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AN ANALYTICAL SOLUTION TO THE PROBLEMS OF THREE-PHONON INTERACTION AND SECOND HARMONIC GENERATION IN A SOLID PLATE

by

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A Green's function approach is used to determine the conditions under which it is possible to generate second harmonics of Lamb waves on solid plates and to have two Lamb waves interact to produce a third phonon (another Lamb mode).

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This Technical Report consists of the Ph.D. thesis of T. S. Chao and deals with nonlinear propagation characteristics of Lamb waves on solid plates. This topic is one aspect of the investigation of ultrasonic reflection from and transmission through solid plates immersed in a liquid. The nonlinear processes described here play an important role whenever the incident ultrasonic beam is of sufficiently high amplitude so that harmonic generation needs to be considered together with the possible generation of new ultrasonic waves on the plate.

The work reported here was supervised by Dr. Joseph A. McClure of this Department. He was the thesis mentor of the author of this Technical Report. The project was supported through Contract N00014-78-C-0584.

Walter G. Mayer

Principal Investigator

August 2, 1982

CHAPTER I

INTRODUCTION

The propagation of mechanical disturbances through solid or fluid media is a nonlinear phenomenon whenever the disturbance, or wave, has a finite amplitude. The term finite amplitude refers to any disturbance where the particle displacement is not infinitesmally small and where the superposition principle no longer holds. One manifestation of finite amplitude effects is the fact that two waves present in one section of volume of the medium will affect one another, i.e., there will be an interaction between them. Two such interactions are the three-phonon interaction and harmonic generation.

In the three-phonon interaction, two noncollinear finite amplitude waves, which we shall call the primary waves, intersect in such directions that they interact to produce a third, or resultant wave. Rollins and Taylor[1] have observed three-phonon interactions for bulk waves in solids which were chosen large enough so that boundary effects could be neglected. Tanski[2] has extended the concept to three-phonon interaction of surface waves on crystals with one defined boundary. A three-phonon interaction was detected by Brower[3] on an isotropic plate with two defined boundaries.

The analytical solution to the problem of the non-linear three-phonon interaction in a bulk isotropic medium has been obtained by Jones and Kobett[4] by a perturbative approach and Green's function techniques. For the surface wave three-phonon interaction on crystals, Tanski[2] has suggested several possible approaches to obtain an approximate solution. The first goal of the present work is to find the solutions for the three-phonon interaction in an isotropic plate with two boundary conditions.

In the harmonic generation interaction, specifically second-harmonic generation, an acoustic wave generates higher harmonics while propagating in a solid medium. Second-harmonic generation has been experimentally observed for bulk waves in an isotropic medium by Viktorov[5], and Løpen[6] has studied second-harmonic generation of acoustic surface waves on crystalline solids. Brower[3] also observed the second-harmonic generation in Lamb modes in an isotropic plate.

Different solutions to the problem of second-harmonic generation in a bulk isotropic medium have been obtained by Hikata[7] and Viktorov[5]. The second-harmonic generation problem on crystals has been solved approximately by

Lapen[6], who used a standard perturbative technique. The second goal of this thesis is to find the solution for second-harmonic generation in an isotropic plate.

In this study a theoretical investigation of the acoustical nonlinear effects for an isotropic plate will be presented. This work will take a new approach for the solution to the nonlinear equations of motion for an isotropic plate. The approach involves a perturbative approximation to the nonlinear equations of motion and a subsequent solution by the Green's function technique. The Green's function is constructed via an eigenfunction expansion. Chapter II contains a review of the three-phonon interaction and second-harmonic generation in a bulk medium and on a crystal surface. Chapter III has three parts, the first of which is a review of Green's function technique. In the second part the orthogonalization relation of eigenfunctions for a plate will be derived. The last section contains the derivations leading to a formal solution for the nonlinear equations of an isotropic plate. This solution for the special cases of the three-phonon interaction and second-harmonic generation will be investigated in Chapter IV along with bounded wave considerations. The last chapter contains the conclusions and suggestions for future work.

CHAPTER II

THEORY

A. Acoustic Nonlinear Equations of Motion

The equations describing the motion of the particle in an isotropic solid may be found in many textbooks[14]. The general equation of motion is

$$\rho_o \frac{\partial^2 U_i}{\partial t^2} = C_{ik\ell m} \frac{\partial W_{\ell m}}{\partial x_k} + \frac{1}{2} C_{ikpqrs} \frac{\partial W_{pq} W_{rs}}{\partial x_k} , \qquad \text{II-1}$$

where ρ_o is the density of the solid; $C_{ik\ell m}$ and C_{ikpqrs} are the second and third order elastic constants, respectively. The quantity $W_{\ell m}$ is the strain tensor and may be expressed as

$$W_{\ell m} = \frac{1}{2} (U_{m\ell} + U_{\ell m} + U_{n\ell} U_{nm})$$

and $$U_{\ell m} = \frac{\partial U_\ell}{\partial x_m} .$$

Here $\vec{U}$ is the displacement vector. The general summation rule of repeated indices is used here and in all that follows. The lowest order nonlinear approximation for Eq. II-1 is obtained when terms up to the second order in $U_{\ell m}$ are kept. The resulting equations of motion is

$$\rho_o \frac{\partial^2 U_i}{\partial t^2} = \frac{1}{2} C_{ik\ell m} \frac{\partial}{\partial x_k} (U_{\ell m} + U_{m\ell} + U_{n\ell} U_{nm}) + \frac{1}{8} C_{ikpqrs} \frac{\partial}{\partial x_k} (U_{pq} U_{rs} + U_{pq} U_{sr} + U_{qp} U_{rs} + U_{qp} U_{sr}). \qquad \text{II-2}$$

The nonlinearity of an elastic medium as expressed by Eq. II-2 gives rise to interactions.

Eq.II-2 may be reduced to

$$\rho_o \frac{\partial^2 U_i}{\partial t^2} - \mu \frac{\partial^2 U_i}{\partial x_k \partial x_k} - (K+\frac{\mu}{3}) \frac{\partial^2 U_\ell}{\partial x_\ell \partial x_i} = (\mu+\frac{A}{4}) (\frac{\partial^2 U_\ell}{\partial x_k \partial x_k} \frac{\partial U_\ell}{\partial x_i} + \frac{\partial^2 U_\ell}{\partial x_k \partial x_k} \frac{\partial U_i}{\partial x_\ell} + 2 \frac{\partial^2 U_i}{\partial x_\ell \partial x_k} \frac{\partial U_\ell}{\partial x_k}) + (K+\frac{\mu}{3}+\frac{A}{4}+B) (\frac{\partial^2 U_\ell}{\partial x_i \partial x_k} \frac{\partial U_\ell}{\partial x_k} + \frac{\partial^2 U_k}{\partial x_k \partial x_k} \frac{\partial U_i}{\partial x_\ell}) + (K-\frac{2}{3}\mu+B)(\frac{\partial^2 U_i}{\partial x_k \partial x_k} \frac{\partial U_\ell}{\partial x_\ell}) + (\frac{A}{4}+B) (\frac{\partial^2 U_k}{\partial x_\ell \partial x_k} \frac{\partial U_\ell}{\partial x_i} + \frac{\partial^2 U_\ell}{\partial x_i \partial x_k} \frac{\partial U_k}{\partial x_\ell}) + (B+2C)(\frac{\partial^2 U_k}{\partial x_i \partial x_k} \frac{\partial U_\ell}{\partial x_\ell}) \qquad \text{II-3}$$

for an isotropic solid, where

$$\mu = \frac{1}{2} C_{ikik}$$

$$K = \frac{1}{3} C_{ikik} + C_{\ell\ell\ell\ell}$$

$$A = \frac{1}{2} C_{iki_{\ell}k_{\ell}}$$

$$B = \frac{1}{6} C_{ikik_{\ell\ell}}$$

$$C = \frac{1}{2} C_{\ell\ell i_{\ell\ell i}} .$$

Here K is the compression modulus, μ is the shear modulus, and A, B, and C are third order elastic constants.

In practical applications a perturbative approach to this equation is used in which U is written as

$$\vec{U} = \vec{U}_o + \vec{U}' \quad . \qquad \text{II-4}$$

Here $\vec{U}_o$ is the solution with the right hand side of Eq.II-3 equal to zero, i.e., the linear homogeneous equation. Thus, when we substitute Eq.II-4 into Eq.II-3, $\vec{U}_o$ disappears from the left hand side leaving only $\vec{U}'$ on the left. On the right hand side since $\vec{U}'$ is small compared to $\vec{U}_o$, it is neglected from terms on the right. So the right hand side of Eq.II-3 contains terms in $\vec{U}_o$ only. Now, the nonlinear partial differential equations are

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reduced to inhomogeneous linear partial differential equations as

$$\rho_o \frac{\partial^2 U_i'}{\partial t^2} - \mu \frac{\partial^2 U_i'}{\partial x_k \partial x_k} - (K+\frac{\mu}{3}) \frac{\partial^2 U_\ell'}{\partial x_\ell \partial x_i} = P_i \qquad \text{II-5}$$

where the vector $\vec{P}$ is acting as the source term for the generated wave. $\vec{P}$ is determined by substituting $\vec{U}_o$ in the right hand side of Eq.II-3.

B. Solving the Acoustic Equations for the Three-phonon Interaction

In this section three-phonon interactions in isotropic bulk media and on crystalline surfaces will be discussed briefly.

1. Acoustic Waves in a Bulk Isotropic Medium

Jones and Kobett[4] have approximately solved the non-linear equations of motion, Eqs.II-5, for a bulk medium using the perturbative approach and a Green's function technique. They take for $\vec{U}_o$

$$\vec{U}_o = \vec{A}_o \cos(\omega_1 t - \vec{k}_1 \cdot \vec{r}) + \vec{B}_o \cos(\omega_2 t - \vec{k}_2 \cdot \vec{r}) \qquad \text{II-6}$$

and this combination of two primary waves generates the three-phonon interaction.

From the homogeneous solutions, which are both shear and longitudinal plane waves, Jones and Kobett construct a Green's function $\overleftrightarrow{G}(\vec{r}, \vec{r}', \omega)$ for Eq.II-5. The solution, after taking the time Fourier transform of Eq.II-5, in the infinite region can be written as

$$\vec{U}'(\vec{r},\omega) = \int_V \overleftrightarrow{G}(\vec{r},\vec{r}',\omega)\, \vec{q}(\vec{r}',\omega)\, dv \qquad \text{II-7}$$

where $\overleftrightarrow{G}$ is a tensor of rank two and consists of longitudinal and transverse Green's function ($\overleftrightarrow{G}_l$, $\overleftrightarrow{G}_t$). $\vec{q}$ in Eq.II-7 is expressed as

$$\vec{q} = \frac{\vec{F}}{-\rho_c} \quad .$$

The solution is quite complicated and the original paper should be consulted for details. However, they find that under a certain resonance condition the solution $\vec{U}'$ is an independently propagating wave. Hence a third-phonon is generated. The resonance condition is a statement of energy and momentum conservation:

$$\omega_1 + \omega_2 = \omega_3 \qquad \text{II-8}$$

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 \quad . \qquad \text{II-9}$$

The resonance condition also gives the limit to the ratio of the primary wave frequencies. In this limit we can choose an angle between the primary waves vectors k_1 and $\vec{k}_2$ so that we get a generated wave in the direction

$$\vec{k}_3 = \vec{k}_1 + \vec{k}_2 \quad .$$

Another important result is that the amplitude of $\vec{U}'$ is proportional to the product of the primary wave amplitudes.

$$\vec{U}' \propto A_o\, B_o \qquad \text{II-10}$$

2. Acoustic Surface Waves on Crystals

The linear equations have been solved by Farnell[8] for an acoustic wave propagating on the surface of single crystal sample. Tanski[2] has suggested several possible approaches for calculating the magnitudes of surface wave interactions. The equations of motion given by Eq.II-2 also describe the nonlinear interaction of surface waves, provided the solutions satisfy a stress-free boundary at $x_3 = 0$,

$$C_{i3\ell m}\,(U_{\ell m}+U_{m\ell}+U_{n\ell}U_{nm})+\frac{1}{2}\,C_{i3pqrs}\,(U_{pq}U_{rs}+U_{pq}U_{sr}+U_{qp}U_{rs}+U_{qp}U_{sr}) = 0\,. \qquad \text{II-11}$$

Several approaches may be taken for solving Eqs.II-2 and II-11 which describe the nonlinear interaction of crystal surface waves. The first two approaches suggested by Tanski[2] both follow the perturbative method used in the isotropic bulk medium case. The third approach assumes that the full solution is composed of three surface waves, two primary and the resultant. The solution is constructed such that the conservation criteria or resonance conditions are satisfied. This gives the important result that the acoustic wave interaction for plane waves in an isotropic bulk medium and surface waves on a crystalline surface have common properties. In addition to obeying the conservation criteria for generation of a third wave, the amplitude of the resulting wave is proportional to the product of the primary amplitudes.

C. Solving the Acoustic Equations for Harmonic Generation

When a sinusoidal acoustic wave of a given frequency and sufficient amplitude is introduced into a nonlinear or anharmonic solid, the fundamental wave distorts as it propagates, so that second and higher harmonics of the fundamental frequency will be generated. In this section second-harmonic generation in an isotropic bulk medium and crystalline surface waves will be discussed briefly.

1. Acoustic Second-harmonic Generation in an Isotropic Bulk Medium

The nonlinear equations for an isotropic bulk solid are given by Eq.II-3. Suppose that a compressional wave (longitudinal wave) is introduced into an isotropic bulk medium. The amplitude $\vec{U}_1$ may be written as

$$\vec{U}_1 = \vec{A}_o \sin(k_1 x - \omega_1 t)$$

and in this case Eq.II-3 reduces to

$$\frac{\partial^2 U_1}{\partial t^2} - \alpha \frac{\partial^2 U_1}{\partial x^2} = \beta \frac{\partial^2 U_1}{\partial x^2} \frac{\partial U_1}{\partial x}$$

where $\alpha = K + \frac{4}{3}$

$\beta = 4\mu + 2A + 3K + 6B + 2C$

This nonlinear differential equation has been solved approximately by Hikata et al[7]. They show that the amplitude of the second-harmonic, A_2, is

$$A_2 = \frac{1}{8} \frac{\beta}{\alpha} (A_o k_1)^2 x \quad , \qquad \text{II-12}$$

where x is the distance propagated by the acoustic wave.

Viktorov[5] also investigates second-harmonic generation in a bulk medium. Differentiating Eq.II-6 with respect to x, y, and z, then adding these equations, he obtains

$$\rho_o \frac{\partial^2(\vec{\nabla}\cdot\vec{U}')}{\partial t^2} - (K+\tfrac{4}{3}\mu)\, \nabla^2(\vec{\nabla}\cdot\vec{U}') = \vec{\nabla}\cdot\vec{P} \qquad \text{II-13}$$

Similary, he also derives the equation

$$\rho_o \frac{\partial^2(\vec{\nabla}x\vec{U}')}{\partial t^2} - \mu\nabla^2\, (\vec{\nabla}x\vec{U}') = \vec{\nabla}x\vec{P}. \qquad \text{II-14}$$

Now we know that Eqs.II-13 and II-14 characterize second-harmonic generation waves. They distinguish the longitudinal and transverse secondary waves, and thus they may be used to analyze the possibility of longitudinal and transverse waves growing.

An analysis of second-harmonic generation in an isotropic bulk medium reveals three important results:

(1). A necessary condition for acoustic second-harmonic generation is that no dispersion of phase velocity is permissible. The velocities of the primary and secondary waves must be the same; that is

$$V_2(2\omega_1) = V_1(\omega_1) \quad . \qquad \text{II-15}$$

where V_2 is the velocity of the second harmonic. (2). If a secondary plane wave is generated and $V_2 = V_1$, then the amplitude of the second-harmonic wave is

$$A_2 \propto A_o^2 x \quad . \qquad \text{II-16}$$

(3). If a secondary plane wave does not meet the necessary condition, (Eq.II-15), the amplitude will remain small.

2. Acoustic Surface Wave Second-harmonic Generation in Crystals

The linear equation, Eq.II-2, with right hand side terms neglected is solved for surface waves in crystals by Verevkina et al[9]. The solutions are subject to stress-free surface boundary condition of an anisotropic medium. The nonlinear equations of motion have been solved approximately by Løpen[6], using the perturbative approach discussed in Sec.II B.1. From the solution we know that the amplitude of the second-harmonic displacement vector is proportional to the interaction length and the square of the primary amplitude. The necessary conditions, Eqs.II-15, II-16, for acoustic second-harmonic generation to occur hold in this case also.

CHAPTER III

SOLVING THE ACOUSTIC EQUATIONS FOR AN ISOTROPIC PLATE

In the preceding chapter, we have reviewed the acoustic nonlinear interactions in a bulk medium and on a crystal surface. We will devote this chapter to solving the nonlinear equations of motion for an isotropic solid plate, a more complicated case, through a perturbative approximation method and Green's function Technique.

The basic equation of motion for the isotropic plate is given by Eq.II-3. In practical applications the displacement vector, $\vec{U}$, is small and the nonlinear terms in $\vec{U}$ will be considerably smaller. Thus, a perturbative approach may be used to solv Eq.II-3. We set

$$\vec{U} = \vec{U}_o + \vec{U}' \quad , \qquad \text{III-1}$$

where $\vec{U}_o$ is the solution with the nonlinear terms or right hand side of Eq.II-3 set equal to zero. $\vec{U}'$ is considered a small correction due to the right hand side of Eq.II-3. Both $\vec{U}_o$ and $\vec{U}'$ are subject to boundary constraints.

A. The Lamb Modes

Solutions $\vec{U}_o$ to the homogeneous linear equations of motion for an isotropic solid plate with stress-free boundary conditions are Lamb (plate) modes conventionally denoted $\vec{L}_n$. It is convenient to describe them with a coordinate system as shown in Fig. 1, where the plate surfaces are at $z = \pm d$. The plate displacement amplitudes, $\vec{L}_n$, may be symmetric or antisymmetric with respect to the plane z = 0 (median plane), shown in Fig. 2. $\vec{L}_n$ for the symmetric case is given by

$$\vec{L}_{ns} = Ak_{ns} \left(\frac{\cosh q_{ns} z}{\sinh q_{ns} d} - \frac{2q_{ns} s_{ns}}{k_{ns}^2 - s_{ns}^2} \frac{\cosh s_{ns} z}{\sinh s_{ns} d} \right) \exp(i) (k_{ns} x - \omega t - \frac{\pi}{2}) \hat{x} - Aq_{ns} \left(\frac{\sinh q_{ns} z}{\sinh q_{ns} d} - \frac{2k_{ns}^2}{k_{ns}^2 + s_{ns}^2} \frac{\sinh s_{ns} z}{\sinh s_{ns} d} \right) \exp i(k_{ns} x - \omega t) \hat{z} \qquad \text{III-2a}$$

An antisymmetric plate mode is described by

$$\vec{L}_{na} = Bk_{na}\left(\frac{\sinh q_{na}z}{\cosh q_{na}d} - \frac{2q_{na}s_{na}}{k_{na}^2 + s_{na}^2}\,\frac{\sinh s_{na}z}{\cosh s_{na}d}\right)\exp i\left(k_{na}x-\omega t-\frac{\pi}{2}\right)\hat{x} - Bq_{na}\left(\frac{\cosh q_{na}z}{\cosh q_{na}d} - \frac{2k_{na}^2}{k_{na}^2 + s_{na}^2}\,\frac{\cosh s_{na}z}{\cosh s_{na}d}\right)\exp i(k_{na}x-\omega t)\,\hat{z} \qquad \text{III-2b}$$

where A and B are arbitary constants. $q_{ns,na}$ is given by

$$q_{ns,na} = \sqrt{k_{ns,na}^2 - k_\ell^2}$$

where k_ℓ is the bulk longitudinal wave number; and $s_{ns,na}$ is given by

$$s_{ns,na} = \sqrt{k_{ns,na}^2 - k_t^2}$$

where k_t is the bulk shear wave number. The $k_{ns,na}$ are the wave numbers of the symmetric and antisymmetric Lamb modes, respectively. For an isotropic plate, the solutions to Eq.II-3 are subject to two boundary constraints; both plate surfaces are to be stress-free. The stress conditions are

$$\sigma_{zz} \equiv \sigma_{33} = 0$$

$$\sigma_{xz} \equiv \sigma_{13} = 0$$

on the planes $z = \pm d$. In the linear approximation these stress-free conditions reduce to

$$\sigma_{zz} = (\lambda + 2\mu) \frac{\partial U_z}{\partial z} + \frac{\partial U_x}{\partial x} = 0$$

$$\sigma_{xz} = \frac{\mu}{2} \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right) = 0 \quad .$$

The particular modes that can exist at a frequency f in a plate of given thickness d are determined by these boundary conditions. In Fig. 3, a graph of variations in phase velocity as a function of fd for symmetric and antisymmetric Lamb modes in vaccum for an unloaded brass plate is shown.

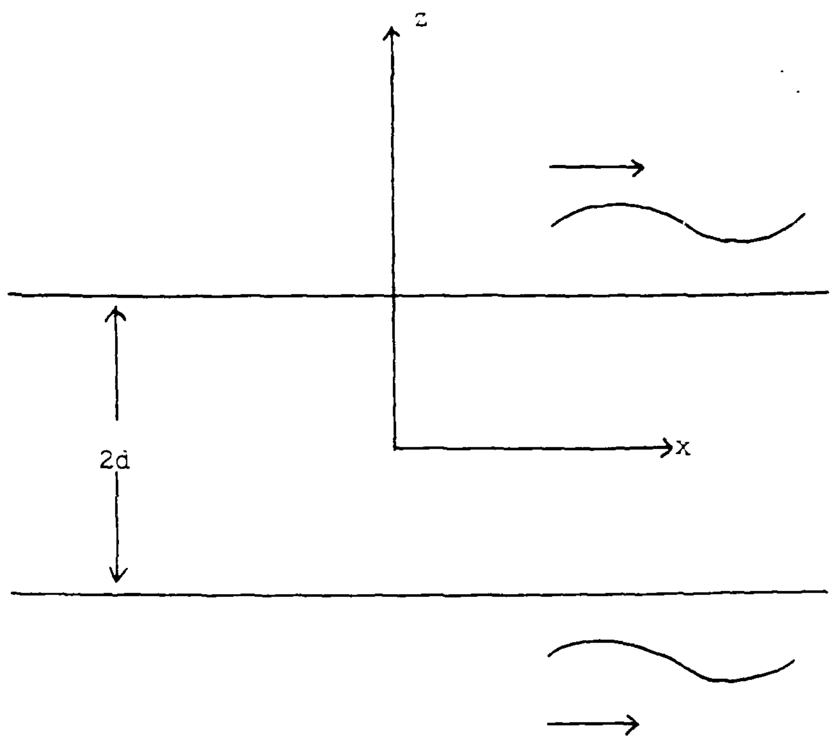


Figure 1. The coordinate system for Lamb(Plate) mode, 2d is the thickness of the plate.

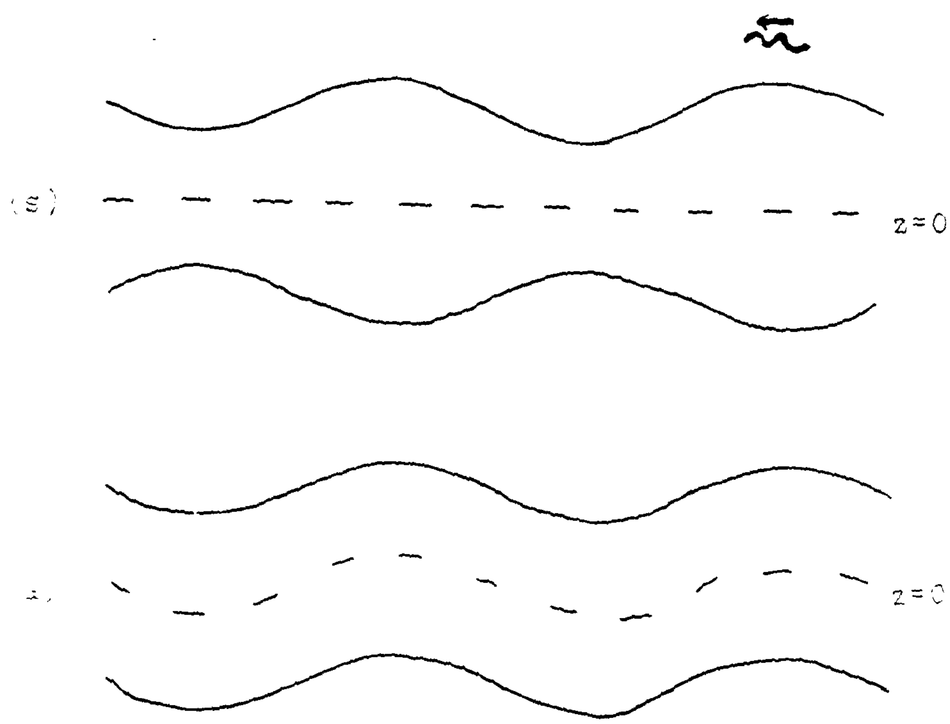


Figure 2. Plate deformations for symmetric (s) and antisymmetric (a) Lamb modes.

B. The Nonlinear Source Term

Substituting Eq.III-1 into Eq.II-3, we get

$$\rho_o \frac{\partial^2 (U_{oi}+U'_i)}{\partial t^2} - \mu \frac{\partial^2 (U_{oi}+U'_i)}{\partial x_k \partial x_k} - (K+\frac{\mu}{3}) \frac{\partial^2 (U_{o\ell}+U'_\ell)}{\partial x_\ell \partial x_i} = F_i, \qquad \text{III-3}$$

where $\overrightarrow{F_i}$ represents the right hand side terms of Eq.II-3. Neglecting $\overrightarrow{U'}$ on the right hand side of Eq.III-3, we obtain a set of linear inhomogeneous equations for $\overrightarrow{U'}$, i.e.

$$\rho_o \frac{\partial^2 U'_i}{\partial t^2} - \mu \frac{\partial^2 U'_i}{\partial x_k \partial x_k} - (K+\frac{\mu}{3}) \frac{\partial^2 U'_\ell}{\partial x_\ell \partial x_i} = F_{io} \quad . \qquad \text{III-4}$$

Here $\overrightarrow{F_{io}}$, which we will refer to as the source term, depends only on U_o.

$$F_{io} = (\mu+\frac{A}{4})(\frac{\partial^2 U_{o\ell}}{\partial x_k \partial x_k} \frac{\partial U_{o\ell}}{\partial x_i} + \frac{\partial^2 U_{o\ell}}{\partial x_k \partial x_k} \frac{\partial U_{oi}}{\partial x_\ell} + 2 \frac{\partial^2 U_{oi}}{\partial x_\ell \partial x_k} \frac{\partial U_{o\ell}}{\partial x_k}) + (K+$$

$$\frac{\mu}{3}+\frac{A}{4}+B)(\frac{\partial^2 U_{o\ell}}{\partial x_i \partial x_k} \frac{\partial U_{o\ell}}{\partial x_k} + \frac{\partial^2 U_{ok}}{\partial x_k \partial x_k} \frac{\partial U_{oi}}{\partial x_\ell}) + (K - \frac{2}{3}\mu + B)(\frac{\partial^2 U_{oi}}{\partial x_k \partial x_k} \frac{\partial U_{o\ell}}{\partial x_\ell}) +$$

$$(\frac{A}{4}+B)(\frac{\partial^2 U_{ok}}{\partial x_\ell \partial x_k} \frac{\partial U_{o\ell}}{\partial x_i} + \frac{\partial^2 U_{o\ell}}{\partial x_i \partial x_k} \frac{\partial U_{ok}}{\partial x_\ell}) + (B+2C)(\frac{\partial^2 U_{ok}}{\partial x_i \partial x_k} \frac{\partial U_{o\ell}}{\partial x_\ell})$$

To investigate the three-phonon interaction, two primary waves must be present. Hence $\vec{U}_c$ is composed of two initial Lamb waves,

$$\vec{U}_c = A_o \vec{L}_1 + B_o \vec{L}_2 \quad , \qquad \text{III-5}$$

where A_o and B_o are constants. For second-harmonic generation only one wave is present initially and U_o takes the form

$$\vec{U}_o = A_o \vec{L} \quad . \qquad \text{III-6}$$

Comparing Eq.III-6 to Eq.III-5, we see that second-harmonic generation is a special case of three-phonon interaction.

Now the source term $\vec{P}_{io}$ is composed of a sum of products of the two primary waves in the three-phonon interaction problem. In the second-harmonic generation case, $\vec{P}_{io}$ is composed of a sum of products of the initial wave with itself. The inhomogeneous plate equations, Eq. III-4, may be written in vector form as

$$\frac{\partial^2 \vec{U}'(\vec{r},t)}{\partial t^2} - C_\ell^2 \vec{\nabla}\left(\vec{\nabla}\cdot\vec{U}'(\vec{r},t)\right) + C_t^2 \vec{\nabla}x \vec{\nabla}x\vec{U}'(\vec{r},t) = 4\pi\vec{q}, \qquad \text{III-7}$$

where $4\pi\vec{q} = \vec{F}_{1o}/\rho_o$. The shear and longitudinal wave velocities are given by $C_t = \sqrt{\mu/\rho_o}$ and $C_l = \sqrt{(K+\frac{4}{3})/\rho_o}$, respectively.

Taking the time Fourier transform of Eq.III-7 we obtain

$$-\omega^2 \vec{U}'(r,\omega)-C_l^2\vec{\nabla}\left[\vec{\nabla}\cdot\vec{U}'(\vec{r},\omega)\right]+C_t^2\,\vec{\nabla}x\vec{\nabla}x\vec{U}'(\vec{r},\omega)=4\pi\vec{q}(\vec{r},\omega), \qquad \text{III-8}$$

where the Fourier transforms are

$$\vec{U}(\vec{r},\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\,\vec{U}(\vec{r},t)\,dt$$

and $$\vec{U}(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,U(\vec{r},\omega)\,d\omega \quad .$$

Using a Green's function method the solution to Eq. III-8 may be expressed as

$$\vec{U}'(\vec{r},\omega) = \int_V \overleftrightarrow{G}(\vec{r},\vec{r}',\omega)\,\frac{\vec{F}_{1o}}{4\pi\rho_o}\,dv \qquad \text{III-9}$$

Here the form of Green's function is dependent upon the boundary conditions. Two possible methods for constructing Green's function are an eigenfunction expansion and the method of images. Vivoli and Filippi[10] have obtained

$\vec{\vec{G}}$ for thin plates using the image technique. The main contribution of this thesis is to construct a Green's function for an arbitrary plate by using the eigenfunction expansion method.

C. Green's Function Technique by Eigenfunction Expansion Representation

As a brief introduction to the eigenfunction expansion representation of the Green's function, consider O, a linear differential operator with a complete orthonormal set of eigenfunctions, $\{\vec{y}_n(\vec{r})\}$, where

$$\hat{O}\,\vec{y}_n(\vec{r}) = \lambda_n\,\vec{y}_n(\vec{r}) \quad .$$

Suppose we are given the inhomogeneous equation

$$\hat{O}\,\vec{v}(\vec{r}) - \lambda\,\vec{v}(\vec{r}) = \vec{w}(\vec{r}) \,, \qquad \text{III-10}$$

where $\vec{v}$ is a vector to be determined, $\vec{w}$ is a known vector, and λ is a constant not equal to λ_n for any n. Writting

$$\vec{v} = \sum_n v_n\,\vec{y}_n$$

and
$$\hat{O}\,\vec{v} = \sum_n v_n\,\lambda_n\,\vec{y}_n$$

equation III-10 becomes

$$\sum_n v_n\,(\lambda_n - \lambda)\,\vec{y}_n = \vec{w} \quad .$$

Taking the scalar product of both sides with $\vec{y}_m$, we obtain

$$v_m(\lambda_m - \lambda) = \int \vec{y}_m^{\,*}(\vec{r}\,') \cdot \vec{w}(\vec{r}\,')\, d\vec{r}\,'$$

This gives

$$v_m = \frac{\int \vec{y}_m^{\,*}(\vec{r}\,') \cdot \vec{w}(\vec{r}\,')\, d\vec{r}\,'}{\lambda_m - \lambda}$$

and

$$\vec{v} = \sum_n \frac{\vec{y}_n(\vec{r}) \int \vec{y}_n^{\,*}(\vec{r}\,') \cdot \vec{w}(\vec{r}\,')\, d\vec{r}\,'}{\lambda_n - \lambda}$$

$$= \int \sum_n \frac{\vec{y}_n(\vec{r})\, \vec{y}_n^{\,*}(\vec{r}\,') \cdot \vec{w}(\vec{r}\,')\, d\vec{r}\,'}{\lambda_n - \lambda}$$

$$= \int \overleftrightarrow{G}(\vec{r},\vec{r}\,') \cdot \vec{w}(\vec{r}\,')\, d\vec{r}\,'$$

Thus the Green's function for the differential equation III-10 is

$$\overleftrightarrow{G}(\vec{r},\vec{r}\,') = \sum_{\lambda \neq \lambda_n} \frac{\vec{y}_n(\vec{r})\, \vec{y}_n^{\,*}(\vec{r}\,')}{\lambda_n - \lambda} \qquad \text{III-11}$$

It is important to note that a necessary condition for the eigenfunction expansion representation of the Green's function is that the set of eigenfunctions $\{\vec{y}_n\}$ be both orthogonal and complete.

For an isotropic plate, the solutions to the homogeneous differential equations of motion, Eq.III-8 with $\vec{q}(\vec{r},\omega)$ set to zero, with stress-free boundary conditions are the Lamb modes of Eq.III-2. To obtain a Green's function, Eq.III-11, for Eq.III-4 one needs to consider the properties of orthonormality and completeness of these Lamb modes, $\{\vec{L}_n\}$. Both these properties present difficulties. The key problem is that the Lamb modes are not orthogonal. However, an orthogonality relation for the Lamb modes has been obtained. A derivation of this result is given in Section D.

The second difficulty is a serious one and yet to be solved. For stress-free constraints completeness has to be assumed and is not yet provable. Various authors[11] have in the past assumed completeness of the eigenfunctions (plate modes) with very good results which are in agreement with experiments. Therefore, completeness will also be assumed in this work. A discussion of completeness is given in Section D also.

I. Eigenfunction Orthogonalization and Completeness of the Lamb Modes

In this section, following a derivation of Fraser's[12], we will find orthogonality relations and discuss completeness for the Lamb modes with various boundary conditions. Fraser, using earlier work by Fama[15] on elastic vibration of a circular cylinder, obtained these orthogonality relations by considering certain adjoint differential equations. The properties of such equations have been studied in detail by Naimark[13]. The orthogonalized plate eigenfunctions which we will thus obtain are solutions of the homogeneous equations III-14 and III-15 and will be used to construct a Green's function for the inhomogeneous nonlinear equation II-3. First we must cast the plate equations in a suitable eigenfunction form.

1. The Eigenvalue Equations

Consider a plate bounded by surfaces $z = \pm d$, and assume the motions are independent of the y coordinate and that the y component of the displacement is equal to zero. Let $\bar{x}$ and $\bar{z}$ be dimensional variables, we introduce dimensionless variables

$$t = \frac{C_\ell \bar{t}}{d}, \quad (x,z) = \frac{(\bar{x},\bar{z})}{d}, \text{ and } \quad (U,W) = \frac{(\bar{U},\bar{W})}{h} \qquad \text{III-12}$$

where C_ℓ is the velocity of dilatation waves in the infinite medium, h is the average displacement amplitude, and U, W are displacements in x, z coordinates, respectively. We may also expand all dependent variables in the form

$$\begin{bmatrix} U(x, z, t) \\ W(x, z, t) \end{bmatrix} = \sum_n a_n \begin{bmatrix} U_n(z) \\ W_n(z) \end{bmatrix} e^{i(k_n x - \omega t)} \qquad \text{III-13}$$

where ω is the angular frequency of vibration of the plate, and the wave number k_n is the eigenvalue for the eigenfunctions $U_n(z)$, $W(z)$. Now we may derive the adjoint differential equations to obtain the orthogonality relation for the Lamb modes.

The homogeneous displacement equations of motion, Eq. II-3 with right hand side equal to zero, in components form are:

x component:

$$\rho_0 \frac{\partial^2 \bar{U}}{\partial \bar{t}^2} - \mu \left(\frac{\partial^2 \bar{U}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{U}}{\partial \bar{z}^2}\right) - \left(K + \frac{\mu}{3}\right) \left(\frac{\partial^2 \bar{U}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{W}}{\partial \bar{z} \partial \bar{x}}\right) = 0 \qquad \text{III-14}$$

z component:

$$\rho_o \frac{\partial^2 \overline{W}}{\partial \overline{t}^2} - \mu \left(\frac{\partial^2 \overline{W}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{W}}{\partial \overline{z}^2}\right) - \left(K+\frac{\mu}{3}\right) \left(\frac{\partial^2 \overline{U}}{\partial \overline{x} \partial \overline{z}} + \frac{\partial^2 \overline{W}}{\partial \overline{z}^2}\right) = 0 \qquad \text{III-15}$$

Substituting Eq.III-12 into Eq.III-14, we obtain

$$\rho_o \frac{\partial^2 (hU)}{\partial \left(\frac{dt}{C_\ell}\right)^2} - \mu\left\{\frac{\partial^2 (hU)}{\partial (dx)^2} + \frac{\partial^2 (hU)}{\partial (dz)^2}\right\} - \left(K+\frac{\mu}{3}\right)\left\{\frac{\partial^2 (hU)}{\partial (dx)^2} + \frac{\partial^2 (hW)}{\partial (dz)\partial (dx)}\right\} = 0 \quad .$$

This equation may be rewritten as

$$\rho_o C_\ell^2 \frac{\partial^2 U}{\partial t^2} - \mu \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right) - \left(K+\frac{\mu}{3}\right) \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 W}{\partial z \partial x}\right) = 0 . \qquad \text{III-16}$$

Substituting Eq.III-13 into Eq.III-16, we obtain

$$\sum_n a_n \left\{\rho_o C_\ell^2 U_n(z)(-i\omega)^2 - \mu \left[U_n(z)(ik_n)^2 + \frac{\partial^2 U_n}{\partial z^2}\right] - \left(K+\frac{\mu}{3}\right) \left[U_n(z)(ik_n)^2 + \frac{\partial W}{\partial z} (ik_n)\right] \right\} e^{i(k_n x-\omega t)} = 0 \quad .$$

This equation will be satisfied provided the factor in brackets vanishes for each n, that is,

$$\rho_c C_\ell^2 (-\omega^2) U_n(z) - \mu\{U_n(z)(-k_n^2) + \frac{\partial^2 U_n}{\partial z^2}\} - (K+\frac{\mu}{3})$$

$$\{U_n(z)(-k_n^2) + \frac{\partial W}{\partial z} ik_n\} = 0 \quad ,$$

which may be rewritten as

$$\frac{1-c^2}{c^2} ik_n \Delta_n + \frac{\partial^2 U_n}{\partial z^2} + \frac{\omega^2}{c^2} U_n = k_n^2 U_n \quad . \qquad \text{III-17}$$

Using the same process, Eq.III-15 can be reduced to

$$\frac{1-c^2}{c^2} \frac{\partial \Delta_n}{\partial z} + \frac{\partial^2 W_n}{\partial z^2} + \frac{\omega^2}{c^2} W_n = k_n^2 W_n \quad , \qquad \text{III-18}$$

where $c = {}^{C}t/C_\ell$.

In these equations we have introduced the dilatation

$$\Delta_n = \frac{d\bar{\Delta}_n}{h} = ik_n U_n + \frac{\partial W_n}{\partial z} \qquad \text{III-19}$$

and we will also have need of the nonzero component of the rotation vector

$$\Omega_n = \frac{d\bar{\Omega}_n}{h} = ik_n W_n - \frac{\partial U_n}{\partial z} \quad . \qquad \text{III-20}$$

A useful equation for Ω_r can be obtained by taking the derivative with respect to z and changing sign of Eq.III-17, multiplying ik_n to Eq.III-18, then add those derived equations, we obtain

$$\frac{\partial^2 \Omega_n}{\partial z^2} + \left(\frac{\omega^2}{c^2} - k_n^2 \right) \Omega_n = 0 \quad . \qquad \text{III-21}$$

Similarly, we obtain for Δ_n

$$\frac{\partial^2 \Delta_n}{\partial z^2} + \left(\omega^2 - k_n^2 \right) \Delta_n = 0 \qquad \text{III-22}$$

We wish to consider the vibrations of the plate as an eigenvalue problem. However the difficulty with Eqs. III-17 and III-18 is that both k_n and k_n^2 appear in Eqs. III-17, III-19, and III-20. To get around this difficulty instead using U_n and W_n as dependent variables, Fraser[12] chose either W_n and Δ_n, or U_n and Ω_n. He thus obtains two pairs of equations in which the eigenvalue k_n^2 appear linearly and further, these equations are adjoint to each other. Take Eqs.III-18 and III-21 as they stand as the (W_n, Δ_n) equations. Using Eq.III-20 to eliminate W_n from Eq.III-19 then we obtain an expression for Δ_n in terms of Ω_n and U_n, which, when substituted into Eq.III-17 gives us

$$(1-c^2)\,\frac{\partial\Omega_n}{\partial z} + \frac{\partial^2 U_n}{\partial z^2} + \omega^2 U_n = k_n^2 U_n \qquad . \qquad \text{III-23}$$

The (U_n, Ω_n) equations are given by Eqs.III-22 and III-23.

In order to express Eqs.III-18, 22 and Eqs.III-21, 23 in eigenvalue form, we define the following two component vectors:

$$\vec{R}_n = \begin{bmatrix} W_n \\ \dfrac{-\Delta_n}{c^2} \end{bmatrix} \quad , \quad \vec{Q}_n = \begin{bmatrix} \Omega_n \\ U_n \end{bmatrix} \quad , \qquad \text{III-24}$$

and

$$\hat{L}(\vec{R}_n) = \begin{bmatrix} (1-c^2)\,\dfrac{\partial}{\partial z}\left(\dfrac{\Delta_n}{c^2}\right) + \left(\dfrac{\partial^2 W_n}{\partial z^2}\right) + \dfrac{\omega^2}{c^2} W_n \\ -\dfrac{\partial^2}{\partial z^2}\left(\dfrac{\Delta_n}{c^2}\right) - \omega^2 \dfrac{\Delta_n}{c^2} \end{bmatrix} , \qquad \text{III-25}$$

$$\hat{M}(\vec{Q}_n) = \begin{bmatrix} \dfrac{\partial^2 \Omega_n}{\partial z^2} + \dfrac{\omega^2}{c^2}\,\Omega_n \\ (1-c^2)\,\dfrac{\partial\Omega_n}{\partial z} + \dfrac{\partial^2 U_n}{\partial z^2} + \omega^2 U_n \end{bmatrix} . \qquad \text{III-26}$$

Eqs.III-18, 21, 22, and 23 may then be compactly expressed as

$$\hat{L}(\vec{R}_n) = k_n^2 \vec{R}_n \quad , \qquad \text{III-27}$$

$$\hat{M}(\vec{Q}_n) = k_n^2 \vec{Q}_n \quad . \qquad \text{III-28}$$

These equations are adjoints of one another according to the definition of Naimark[13].

2. Orthogonality Relations for Different Boundary Conditions

An equation useful in considering orthogonality relations can be obtain by multiplying Eq.III-27 by $\vec{Q}_n$ and Eq.III-28 by R_n.

$$(k_m^2 - k_n^2) \int_{-1}^{+1} \vec{R}_m \cdot \vec{Q}_n \, dz = \int_{-1}^{1} \{\hat{L}(\vec{R}_m)\cdot\vec{Q}_n - \hat{M}(\vec{Q}_n)\cdot\vec{R}_m\} \, dz$$

$$= \Big[\text{boundary terms}\Big]_{-1}^{1} \quad . \qquad \text{III-29}$$

Here

$$\Big[\text{boundary terms}\Big]_{-1}^{1} = \Big|(1-c^2)\,\Omega_n\,\left(\frac{\Delta_m}{c^2}\right) + \Omega_n \frac{\partial W_n}{\partial z} - W_m \frac{\partial \Omega_n}{\partial z}$$

$$-U_n \frac{\partial}{\partial z}\left(\frac{\Delta_m}{c^2}\right) + \frac{\partial U_n}{\partial z}\left(\frac{\Delta_m}{c^2}\right)\Big|_{-1}^{1}$$

are easily obtained by partial integration. When these boundary terms vanish, a conventional orthogonality relation results

$$\int_{-1}^{1} \vec{R}_m \cdot \vec{Q}_n \, dz = \int_{-1}^{1} \left(W_m \Omega_n - \frac{1}{c^2}\Delta_m U_n\right) dz = 0 \qquad \text{III-30}$$

$$\text{for } n \neq m$$

To obtain an orthogonality relation for the stress-free plate, of particular interest here, it is useful to

Introduce the stresses

$$\sigma_{ij}(x,z,t) = \frac{\bar{\sigma}_{ij}}{n\rho c_T^2} = \sum_n a_n \sigma_{nij}(z)\, e^{i(k_n x-\omega t)}$$

where ρ is the density of the plate material. In terms of Δ_n, Ω_n, U_n and W_n, these are either

$$\sigma_{xxn} = \Delta_n - 2c^2 \frac{\partial W_n}{\partial z} \qquad \text{III-31}$$

$$\sigma_{xzn} = c^2 (ik_n)^{-1} \left[c^{-2} \frac{\partial \Delta_n}{\partial z} - (2k_n^2 - \frac{\omega^2}{c^2})\, W_n \right] \qquad \text{III-32}$$

$$\sigma_{zzn} = (1-2c^2)\, \Delta_n + 2c^2 \frac{\partial W_n}{\partial z} \qquad \text{III-33}$$

or

$$\sigma_{xxn} = c^2 (ik_n)^{-1} \left\{ (1-2c^2) \frac{\partial \Omega_n}{\partial z} + \left[(1-2c^2) \frac{\omega^2}{c^2} - 2k_n^2 \right] U_n \right\} \qquad \text{III-34}$$

$$\sigma_{xzn} = c^2 \left(\Omega_n + 2 \frac{\partial U_n}{\partial z} \right) \qquad \text{III-35}$$

$$\sigma_{zzn} = c^2 (ik_n)^{-1} \left[\frac{\partial \Omega_n}{\partial z} + (2k_n^2 - \frac{\omega^2}{c^2})\, U_n \right] \qquad \text{III-36}$$

To find an orthogonality relation in terms of stress, we use Eqs.III-31 and III-35 to eliminate Δ_m and Ω_n from Eq.III-30.

$$\int_{-1}^{1} \vec{F}_m \cdot \vec{u}_n \, dz = \int_{-1}^{1} \frac{1}{2}(W_m \sigma_{xzn} - \sigma_{xxm} U_n)\, dz + 2 \left[W_m U_n\right]_{-1}^{1}. \quad \text{III-37}$$

Comparison with Eq.III-29 shoes that

$$\int_{-1}^{1} (W_m \sigma_{xzn} - \sigma_{xxm} U_n)\, dz = 0 \quad \text{for } n \neq m \qquad \text{III-38}$$

whenever

$$\left[\text{boundary terms}\right]_{-1}^{1} = 2\,(k_n^2 - k_m^2)\left[W_m U_n\right]_{-1}^{1}$$

III-39

Eq.III-38 is an important result and will enable us to carry through a formal solution of the inhomogeneous equation for the stress-free plate.

Fraser[12] considers four sets of possible homogeneous boundary conditions at $z = \pm 1$ for the plate: (1) $U=W=0$, (2) $U=\sigma_{zz}=0$, (3) $W=\sigma_{xz}=0$ and (4) $\sigma_{zz}=\sigma_{xz}=0$. It is the last case, the vanishing of both stress at the surface, which is of particular interest in this investigation. Although cases (1)-(3) will not be used in our calculations, we will carry through the orthogonality argument for them in anticipation of possible future need.

In cases (1)-(3), it is easy to show that

$$\left[\text{boundary terms}\right]_{-1}^{1} = 0$$

so that the conventional orthogonality relation Eq.III-30 hold. The constrains for the Lamb modes are stress-free boundaries, case (4), and it is straightforward to show Eq.III-39 holds in this instance. Thus Eq.III-38 is the orthogonality relation for the Lamb modes.

3. Completeness with Different Boundary Conditions

Casting our original pair of equations Eqs.III-14 and III-15 into the adjoint eigenvalue form Eqs.III-27 and III-28 accomplishes several things. A set of eigenfunction solutions is found and orthogonality relations, Eqs. III-30 and III-38, for them are proved. Secondly, systems of such adjoint equations have been extensively investigated by Naimark[13]. In particular, completeness of these sets under various boundary conditions has been studied.

In case (2) and (3) the boundary conditions are regular according to the definition of Naimark, and his Theorem Sec.2.1 can be invoked to assert that form

$$\hat{L}(\vec{R}) - k^2 \vec{R} = 0$$

ensures that the eigenvalues k_n^2 are denumerable, and the system of eigenfunctions $\{\vec{R}_n\}$ is complete. The completness theorem clearly holds also for the vectors $\vec{Q}_n$.

In case (1) the boundary conditions are not regular, and in case (4) the boundary conditions contain the eigenvalue k_n^2 so that Naimark's theorem is not relevant. Fama[15] who has applied Naimark's theorem to thr orthogonalized solutions for a vibrating cylinder believes that for these cases the conditions of the theorem may be too strong. Because a number of calculations assuming completeness in case (4) have yield good results for the cylinder, Fama conjectures that this set of solutions is essentially complete. Based on close analogy between the plate and the cylinder, we too will assume completeness for the case (4) boundary conditions.

4. Expansion of an Arbitrary Function

In Section III-C, we briefly summarized the use of a complete set of orthogonal eigenfunction in constructing, through the Green's function, a formal solution to an inhomogeneous differential equation. Essential to that argument is the expansion of an arbitrary function in terms of the set of eigenfunctions. The property of completeness insures the existence of the expansion and orthogonality is necessary to find the coefficients in that expansion. In the preceding sections we have derived from the Lamb modes a new set of solutions for the stress-free plate. We have developed orthogonality relations for this set and have argued that they are complete. In this section we will see how to expand an arbitrary function in terms of this set.

From Eq.III-24, the new plate functions are

$$\vec{R}_n = \begin{bmatrix} W_n \\ -\frac{\Delta_n}{c^2} \end{bmatrix} \equiv \begin{bmatrix} R_{1n} \\ R_{2n} \end{bmatrix} , \quad \vec{Q}_n = \begin{bmatrix} \Omega_n \\ U_n \end{bmatrix} \equiv \begin{bmatrix} Q_{1n} \\ Q_{2n} \end{bmatrix} \qquad \text{III-40}$$

in which Δ_n and Ω_n are given in terms of U_n and W_n by Eqs.III-19 and III-20 respectively. The orthogonality relation for cases (1)-(3) is

$$\int_{-1}^{1} \vec{R}_m \cdot \vec{Q}_n \, dz = \int_{-1}^{1} (W_m \Omega_n - \frac{1}{c^2} \Delta_n U_n) \, dz$$

$$= \int_{-1}^{1} (R_{1m} Q_{1n} + Q_{2n} R_{2m}) \, dz = 0 \text{ if } m \neq n \qquad \text{III-41}$$

and for case (4)

$$\int_{-1}^{1} (W_m \sigma_{xzn} - \sigma_{xxm} U_n) \, dz = \int_{-1}^{1} \{ W_m c^2 (\Omega_n + 2 \frac{\partial U_n}{\partial z}) - (\Delta_m - 2c^2$$

$$\frac{\partial W_m}{\partial z}) U_n \} \, dz = c^2 \int_{-1}^{1} \{ R_{1m} (Q_{1n} + 2 \frac{\partial Q_{2n}}{\partial z}) + (R_{2m} +$$

$$2 \frac{\partial R_{1m}}{\partial z}) Q_{2n} \} \, dz = 0 \quad \text{if } m \neq n \qquad \text{III-42}$$

in which Eqs. III-31 and III-35 have been used.

$$\vec{P}(z) = \begin{bmatrix} P_1(z) \\ P_2(z) \end{bmatrix}$$

in which P_1 and P_2 are arbitrary functions of z. Because the set $\{\vec{R}_n(z)\}$ is complete we may write

$$\vec{P} = \sum_n a_n \vec{R}_n = a_n \begin{bmatrix} R_{1n} \\ R_{2n} \end{bmatrix} \qquad \text{III-43}$$

and the coefficients a_n can be found using the orthogonality relations. For cases (1)-(3) we have from Eq. III-41 that

$$\int_{-1}^{1} (P_1 Q_{1k} + P_2 Q_{2k})\, dz = \sum_n a_n \int_{-1}^{1} (R_{1n} Q_{1k} + R_{2n} Q_{2k})\, dz$$

$$= a_k \int_{-1}^{1} (R_{1k} Q_{1k} + R_{2k} Q_{2k})\, dz$$

whence

$$a_k = \frac{\int_{-1}^{1} (P_1 Q_{1k} + P_2 Q_{2k})\, dz}{\int_{-1}^{1} (R_{1k} Q_{1k} + R_{2k} Q_{2k})\, dz} = \frac{\int_{-1}^{1} (P_1 \Omega_k + P_2 U_k)\, dz}{\int_{-1}^{1} (W_k \;_k - \frac{1}{c^2} \Delta_k U_k)\, dz} \qquad \text{III-44}$$

Similarly in case (4), we use the orthogonality relation Eq.III-42 to obtain

$$a_k = \frac{\int_{-1}^{1}\{P_1(Q_{1k}+2\frac{\partial Q_{2k}}{\partial z}) + (P_2+2\frac{\partial P_1}{\partial z})\,Q_{2k}\}\,dz}{\int_{-1}^{1}\{R_{1k}(Q_{1k}+2\frac{\partial Q_{2k}}{\partial z}) + (R_{2k}+2\frac{\partial R_{1k}}{\partial z})\}\,dz}$$

$$= \frac{\int_{-1}^{1}\{P_1(\;_k+2\frac{\partial U_k}{\partial z}) + (P_2+2\frac{\partial P_1}{\partial z})\,U_k\}\,dz}{\int_{-1}^{1}\{W_k(\;_k+2\frac{\partial U_k}{\partial z}) + (\frac{-\;_k}{c^2}+2\frac{\partial W_k}{\partial z})\,U_k\}\,dz} \qquad \text{III-45}$$

The U_k and W_k are known functions, the Lamb modes Eq.III-2, and Ω_k and Δ_k are readily calculated from them. Thus given $P_1(z)$ and $P_2(z)$, the components of $\vec{P}$, the integrals in Eq.III-44 and Eq.III-45 can be evaluated and the coefficients a_k determined.

E. Formal Solution of the Inhomogeneous Plate Equations

In this section, we will use the eigenfunctions $\overrightarrow{F}_n$ and $\overrightarrow{Q}_n$ to solve the inhomogeneous nonlinear equations, Eqs.III-6, for an isotropic plate. In component form, we may write Eq.III-6 as x component:

$$\rho_o \frac{\partial^2\overline{U}'}{\partial\overline{t}^2} - \mu\left(\frac{\partial^2\overline{U}'}{\partial\overline{x}^2} + \frac{\partial^2\overline{U}'}{\partial\overline{z}^2}\right) - \left(K+\frac{\mu}{3}\right)\left(\frac{\partial^2\overline{U}'}{\partial\overline{x}^2} + \frac{\partial^2\overline{W}'}{\partial\overline{z}\partial\overline{x}}\right) = \overline{P}_{xo} \quad \text{III-46}$$

and z component:

$$\rho_o \frac{\partial^2\overline{W}'}{\partial\overline{t}^2} - \mu\left(\frac{\partial^2\overline{W}'}{\partial\overline{x}^2} + \frac{\partial^2\overline{W}'}{\partial\overline{z}^2}\right) - \left(K+\frac{\mu}{3}\right)\left(\frac{\partial^2\overline{U}'}{\partial\overline{x}\partial\overline{z}} + \frac{\partial^2\overline{W}'}{\partial\overline{z}^2}\right) = \overline{P}_{zo} \quad \text{III-47}$$

In Section D, we manipulated the homogeneous versions of these equations, Eqs.III-14 and Eqs.III-15, into the homogeneous adjoint form Eqs.III-27, III-28. In this section, after dealing with the $\overline{x}$ and $\overline{t}$ dependent variables, we will carry through essentially the same manipulations on Eqs.III-46 and III-47 to obtain an analogous inhomogeneous adjoint form.

If we assume an exponential dependence in x and t and take

$$\overline{U}'(\overline{x},\overline{z},\overline{t}) = \overline{U}'(z)\, e^{i(k'\cos\theta'\overline{x}-\omega'\overline{t})}$$

$$\overline{W}'(\overline{x},\overline{z},\overline{t}) = \overline{W}'(z)\, e^{i(k'\cos\theta\overline{x}-\omega'\overline{t})} \qquad \text{III-48}$$

Equation III-46 becomes

$$\rho_o C_\ell^2 \{U'(i\omega')^2\} - \mu\{U'(ik'\cos\theta')^2 + \frac{\partial^2 U'}{\partial z^2}\} - (K+\frac{\mu}{3})$$

$$\{U'(ik'\cos\theta')^2 + \frac{\partial W'}{\partial z}(ik'\cos\theta')\}\, e^{i(k'\cos\theta' x-\omega' t)}$$

$$= \overline{F}_{xo}(x,z,t) \qquad \text{III-49}$$

Later when we examine the source function $P_{xo}(x,z,t)$ we will find it to have an x, t dependence of the form

$$\overline{F}'_{xo}(z)\, e^{i(kx-\omega'' t)}$$

and the validity of Eq.III-49 at all x andt will imply that

$$k'\cos\theta' = k\ , \qquad \omega'' = \omega'$$

Thus the exponentials can be divided out, and we are left with only the z-dependence of Eq.III-49.

In anlogy to Eqs.III-19 and III-20, we define

$$\Delta' = ik'U' + \frac{\partial W'}{\partial z} \qquad \text{III-50}$$

$$\Omega' = ik'W' - \frac{\partial U'}{\partial z} \qquad \text{III-51}$$

in terms of which, after some manipulation, Eq.III-49 becomes

$$(1-c^2)\,\frac{\partial\Omega'}{\partial z} + \frac{\partial^2 U'}{\partial z^2} + W'^2 U' - k'^2\cos^2\theta' U' = -\frac{P'_{xo}}{\rho_o\, C_\ell^2} \qquad \text{III-52}$$

Similarly, Eq.III-47 can be written as

$$\frac{1-c^2}{c^2}\,\frac{\partial\Delta'}{\partial z} + \frac{\partial^2 W'}{\partial z^2} + \frac{\omega'^2}{c^2}W' - k'^2\cos^2\theta' W' = -\frac{P'_{zo}}{\rho_o C_\ell^2 c^2} \qquad \text{III-53}$$

Also, in analogy to Eqs.III-21 and III-22, we can obtain equations for Ω' and ' individually:

$$\frac{\partial^2\Omega'}{\partial z^2} + \left(\frac{\omega'^2}{c^2} - k'^2\cos^2\theta'\right)\Omega' = \frac{1}{\rho_o C_\ell^2}\left(-\frac{\partial P'_{zo}}{\partial z} - ik'\cos\theta' P'_{xo}\right)$$

III-54

$$\frac{\partial^2\Delta'}{\partial z^2} + (\omega'^2 - k'^2\cos^2\theta')\,\Delta' = \frac{1}{\rho_o C^2}\left(-\frac{\partial P'_{zo}}{\partial z} - ik'\cos\theta' P'_{xo}\right)$$

III-55

In order to express Eqs.III-53,55 and Eqs.III-52, 54 in eigenvalue form we define

$$\hat{L}(\vec{R}') = \begin{bmatrix} (1-c^2)\,\frac{\partial}{\partial z}\left(\frac{\Delta'}{c^2}\right) + \frac{\partial W'}{\partial z^2} + \frac{\omega'^2}{c^2}W' \\ -\frac{\partial^2}{\partial z^2}\left(\frac{\omega'}{c^2}\right) - \frac{\omega'^2}{c^2}\Delta' \end{bmatrix} \qquad \text{III-56}$$

$$\hat{M}(\vec{Q}') = \begin{bmatrix} \frac{\partial^2\Omega'}{\partial z^2} + \frac{\omega'^2}{c^2}\Omega' \\ (1-c^2)\frac{\partial\Omega'}{\partial z} + \frac{\partial^2 U'}{\partial z^2} + \omega'^2 U' \end{bmatrix} \qquad \text{III-57}$$

Then these pairs of equations may, respectively, be written as

$$\hat{L}(\vec{R}') - k'^2\cos^2\theta'\vec{R}' = \frac{1}{\rho_o C_\ell^2 c^2}\begin{bmatrix} -P'_{zo} \\ \frac{\partial P'_{zo}}{\partial z} + ik'\cos\theta' P'_{xo} \end{bmatrix} \equiv \vec{P}_I \qquad \text{III-58}$$

$$\hat{M}(\vec{Q}') - k'^2\cos^2\theta'\vec{Q}' = \frac{1}{\rho_o C_\ell^2}\begin{bmatrix} \frac{1}{c^2}\left(\frac{\partial P'_{xo}}{\partial z} - ik'\cos\theta' P'_{zo}\right) \\ -P'_{xo} \end{bmatrix} \equiv \vec{P}_{II} \qquad \text{III-59}$$

For convenience and to emphasize the analogy with Eqs. III-27 and III-28, we denote the source vectors on the right side by $\vec{P}_I$ and $\vec{P}_{II}$ so that Eqs.III-58 and III-59 become finally

$$\hat{L}(\vec{R}') - k'^2\cos^2\theta'\vec{R}' = \vec{P}_I \qquad \text{III-60}$$

$$\hat{M}(\vec{Q}') - k'^2\cos^2\theta'\vec{Q}' = \vec{P}_{II} \qquad \text{III-61}$$

Considering Eqs.III-27 and III-28

$$\hat{L}(\vec{R}_n) - k_n^2 \vec{R}_n = 0 \qquad \text{III-27}$$

$$\hat{M}(\vec{Q}_n) - k_n^2 \vec{Q}_n = 0 \qquad \text{III-28}$$

and Eqs.III-60, 61 together, we may now easily obtain a series solution to the inhomogeneous equation following the procedure sketched in Section III.E. We have argued in Section III.D.3 that the function $\vec{R}_n$ are complete so that both the given source function $\vec{P}_I$ and the desered solution $\vec{R}'$ may be expanded in terms of them.

$$\vec{P}_I = \sum_n a_n \vec{R}_n \ , \qquad \vec{R}' = \sum_n b_n \vec{R}_n \qquad \text{III-62}$$

Inserting Eqs.III-62 into III-60

$$\hat{L}\left(\sum_n b_n \vec{R}_n\right) - k'^2 \cos^2\theta' \left(\sum_n b_n \vec{R}_n\right) = \sum_n a_n \vec{R}_n$$

$$\sum_n b_n \left\{\hat{L}(\vec{R}_n) - k'^2 \cos^2\theta' \, \vec{R}_n\right\} = \sum_n a_n \vec{R}_n$$

and eliminating $\hat{L}(\vec{R}_n)$ through Eq.III-27, yields

$$\sum_n b_n \left(k_n^2 - k'^2 \cos^2\theta' \right) \vec{R}_n = \sum_n a_n \vec{R}_n \ .$$

Orthogonality implies

$$b_n = \frac{a_n}{k_n^2 - k'^2\cos^2\theta'} \qquad \text{III-63}$$

The coefficients a_n for the known function $\overrightarrow{P_I}$ may be obtained from Eq.III-44 when the boundary conditions correspond to cases (1)-(3)

$$b_n = \frac{1}{k_n^2 - k'^2\cos^2\theta'} \frac{\int_{-1}^{1} (P_{I1}\Omega_n + P_{I2}U_n)\, dz}{\int_{-1}^{1} (W_n\Omega_n - \frac{1}{c^2}\Delta_n U_n)\, dz} \qquad \text{III-64}$$

or from Eq.III-45 for case (4)

$$b_n = \frac{1}{k_n^2 - k'^2\cos^2\theta'} \frac{\int_{-1}^{1} \left[P_{I1}(\Omega_n + 2\frac{\partial U_m}{\partial z}) + (P_{I2} + 2\frac{\partial P_{I1}}{\partial z})U_n\right] dz}{\int_{-1}^{1} \left[W_n(\Omega_m + 2\frac{\partial U_m}{\partial z}) + (\frac{-\Delta_n}{c^2} + 2\frac{\partial W_n}{\partial z})U_n\right] dz} .$$

III-65

Finally, we have the solution R' of the inhomogeneous Eq.III-60 as

$$\overrightarrow{R'}(z) = \sum_n \left\{\frac{1}{k_n^2 - k'^2\cos^2\theta'} \frac{\int_{-1}^{1} \left[(P_{I1}\Omega_n + P_{I2}U_n)\right] dz}{\int_{-1}^{1} \left[(W_n\Omega_n - \frac{1}{c^2}\Delta_n U_n)\right] dz}\right\} \overrightarrow{R_n}$$

III-66

for cases (1)-(3) and

$$\vec{R}'(z) = \sum_n \left\{ \frac{\vec{F}_n}{k_n^2 - k'^2 \cos^2\theta'} \; \frac{\int_{-1}^{1}\left[P_{I1}\left(\Omega_n + 2\frac{\partial U_n}{\partial z}\right) + \left(P_{I2} + 2\frac{\partial P_{I1}}{\partial z}\right)U_n\right]dz}{\int_{-1}^{1}\left[W_n\left(\Omega_n + 2\frac{\partial U_n}{\partial z}\right) + \left(\frac{\Omega_n}{c^2} + 2\frac{\partial W_n}{\partial z}\right)U_n\right]dz} \right\} \quad \text{III-67}$$

in case (4). Similar results may be obtained for $\vec{Q}'(z)$.

F. Green's Functions for the Nonlinear Plate Equations

Although Eq.III-66 or III-67 as it stands is a solution for the nonlinear plate equation, it is useful to construct Green's functions for the plate. For boundary conditions cases (1)-(3) where

$$\int_{-1}^{1} \vec{R}_m \cdot \vec{Q}_n \, dz = 0 \ , \quad \text{for } n \neq m \qquad \text{III-41}$$

is the orthogonality relation, the Green's function can be written down immediately. They are the dyads

$$\overleftrightarrow{G}(z,z') = \sum_n \frac{\vec{R}_n(z)\,\vec{Q}_n^{*}(z')}{T_n\,(k_n^2 - k'^2\cos^2\theta')} \qquad \text{III-68}$$

$$\overleftrightarrow{H}(z,z') = \sum_n \frac{\vec{Q}_n(z)\,\vec{R}_n^{*}(z')}{T_n\,(k_n^2 - k'^2\cos^2\theta')} \qquad \text{III-69}$$

in which

$$T_n = \int_{-1}^{1} \vec{R}_n^{*}(z) \cdot \vec{Q}_n(z) \; dz \qquad \text{III-70}$$

The solutions of Eq.III-60 and Eq.III-61, with the x and t dependence, may now be written as

$$\vec{R}' = \int_{-1}^{1} \overleftrightarrow{G}(z,z')\, \vec{P}_{I}(z')\, dz'\, e^{-i(k'\cos\theta' x-\omega' t)} \qquad \text{III-71}$$

$$\vec{Q}' = \int_{-1}^{1} \overleftrightarrow{H}(z,z')\, \vec{P}_{II}(z')\,dz'\, e^{-i(k'\cos\theta' x-\omega' t)} \qquad \text{III-72}$$

For stress-free boundaries, case (4), an orthogonality relation is

$$\int_{-1}^{1} (W_m \sigma_{xzn} - \sigma_{xxm} U_n)dz = c^2 \int_{-1}^{1} \{R_{1m}(Q_{1n}+2\frac{\partial Q_{2n}}{\partial z}) +$$

$$(R_{2m}+2\frac{\partial R_{1m}}{\partial z})Q_{2n}\}\, dz = 0 \text{ if } n \neq m \qquad \text{III-42}$$

as was shown earlier. This may be expressed as an ordinary vector product like Eq.III-41 by defining new vectors

$$\vec{I}_n = \begin{bmatrix} W_n \\ -\sigma_{xxn} \end{bmatrix} = \begin{bmatrix} R_{1n} \\ c^2(R_{2n}+2\frac{\partial R_{1n}}{\partial z}) \end{bmatrix}$$

$$\vec{J}_n = \begin{bmatrix} \sigma_{xzn} \\ U_n \end{bmatrix} = \begin{bmatrix} 2(Q_{1n}+2\,\frac{\partial Q_{2n}}{\partial z}) \\ Q_{2n} \end{bmatrix} \qquad \text{III-73}$$

so that Eq.III-42 may be written as

$$\int_{-1}^{1} \vec{I}_n \cdot \vec{J}_m = 0 \qquad \text{for } n \neq m \qquad \text{III-74}$$

It is straightforward to show in terms of I_n and J_n, the appropriate Green's functions are

$$\overleftrightarrow{G}(z,z') = \sum_n \frac{\vec{R}_n(z)\,\vec{J}_n^{\,*}(z')}{T_n'\,(k_n^2 - k'^2\cos^2\theta')} \qquad \text{III-75}$$

$$\overleftrightarrow{H}(z,z') = \sum_n \frac{\vec{Q}_n(z)\,\vec{I}_n^{\,*}(z')}{T_n'\,(k_n^2 - k'^2\cos^2\theta')} \qquad \text{III-76}$$

where

$$T_n' = \int_{-1}^{1} \vec{I}_n \cdot \vec{J}_n \; dz$$

and , finally, the solutions of the inhomogeneous Eqs. III-60 and III-61 for stress-free boundaries are

$$\vec{R}' = \int_{-1}^{1} \overleftrightarrow{G}(z,z')\,\vec{P}_I(z')\,dz'\; e^{-i(k'\cos\theta' x - \omega' t)} \qquad \text{III-77}$$

$$\vec{Q}' = \int_{-1}^{1} H(z,z')\,P_{II}(z')dz'\; e^{-i(k'\cos\theta' x - \omega' t)} \qquad \text{III-78}$$

CHAPTER IV

CALCULATIONS AND RESULTS

In the preceding chapter, we formed a formal solution for the nonlinear equations describing the vibrations of a homogeneous plate. This solution takes the form of an infinite series Eqs.III-66 and III-67, and may also be expressed in Green's function form Eqs.iii-77 and III-78. Formal, infinite series solutions are pratically useful either when they can be reduced to just a few terms for calculational purposes or when some general property can be discussed by the examination of a single term. We will consider both these possibilities for nonlinear plate solution formed in Chapter III.

Although an extensive numerical calculation is now possible, we will restrict ourselves here to a consideration of the symmetry properties of the solutions arising in the three-phonon process and in harmonic generation. Experimental measurements on symmetry properties with which we can compare our results have been taken by Brower[3].

Under certain experimental conditions, a mechanism exists for isolating a single term or limited group of terms in our infinite series plate solutions formed in the last chapter. From either the series or Green's function forms, it is seen that the series coefficients contain a factor

$$(k_n^2 - k'^2 \cos^2 \theta')^{-1} \qquad \text{IV-1}$$

in which k' and cosθ' are determined by the momentum and relative angle of the two input pump modes. k_n is the magnitude of one of the allowed modes of the plate as discussed in Sec. II.A and illustrated in Fig. 3. Thus when the combined momentum of the pump modes is close to that of an allowed plate mode so that

$$k'^2 \cos^2 \theta' \cong k_n^2$$

and the factor (Eq.IV-1) is large, the nth term can be expected to dominate the series.

In this chapter, we will study the properties of a single term in the series solution with two objectes in mind. First to determine the symmetry characteristics of this term associates with the pump modes. These results are of immediate interest in the present investigation. And secondary to simplify and partially evaluate the term in anticipation of future numerical evaluation.

In the following sections five cases will be studied.

These are the three-phonon process for combinations of the two pump modes which are antisymmetric-antisymmetric, antisymmetric-symmetric, symmetric-symmetric and harmonic generation for symmetric and antisymmetric pump modes. Noting that these processes are all generally quite similar we will treat the first two in great detail. The last three-phonon case followes easily. And recalling that harmonic generation can be regarded as a special case of three-phonon interaction with collinear pump modes of the same frequency, results for these processes can be written down without much additional work.

A. Three-Phonon Interaction in an Isotropic Plate

To investigate the three-phonon portion of the interaction, two primary waves must be present. Hence $\vec{U}_o$ described in Chapter III is now composed of two initial Lamb modes.

$$\vec{U}_o = A_o \vec{L}_1 + B_o \vec{L}_2 \qquad \text{IV-2}$$

where A_o and B_o are constants, and $\vec{L}_n$ $(n=1,2)$ are solutions to the linear homogeneous equations of motion with the stress-free boundary conditions.

In the following subsections, we will study three combinations of the two pump modes bases on their symmetry characteristics.

1. Three-phonon Interaction of Two Antisymmetric Lamb Modes

For the three-phonon interaction of two antisymmetric Lamb modes, $\vec{U}_o$ is expressed as

$$\vec{U}_o = A_o \vec{L}_{1a} + B_o \vec{L}_{2a}$$

$$\text{or} \quad \vec{U}_o = -\frac{A_o}{h} Ak_{1a}E_{1a}i(I'_{1a}-C'_{1a}H'_{1a}) - \frac{B_o}{h} Ak_{2a}E_{2a}i(I'_{2a}-C'_{2a}H'_{2a})\hat{x} - \frac{A_o}{h} Aq_{1a}E_{1a}(I_{1a}-C_{1a}H_{1a}) - \frac{B_o}{h} Aq_{2a}E_{2a}(I_{2a}-C_{2a}H_{2a})\hat{z}$$

where

$$I'_{na} = \frac{\sinh q_{na} z}{\cosh q_{na} d} \qquad I_{na} = \frac{\cosh q_{na} z}{\cosh q_{na} d}$$

$$H'_{na} = \frac{\sinh s_{na} z}{\cosh s_{na} d} \qquad H_{na} = \frac{\cosh s_{na} z}{\cosh s_{na} d}$$

$$C'_{na} = \frac{2q_{na}s_{na}}{k^2_{na}+s^2_{na}} \qquad C_{na} = \frac{2k^2_{na}}{k^2_{na}+s^2_{na}}$$

$$E_{na} = e^{i(k_{na} \cos\theta x - t)}$$

Disregarding "self-interaction" terms, terms with A_o^2 or B_o^2, $\vec{P}_o$ may be written in component form, with all subscripts a omitted.

$$P_{ox} = \frac{d^2}{h^2}\,(e_1 I_1' I_2' + e_2 I_1 I_2 + e_3 I_1' H_2' + e_4 I_1 H_2 + e_5 H_1' I_2' +$$

$$e_6 H_1 I_2 + e_7 H_1' H_2' + e_8 H_1 H_2) \qquad \text{IV-3}$$

$$P_{oz} = \frac{d^2}{h^2}\,(g_1 I_1 I_2' + g_2 I_1' I_2 + g_3 I_1 H_2' + g_4 I_1' H_2 + g_5 I_2' H_1 +$$

$$g_6 I_2 H_1' + g_7 H_1 H_2' + g_8 H_1' H_2) \qquad \text{IV-4}$$

in which the constants $e_1 - e_8$ and $g_1 - g_8$ are complicated expression given in Appendix I. We chose $\theta_1 = 0$ and $\theta_2 = 0$ for E_{na}, the configuration of the three-phonon interaction is shown in Fig. 4.

Next we need $\overrightarrow{P_I}$ and $\overrightarrow{P_{II}}$, Eqs.III-58 thru III-61, which are combinations and derivatives P_{ox} and P_{oz}. It is the components of $\overrightarrow{P_I}$ and $\overrightarrow{P_{II}}$ which actually enter our solutions, Eqs.III-71, 72 and III-77, 78.

$$\overrightarrow{P_I} = \begin{pmatrix} P_{I1} \\ P_{I2} \end{pmatrix} = \frac{1}{\rho_o C_\ell^2 c^2} \begin{pmatrix} -P_{oz} \\ \dfrac{\partial P_{oz}}{\partial z} + ik'\cos\theta' P_{ox} \end{pmatrix}$$

$$\vec{F}_{II} = \begin{pmatrix} F_{II1} \\ F_{II2} \end{pmatrix} = \frac{1}{\rho_o c_\ell^2} \begin{bmatrix} \frac{1}{c^2}\left(\frac{\partial P_{ox}}{\partial z} - ik'\cos\theta' \, P_{oz}\right) \\ -P_{ox} \end{bmatrix}$$

And for stress-free boundaries, case 4, the orthogonality relation is Eq.III-47

$$\int_{-1}^{1} \vec{I}_n \cdot \vec{J}_m = 0 \quad \text{for } n \neq m \quad .$$

Also we select the appropriate Green's functions, Eqs.III-75 and III-76.

$$\overleftrightarrow{G}(z,z') = \sum_n \frac{\vec{R}_n(z)\,\vec{J}_n^*(z')}{T_n'(k_n^2 - k'^2\cos^2\theta')}$$

$$\overleftrightarrow{H}(z,z') = \sum_n \frac{\vec{Q}_n(z)\,\vec{I}_n^*(z')}{T_n'(k_n^2 - k'^2\cos^2\theta')}$$

Since Lamb modes, $\vec{L}_n$, may be symmetric or antisymmetric with respect to the median plane, so the Green's function, $\overleftrightarrow{G}$, is composed of a sum of symmetric and antisymmetric Green's functions, $\overleftrightarrow{G}_s$, $\overleftrightarrow{G}_a$. Hence Green's functions for a stress-free isotropic plate may be expressed as

$$\overleftrightarrow{G} = \overleftrightarrow{G}_s + \overleftrightarrow{G}_a = \sum_{ns} \frac{\vec{F}_{ns}\, \vec{J}^{*}_{ns}}{T'_{ns}(k^2_{ns} - k'^2\cos^2\theta')} + \sum_{na} \frac{\vec{F}_{na}\, \vec{J}^{*}_{na}}{T'_{na}(k^2_{na} - k'^2\cos^2\theta')} \quad \text{IV-5}$$

$$\overleftrightarrow{H} = \overleftrightarrow{H}_s + \overleftrightarrow{H}_a = \sum_{ns} \frac{\vec{Q}_{ns}\, \vec{I}^{*}_{ns}}{T'_{ns}(k^2_{ns} - k'^2\cos^2\theta')} + \sum_{na} \frac{\vec{Q}_{na}\, \vec{I}^{*}_{na}}{T'_{na}(k^2_{na} - k'^2\cos^2\theta')} \quad \text{IV-6}$$

Noting that the orthogonality relations for eigenfunctions with mixed symmetric characteristics have vanished.

$$\int_{-1}^{1} \vec{I}_{ns} \cdot \vec{J}^{*}_{ma} = \int_{-1}^{1} \vec{I}_{na} \cdot \vec{J}^{*}_{ms} = 0 \quad \text{for all } m$$

These results assure us that there is no Green's function with mixed symmetric characteristic. Finally we may write the solutions of the inhomogeneous equations as

$$\vec{F}'\, e^{i(k'\cos\theta' x - \omega' t)} = \int_{-1}^{1} (\overleftrightarrow{G}_s + \overleftrightarrow{G}_a)\, \vec{P}_I\, dz' \quad \text{IV-7}$$

$$\vec{Q}'\, e^{i(k'\cos\theta' x - \omega' t)} = \int_{-1}^{1} (\overleftrightarrow{H}_s + \overleftrightarrow{H}_a)\, \vec{P}_{II}\, dz' \quad \text{IV-8}$$

To evaluate the integral on the right hand sides of Eqs.IV-7, 8, first we need consider the integral relations of functions (f_n, g_n) with symmetric characteristics. For two symmetric or antisymmetric functions, we have the relations

$$\int_{-a}^{a} f_1(\text{sym. wrt. } z)\, f_2(\text{sym. wrt. } z)\, dz \neq 0$$

$$\int_{-a}^{a} g_1(\text{antisym. wrt. } z)\, g_2(\text{antisym. wrt. } z) \neq 0,$$

and for a symmetric and an antisymmetric functions, the integral vanishes.

$$\int_{-a}^{a} f_n(\text{sym. wrt. } z)\, g_n(\text{antisym. wrt. } z)\, dz = 0$$

Hence we obtain, from Eqs.IV-7, 8,

$$\vec{R}' e^{i(k'\cos\theta' x-\omega' t)} = \sum_{ns} \frac{\vec{R}_{ns}(hBk_{ns})(\text{constant } 1)}{T'_{ns}(k_{ns}^2-k'^2\cos^2\theta')\ {}_oC^2 d^2} \qquad \text{IV-9}$$

$$\vec{Q}' e^{i(k'\cos\theta' x-\omega' t)} = \sum_{ns} \frac{\vec{Q}_{ns}(hB)(\text{constant } 2)}{T'_{ns}(k_{ns}^2-k'^2\cos^2\theta')\ {}_oC^2 d^2} \qquad \text{IV-10}$$

in which

$$(\text{constant } 1)=\int_{-1}^{1}\{(q_{ns}C_{ns}-s_{ns}C'_{ns})iH_{ns}P_{oz}+(k'P_{ox}+iP'_{oz})(I'_{ns}-C'_{ns}H'_{ns})\}\, dz'$$

$$(\text{constant } 2) = \int_{-1}^{1} \{ (I_{ns} - C_{ns} H_{ns}) \left[-ik' q_{ns} P_{oz} + q_{ns} (1-2c^2) P'_{ox} \right]$$

$$+ \left[k_{ns}^2 + (1-2c^2) q_{ns}^2 \right] P_{ox} I'_{ns} - \left[k_{ns}^2 C'_{ns} - \right.$$

$$\left. (1-2c^2) q_{ns} C_{ns} s_{ns} \right] P_{ox} H'_{ns} \} \, dz'$$

and $C_{ns} = \dfrac{2k_{ns}^2}{k_{ns}^2 + s_{ns}^2}$ $\qquad C'_{ns} = \dfrac{2q_{ns} s_{ns}}{k_{ns}^2 + s_{ns}^2}$

$$H_{ns} = \frac{\sinh s_{ns} z}{\sinh s_{ns} d} \qquad H'_{ns} = \frac{\cosh s_{ns} z}{\sinh s_{ns} d}$$

$$I_{ns} = \frac{\sinh q_{ns} z}{\sinh q_{ns} d} \qquad I'_{ns} = \frac{\cosh q_{ns} z}{\sinh q_{ns} d}$$

The immediate results are that the resultant wave is symmetric with respect to the median plane and because P_{ox} and P_{oz} are all proportional to the product of the primary wave amplitudes. The solutions are summations over ns and have $(k_{ns}^2 - k'^2 \cos^2\theta')$ in the denominator, if we consider $k'\cos\theta'$ close to an allowed mode k_{ns}, the summations may be represented by one term with $k_{ns} \cong k'\cos\theta'$. Finally we may find by comparing the propagating terms on both sides of Eqs.IV-9, 10,

$$k'\cos\theta' = k_1 + k_2\cos\theta \quad \text{and} \quad \omega' = \omega_1 + \omega_2$$

or in vector form, we write

$$\vec{k}' \equiv \vec{k}_3 = \vec{k}_1 + \vec{k}_2$$

Those two equivalents are the energy and momentum conservations.

The momentum conservation also limits the frequency range of the generated third phonon, the limits are

$$\left(\frac{\omega_1}{V_1} - \frac{\omega_2}{V_2}\right) < \frac{\omega_3}{V_3} < \left(\frac{\omega_1}{V_1} + \frac{\omega_2}{V_2}\right) \quad .$$

I θ is the angle between $\vec{k}_1$ and $\vec{k}_2$, and (ω_3, V_3) lies in this limits, then the pump waves will yield a generated, or third, wave in the direction $\vec{k}_1 + \vec{k}_2 = \vec{k}_3$, $\vec{k}_3$ is an allowed mode in the limits.

2. Three-phonon Interaction of a Symmetric and Antisymmetric Lamb Modes

We may write $\vec{U}_o$ for a symmetric and an antisymmetric Lamb modes three-phonon interaction as

$$\vec{U}_o = A_o \vec{L}_{1s} + B_o \vec{L}_{2a}$$

$$\text{or}\quad \vec{U}_o = \left[b_1(I'_{1s} - C'_{1s}H'_{1s}) + b_2(I'_{2a} - C'_{2a}H'_{2a})\right]\hat{x} -$$

$$\left[b_3(I_{1s} - C_{1s}H_{1s}) + b_4(I_{2a} - C_{2a}H_{2a})\right]\hat{z}$$

in which $\quad b_1 = -A_oAk_{1s}E_{1s}i \qquad b_2 = -iB_oAk_{2a}E_{2a}$

$$b_3 = -A_oAq_{1s}E_{1s} \qquad b_4 = -B_oAq_{2a}E_{2a} \quad .$$

Source term $\vec{F}_o$ may be expressed in component form

$$F_{ox} = e_1I'_{1s}I'_{2a} + e_2I_{1s}I_{2a} + e_3I'_{1s}H'_{2a} + e_4I_{1s}H_{2a} + e_5H'_{1s}I'_{2a} +$$

$$e_6H_{1s}I_{2a} + e_7H'_{1s}H'_{2a} + e_8H_{1s}H_{2a}$$

$$\text{and}\quad F_{oz} = g_1I_{1s}I'_{2a} + g_2I'_{1s}I_{2a} + g_3I_{1s}H'_{2a} + g_4I'_{1s}H_{2a} + g_5I'_{2a}H_{1s}$$

$$+g_6I_{2a}H'_{1s} + g_7H_{1s}H'_{2a} + g_8H'_{1s}H_{2a} \quad .$$

The process to find the solutions for this case (symmetric-antisymmetric) is similar to the previous case (antisymmetric-antisymmetric), we need only write down the results without any more work.

$$\vec{R}'e^{i(k'\cos\theta' x - \omega' t)} = \sum_{na} \frac{\vec{R}_{na}\,(Bhk_{na})\,(\text{constant } 3)}{T'_{na}(k^2_{na} - k'^2\cos^2\theta')\,\rho_o c_1^2 d^2} \qquad \text{IV-11}$$

$$\vec{Q}' e^{i(k'\cos\theta' x-\omega' t)} = \sum_{na} \frac{\vec{Q}_{na}(En)\,(\text{constant } 4)}{T'_{na}(k_{na}^2-k'^2\cos^2\theta')\,\rho_0 c_t^2 d^2} \qquad \text{IV-22}$$

in which

$$(\text{constant } 3)=\int_{-1}^{1}\{(q_{na}C_{na}-s_{na}C'_{na})iH_{na}P_{oz} + (I'_{na}-C'_{na}H'_{na})$$

$$(k'P_{ox}+iP'_{oz})\}\ dz'$$

$$(\text{constant } 4)=\int_{-1}^{1}\{(I_{na}-C_{na}H_{na})\ \left[ik' q_{na}P_{oz}+(2c^2-1)q_{na}P'_{ox}\right]$$

$$+P_{ox}\left[k_{na}^2(I'_{na}-C'_{na}H'_{na})+(2c^2-1)(q_{na}^2 I'_{na}-s_{na}C_{na}q_{na}H'_{na}\right]\}\ dz'$$

From above two equations, we find that $\vec{R}'$ and $\vec{Q}'$ are antisymmetric with respect to z, and their amplitudes are proportional to the product of the primary wave amplitudes, also by comparing the propagating terms on both sides, we obtain

$$\vec{k}' = \vec{k}_1 + \vec{k}_2, \qquad \omega' = \omega_1 + \omega_2 .$$

Hence a symmetric and an antisymmetric Lamb modes may interact to generate an antisymmetric third phonon.

3. Three-phonon Interaction of Two Symmetric Lamb Modes

For the three-phonon interaction of two symmetric

Lamb modes, $\vec{U}_o$ is expressed as

$$\vec{U}_o = A_o \overrightarrow{L_{1s}} + B_o \overrightarrow{L_{2s}}$$

$$\text{or } \vec{U}_o = \left[\frac{b_1}{h}\left(I'_{1s} - C'_{1s}H'_{1s}\right) + \frac{b_2}{h}\left(I'_{2s} - C'_{2s}H'_{2s}\right)\right]\hat{x}$$

$$+ \left[\frac{b_3}{h}\left(I_{1s} - C_{1s}H_{1s}\right) + \frac{b_4}{h}\left(I_{2s} - C_{2s}H_{2s}\right)\right]\hat{z} \quad .$$

Source term $\vec{P}_o$ may be written in component form, with all subscripts s omitted,

$$P_{ox} = \frac{h^2}{d^2}\left(e_1 I'_1 I'_2 + e_2 I_1 I_2 + e_3 I'_1 H'_2 + e_4 I_1 H_2 + e_5 H'_1 I'_2 + e_6 H_1 I_2 + \right.$$

$$\left. e_7 H'_1 H'_2 + e_8 H_1 H_2\right)$$

$$P_{oz} = \frac{h^2}{d^2}\left(g_1 I_1 I'_2 + g_2 I'_1 I_2 + g_3 I_1 H'_2 + g_4 I'_1 H_2 + g_5 I'_2 H_1 + g_6 I_2 H'_1 + \right.$$

$$\left. g_7 H_1 H'_2 + g_8 H'_1 H_2\right) \quad .$$

The solutions for this case (symmetric-symmetric) are

$$\vec{R}' e^{i(k'\cos\theta' x - \omega' t)} = \sum_{ns} \frac{\vec{R}_{ns}\,(Bhk_{ns})\,(\text{constant } 5)}{T'_{ns}\left(k_{ns}^2 - k'^2\cos^2\theta'\right)\,\rho_o C_\ell^2 d^2} \qquad \text{IV-13}$$

$$\vec{Q}' e^{i(k'\cos\theta' x - \omega' t)} = \sum_{ns} \frac{Q_{ns}\,(Bh)\,(\text{constant } 6)}{T'_{ns}(k_{ns}^2 - k'^2\cos^2\theta')\,\rho_o C_i^2 d^2} \qquad \text{IV-14}$$

in which

$$(\text{constant } 5) = \int_{-1}^{1} \{(qC - sC')iHP_{oz} + (K'P_{ox} + iP'_{oz})(I' - C'H')\}\, dz'$$

$$(\text{constant } 6) = \int_{-1}^{1} \{ (I - CH) \left[-ik'qP_{oz} + q(1-2c^2)P'_{ox}\right] + \left[k_n^2 + (1-2c^2)q^2\right] P_{ox}I' - \left[k_n^2 C' + (1-2c^2)qCs\right] P_{ox}H' \}\, dz'$$

From above two equations, we find that $\vec{R}'$ and $\vec{Q}'$ are symmetric with respect to z, their amplitudes are proportional to the product of the primary wave amplitudes, and the conservation criteria also hold. Hence the three-phonon interaction for two symmetric Lamb modes will generate a third symmetric mode.

Now we may conclude that the three-phonon interaction for case (1) (two antisymmetric pump modes) or case (3) (two symmetric pump modes) will creat a symmetric resultant wave, and for case (2) (one symmetric and one antisymmetric pump modes) will create an antisymmetric resultant wave subject to conservation criteria laws.

B. Second Harmonic Generation in an Isotropic Plate

To investigate the second harmonic generation, we need only one pump mode, hence $\vec{U}_o$ is now expressed as

$$\vec{U}_o = A \vec{L}$$

where A is a constant, and L is the Lamb mode. By comparision, this expression is a special case of Eq.IV-2, hence the second harmonic generation can be treated as a special case of three-phonon interaction with collinear pump modes of the same frequency.

1. Symmetric Pump Mode Harmonic Generation

For the symmetric Lamb mode second harmonic generation, $\vec{U}_o$ is expressed as

$$\vec{U}_o = L_{ns} = \frac{-A}{h} ik_{ns}(I'_{ns}-C'_{ns}H'_{ns})E_{ns}\,\hat{x} - \frac{A}{h} q_{ns}(I_{ns}-C_{ns}H_{ns})E_{ns}\,\hat{z}$$

where A is a constant and

$$E_{ns} = e^{i(k_{ns}x-\omega t)}$$

$\vec{L}_{ns}$ are symmetric solutions for the linear homogeneous equations of motion with stress-free boundary conditions at $z = \pm 1$ of an isotropic plate. Considering "self-interacting" terms only, the source term $\vec{P}_o$ may be written in component form as, with all subscripts ns omitted,

$$P_{ox} = \frac{d^2}{h^2} A^2 E^2 \, i(b_1 I'^2 + b_2 I'H' + b_3 H'^2 + b_4 I^2 + b_5 IH + b_6 H^2) \qquad \text{IV-15}$$

$$P_{oz} = \frac{d^2}{h^2} A^2 E^2 \, (b_7 I\ I' + b_8 H'I + b_9 I'H + b_{10} HH') \qquad \text{IV-16}$$

in which the constants b_1-b_{10} are complicated expression given in Appendix II.

The process to find the solutions for the harmonic generation is similar to (also simpler than) the three-phonon cases, we may just write down the results.

$$\vec{R}' e^{i(k'x-\omega' t)} = \sum_{ns} \frac{\vec{P}_{ns} \, (ik_{ns} BA^2E^2)(\text{constant } 1)}{T'_{ns}(k_{ns}^2 - k'^2) \, h \rho_o C_\ell^2} \qquad \text{IV-17}$$

$$\vec{Q}' e^{i(k'x-\omega' t)} = \sum_{ns} \frac{\vec{Q}_{ns} (A^2E^2 iB)(\text{constant } 2)}{T'_{ns}(k_{ns}^2 - k'^2) \, \rho_o C_\ell^2 h} \qquad \text{IV-18}$$

in which

$$(\text{constant } 1) = \int_{-1}^{1} (C_8 I'^3 + C_2 H'^3 + C_3 I'^2 H' + C_4 I'H'^2 + C_5 H'H^2$$

$$+ C_6 I'H^2 + C_7 IHH' + C_1 I'IH) \, dz'$$

$$(\text{constant } 2) = \int_{-1}^{1} (C_9 I'^3 + C_{10} I'H' + C_{11} I'H'^2 + C_{12} H'^3 + C_{13} I^2 I' + C_{14} I'IH + C_{15} I'H^2 + C_{16} IHH' + C_{17} H^2 H')\, dz'$$

the constants C_1-C_{17} are complicated expression given in Appendix II.

From above equations, we find $\vec{R}'$ and $\vec{Q}'$ are symmetric with respect to z, and their amplitudes are proportional to A^2 (the square of the initial amplitude). The solutions are also summations over ns and have $(k_{ns}^2 - k'^2)$ in the denominator, if we consider k' close to an allowed mode k_{ns}, the summations may both be reduced to one term with $k' = k_{ns}$. Finally by comparing the propagating terms on both sides, we obtain

$$k' = 2k_{ns} \quad \text{and} \quad \omega' = 2\omega .$$

Those are the statements of energy and momentum conservations, furthermore those conditions may act as the necessary conditions for Lamb mode second harmonic generation. In compact form, same as Eq.II-15,

$$\vec{V}_2(2\omega) = \vec{V}_1(\omega)$$

where $\vec{V}_n$ $(n = 1, 2)$ is velocity.

2. Antisymmetric Pump Mode Harmonic Generation

The procedure to find the second harmonic generation for an antisymmetric initial Lamb mode is the same as the procedure for the symmetric case. The $\vec{U}_o$ now is expressed as

$$\vec{U}_o = \vec{L}_{na} = \frac{A}{h} k_{na} (I'_{na} - C'_{na} H'_{na}) (-iE_{na}) \hat{x}$$

$$- \frac{A}{h} q_{na} E_{na} (I_{na} - C_{na} H_{na}) \hat{z}$$

The source term $\vec{P}_o$ may be written as, considering "self-interaction" terms only and with all subscripts na omitted,

$$P_{ox} = \frac{d^2}{h^2} A^2 E^2 i(b_1 I'^2 + b_2 I'H' + b_3 H'^2 + b_4 I^2 + b_5 IH + b_6 H^2) \qquad \text{IV-19}$$

$$P_{oz} = \frac{d^2}{h^2} A^2 E^2 (b_7 II' + b_8 IH' + b_9 I'H + b_{10} HH') \qquad \text{IV-20}$$

The solutions for this case are

$$\vec{R}' e^{i(k'x - \omega' t)} = \sum_{ns} \frac{\vec{R}_{ns} BA^2 E^2_{ns} ik_{ns} \text{ (constant 3)}}{T'_{ns}(k^2_{ns} - k'^2) \rho_o C^2_l h}$$

$$\vec{Q}' e^{i(k'x-\omega' t)} = \sum_{ns} \frac{\vec{Q}_{ns} B A^2 E_{ns}^2 \, i(\text{constant } 4)}{F'_{ns}(k_{ns}^2 - k'^2)\, \rho_o C_2^2 h}$$

in which

$$(\text{constant } 3) = \int_{-1}^{1} (C_8 I'^3 + C_2 H'^3 + C_3 I'^2 H' + C_4 I H'^2 + C_5 H' H^2 + C_6 I' H^2 + C_7 I H H' + C_1 I' I H)\, dz'$$

$$(\text{constant } 4) = \int_{-1}^{1} (C_9 I'^3 + C_{10} I' H' + C_{11} I' H'^2 + C_{12} H'^3 + C_{13} I^2 I' + C_{14} I' I H + C_{15} I' H^2 + C_{16} I H H' + C_{17} H^2 H')\, dz'$$

We may find from above equations that R' and Q' are symmetric with respect to z, and their amplitudes are proportional to A^2. By comparing the propagating terms on both sides gives

$$k' = 2\, k_{ns}, \quad \text{and} \quad \omega' = 2\,\omega$$

Hence the second harmonic generation by an antisymmetric Lamb mode has symmetric characteristics.

We can conclude now that a symmetric or an antisymmetric pump mode will create a symmetric second-harmonic but no asymmetric harmonic.

C. Bounded Pump Mode Approximation

In the preceding sections, we have discussed several cases of three-phonon interaction and second-harmonic generation with plane pump waves. In reality, however, plane waves do not exist, hence the three-phonon interaction or second-harmonic generation is generated in a "exciting region" by bounded pump wave(s). To introduce the bounded pump waves in the nonlinear equations of motion, we have to multiply a function g(x) on source term $\vec{P}_o$. g(x) is zero outside the "exciting region", so it can modify the source P_o to have nonlinear interaction only within a region. Thus the modified eigenfunction equations may be expressed as

$$\hat{L}'(\vec{R}')e^{i(k'\cos\theta' x-\omega' t)}=\vec{P}_{\ell}'(z)g(x)e^{i\left[(k_1+k_2\cos\theta)x-(\omega_1+\omega_2)t\right]}$$

IV-21

and

$$\hat{M}(\vec{Q}')e^{i(k'\cos\theta' x-\omega' t)}=\vec{P}_m'(z)g(x)e^{i\left[(k_1+k_2\cos\theta)x-(\omega_1+\omega_2)t\right]}$$

IV-22

where $\overrightarrow{P'_\ell}(z) = \frac{1}{\rho_o C_\ell^2 c^2} \begin{bmatrix} -P_{oz} \\ \frac{\partial P_{oz}}{\partial z} + ik'\cos\theta' P_{ox} \end{bmatrix}$

$$\overrightarrow{P'_m}(z) = \frac{1}{\rho_o C_\ell^2} \begin{bmatrix} \frac{1}{c^2}\left(\frac{\partial P_{ox}}{\partial z} - ik'\cos\theta' P_{oz}\right) \\ -P_{ox} \end{bmatrix}$$

$$\hat{L}'(\overrightarrow{R'}) = \hat{L}(\overrightarrow{R'}) - k'^2\cos^2\theta'\overrightarrow{R'}$$

$$\hat{M}'(\overrightarrow{Q'}) = \hat{M}(\overrightarrow{Q'}) - k'^2\cos^2\theta'\overrightarrow{Q'}$$

For second-harmonic generation case, we set $\theta'=0$, $\theta=0$, and $k_1=k_2$ in .qs.IV-21 and IV-22.

Taking the Fourier transform of Eqs.IV-21, one obtains

$$\int_{-\infty}^{\infty} \hat{L}'(\overrightarrow{R'}) e^{i(k'\cos\theta'+k)x} e^{-i\omega' t}\, dx = \int_{-\infty}^{\infty} \overrightarrow{P'_\ell}(z) g(x)$$

$$e^{i(k_1+k_2\cos\theta+k)x}\, e^{-i(\omega_1+\omega_2)t} \quad dx$$

or

$$\hat{L}'(\vec{R}')\,\delta\,(k'\cos\theta'+k)\;e^{-i\omega' t} = \vec{P}_{\ell}'(z)g(k_1+k_2\cos\theta+k)e^{-i(\omega_1+\omega_2)t}$$

IV-23

where $\hat{g}(k_1+k_2\cos\theta+k)$ is the Fourier transform of g(x), and $\delta(k'\cos\theta'+k)$ is zero when $k'\cos\theta'\neq -k$. For $k\neq -k'\cos\theta'$, we have $\tilde{g}(k_1+k_2\cos\theta+k)=0$, because $\delta(k'\cos\theta'+k)=0$. For $k=k'(-\cos\theta')$, Eq.IV-23 may be rewritten as

$$\hat{L}'(\vec{R}')\,\delta\,(0)e^{-i\omega' t} = \vec{P}_{\ell}'(z)\tilde{g}(k_1+k_2\cos\theta-\mathbf{k}'\cos\theta')e^{-i(\omega_1+\omega_2)t}$$

IV-23,a

where

$$\tilde{g}(k_1+k_2\cos\theta-k'\cos\theta')=\int_{-\infty}^{\infty} e^{i(k_1+k_2\cos-k'\cos\theta')x}g(x)dx$$

IV-24

The integrand of Eq.IV-34 oscillates as one integrates over x. The frequency of oscillation is determined by the coefficient of x in the exponential term, so the frequency is given by $(k_1+k_2\cos\theta-k'\cos\theta')$. The integrand will oscillate unless

$$k_1+k_2\cos\theta = k'\cos\theta'$$

This condition is called a resonance condition. We may rewrite Eq.IV-24 as

$$\tilde{g}_r(k_1+k_2\cos\theta-k'\cos\theta')=\tilde{g}(0)=\int_{-\infty}^{\infty} g(x)dx = \int_0^x g(x)dx \ .$$

Hence $$\tilde{g}(k_1+k_2\cos\theta-k'\cos\theta') = \begin{cases} 0 & k=-k'\cos\theta' \\ \tilde{g}(0) & k_1+k_2\cos\theta=k'\cos\theta' \\ & =-k \end{cases} \qquad \text{IV-25}$$

If we assume $g(x)$ a step function, then $\tilde{g}(0)=Kx$, where K is a constant. We may then also rewrite Eq.IV-25 as

$$\tilde{g}(k_1+k_2\cos\theta-k'\cos\theta') = Kx \quad \delta(k_1+k_2\cos\theta+k) \qquad \text{IV-26}$$

Substituting Eq.IV-26 into Eq.IV-23 and taking an inverse Fourier transform, we may obtain

$$\int_{-\infty}^{\infty} \hat{L}'(\vec{R}') \ \delta(k'\cos\theta'+k) \ e^{-i(k'x+\omega't)}dk =$$

$$\int_{-\infty}^{\infty} \vec{P}'_{\ell}(z)Kx \quad (k_1+k_2\cos\theta+k) \ e^{-i\{kx+(\omega_1+\omega_2)t\}} \ dk$$

or $$\hat{L}'(\vec{R}')e^{i(k'\cos\theta'-\omega't)}=\vec{P}'_{\ell}(z)Kxe^{i\{(k_1+k_2\cos\theta)x-(\omega_1+\omega_2)t\}}$$

IV-27

Following the same procedure, we may transform Eq.IV-22 to

$$\hat{K}'(\vec{q}')e^{i(k'\cos\theta'-\omega' t)}=\vec{F}_m'(z)Kxe^{i[(k_1+k_2\cos\theta)x-(\omega_1+\omega_2)t]} \qquad \text{IV.-28}$$

By solving Eqs.IV-27 and IV-28 through a Green's function technique, we find the resultant wave is proportional to the interacting volume for three-phonon interaction or propagating distance for the second-harmonic generation, respectively.

Now, we know that symmetric or antisymmetric Lamb mode will generate only symmetric second-harmonic wave with amplitude of the resultant wave being proportional to A^2, the square of the initial wave amplitude. We also find that two symmetric or two antisymmetric Lamb modes interact will generate a symmetric third phonon, and one symmetric and one antisymmetric Lamb mode interact will give an antisymmetric third phonon. The amplitude of the third phonon is proportional to the product of the two primary wave amplitudes. In reality, an "exciting region" caused by bounded pump wave will give a volume or distance dependent resultant wave amplitude.

CHAPTER V

COMPARISON WITH EXPERIMENT AND CONCLUSIONS

In the preceding chapter, we have evaluated a single term of our general series solution for the nonlinear vibrations of the plate. This was done with special emphasis on symmetry properties and all possible combinations of symmetric-antisymmetric pump modes were considered for both the three-phonon interaction and harmonic generation. One of the reasons for doing this, as explained in Sec. IV.A, was to facilitate comparison with the experimental results of Brower[3].

A. Three-phonon Interaction

Two important results have been obtained for the three-phonon interaction in a bulk medium or a single crystal. These are the necessity of the conservation criteria and the fact that the generated wave is proportional to the product of the primary wave amplitudes. We have found these results, Eq.II-8 and II-9, to also be true for the Lamb mode three-phonon interaction in the plate.

Brower has demonstrated that two input Lamb modes generate a third phonon in a brass plate and that the amplitude of this phonon is proportional to the product of those of the pump modes. His work relies heavily on the use of the dispersion curves, Fig.3, for an unloaded plate. He considers two different sets of pump modes and finds (1) the A_1 mode at 5MHz and the A_2 mode at 7MHz generate the S_3 mode at 12MHz, (2) the A_1 mode at 5MHz and the S_1 mode at 7MHz generate the A_3 mode at 12MHz. We have shown in our calculations of Secs. IV.A.1 and A.2 that antisymmetric-antisymmetric or symmetric-symmetric pump modes generate a symmetric third phonon. This is consistent with case (1), (A_1, A_2, S_3) of Brower's observations. Further, we found in Sec.IV.A.3 that the antisymmetric-symmetric combination creates an antisymmetric third mode, consistent with case (2), (A_1, S_1, A_3) of the experimental findings. Thus our theoretical results for the three-phonon interaction are in agreement with Brower's experiments.

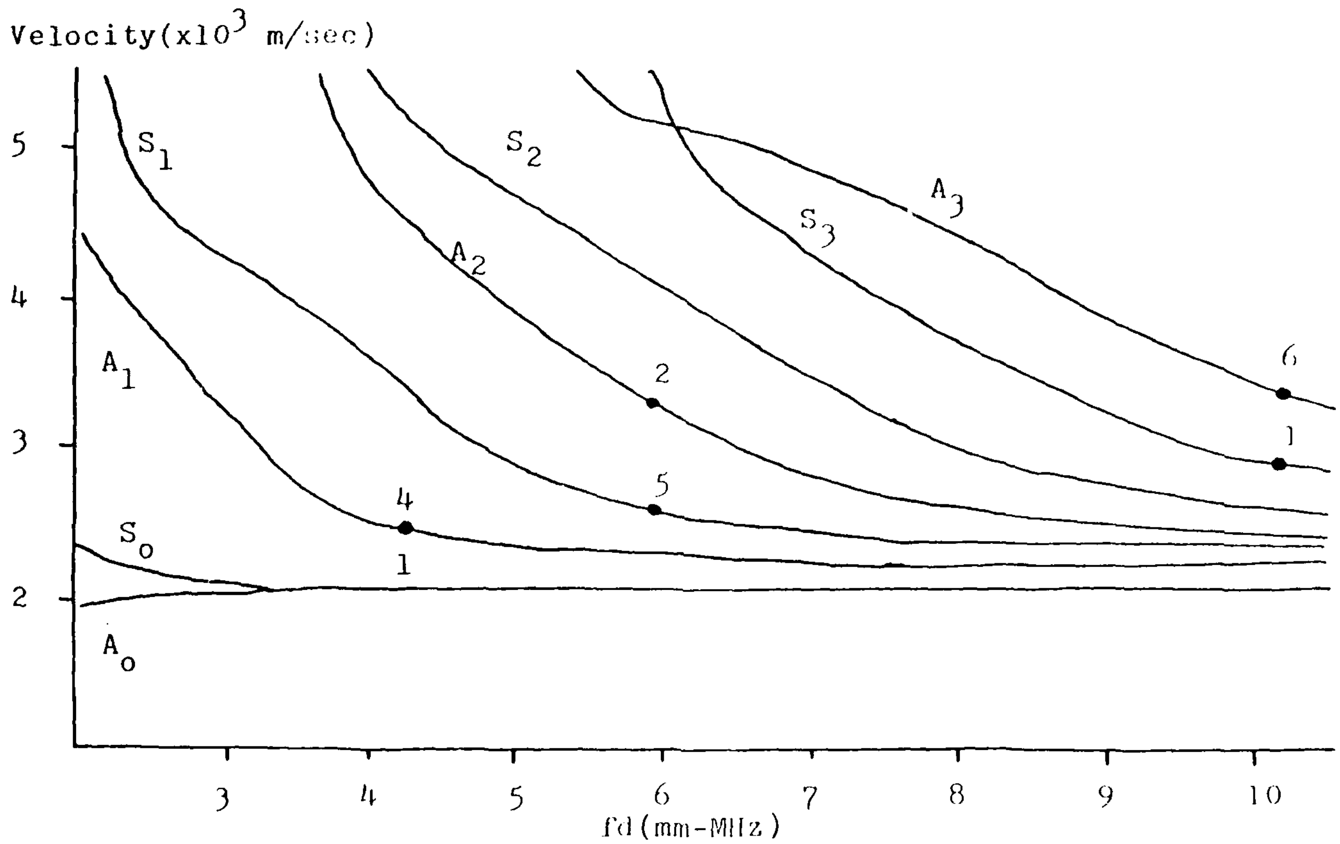


Figure 3.a Dispersion curves for symmetric and antisymmetric Lamb modes in an unloaded brass plate with two three-phonon interactions, points(1, 2, 3) and (4, 5, 6).

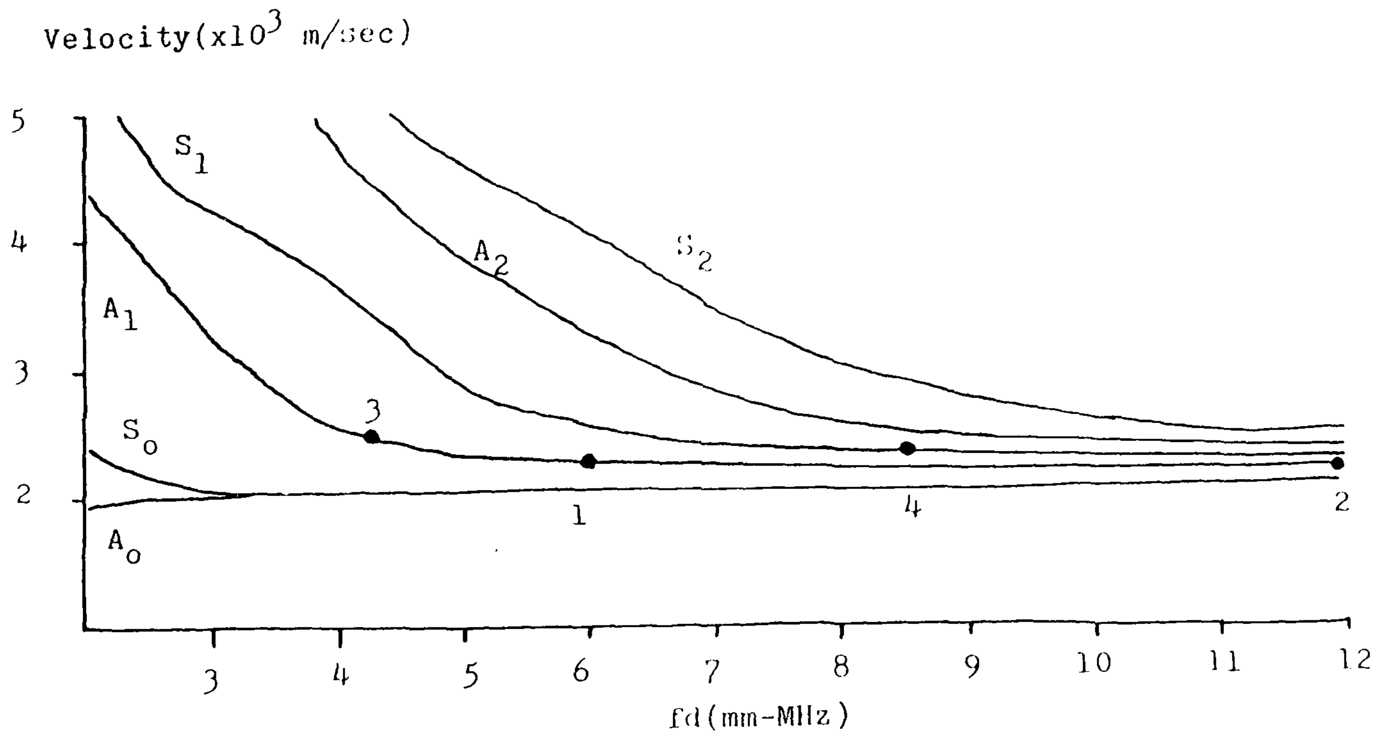
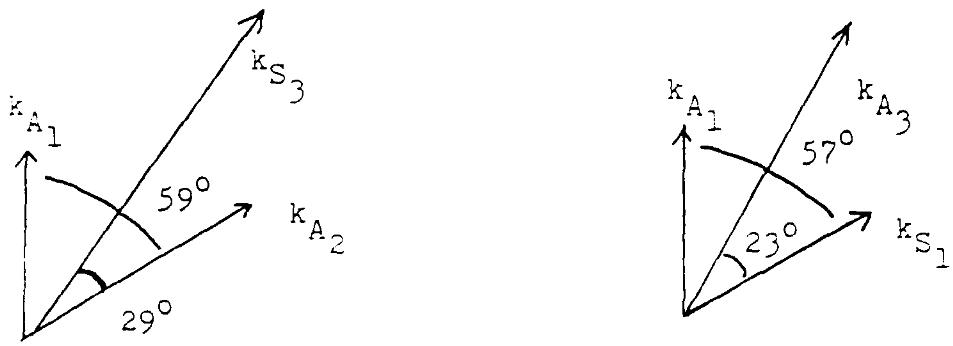


Figure 3.b Dispersion curves for symmetric and antisymmetric Lamb modes in an unloaded brass plate with two second-harmonic generations, points (1, 2) and (3, 4).



(a) (b)

Figure 4. (a). Two antisymmetric Lamb modes configuration (A_1, A_2, S_3).

(b). Symmetric and antisymmetric Lamb modes configuration (A_1, S_1, A_3).

B. Harmonic Generation

Studies of second-harmonic generation in bulk media and single crystals have yielded two important results. One is the necessary condition

$$V_2(2\omega) = V_1(\omega),$$

that is, the second-harmonic must have the same velocity as the initial pump wave. The other is that the amplitude of the generated harmonic varies as

$$A_2 \propto A_1^2 x$$

where A_1 is the initial pump amplitude and x is the distance the primary wave has traveled in the medium. The first result is true for our plate solutions of Lamb mode second-harmonic generation. The second was obtained in Sec. IV.C, where we modify the basic plane wave approach to take account of the bounded volume of interaction.

In Viktorov's[5] analysis, it is stated that only for the zero order symmetric pump mode (S_0) at thin plate frequency-thickness, fd   1, is it possible to have second-harmonic generation. This is because for small fd's the velocity in S_0 mode is not dispersive. Later, however, Brower pointed out that harmonic generation in Lamb modes is indeed possible for particular sets of

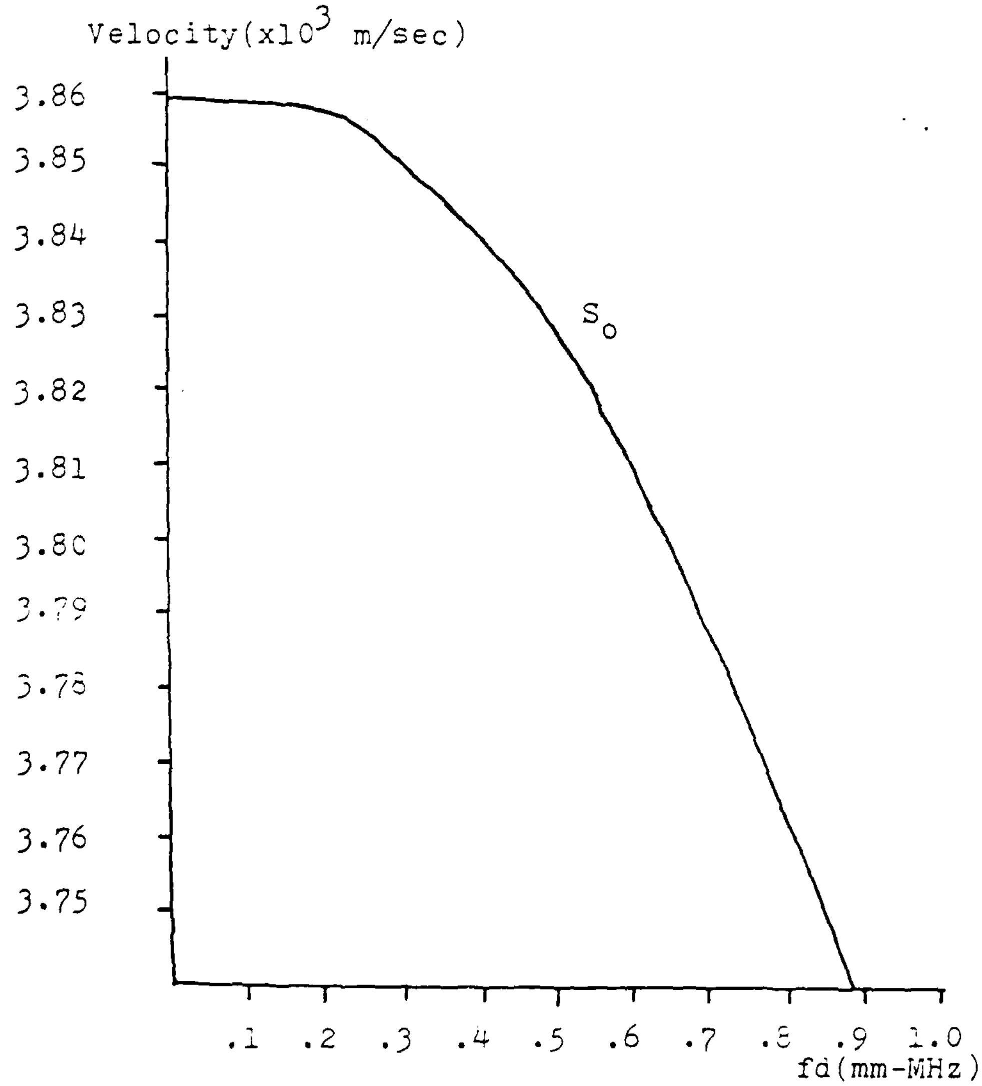


Figure 5. Dispersion curve for the S_o mode at small fd's.

dispersion curves. He demonstrated experimentally that for A_1 mode of a brass plate with thickness d=0.85mm at f=7MHz and f=5MHz harmonic generation occurs to produce the A_2 mode. Thus the S_0 mode at thin plate fd's is not the only mode to permit harmonic generation.

In Sec.IV-B, we found that either a symmetric or anti-symmetric Lamb mode will generate only a symmetric second-harmonic, a result inconsistent with Brower's observations of (A_1, A_2) for the isotropic brass plate. But by examining the dispersion curves for the brass plate carefully, we find that in the region of interest the curve of the antisymmetric and symmetric modes are very close. Therefore we conjecture that what Brower observed was actually a symmetric rather than an antisymmetric second-harmonic.

C. Conclusions

The amplitude of mechanical vibrations in solids is not infinitesimal, and, therfore, nonlinear equations of motion are required to describe the disturbance. The principal of superposition no longer holds and hence non-linear interactions may occur. Two possible such interactions in an isotropic plate are the three-phonon interaction and second-harmonic generation.

A new approach is presented to solve the nonlinear equations of motion for the plate. This approach involves the combination of the perturbative and the Green's function techniques. The Green's function is constructed by the eigenfunction expansion method after a suitable orthogonalization relation for the Lamb modes has been derived.

It is found that a second-harmonic acoustic wave generated by either a symmetric or antisymmetric initial Lamb mode is symmetric with respect to the median plane. The amplitude of the generated wave is proportional to the square of the pump wave amplitude and is given by a summation over plate eigenfunctions of terms which have denominators of the form $(k_{ns}^2 - k'^2)$. In approximation, the summation may be well represented by one term when $k_{ns} = k'$.

The three-phonon interaction of two noncollinear symmetric or antisymmetric Lamb modes which satisfies the conservation criteria will produce a third mode with symmetric characteristic. Similarly, two primary modes one symmetric the other antisymmetric, will produce an antisymmetric third wave. The amplitude of the third wave is proportional to the product of the pump mode amplitudes and is expressed by a summation over plate eigenfunctions. When the total momentum of the two inputs waves is close to that of a plate mode, the summation may again be approximated by a single term.

The approach presented in this thesis is appicable not only to the plate with stress-free surfaces but to certain other plate boundary conditions as well. By finding the appropriate orthogonalization relations for the plate modes, a Green's function satisfying different boundary constraints can be constructed. The third phonon of the three-phonon interaction may also have the following conditions.

$$\vec{k}_3 = \vec{k}_1 - \vec{k}_2 , \qquad \omega_3 = \omega_1 - \omega_3$$

We will leave those as futhur investigation.

APPENDIX I

THE CONSTANTS e_1-e_8 AND g_1-g_8 IN Sec.IV.A.1.

$$e_1=d_1+q_1(d_2q_1+d_{16}+d_{20})+q_2(d_3q_2+d_{17}+d_{19})+q_1q_2(d_5q_1+d_6q_2+d_{18})$$

$$e_2=d_{15}+q_1(d_7+d_{12}+d_{13}q_1)+q_2(d_8+d_{11}+d_{14}q_2)+q_1q_2(d_4+d_9q_1+ d_{10}q_2)$$

$$e_3=-C_2'(d_1+d_2q_1^2+d_3s_2^2+d_6s_2^2q_1+d_{16}q_1+d_{20}q_1)-C_2s_2(d_5q_1^2+d_{17}+d_{19})$$

$$e_4=-C^2(d_{14}s_2^2+d_{15})-C_2q_1(d_7+d_{10}s_2^2+d_{12}+d_{13}q_1)-C_2's_2(d_4q_1+d_8 +d_9q_1^2+d_{11})$$

$$e_5=-d_1C_1'-C_1's_1^2(d_2+d_5q_2)-C_1'q_2(d_3q_2+d_{17}+d_{19})-C_1s_1(d_6q_2^2+d_{16} +d_{18}+d_{20})$$

$$e_6=-C_1's_1(d_4q_2+d_7+d_{10}q_2^2+d_{12})-q_2C_1(d_8+d_9s_1^2+d_{11})-C_1(d_{13}s_1^2+ d_{14}q_2^2+d_{15})$$

$$e_7=C_1'C_2'(d_1+d_2s_1^2+d_3s_2^2)+C_1'C_2s_2(d_5s_1^2+d_{17}+d_{19})+C_2'C_1s_1(d_6s_2^2+ d_{16}+d_{20})+d_{18}C_1C_2s_1s_2$$

$$e_3=d_4C_1'C_2's_1s_2+C_1'C_2s_1(d_7+d_{10}s_2^2+d_{12})+C_1C_2's_2(d_8+d_9s_1^2+d_{11})+$$

$$C_1C_2(d_{13}s_1^2+d_{14}s_2^2+d_{14}s_2^2+d_{15})$$

in which

$d_1=-ia_1b_1b_2k_1k_2\cos\theta(k_1+k_2\cos\theta)$ $\quad$ $d_2=a_2b_1b_2ik_2\cos\theta$

$d_4=ia_3b_1b_2(k_1+k_2\cos\theta)$ $\quad$ $d_3=ik_1a_2b_1b_2$

$d_5=a_2b_1b_4$ $\quad$ $d_6=a_2b_2b_3$

$d_7=-k_1k_2\cos\theta a_4b_1b_4$ $\quad$ $d_8=-k_1k_2\cos 0a_4b_2b_3$

$d_9=(\frac{a_3}{2})b_2b_3$ $\quad$ $d_{10}=(a_3/2)b_1b_4$

$d_{11}=(-a_4/2)b_2b_3k_1^2$ $\quad$ $d_{12}=(-a_4/2)b_1b_4k_2^2\cos^2\theta$

$d_{13}=(a_4/2)b_3b_4ik_2\cos\theta$ $\quad$ $d_{14}=(a_4/2)b_3b_4ik_1$

$d_{15}=(a_3/2)b_3b_4(-ik_1k_2\cos\theta)(k_1+k_2\cos\theta)$

$d_{16}=a_5b_2b_3(-k_1k_2\cos\theta)$ $\quad$ $d_{17}=-a_5b_1b_4k_1k_2\cos\theta$

$d_{18}=ia_5b_3b_4(k_1+k_2\cos\theta)$ $\quad$ $d_{19}=-a_6b_1b_4k_1^2$

$d_{20}=-a_6b_2b_3k_2^2\cos\theta$

in which

$b_1 = -iA_oAk_1E_1$ $\qquad$ $b_2 = -iB_oAk_2E_2$

$b_3 = -A_oAq_1E_1$ $\qquad$ $b_4 = -B_oAq_2E_2$

$$\varepsilon_1 = q_1^2(f_1q_2+f_{19})+q_2^2(f_{10}+f_{13}q_1)+q_1q_2(f_6+f_{17})+q_2(f_2+f_4)+$$
$$q_1(f_{15}+f_{16})+f_7+f_{12}$$

$$\varepsilon_2 = q_1^2(f_9+f_{13}q_2)+q_2^2(f_1q_1+f_{20})+q_1q_2(f_5+f_{18})+q_1(f_3+f_4)+$$
$$q_2(f_{14}+f_{16})+f_8+f_{11}$$

$$\varepsilon_3 = -C_2s_2(f_1q_1^2+f_2+f_4+f_6q_1+f_{17}q_1)-C_2'(f_7+f_{12})-C_2's_2^2(f_{10}+$$
$$f_{13}q_1)-C_2'q_1(f_{15}+f_{16}+f_{19}q_1)$$

$$\varepsilon_4 = -C_2s_2^2(f_1q_1+f_{20})-C_2q_1(f_3+f_4)-C_2's_2(f_5q_1+f_{13}q_1^2+f_{14}+f_{16}+$$
$$f_{18}q_1)-C_2(f_8+f_9q_1^2+f_{11})$$

$$\varepsilon_5 = -C_1s_1^2(f_1+f_{19})-C_1q_2(f_2+f_4)-C_1(f_7+f_{10}q_2^2+f_{12})-C_1's_1(f_{13}q_2^2+$$
$$f_{15}+f_{16}+f_{17}q_2)-f_6C_2s_2q_1$$

$$\varepsilon_6 = -C_1s_1(f_1q_2^2+f_3+f_4+f_5q_2+f_{18}q_2)-C_1'(f_8+f_{11}+f_{20}q_2^2)-$$
$$C_1's_1^2(f_9+{}_{13}q_2)-C_1'q_2(f_{14}+f_{16})$$

$$\varepsilon_7 = C_1C_2s_2(f_1s_1^2+f_2+f_4)+C_1'C_2s_1s_2(f_6+f_{17})+C_1C_2'(f_7+f_{10}s_2^2+$$
$$f_{12})+C_1'C_2's_1(f_{13}s_2^2+f_{15}+f_{16})$$

$$\varepsilon_8 = C_1C_2s_1(f_1s_2^2+f_3+f_4)+C_1C_2's_1s_2(f_5+f_{18})+C_1'C_2(f_8+f_9s_1^2+$$
$$f_{11}+f_{20}s_2^2)+C_1'C_2's_2(f_{13}s_1^2+f_{14}+f_{16})$$

in which

$f_1=a_1b_3b_4$

$f_2=a_2b_3b_4$

$f_3=a_2b_3b_4(-k_2^2\cos^2\theta)$

$f_4=-a_3b_3b_4k_1k_2\cos\theta$

$f_5=ia_4b_3b_2k_1$

$f_6=ia_4b_4b_1k_2\cos\theta$

$f_7=-ia_2b_3b_2k_1^2k_2\cos\theta$

$f_8=-ia_2b_1b_4k_1k_2^2\cos^2\theta$

$f_9=(a_4/2)b_1b_4ik_2\cos\theta$

$f_{10}=(a_4/2)b_2b_3ik_1$

$f_{11}=(-a_3/2)b_1b_4ik_1^2k_2\cos\theta$

$f_{12}=(a_3/2)b_2b_3(-ik_1k_2^2\cos^2\theta)$

$f_{13}=(a_3/2)b_1b_2$

$f_{14}=(-a_4/2)b_1b_2k_1^2$

$f_{15}=(-a_4/2)b_1b_2k_2^2\cos^2\theta$

$f_{16}=-a_5b_1b_2k_1k_2\cos\theta$

$f_{17}=a_5b_1b_4ik_1$

$f_{18}=a_5b_2b_3ik_2\cos\theta$

$f_{19}=a_6b_2b_3ik_2\cos\theta$

$f_{20}=a_6b_1b_4ik_1$

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APPENDIX II

THE CONSTANTS b_1-b_{10} AND C_1-C_{17} IN Sec.IV.B.1.

$b_1=a_1k^5-(a_2+a_5)k^3q^2-a_6k^2q^3+(a_2+a_5)kq^4$

$b_2=-2a_1C'k^5+(a_2+a_5)C'k^3q^2+a_2C'k^3s^2+a_5Ck^3qs+a_6C'k^2(qs^2+$

$q^3)-(2a_5+a_2)Ckq^3s-a_2C'kq^2s^2$

$b_3=a_1C'^2k^5-a_2C'^2k^3s^2-a_5CC'k^3qs-a_6C'^2k^2qs^2+a_2C'^2kqs^3+$

$a_5kq^2s_2$

$b_4=(-a_3+2/3\ a_4)k^3q^2-(a_3/2)k^2q^2+(a_3/2\ -a_4/2)kq^4$

$b_5=(a_3-\ 3/2\ a_4)Ck^3q^2-(3/2)a_4C'qs+(-a_3+a_4)\frac{1}{2}kq^2s^2+$

$\frac{1}{2}a_4Ckq^4-\frac{1}{2}a_3C'kq^3s$

$b_6=-a_3C'^2k^3s^2-\frac{1}{2}a_3C^2k^3q^2+(3/2)a_4CC'k^3qs+\frac{1}{2}a_3CC'kqs^3-\frac{1}{2}a_4C^2kq^2$

s^2

$b_7=a_1q^5+(a_2+\frac{1}{2}a_3+\frac{1}{2}a_4+a_5)k^4q-(a_2+3/2\ a_3+3/2\ a_4+a_5+a_6)k^2q^3$

$b_8=-(a_2+\frac{1}{2}a_3+\frac{1}{2}a_4+a_5)C'k^4q+(a_2+a_3+a_4+a_5)k^2q^2sC+(\frac{1}{2}a_3+\frac{1}{2}a_4)$

$k^2s^2qC'+a_6k^2q^3C'-a_1q^4sC$

$$b_9=-(a_2+\tfrac{1}{2}a_3)k^4qC-(\tfrac{1}{2}a_4+a_5)k^4sC'+(\tfrac{1}{2}a_3+a_4+a_5)k^2q^2sC'+$$

$$(a_2+a_3+\tfrac{1}{2}a_4)Ck^2q^3+a_6k^2qs^2C-a_1q^3s^2C$$

$$b_{10}=(a_2+\tfrac{1}{2}a_3)k^4qCC'+(\tfrac{1}{2}a_4+a_5)k^4sC'^2-(3/2\ a_4+a_5+a_6)k^2qs^2CC'-$$

$$(a_2+a_3)k^2q^2sC^2-\tfrac{1}{2}a_3k^2C'^2+a_1q^2s^3C^2$$

in which

$a_1=4a+e+2d+2b+c$ $\qquad$ $a_2=2a+c$

$a_3=2a+2b$ $\qquad$ $a_4=2a+2d$

$a_5=b+d+e$ $\qquad$ $a_6=c+e$

in which

$a=\mu+\tfrac{1}{4}A$ $\qquad$ $b=K+\ \mu/3+\tfrac{1}{4}A+B$

$c=K-2/3\ \mu+B$ $\qquad$ $d=\tfrac{1}{4}A+B$

$e=B+2C$

$C_1=b_5k'+b_8s+b_9q+b_7qC-b_7sC'$ $\quad$ $C_2=-C'(b_3k'+b_{10}s)$

$C_3=b_2k'+b_6q+b_9s-C'b_1k'-C'b_7q$

$C_4=b_3k'+b_{10}s-C'b_2k'-C'b_8q-C'b_9s$

$C_5=b_{10}qC-2b_{10}sC'-C'b_6k'$ $\quad$ $C_6=b_6k'+b_{10}s+b_9qC-b_9sC'$

$C_7=b_8qC-2b_8sC'-C'b_5k'-C'b_9q$ $\quad$ $C_8=b_1k'+b_7q$

$C_9=b_1k^2+91-2c^2)b_1q^2$

$C_{10}=b_2\{k^2+(1-2c^2)q^2\}-b_1\{k^2C'+(1-2c^2)qCs\}$

$C_{11}=nb_3-b_2m$ $\quad$ $C_{12}=-b_3m$

$C_{13}=nb_4+q\{-b_7k'+(1-2c^2)(2b_1q+2b_4q)\}$

$C_{14}=b_5n+q\{-b_9k'+(1-2c^2)(b_2s+b_5q)\}-Cq\{-b_7k'+(1-2c^2)$
$(2b_1q+2b_4q)\}$

$C_{15}=b_6n-Cq\{-b_9k'+(1-2c^2)(b_2s+b_5q)\}$

$C_{16}=-b_5m+q\{-b_{10}k'+(1-2c^2)(2b_3s+2b_6s)\}-qC\{-b_8k'+$
$(1-2c^2)(b_2q+b_5s)\}$

$C_{17}=-b_6m-Cq\{-b_{10}k'+(1-2c^2)(2b_3s+2b_6s)\}$

in which

$$n = k^2 + (1-2c^2)\, q^2$$

$$m = k^2C' + (1-2c^2)qCs$$

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