) = \int_0^R H_n(r, s; R) a(s) s ds$$

the equation (5.3) can be rewritten in operator notation as

$$(\underline{I}/2\pi + \underline{\phi}_n) (\underline{I} - \underline{H}_n) = \underline{I}/2\pi \quad (5.4)$$

This shows that \underline{H}_n is the resolvent operator associated to $\underline{\phi}_n$. Furthermore, by setting $r = R$ in (5.3), we can identify

$$H_n(R, s; R) = g_n(R, s) \quad (5.5)$$

To compute the kernel $H_n(\cdot, \cdot; R)$, instead of solving directly the integral equation (5.3), we can use the Bellman-Krein identity [21]

$$\frac{1}{R} \frac{\partial}{\partial R} H_n(r, s; R) = - H_n(r, R; R) g_n(R, s) \quad (5.6)$$

which is obtained by taking the partial derivative of (5.3) with respect to R and by comparing the resulting equation to (2.26). By solving first the vibrating membrane equation (2.27) for $g_n(\cdot, \cdot)$ we can use this identity to compute $H_n(\cdot, \cdot; R)$ for increasing values of R . In the special case when $0 \leq r \leq R$, the identity (5.6) becomes

$$\frac{1}{R} \frac{\partial}{\partial R} H_n(r, s; R) = - g_n(R, r) g_n(R, s) \quad (5.7)$$

and by integration, it can be expressed in operator form as

$$(\underline{I} - \underline{H}_n) = (\underline{I} - \underline{q}_n^*) (\underline{I} - \underline{q}_n) \quad (5.8)$$

where \underline{q}_n is the lower triangular Volterra operator on $L_2(rdr; [0, R])$ given by

$$\underline{q}_n: a(r) \rightarrow \int_0^r q_n(r, s) a(s) s ds$$

and where g_n^* is the dual operator of g_n , i.e. its kernel is given by

$$g_n^*(r,s) = g_n(s,r) \quad (5.9)$$

for all $0 \leq r, s \leq R$. The operator factorization (5.8) in terms of upper times lower triangular operators is of the type considered by Gohberg and Krein [22].

Comments

1) To implement numerically the previous method for computing $H_n(\cdot, \cdot; R)$, the interval $[0,R]$ can be divided into N subintervals of length $\Delta = R/N$ and the Bellman-Krein identity (5.7) and the vibrating membrane equation (2.27) can be discretized. The resulting recursions enable us to compute $H_n(i\Delta, j\Delta; R)$ for all $0 \leq i, j \leq N$ with only $O(N^3)$ operations, whereas by discretizing the integral equation (5.3) and by solving the corresponding system of linear equations, we would need $O(N^4)$ operations. This shows that the vibrating membrane equations of Section II provide an efficient solution of the estimation problem over a finite disk.

2) The expression (5.1) - (5.2) for the estimate of $z(r,\theta)$ given Y^R is different from the one that was obtained in Theorem 2. To relate these two expressions, note that the integrated form of the Bellman-Krein identity (5.6) is

$$H_n(r,s; R) = H_n(r, s; s) - \int_s^R H_n(r, u; u) g_n(u,s) u du \quad (5.10)$$

By substituting this relation inside (5.2), we find that

$$\hat{z}_n(r|R) = \int_0^R H_n(r, s; s) dv_n(s) \quad (5.11)$$

so that

$$E[z(r,\theta) dv_n^*(s)] = \frac{1}{2\pi} H_n(r, s; s) \exp jn\theta ds \quad (5.12)$$

and by taking (5.12) into account inside (4.1) and (5.11), we obtain the expression (3.1) for the estimate of $\xi = z(r, \theta)$.

The symmetric smoothing problem

A special case of interest is when we want to estimate the random field $z(\underline{0})$ at the origin of the disk D_R . In this case, the geometry of estimation is circularly symmetric, and it will be shown below that the associated smoothing filter can be used to approximate the smoothing filter based on observations over the whole plane. For this problem, we have $z(\underline{0}) = z_0(0)$ and $z_n(0) = 0$ for $n \neq 0$, so that the estimate of $z(\underline{0})$ given Y^R needs only to be based on Y_0^R , i.e.

$$E[z(\underline{0}) | Y^R] = \hat{z}_0(0 | R) \quad . \quad (5.13)$$

By denoting

$$\begin{aligned} \hat{z}_0(0 | R) &= \int_0^R f(R, r) dy_0(r) \\ &= \frac{1}{2\pi} \int_{D_R} f(R, r) dy(r, \theta) \end{aligned} \quad (5.14)$$

we find that

$$f(R, r) = H_0(0, r; R) \quad (5.15)$$

and by using (5.6) one gets

$$\frac{1}{R} \frac{\partial}{\partial R} f(R, r) = - f(R, R) g_0(R, r) \quad (5.16)$$

with the boundary condition

$$f(R, R) = g_0(R, 0) \quad . \quad (5.17)$$

This shows that to compute $E[z(\underline{0})|Y^R]$ we need only to solve simultaneously (5.16) and the vibrating membrane equation (2.27) for $g_0(\cdot, \cdot)$. After dividing the interval $[0, R]$ into N subintervals and discretizing these equations, this procedure requires only $O(N^2)$ operations whereas the method proposed in [9], [10] would require $O(N^3)$ operations. In some sense, the relations (5.16) and (2.27) constitute a generalization of the Levinson recursions [23], [24]: $g_0(\cdot, \cdot)$ and $f(\cdot, \cdot)$ correspond respectively to the forwards and backwards predictors of [24]. The difference is that here the domain of observation grows radially.

The result described above can be used as follows: assume that some observations of an isotropic random field are given over a large domain, e.g. the whole plane. In this case, we could construct an optimum smoothing filter based on all observations, but to estimate the field everywhere, a very large amount of computations would be required. An approximate smoothing filter can be obtained by computing $f(R, \cdot)$ for increasing values of R until R is such that the mean-square error

$$g_0(R, R) = E[(z(\underline{0}) - E[z(\underline{0})|Y^R])^2]$$

is sufficiently small. Then at every point (r, θ) we can construct an estimate of $z(r, \theta)$ by using the filter $f(R, \cdot)$ and the observations located in a radius R of (r, θ) . In most cases, R will be much smaller than the size of the domain of observation, but the estimate $\hat{z}_R(r, \theta)$ obtained by the previous procedure will not be very different from the optimum estimate based on all observations.

VI. Conclusion

In this paper we have obtained a set of vibrating membrane equations for estimating a two-dimensional isotropic random field given some observations of

this field over a finite disk. These equations depend recursively on the radius of the observation disk and they provide an efficient method for constructing the random field estimates. They also generalize the vibrating string equations of [1] - [5] and the Levinson recursions [23] which had been introduced to solve the corresponding problem in one dimension. By mapping this estimation problem to the spectral domain, we have also shown that the estimation procedure described here is related to the Gelfand-Levitan solution of the inverse scattering problem of quantum mechanics for circularly symmetric potentials.

The analysis developed in this paper can be extended easily to higher dimensional isotropic random fields provided that instead of expanding the random field in Fourier series, we expand it in spherical harmonics as shown in [16]. The isotropy of the random field has been reflected here by the fact that the vibrating membrane equations are circularly symmetric. If the observed random field is only homogeneous, it is not clear whether some non-circularly symmetric membrane equations could be derived to solve the linear estimation problem. Note however that this type of equation appears in inverse scattering studies [14] for noncircularly symmetric potentials. Finally, we note that the connection between linear estimation and inverse scattering problems has only been discussed briefly. A more detailed study of this relation will be given in [17] (see also [6] - [8]). It seems likely that the study of the connections between estimation theory and inverse scattering problems would be beneficial to both fields: note for example that the Kalman filter could be a valuable tool to solve problems such as those discussed in [25].

Appendix A

Proof of Theorem 1

To show that $g_n(\cdot, \cdot)$ is twice differentiable, we note that since $k(\cdot)$ is

twice differentiable, $\phi_n(\cdot, \cdot)$ is also twice differentiable. Then (2.26) can be rewritten as

$$\frac{1}{2\pi} g_n(R, r) = \phi_n(R, r) - \int_0^R g_n(R, s) \phi_n(s, r) ds \quad (A.1)$$

where the right-hand side of (A.1) is once, then twice differentiable, so that $g_n(\cdot, \cdot)$ is twice differentiable.

Now, apply the displacement operator δ_n to (2.26). By using the displacement property (2.13) of ϕ_n , we obtain

$$\begin{aligned} 0 &= \int_0^R ((\Delta_n(R) g_n(R, s)) \phi_n(s, r) - g_n(R, s) (\Delta_n(r) \phi_n(s, r))) ds \\ &+ \left(\frac{d}{dR} (Rg_n(R, R)) + \frac{\partial}{\partial R} (Rg_n(R, s)) \Big|_{s=R} \right) \phi_n(R, r) \\ &+ Rg_n(R, R) \frac{\partial}{\partial R} \phi_n(R, r) + \frac{1}{2\pi} \delta_n g_n(R, r) \end{aligned} \quad (A.2)$$

By taking again into account the displacement property of ϕ_n , and integrating by parts, one gets

$$\begin{aligned} &\int_0^R g_n(R, s) (\Delta_n(r) \phi_n(s, r)) ds \\ &= \int_0^R (\Delta_n(r) g_n(r, s)) \phi_n(s, r) ds + sg_n(R, s) \frac{\partial}{\partial s} \phi_n(s, r) \Big|_{s=R} \\ &- s \frac{\partial}{\partial s} g_n(R, s) \phi_n(s, r) \Big|_{s=0} \end{aligned}$$

so that (A.2) becomes

$$\begin{aligned} 0 &= \int_0^R (\delta_n g_n(R, s)) \phi_n(s, r) ds + \frac{1}{2\pi} \delta_n g_n(R, r) \\ &- g_n(R) \phi_n(R, r) \end{aligned} \quad (A.3)$$

where $q_n(R)$ is given by (2.28). This shows that $\hat{\delta}_n g_n(R,r)$ is the solution of the Fredholm equation (A.3). This equation is identical to the one that would be obtained by multiplying (2.26) by $q_n(R)$. Therefore, by linearity, and by noting that the solution of (2.26) is unique, we have

$$\hat{\delta}_n g_n(R,r) = q_n(R) g_n(R,r) \quad . \quad (A.4)$$

To obtain the boundary conditions (2.29), use (2.26) and observe the $\phi_n(\cdot, \cdot)$ satisfies some similar conditions.

Proof of Theorem 4

By operating respectively with $\frac{\partial}{\partial r} - \frac{n}{r}$ and $\frac{\partial}{\partial s} + \frac{(n+1)}{s}$ on the integral equations satisfied by g_n and g_{n+1} , we obtain

$$\begin{aligned} \left(\frac{\partial}{\partial r} - \frac{n}{r}\right) \phi_n(r,s) &= \int_0^r \left(\frac{\partial}{\partial r} - \frac{n}{r}\right) g_n(r,u) \hat{\delta}_n(u,s) \, u \, du \\ &+ \frac{1}{2\pi} \left(\frac{\partial}{\partial r} - \frac{n}{r}\right) \phi_n(r,s) + r g_n(r,r) \phi_n(r,s) \end{aligned} \quad (A.5)$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) \phi_{n+1}(r,s) &= \int_0^r g_{n+1}(r,u) \left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) \phi_{n+1}(u,s) \, u \, du \\ &+ \frac{1}{2\pi} \left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) \phi_{n+1}(r,s) \end{aligned} \quad (A.6)$$

Then, if we take into account the property (4.1a) of ϕ_n and ϕ_{n+1} and integrate by parts, we get

$$\begin{aligned} &\int_0^r g_{n+1}(r,u) \left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) \phi_{n+1}(u,s) \, u \, du \\ &= \int_0^r \left(\left(\frac{\partial}{\partial u} + \frac{(n+1)}{u}\right) g_{n+1}(r,u)\right) \phi_n(u,s) \, u \, du - r g_{n+1}(r,r) \phi_n(r,s) \quad . \end{aligned} \quad (A.7)$$

Now, denote

$$c_n(r,s) = \left(\frac{\partial}{\partial r} - \frac{n}{r}\right) c_n(r,s) + \left(\frac{\partial}{\partial s} + \frac{(n+1)}{s}\right) g_{n+1}(r,s) . \quad (A.8)$$

When (A.7) is substituted inside (A.6), by adding the resulting equation to (A.5) we find that $c_n(\cdot, \cdot)$ satisfies the integral equation

$$0 = \int_0^r c_n(r,u) \phi_n(u,s) u du + \frac{1}{2\pi} c_n(r,s) + a_n(r) \phi_n(r,s) \quad (A.9)$$

where $a_n(r)$ is given by (4.4). This equation can be viewed as obtained from (2.26) by multiplication by $-a_n(r)$. Therefore, by linearity, and since the solution of (2.26) is unique, we have

$$c_n(r,s) = -a_n(r) g_n(r,s) . \quad (A.10)$$

The identity (4.3b) can be derived similarly.

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