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SYSTEM OPTIMIZATION BY PERIODIC CONTROL

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Research results obtained under the grant are summarized. Contributions to periodic control include: theory, computational methods and applications to aircraft cruise. The theory centers around necessary or sufficient conditions for optimality and gives information on whether or not periodic operation of a dynamic system gives better performance than steady-state operation. The treatment is comprehensive and includes new second-order conditions which have simplified assumptions and incorporate control constraints. Some of these results follow from a new approach to the derivation of higher-order (CONTINUED)

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ITEM #20, CONTINUED: necessary conditions. The approach does not require normality assumptions and has provided other new results, including second-order necessary conditions in optimal control. A method for computing periodic optima is described. It addresses difficulties observed in other approaches and has proved effective in example problems. Optimal aircraft cruise (specific range, endurance, peak altitude) was studied as an application of theoretical and computational techniques. Under special circumstances (e.g., altitude constraints, low wing loading and drag, high thrust limits), it appears that periodic cruise is significantly better than steady-state cruise. Some research was also done on the theory of nonlinear systems. It includes: functional expansions for input-output maps, conditions for realizability, a backward shift approach to internally bilinear realizations and canonical forms for minimal-order realizations of two-power input-output maps.

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ABSTRACT

Research results obtained under the grant are summarized. Contributions to periodic control include: theory, computational methods and applications to aircraft cruise. The theory centers around necessary or sufficient conditions for optimality and gives information on whether or not periodic operation of a dynamic system gives better performance than steady-state operation. The treatment is comprehensive and includes new second-order conditions which have simplified assumptions and incorporate control constraints. Some of these results follow from a new approach to the derivation of higher-order necessary conditions. The approach doesn't require normality assumptions and has provided other new results, including second-order necessary conditions in optimal control. A method for computing periodic optima is described. It addresses difficulties observed in other approaches and has proved effective in example problems. Optimal aircraft cruise (specific range, endurance, peak altitude) was studied as an application of theoretical and computational techniques. Under special circumstances (e.g., altitude constraints, low wing loading and drag, high thrust limits), it appears that periodic cruise is significantly better than steady-state cruise. Some research was also done on the theory of nonlinear systems. It includes: functional expansions for input-output maps, conditions for realizability, a backward shift approach to internally bilinear realizations and canonical forms for minimal-order realizations of two-power input-output maps.

INTRODUCTION

Many dynamic systems, such as jet engines and aircraft in cruise, are operated in an optimum steady-state mode. Sometimes it is possible to improve the performance of these systems still further by using time-dependent periodic controls. Such improvement is based on the exploitation of system nonlinearities which are only active when the system is in motion. The first examples of improvement were noted in the field of chemical process control. This led to the first paper on optimal periodic control [a] and a subsequent, rapid development of a general theory. See [b, c, d] for surveys of the theoretical results through 1975. Early applications to vehicle and aircraft cruise appear in [e, f, g] .

This report describes research which was carried out over the period October 1, 1976 to January 31, 1982 under AFOSR Grant Number 77-3158. The original objectives of this grant were to study the theory of periodic control and develop methods for the computation of periodic optima. Considerable progress has been made in these directions. In addition, the investigations have led into several other areas. The most important of these are optimization and nonlinear systems. The Bibliography gives a chronological listing of journal articles, conference papers, articles submitted for publication and articles in preparation. In the sections which follow, these contributions are reviewed. The emphasis is on a general description of the results and their relation to applications and prior research. For the items still in preparation, a somewhat more detailed account is given.

Personnel associated with the research effort included: Elmer G.
Gilbert (principal investigator); Arthur E. Frazho (post-doctoral researcher,
1976-1978), Daniel T. Lyons (PhD student, 1976-1980), Dennis S. Bernstein
(PhD student, 1977-1982). The activities carried out by Lyons and Bernstein
contributed significantly to their PhD degrees (see items [15,22] of the
Bibliography). The papers [3,4] received the O. H. Schuck Award of the
American Automatic Control Council for the best contribution at the 1977
Joint Automatic Control Conference. The journal article [6] received
special recognition by the IEEE Control Systems Society.

There have been a number of interactions with the professional and
academic community. Spoken papers include items [1,2,3,4,7,8,11,12,13,
17,18,19] and a presentation by Gilbert at the 1979 Optimization Days
Conference in Montreal. Seminars were presented by Gilbert at the
following universities: Michigan, Minnesota, Purdue, Washington (St. Louis),
Rochester, Johns Hopkins. Several conversations were held with Jason L.
Speyer who has been concerned with aircraft cruise and the computation of
periodic optima. At the December 1981 Conference on Decision and Control
Gilbert participated in an informal meeting of researchers working in
periodic control (J. Speyer, P. Dorato, S. Bittanti, R. Evans, D.
Rodabaugh). Presently, there is a study of periodic aircraft cruise at the
Lockheed-California Company. Researchers there (E. Shapiro and D.
Rodabaugh) have been motivated by work done under this grant [13,15,18],
and are trying to extend it to specific aircraft. They have been in touch
with both Gilbert and Lyons.

THEORY OF PERIODIC CONTROL

A general formulation for an optimal periodic control problem

(OPC) is:

(1) $\dot{x}(t) = f(x(t), u(t))$, $t \in [0, \tau]$,

(2) $u(t) \in U \subset R^m$, $x(0) = x(\tau) \in R^n$, $\tau \in (0, T] \subset R$,

(3) $y = \dfrac{1}{\tau} \displaystyle\int_0^\tau \tilde{f}(x(t), u(t))\, dt \in R^\ell$,

(4) $g_i(y) \leq 0$, $i = -j, \ldots, -1$,

        $g_i(y) = 0$, $i = 1, \ldots, k$,

(5) $J = g_0(y)$.

Here (1) describes the dynamic system and (2) gives the constraints. The

vector y is the average over the period τ of the quantities of interest,

$\tilde{f}(x(t), u(t))$, in the optimization problem. The objective is to minimize J

by choice of $u(\cdot), x(\cdot)$ and τ subject to the additional constraints (4).

The related optimal steady-state problem (OSS) is obtained when

$x(\cdot)$ and $u(\cdot)$ are assumed to be constant:

(6) $0 = f(x, u)$,

(7) $u \in U$, $x \in R^n$,

(8) $y = \tilde{f}(x, u)$.

As in OPC the cost and additional constraints are given by (4) and (5). The

problem OSS is simpler than OPC because it is finite-dimensional. Since OSS

is a specialization of OPC it may happen that OPC yields a lower cost than

the optimal cost in OSS. When this does happen it is said that OPC is

proper.

Much of the early literature (see [b,c,d]) concerns necessary
conditions for optimality in OPC and OSS and tests for proper. The
portion of this research area related to first-order necessary conditions
is brought together in a very general, unified treatment in [1,5] . These
papers include: a precise statement of necessary conditions, a set of tests
for proper (based on the maximum principle and relaxed controls), rela-
tionships between tests for proper and a proof that the set of tests cannot
be made larger. Most of the research for [1,5] was completed before
the grant began.

Some sufficient conditions for steady-state optimality in OPC are
given in [10] . They require that $f(x,u) = Ax + h(u)$ and that $g_i(y)$,
$i = -j, \ldots, k,$ and $\bar{f}(x,u)$ satisfy certain convexity assumptions. Care is
taken to establish close connections to the results of [5] . The sufficient
conditions give some indication of circumstances under which OPC cannot
be proper. For example, OPC cannot be proper if the plant is modelled
by linear differential equations and the cost functional is convex. This
underlines the importance of understanding the effects of nonlinearities in
periodic control problems.

One approach to a fuller understanding of the nonlinearities is
second-order theory. As first shown in [h] this leads to frequency-domain
conditions called Π tests. The paper [14] makes three contributions in
this direction: (1) it gives a Π test for OPC, which is a more general
problem than the one considered in [h] , (2) it shows that "normality"
conditions must be added if the results of [h] are to be valid, (3) it

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explores fully the relationships between necessary and sufficient conditions for optimality in OSS and OPC. With respect to (3), the attack is similar in spirit to the treatment of first-order conditions presented in [5].

The results in [14] have two limitations: they require normality conditions which may be difficult to verify or may not be satisfied, and the control set U must be open. The latter limitation is troublesome in practical applications because optima often occur on the boundaries of closed constraint sets. The investigation of the limitations (mostly by Bernstein) led ultimately to a new theory of higher-order necessary conditions in optimization and optimal control. This theory is reviewed in the next section. A consequence of the theory is a new Π test for proper. This test will be developed fully in [23]; here, only the test itself will be described.

Let the general notation and assumptions in [14] hold. Let U be a convex (not necessarily open) set. Define

(9) $\quad \Gamma = \{(\bar{\lambda}, \bar{\mu}, \bar{\alpha}) : (\bar{\lambda}, \bar{\mu}, \bar{\alpha}) \in R^n \times R^\ell \times R^{j+k+1} \text{ satisfies (10)}\}$,

where

(10) $\quad \bar{H}_x, \bar{H}_y = 0$

$\quad \bar{H}_u u \geq 0, \quad u + \bar{u} \in U$,

$\quad \bar{\alpha}_i \geq 0, \quad i = -j, \ldots, 0$,

$\quad \bar{\alpha}_i \bar{g}_i = 0, \quad i = -j, \ldots, -1$,

$\quad (\bar{\lambda}, \bar{\mu}, \bar{\alpha}) \neq 0$.

Γ is the set of multipliers that satisfy the first-order necessary conditions

for optimality in OSS. Notice that (10) becomes (3.3) in [14] when U is open. Define $\hat{I}$ and $\overline{I}$ by

(11) $\hat{I} = \{1, \ldots, k\}$, $\overline{I} = \{i : i < 0, \overline{g}_i = 0\} \cup \{0, \ldots, k\}$.

Theorem. Assume $(\overline{x}, \overline{u})$ solves OSS. Then Γ is not empty. Let $x \in R^n$, $u \in R^m$, $\eta \in \mathbb{C}^m$ and $\omega \geq \frac{2\pi}{T}$, $\omega \notin \Omega(\overline{A})$ satisfy (3.5) of [14] and

(12) $\overline{u} + u + \operatorname{Re} \eta\, e^{j\omega t} \in U$, $t \in [0, \frac{2\pi}{\omega}]$.

Then if

(13) $y' \overline{H}_{yy} y + x' \overline{H}_{xx} x + 2x' \overline{H}_{xu} u + u' \overline{H}_{uu} u + \eta^* \Pi(\omega) \eta < 0$,
$$\text{for all } (\overline{\lambda}, \overline{\mu}, \overline{\alpha}) \in \Gamma$$

OPC is proper.

A few comments are in order. Because $\overline{H}_{yy}$, $\overline{H}_{xx}$, $\overline{H}_{xu}$, $\overline{H}_{uu}$ and $\Pi(\omega)$ depend on $(\overline{\lambda}, \overline{\mu}, \overline{\alpha})$ the appearance of Γ in (13) is significant. Condition (13) cannot hold if $\eta = 0$ (this is an easy consequence of the second-order necessary conditions for optimality in OSS). If $\overline{u}$ is in the boundary of U , it is usually not possible to satisfy (12) unless $u \neq 0$. If $U = R^m$ (as in [14]) the choice $\eta \neq 0$, $u = 0$, $x = 0$, $y = 0$ is possible and $\Pi(\omega) < 0$ for all $(\overline{\lambda}, \overline{\mu}, \overline{\alpha}) \in \Gamma$ becomes a test for proper. This is like the condition (5.3) in Theorem 5.1 of [14] , except there Γ does not appear. This is because the normality assumption in Theorem 5.1 implies Γ is a single ray with $\overline{\alpha}_o > 0$. Thus by setting $\overline{\alpha}_o = 1$, $(\overline{\lambda}, \overline{\mu}, \overline{\alpha})$ is unique. The appearance of Γ in (13) is the complication which results from the lack of any normality assumption in the above theorem.

When $f(x,u)$ and $\bar{f}(x,u)$ are replaced respectively by $f(t,x,u)$ and $\bar{f}(t,x,u)$, OPC is changed drastically. For example, OSS becomes much more complicated. To maintain periodicity it is necessary to assume that τ is fixed and that $f(t,x,u)$ and $\bar{f}(t,x,u)$ are periodic in t with period τ. An example of this situation is the periodic cruise of a solar-powered aircraft. Then $\tau = 24$ hours and the time-dependence of $f(t,x,u)$ and $\bar{f}(t,x,u)$ accounts for the variation in solar radiation during the day. Little work has been done on such time-dependent systems.

The note [17] concerns a class of mechanical systems which are time-dependent. The objective is to extract energy from the systems by means of periodically-varying applied forces. A characterization of the optimal control is obtained and, when the mechanical system is conservative, it takes on an especially simple form. This leads to a feedback control law for mechanizing a near optimal control. The feedback control is a new idea and appears to have practical advantages. The research will be developed more fully in a future paper.

HIGHER-ORDER NECESSARY CONDITIONS IN OPTIMIZATION

The principal contribution of [20] is a systematic approach to higher-order necessary conditions in optimization theory. The approach is based on a very general, abstract optimization problem: Minimize $\phi_0(e)$ subject to

$$(14) \quad e \in E,$$

(15) $\bar{\phi}(e) \le 0$,

(16) $\psi(e) = 0$,

where: $\mathcal{E}$ is a set, $E \subset \mathcal{E}$, k = positive integer, $\psi : \mathcal{E} \to \mathbb{R}^k$,
$\mathfrak{Z}_0$ and $\bar{\mathfrak{Z}}$ are topological vector spaces, $\phi_0 : \mathcal{E} \to \mathfrak{Z}_0$ and
$\bar{\phi} : \mathcal{E} \to \bar{\mathfrak{Z}}$. Let $Z_0 \subset \mathfrak{Z}_0$ and $\bar{Z} \subset \bar{\mathfrak{Z}}$ be closed convex cones with
nonempty interior such that $Z_0 \ne \mathfrak{Z}_0$ and $\bar{Z} \ne \bar{\mathfrak{Z}}$. If $z, \hat{z} \in \mathfrak{Z}_0$, then
$z \le \hat{z}$ means $z - \hat{z} \in Z_0$. Identical notation applies for $z, \hat{z} \in \bar{\mathfrak{Z}}$.

Necessary conditions are derived for this problem under weak
assumptions on ϕ_0 , $\bar{\phi}$, ψ and E. They take the following form. Let
$\bar{e}$ solve the optimization problem. Denote the set of bounded linear maps
from $\mathfrak{Z} = \mathfrak{Z}_0 \times \bar{\mathfrak{Z}}$ into $\mathbb{R}$ by $\mathfrak{Z}^*$. Then there exists $\ell = (\ell_\phi, \ell_\psi) \in \mathfrak{Z}^* \times \mathbb{R}^k$
such that

(17) $\ell \ne 0$,

(18) $\ell_\phi \in Z^*$, $Z^* = \{ \ell_\phi \in \mathfrak{Z}^* : \ell_\phi(\mathfrak{Z}) \le 0, \mathfrak{Z} \in Z_0 \times \bar{Z} \}$

(19) $\ell_\phi (\hat{\phi}(\bar{e}) + Y) = 0$,

(20) $\ell(h) \ge 0$, $h \in K$.

In these conditions $Y \in Z = Z_0 \times \bar{Z}$ is a generalized critical direction and
$K \subset \mathfrak{Z} \times \mathbb{R}^k$ is a convex set which is used together with a map Θ (defined
on a subset of K into E) to form a representation for E. The necessary
conditions and the required assumptions are in the spirit of those given by
Neustadt [i] , but the results are stronger and involve the (new) concept
of a critical direction.

By specializing E and making assumptions on the differentiability
of ϕ_0 , $\bar{\phi}$ and ψ conditions (17) - (20) lead to necessary conditions of

arbitrary order. In particular, Y and K are related respectively
to the intermediate-order and highest-order terms in a series expansion
of the functions ϕ_o, $\bar{\phi}$ and ψ .

One set of results concerns first- and second-order necessary
conditions when $\mathcal{E}$ is a subset of a vector space and ϕ_o, $\bar{\phi}$ and ψ have
directional differentials. The second-order results generalize necessary
conditions due to Warga (Theorem 2.3 of [j]) in the following ways: The
optimization problem is more general, the hypotheses are weaker and a
normality assumption is not required. Another set of results concerns first- ,
second- and third-order necessary conditions when $\mathcal{E}$ is a Banach space and
ϕ_o, $\bar{\phi}$ and ψ have Frechet derivatives of appropriate order. While similar
conditions of first- and second-order have appeared previously, the third-
order conditions appear to be new.

The lack of normality assumptions in all of the above results means
that the multipliers (e.g., (ℓ_ϕ, ℓ_ψ) in (17) - (20)) are not necessarily
unique. This brings up interesting questions which are explored at length
in [21,22] .

The paper [21] applies the preceding theory to the following problem
in optimal control: minimize $J(x(\cdot),u(\cdot))$ subject to

(21) $J(x(\cdot),u(\cdot)) = \phi_o(x(t_1),x(t_2))$,

(22) $\phi_i(x(t_1),x(t_2)) \leq 0$, $i = 1,\ldots,j$,

(23) $\psi(x(t_1),x(t_2)) = 0$,

(24) $\dot{x}(t) = f(t,x(t),u(t))$, $t \in [t_1,t_2]$.

Here $x(t) \in \mathbb{R}^n$, $u(t) \in U_o \subset \mathbb{R}^m$, $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\phi_i: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\psi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$. Under suitable assumptions on f, ϕ_i and ψ and the class of control functions, second-order necessary conditions for optimality are derived and expressed concisely in terms of Lagrangian and Hamiltonian functions and a multiplier-free characterization of admissible variations. The optimal control problem is more general than problems considered previously [k, ℓ] in the rigorous derivation of second-order necessary conditions. Unlike [k, ℓ] normality assumptions are not required. This leads to an unusual min-max formulation of an accessory minimum problem. A number of questions concerning normality and regularity (uniqueness of multipliers) occur and are examined. The new necessary conditions should be useful in a variety of applications involving optimal control. The generalized II test described in the preceding section is an example.

The necessary conditions of the preceding paragraph are based on "weak" variations. The same approach can be applied to relaxed controls [j]. Under suitable assumptions this gives a new second-order necessary condition for ordinary controls. See Theorem 6.3 of [21].

COMPUTATIONAL METHODS

The computation of solutions of OPC is difficult. This has been observed by Speyer and his coworkers [e, m, n] and by Lyons in his first attempts to optimize aircraft cruise. The computational use of necessary

conditions seems particulary hazardous in view of [m,n] . There it

is shown that the necessary conditions may have rich families of solutions.

Thus, there is a good chance that computational algorithms may yield

a solution of the necessary conditions which is not a solution of OPC.

Another approach is to use a gradient descent algorithm in the space of

controls. But then it is difficult to satisfy the periodicity constraint

$x(0) = x(\tau)$ precisely.

The following approach addresses these difficulties and has proved

to be effective. The state function $x(t)$ is represented by a (vector)

polynomial spline function. The periodicity constraint is satisfied exactly

by making the spline function periodic. This establishes a one-to-one

correspondence between the spline and the values of the spline at its

joints. The differential equation (1) is solved by choice of the control

function $u(t)$. This step requires that f has a special structure so when $x(t)$

is given there exists a $u(t)$ which solves (1). Fortunately, the structure

is often present. For example, suppose $x(t) = (S(t), V(t))$ where

$S(t), V(t) \in \mathbb{R}^{n/2}$ and (1) can be written

(25) $\quad \dot{S}(t) = V(t) , \quad \dot{V}(t) = F(S(t), V(t)) + u(t)$.

Then if $S(t)$ is a twice-differentiable periodic spline the (periodic) control

which solves the differential equation is given by

(26) $\quad u(t) = \ddot{S}(t) - F(S(t), \dot{S}(t))$.

Once $x(t)$ and $u(t)$ are known, the integral in (3) is evaluated by an accurate

quadrature formula such as Simpson's rule. Control constraints and the

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constraints (4) are incorporated by means of penalty functions. The

resulting optimization problem is finite-dimensional because all the data

$(x(t), u(t), y, g_i, y)$ are determined by the values of the spline at its joints.

The approach just described has proved to be effective in applications

such as those in the next section (see [13, 15, 18]). Its advantages include:

exact satisfaction of the periodicity constraint, an exact solution of the

differential equation (1), control of the smoothness of $x(t)$ by selecting the

type of spline and the location of its joints, simple formulas for computing the

penalized cost and its derivatives with respect to the joint values and an

efficient start-up procedures for the minimization algorithm when the

spline has many joints. Simple problems have been solved by minimi-

zation algorithms which use only the values of the penalized cost. More complex

problems with difficult constraints require gradient descent algorithms

and an augmented Lagrangian approach for the penalty terms.

AIRCRAFT CRUISE

For many years it was assumed that fuel-optimal aircraft cruise

was steady-state. This belief was debated in a series of papers in the

early 1970's (see [e, g] for references and comment). Using techniques

from the theory of optimal control, Speyer [e] gave evidence that

steady-state cruise is not necessarily optimal. This evidence was

supported by the work of Gilbert and Parsons [g] who modelled aircraft

cruise as a problem in periodic control. Using the energy-state model

for aircraft motion and the idea of relaxed steady-state control [5, b, c, d],

they showed that time-dependent periodic control increases the specific range for the F-4 aircraft and a class of subsonic aircraft models. Unfortunately, the results of these papers are not totally convincing. In [e] the improvements are small and in [g] the required "chattering" controls can only be approximated by physically realistic controls.

This was the state of affairs when research under the grant began. Since then, considerable progress has been made. Results on the improvement of specific range have appeared in [13,18]. The PhD dissertation [15] includes additional results, some of which will appear in [24]. The following paragraphs give a brief summary of the work which has been done. Significant improvements in cruise performance appear feasible in certain specialized situations.

Three aircraft cruise problems have been studied: maximum specific range, maximum endurance and maximum peak altitude. In all three problems the same point-mass model is used. In it range is the independent variable. The controls are lift and thrust and the state variables are altitude, velocity and flight-path angle. The basic assumptions are: a classical subsonic lift-drag relation, an exponential atmospheric density, constant thrust-specific fuel consumption, limits on altitude and engine thrust. This model captures the essential nonlinearities and dynamics and is much more realistic than the energy-state model. The model is characterized by three nondimensional parameters: $\delta = \frac{1}{2}$ minimum lift-to-drag ratio, β = constant $\times$ (wing loading)$^{-1}$ $\times$ (air density at reference altitude)$^{-1}$, $\overline{T}_m$ = (maximum engine thrust) $\times$ (engine thrust for optimal steady-state cruise at reference altitude)$^{-1}$.

In all three problems both analytical and computational approaches are used. The analytical studies involve simplifications of the model and employ generalizations of methods from periodic control (relaxed-controls, quasi-steady-state controls, frequency response); the objective is to estimate the effects of parameter change and give insight into the mechanisms which cause improvement. In the computational studies optimal periodic trajectories are determined without making any simplifications of the model. A number of solutions for different values of the key parameters are obtained. Thus, performance trends are established.

In the specific range problem a constraint on the maximum altitude is necessary if improvements in cruise are to occur. Under the best conditions ($\delta \leq .05$, $\beta \leq .05$, $\bar{T}_m \geq 8$, reference altitude = maximum altitude) fuel consumption can be reduced by more than 25%. Optimal periodic cruise requires (approximately) a maximum-range glide followed by a transition to a fuel-efficient, full-thrust climb which recovers the altitude lost in the glide. The periodic cruise is better than optimal steady-state cruise because thrusting takes place at high speeds where energy addition is most efficient.

In the endurance problem a constraint on the minimum altitude is essential for improvement. Under the best conditions ($\delta \leq .05$, $\beta \leq .04$, $\bar{T}_m \geq 8$, reference altitude = minimum altitude) fuel consumption can be reduced by more than 32%. The form of the optimal trajectory is (approximately) a minimum-rate-of-descent glide followed by a transition to a maximum-rate-of-ascent climb.

The maximization of peak altitude is not a special case of OPC. Thus ad hoc techniques, somewhat similar to those used for the range and endurance problems, must be used. The results obtained are less complete and additional physical considerations appear. For example, the dynamic pressure at peak altitude must be constrained. Otherwise, aerodynamic control would be lost at peak altitude. An upper bound on the peak altitude is the loft ceiling. Realistic periodic cruise trajectories which approach this altitude have been obtained.

Another, more complex problem in periodic flight was also investigated. Part of the motivation was to test further the computational techniques of the previous section. The problem concerns the motion of a gliding aircraft in a horizontal wind with vertical shear. Observations of bird flight suggest that *it should be possible* (by periodic flight) to extract energy from the wind shear and stay aloft despite drag losses. Analyses [o, p] based on crude models of motion verify this possibility, but computational efforts on realistic models have not been undertaken.

A reasonably good model [o] for flight through wind shear is

(27) $\dot{V} = -VW \sin \gamma \cos \gamma \cos \beta - C_D V^2 - \sin \gamma$

(28) $\dot{\gamma} = W \sin^2 \gamma \cos \beta + C_L V \cos \psi - V^{-1} \cos \gamma$

(29) $\dot{\beta} = W \tan \gamma \sin \beta + C_L V (\cos \gamma)^{-1} \sin \psi$

(30) $C_D = C_{D_o} + K C_L^2$,

where V = normalized wind speed of aircraft, γ = flight path angle,

β = heading angle, C_D = drag coefficient, C_L = lift coefficient, ψ = bank angle, W = wind shear (constant), K = .02 and C_{D_0} = .03. An optimal periodic control problem is formed by defining the cost

$$(31) \qquad J = - \frac{1}{\tau} \int_0^\tau V \sin \gamma \, dt = \text{average rate of descent} \, .$$

If min J < 0 , it is certainly possible to maintain altitude.

The problem (27) - (31) was solved by the method of the previous section. The state variables V, γ, β were specified by periodic cubic splines and the equations of motion (27) - (29) satisfied by selecting the controls C_D , C_L , ψ . The control constraint (30) was implemented by a penalty function approach. The gradient of J with respect to the spline values at the knots was computed; the implementation employed basis splines and was quite efficient. The resulting finite-dimensional problem (dimension = 15 and 30) was solved numerically by a standard quasi-Newton algorithm. Convergence tended to be slow and was probably caused by the penalty function implementation of (30).

The first computations were for a single period (τ = 10) and W = .25, .5 , 1.0. They all gave physically realistic trajectories with J < 0. Later computations with longer periods led to better values for J, but the trajectories were physically unrealistic. Multiple local minima were also discovered. Additional work will be done in the future and reported in a paper.

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NONLINEAR SYSTEMS THEORY

The research on nonlinear systems theory grew out of attempts
to express the periodic solutions of (1) by means of a functional series,
similar in form to a Volterra series. The objective was to obtain new
conditions and tests for optimality which were of order two and higher.
A variational series approach proved to be effective and was instrumental
in developing ideas which helped lead to some of the results outlined in
the preceding sections [14, 20, 21]. The consequences in the area of
nonlinear systems theory are now described. Applications by other
researchers can be found in the literature (see, e.g., [q, r]).

The papers [4, 6] develop the general theory of variational series
and apply it to the derivation of Volterra series and Volterra-like series
for the input-output maps of nonlinear differential systems. There are
a number of advantages over the Carleman technique proposed earlier by
Brockett and Krener (see, e.g., [s]). The approach also extends to
many other types of dynamical systems, including discrete-time systems.

These ideas are carried further in [3, 9]. By observing the
form of the second-order terms in the variational series for the dynamical
system is easy to obtain necessary and sufficient conditions for the
realizability of input-output maps as finite-dimensional, nonlinear dynamical
systems (both continuous-time and discrete-time). The more involved
algebraic approach of the earlier literature (e.g., [t, u]) is avoided.

Another application of [6] is given in [19]. It elaborates and
combines results in earlier conference papers [2, 12]. The structural

representation in [6] is modified to obtain a class of minimal-order realizations for continuous-time, two-power, input-output maps. Further developments describe the entire class of realizations, show how they are connected to the class of realizable symmetric Volterra transfer functions and give a simple canonical form for the (equivalence) class of .minimal-order realizations. A theory of minimal realizations for internally bilinear systems was known before [v] ; but [2] is believed to be the first published result for a general class of nonlinear systems. Crouch [w] has obtained deep results for a general class of nonlinear systems with polynomial input-output maps. For the (special) two-power case, the details in [19] are more general and complete.

In 1976 Frazho had been working on the use of restricted backward shift realizations in the theory of linear systems. He and Gilbert saw that this formalism provided an alternative view for the theory and realization of internally bilinear dynamical systems. Gilbert discusses the formalism and applies it to linear systems in [8] . He also describes how to obtain a simple characterization for the class of partial realizations. In [7, 11, 16] Frazho developes an elegant statement of the theory for the nonlinear problem. W. J. Rugh was aware of this research in its early stages and has incorporated modification of it in his recent book [r] on nonlinear systems theory. Extensions to systems which are internally state-affine have been published recently by Frazho [x] .

SENSITIVITY OF CHARACTERISTIC ROOTS

The research described in this section is not connected to the other research done under the grant. It was motivated by a conversation with Dr. E. Shapiro of the Lockheed-California Company. He observed that, whatever the design approach, good antopilot systems often have an interesting property: the eigenvectors of the linear closed-loop system are (approximately) mutually orthogonal. He wondered if there was some unknown, underlying reason. Gilbert arrived at a possible explanation based on the sensitivity of characteristic roots. A paper on this work is being prepared [25]. The following is a brief review of the main results.

Let A be an n by n real matrix and let $\lambda_i(A)$ be a real characteristic root of A. If $\lambda_i(A)$ is distinct, $\lambda_i(A)$ is a continuously differentiable function of a_{jk}, j, k = 1, ..., n, the elements of A. There are formulas for the n^2 first partial derivatives of $\lambda_i(A)$. Let the n by n matrix of these partial derivatives by given by

$$(32) \qquad S_i = \left[\frac{\partial \lambda_i}{\partial a_{jk}} \right]$$

and let v_i, $w_i \in R^n$ satisfy the conditions (prime denotes transpose)

$$(33) \qquad A v_i = \lambda_i v_i , \quad A' w_i = \lambda_i w_i , \quad v_i' w_i = 1 , \quad v_i' v_i = 1 .$$

Such vectors are known to exist and (Jacobi's formula [y])

$$(34) \qquad S_i = w_i' v_i$$

Since S_i has n^2 elements it is an overly elaborate characterization for measuring root sensitivity in design. If A is not sparce and all of its elements are subject to variations, the (Euclidean) norm of S_i is a good (scalar) measure for root sensitivity. Let it be given by

$$(35) \quad \delta_i = ||S_i|| = \left(\sum_{j,k=1}^{n} (\frac{\partial \lambda_i}{\partial a_{jk}})^2 \right)^{1/2} .$$

How can δ_i be made small by the choice of A? This question comes up in design because the close-loop matrix A depends on choice of the control law.

The minimization of δ_i over the a_{jk} has a simple answer: $\delta_i(A) \geq 1$ for all A and $\delta_i(A) = 1$ if and only if the eigenvector v_i is orthogonal to all other eigenvectors of A. Though the proofs are more involved, similar results have been derived when λ_i is complex ($\lambda_i = \sigma_i + \sqrt{-1}\omega_i$, $\sigma_i \omega_i \in R$). For example,

$$(36) \quad \delta_i^{\sigma} = ||\frac{\partial \sigma_i}{\partial a_{jk}}||$$

satisfies $\delta_i^{\sigma} \geq \frac{1}{\sqrt{2}}$ and $\delta_i^{\sigma} = \frac{1}{\sqrt{2}}$ if and only if the (complex) eigenvectors corresponding to the characteristic roots $\sigma_i \pm \sqrt{-1}\omega_i$ are orthogonal to all other eigenvectors of A. Analgous results have been obtained for the sensitivities of ω_i and $\sqrt{\omega_i^2 + \sigma_i^2}$.

If all root sensitivites are made minimal, the mutual orthogonality of eigenvectors mentioned in the beginning paragraph of this section occurs. This suggests that root sensitivity may have been an underlying objective in the autopilot designs. The above results may lead to a design procedure for multivariable systems when root sensitivity is incorporated directly. For example, the sensitivity requirements could be achieved by state feedback using eigenvector placement techniques [z].

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