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MRC Technical Summary Report # 2391

STEWARTSON'S TRIPLE DECK

R. E. Meyer

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

June 1982

(Received February 12, 1982)

Approved for public release
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Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

An introduction to triple-deck theory for steady, two-dimensional boundary layers in low-speed flow is presented. It aims to clarify how the rational structure of the theory can rest on few premises and to make its new ideas and challenges more widely accessible.

AMS (MOS) Subject Classifications: 76-02, 76B05, 76D15, 76D10

Key Words: Boundary Layer, Flow Separation

Work Unit Number 2 - Physical Mathematics

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This work was partially supported by the National Science Foundation under Grant No. MCS-8001960.

SIGNIFICANCE AND EXPLANATION

Fluids are within us and all around us, from kitchen to galaxy, and have been studied intensively for over two hundred years. Some central questions, however, have continued to defeat Science. Why do you not inhale the same body of stale air that you have just exhaled? The answer lies in a break-away of fluid streams from solid surfaces which we see everywhere, but cannot yet understand or predict in any detail.

It now appears, however, that some decisive progress towards solving the riddle has been made by a restricted group of aerodynamicists and that we are in the midst of what can really be called a break-through. At the same time, a much wider scientific community appears to have formed a strong interest in access to the subject on account of its importance and also, of dimly perceived, novel challenges that it poses for mathematical analysis. Conversely, it looks as if mathematics could accelerate and widen the break-through. To this end, the following offers an introduction to the main line of the new theory for a wide class of scientists and describes some of its analytical challenges.

Since the earlier, advanced reviews have proven so difficult for all but specialist aerodynamicists, the following is restricted to just the basic theory in its simplest setting. This offers an opportunity, also here seized, of giving a clarified, new development of the main theory from very few premises to illuminate its logical structure.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STEWARTSON'S TRIPLE DECK

R. E. Meyer

1. Introduction

The main concern of boundary-layer theory during its first half-century has been with fluid motion at large Reynolds number Re past streamlined bodies. In this context, the term 'streamlined' is meant to indicate that the motion, except very close to the body and in a thin wake, is almost independent of Re and can therefore be determined closely without reference to the boundary-layer development. The practical scope of the theory is severely limited, however, by such a condition of 'weak interaction' in the sense that the boundary layer and wake have only a minor influence on the rest of the flow.

The primary aim was to understand the aerodynamics of wings at small incidence (Figure 1) and observation soon confirmed the striking success of Prandtl's theory in this context. As the incidence is increased, however, a stage comes where the airfoil "stalls". Indeed, if the airfoil of Figure 1 be rotated through 90° to a position broadside to the incident stream (Figure 2), intuition and experience predict a flow pattern in which the stream has "broken away" entirely from the body at what had been originally its nose and tail. At an intermediate inclination of the airfoil (Figure 3), the incident stream will whet the front part of the upper airfoil side, but will break away from it at some point which inviscid fluid dynamics by itself has not been

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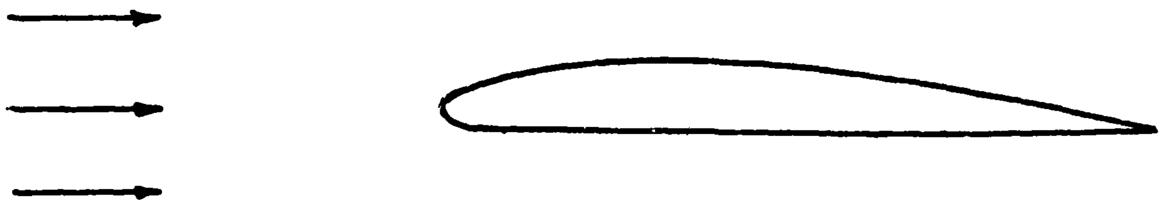


Fig. 1

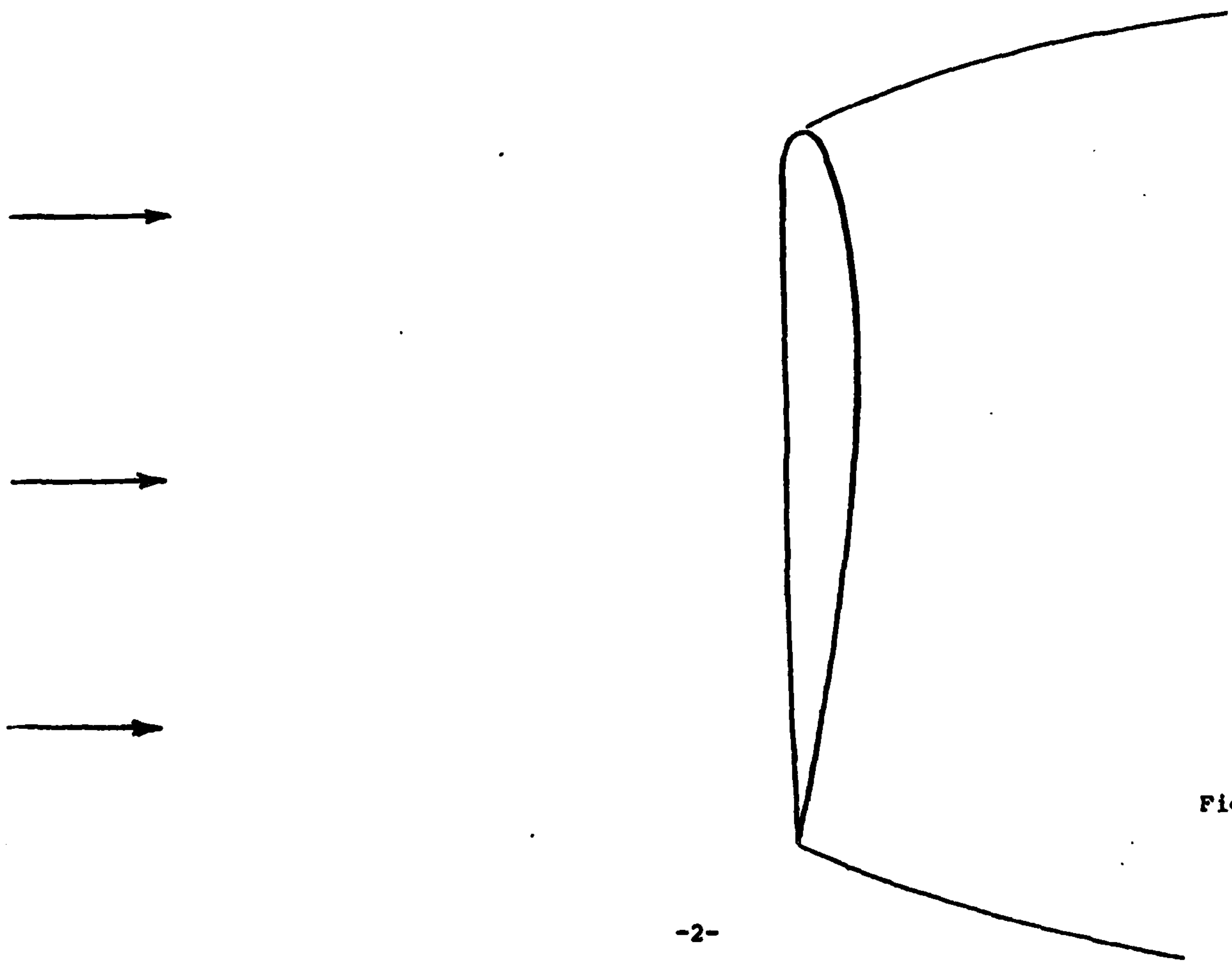


Fig. 2

Fig. 3

able to determine. The boundary layer on the body and the rest of the flow then 'interact strongly' in the sense that no approximation to either is determinable independently of the other.

It is readily apparent from everyday experience that a similar principle of strong interaction applies to many real fluid motions. For instance, human breathing must involve such "breakaway" of a stream in an essential way, because otherwise, we could only inhale the same body of gas that we have just exhaled. Despite all the aeronautical triumphs, therefore, the failure of the first half-century of boundary-layer theory to advance beyond the weak-interaction premise has left the understanding and prediction of most real fluid motions still in a rudimentary stage.

This challenge and need to come to grips with strong interactions has fuelled a cooperative research effort over two decades under the leadership of K. E. Stewartson, which has resulted in a major step forward. The following aims to open a readier access to this step for a wider audience, because one can hardly fail to be impressed by the work achieved, to enjoy the elegance of the new ideas taking shape, and to be intrigued by the novel challenges to mathematics emerging.

The present survey is, of course, far from being the first and owes a particular debt to those of Messiter [1978] and Stewartson [1975]. The early reviews, however, aimed mainly at redirecting research and were addressed to those steeped in the work, in the first place. Later reviews were burdened by the sheer bulk of all the recent results. This was aggravated by the fact that the earlier, clear successes of the theory concerned supersonic aerodynamics, where the theory rests on a rather large body of knowledge commanded normally only by professional aerodynamicists. To avoid this substantial barrier to access, the introduction to follow is limited to low-speed flow, with only a few remarks on the supersonic successes in Section 8. It attempts a survey taking little for granted beyond the Navier-Stokes equations and rudimentary notions of fluid dynamics of wide familiarity. And it aims only at an illumination of the main features of the rational structure of the theory to the degree that they have yet emerged. [Reference, accordingly, is made only to books and review articles, rather than original sources.]

The next section outlines the limit concept underlying Prandtl's boundary-layer theory for weak interaction. Section 3 recalls certain aspects of that theory which are of particular relevance to the present purposes. Those begin to be addressed in Section 4 by the introduction of a triplet of notions which appear essentially sufficient to characterize the new theory. Section 5 shows how their application to the Navier-Stokes equations results in a need for more than two, simultaneous boundary-layer limits. A third, needed limit is developed in Section 6 and shown to complete a definite triple-deck of limit concepts, but all the same, to leave us far short of a definite system of differential equations for a description of the flow. Section 7 outlines how an exploration of the interaction between the limit concepts ties up the loose ends into a boundary layer problem of mathematically quite novel type. It has been explored successfully for supersonic flow (Section 8), but for low-speed flow, turns out to be a dead-end. In Section 9, finally, an attempt is made to sketch Sychev's ideas on how the impasse might be broken to nail down an important step in the understanding of fluid motion.

In a way, the whole present enterprise may be described as an attempt to enable the non-specialist reader to benefit thoroughly from such surveys as Stewartson [1975], Messiter [1978] and Stewartson [1981]. To that end, an

Appendix adds a succinct derivation of the Illingworth-Stewartson transformation to give a first impression of how that transformation removes the barriers to a direct extension of triple-deck theory to compressible and supersonic flow. Its development for unsteady and three-dimensional flow would not yet appear to have reached a stage suitable for an introductory review.

2. The Limit Concept

To fix the ideas, the discussion will be focused on the example of flow past an airfoil much like that indicated in Figure 1. The motion will also be assumed two-dimensional, incompressible and steady in the pilot's frame of reference, to which it will be referred throughout. Let U denote the flight speed (the wing is envisaged as moving into air at rest, far ahead) and x,y, Cartesian coordinates measured in units of a body dimension L from a suitable point on the body surface. Mass conservation is satisfied by deriving the (non-dimensional) velocity components u,v in the respective directions of x and y increasing by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x \tag{1}$$

from a stream function $\psi(x,y)$ measured from the body surface in units of UL. If ρ,μ and $p(x,y)$ denote respectively, the constant density and viscosity and the local pressure measured in units of ρU^2, the overall Reynolds number is $\rho UL/\mu = Re$ and momentum conservation is expressed by the incompressible Navier-Stokes equations,

$$\begin{aligned} \psi_y\psi_{xy} - \psi_x\psi_{yy} &= -p_x + Re^{-1}(\psi_{yxx} + \psi_{yyy}) \\ \psi_x\psi_{xy} - \psi_y\psi_{xx} &= -p_y - Re^{-1}(\psi_{xxx} + \psi_{xyy}) \end{aligned} \tag{2}$$

where subscripts denote partial differentiation, with boundary condition $u = v = 0$ on the body surface.

The notion of limit most commonly employed in Boundary-Layer Theory, and briefly referred to as Lim hereafter, is that of a 'stretching' transformation

$$\begin{aligned} x^* &= x/X(Re), \qquad y^* = y/Y(Re) \\ \psi(x,y) &= \sigma(Re)\, \psi^*(x^*,y^*) \\ p(x,y) &= p_e(x,y) + \pi(Re)\, p^*(x^*,y^*) \end{aligned} \tag{3}$$

such that the limit $Re \to \infty$ of the transformed equations with appropriate boundary conditions have a strict solution $\psi^*(x^*,y^*)$, $p^*(x^*,y^*)$, that is, p^* is differentiable and ψ^* is thrice differentiable for all x^* and y^* in the fluid domain.

Superficially, of course, this notion looks quite improper in the sense of routine mathematics, since it concerns the limit of the differential equation instead of the limit of its solution. 'Proper' mathematical work on such equations, however, has long been proven to be hopeless until an understanding has been gained of the real issues that it must address. Meanwhile, the 'improper' limit notion has proven itself a thoroughly reliable and fruitful guide to the discovery of candidates for the role of approximate solution of (2), which are a greatly needed starting point for rigorous work. Once this key shift of objective towards the <u>discovery of candidates</u> be accepted, the theory to be outlined will be found to have a remarkable degree of logical coherence, even though the present account is not arranged to emphasize that aspect.

One obvious example of such a Lim is, of course, the 'external flow' in which x, y, ψ and p themselves are kept fixed as $Re \to \infty$, so that in (3),

(4) $$X \equiv Y \equiv \sigma \equiv 1, \quad \pi \equiv 0$$

and $\psi = \psi_e(x,y)$, $p = p_e(x,y)$ describe the classical, irrotational flow past the geometrical airfoil shape with circulation determined by the Kutta-Joukowsky condition.

Another familiar example is Prandtl's boundary layer, for which it is convenient to take the direction of x increasing tangential to the airfoil surface in the sense of the local flow and that of y increasing, normal to that surface and pointing into the fluid. Prandtl's Lim is then characterized by the stretching transformation

(5) $$X = 1, \quad Y = \sigma = \pi = Re^{-1/2}$$

of the local coordinates.

To keep confusion between different Lim's at bay, an asterisk will refer to the generic definition (3), but a subscript i will denote its particular case (5), e.g., $\psi_i(x_i,y_i)$ denotes the special case (5) of $\psi^*(x^*,y^*)$, and $v_i = -\partial\psi_i/\partial x_i$, etc. The subscript e only serves to recall the shift of emphasis which interpretation of (4) as a Lim introduces into the aspect of quantities already considered in (2).

The main content of the weak-interaction principle is that the two Lim's (4) and (5) together give a complete picture of the flow, and that brings up an important conceptual point intrinsic to the limit motion. The two Lim's represent different views of the <u>same</u> solution of the Navier-Stokes equations (2), different magnifying glasses by which we focus on different features of the motion so as to gain a greatly increased understanding of what goes on in the fluid by the comparison and correlation of the two views. It follows that the two views must display a <u>consistency</u> consonant with the nature of the solution of (2) which they illuminate. The diffusive (parabolic) structure of (2), in turn, implies that the solution of (2) must be a smooth function of x and y.

A simple illustration of such consistency is provided by the observation that (1) implies v (by contrast to v_i) tends to zero for all y_i under the Prandtl stretch (5) as $Re \to \infty$. Since $v(x,y)$ is smooth, its external-flow limit $v_e(x,y)$ must also be zero at those values of y corresponding to values of y_i, that is, at $y = 0$. The familiar, zero-normal-velocity boundary condition of classical hydrodynamics then re-emerges as a consistency condition implied by the Lim-definition together with the weak-interaction principle.

In turn, the weak-interaction principle is also meant to postulate that the external flow adheres to the airfoil without breakaways and tends to a uniform one at large distances from the body. The condition of zero v_e at the body surface then suffices to determine it uniquely. In particular, the values of p_e and u_e along the body surface are thereby determined. For the pressure, that is already anticipated by the notation of (3) and (4), but for the tangential velocity component u, it entails a further consistency condition: since $u_i = u$ under the Prandtl-stretch (5), the composite picture of the two Lim's can represent a smooth solution of (2) only if $u_i = u_e$ for all x at those y belonging to both Lim's, i.e., as $y_i \to \infty$ but $y_e = y \to 0$ (see (6d) below).

Such consistency implications of the limit notion play a key part in the logical structure of the theory to be outlined below. Their technical name is matching conditions, and their mathematical nature has been explored carefully by Lagerstrom, Kaplun and others [Eckhaus 1979] because matters can get complicated when the limit notion is pushed to high orders of approximation. For the present purposes, however, no such complications arise and it may be sufficient to mention only one further aspect: if consistency of two Lim's turns out impossible, then there are only two alternatives -- one of the Lim's must be rejected as a false lead on the way to discovery of a candidate, or a third Lim must intervene between the first two to reconcile them.

3. Weak Interaction

A fairly full account may be found in many books [e.g., Goldstein 1938, Lagerstrom 1964, Stewartson 1964, Meyer 1971] of how the limit notion and weak-interaction principle lead uniquely from the Navier-Stokes equations (2) to a set of limit-differential equations for the Lim characterized by (5). Those Prandtl equations are

(6a) $$u_i = \partial\psi_i/\partial y_i, \quad v_i = -\partial\psi_i/\partial x_i$$

(6b) $$u_i\partial u_i/\partial x_i + v_i\partial u_i/\partial y_i = -dp_e/dx_i + \partial^2 u_i/\partial y_i^2$$

(6c) $$u_i(x_i,0) = v_i(x_i,0) = 0$$

(6d) $$u_i(x_i,y_i) \to u_e(x_i) \quad \text{as} \quad y_i \to \infty$$

where $u_e(x_i)$ is the surface value of the velocity of the external flow, related to its surface pressure $p_e(x_i)$ by Bernoulli's equation

(7) $$p_e + \frac{1}{2}u_e^2 = \text{const.}$$

This is a simpler system of nonlinear diffusion equations than (2), to be complemented by specification of $u_i(0,y_i)$ for all $y_i > 0$ as an initial condition. It should also be recalled from the discussion of the preceding Section that, from the viewpoint of the Prandtl Lim, u_e and p_e are known quantities in (6b), (6d).

It will be helpful to remark here on three further aspects of Prandtl's Lim which are of particular relevance to the later discussion. While our local Cartesian coordinates have a strictly global interpretation only for the limiting case of an airfoil of the shape of a flat plate, a closer exploration of the influence of surface curvature [e.g. Goldstein 1938] shows it not to affect the theory to the order of approximation considered in this account, as long as the body shape is independent of Re -- as will naturally be envisaged -- and the surface curvature is finite everywhere except at a trailing edge. The underlying reason flows, of course, from the fact that a Lim like

Prandtl's uses only a coordinate system covering distances y_i from the body surface that tend to zero by comparison with the surface's radius of curvature as $Re \to \infty$: when viewed through such a microscope, the body surface looks flat. The result of the curvature estimates is that the x_i, y_i-coordinate system can serve as a semi-local one for all the Lim's to be here considered (except the external flow) in which x^* measures distance along the whole (upper or lower side of the) body surface and y^*, distance normal to it, and that interpretation will be used henceforth. To the order of approximation considered in the following, moreover, the Lim-definition can be applied to (2) without regard to curvature corrections.

Secondly, an interesting facet of the weak interaction emerges from the consideration of consistency to a second approximation. Though $v = -\partial\psi/\partial x$ tends to zero under the Prandtl Lim (5) as $Re \to \infty$, the same does not follow for $v_i = -\partial\psi_i/\partial x_i = Re^{1/2}v$, and that function will be determined whenever Prandtl's equations (6a) - (6d) have a unique solution. Its limit $v_i(x_i,\infty)$ may be finite and non-zero, in which case "zero normal velocity" ceases to be the exact boundary condition for the external flow at finite Re. Instead, the body surface assumes a 'permeable' aspect, from the point of view of the external-flow Lim (4), and looks as if there were a small, normal velocity $v_e(x,0) = Re^{-1/2}v_i(x_i,\infty)$ across it. Its determination is commonly approached by considerations of mass-flow balance. Since (6c), (6d) show u_i to fall from u_e, at large y_i, to zero, at $y_i = 0$, it may be anticipated that $y_i u_e(x_i)$ will generally exceed the stream function $\psi_i(x_i,y_i)$, although both grow with y_i at the same rate as $y_i \to \infty$, by (6d). That suggests consideration of

(8) $$\lim_{\substack{y_i \to \infty \\ y \to 0}} (u_e y - \psi)/u_e = \delta(x_i)$$

which characterizes the mass-flow deficiency of the boundary layer, and $Re^{1/2}\delta(x_1)$ is normally found to be a well-defined, positive function in a weak interaction. In its terms, the surface-value of the normal velocity for the external flow becomes

$$v_e(x,0) = (d\delta/dx_1)u_e(x,0) \tag{9}$$

and the boundary-layer mass-flow deficiency is then seen to be interpretable, from the viewpoint of the external-flow Lim, either as a permeable body surface with volume-outflow rate $v_e(x,0)$ per unit distance along the surface and unit span normal to the flow plane, or as a solid-body surface displaced from $y = 0$ by a distance $\delta(x_1) = 0(Re^{-1/2})$. Aerodynamicists have taken to the latter interpretation and as a result, mass-flow effects in the theory are normally discussed in terms of this displacement thickness δ.

As emphasized in the preceding Section, the known arguments leading to Prandtl's equations (6) prove only that those equations identify the unique candidates for approximate solutions of (2) of the general type contemplated, if there are any, but that has not yet been established rigorously. Historically, the value of Prandtl's equations was first demonstrated by experiments and by technology based on them. Applied mathematicians then explored the equations quantitatively and purer mathematicians turned to the task of proving that their solutions are approximate solutions of the Navier-Stokes equations. When that failed, they discovered that Prandtl's equations themselves have mathematical interest. Oleinik [1963] proved existence and Walter [1970] found a substantially simpler proof. Nichel [1958] established uniqueness and Serrin [1967] and Pelletier [1972] attacked the key question of the asymptotics of solutions of (6). Pelletier was able to show with considerable generality that, in normal circumstances and for $dp_e/dx_1 < 0$, the influence of the initial condition is transient and with increasing x_1, the solutions develop an asymptotic character determined by the functions $p_e(x_i)$ and $u_e(x_i)$ related by (7). In this way, he illuminated the essential character of Prandtl's weak-interaction equations (6) as an inhomogeneous system of partial differential equations forced by the given functions dp_e/dx_i and $u_e(x_i)$ in (6b) and (6d).

All those theorems, however, amount to only half a loaf because they depend on a 'favorable' pressure gradient, $dp_e/dx_i < 0$. When forced by an 'adverse' pressure gradient, $dp_e/dx_i > 0$, the solutions develop differently, and our knowledge is more tentative. As x_i increases, the surface value of $\partial u_i/\partial y_i$ then decreases, and before it ceases to be positive, the solution of (6) tends to develop a singularity which is terminal in the sense that a continuation of the solution beyond it is not possible. Breakaway (Figure 3) involves a reversal of flow direction near the airfoil surface and hence, a negative 'wall shear' $\partial u_i/\partial y_i$, and before the solution of (6) can reach that, weak interaction appears to fail decisively.

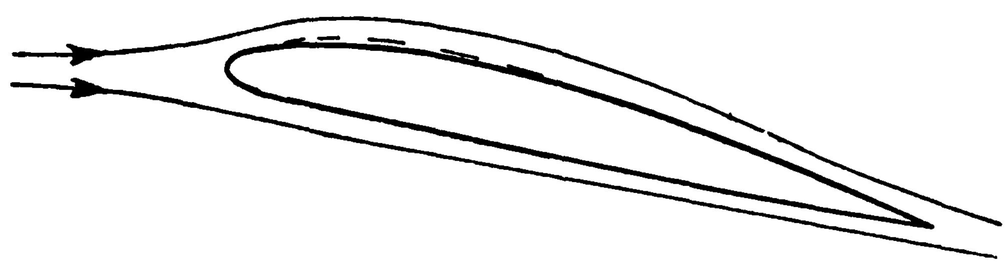
4. Mild Interaction

Strong interaction poses the formidable difficulty of interdependent non-uniqueness of both the external inviscid flow (Figure 3) and the viscous flow in boundary and shear layers. It is fortunate, therefore, that careful experiments have revealed an interval of 'pre-stall' incidences in which recirculation of the fluid is confined to a thin 'bubble' region adjacent to the airfoil surface (Figure 4). For part of this incidence interval, the observed bubble thickness does not greatly exceed the displacement thickness of the weak-interaction boundary layer ahead of the bubble and accordingly, the total mass-flow deficiency near the upper side of the airfoil remains of the same order as in a weak interaction, and the first approximation to the external flow therefore remains the classical potential flow past the geometrical airfoil.

Such observations demonstrate that the key-process of 'separation' -- reversal of flow direction and recirculation of fluid close to the body surface -- can occur without full break-away (Figure 3) and without really strong interaction between boundary layer and external flow. An opportunity thus appears of dividing the difficulties of strong interaction by studying first the flow at pre-stall incidence and in particular, by focusing attention on the bubble ends, where a merely local breakdown of weak interaction seems to occur. This opportunity to explore the local mechanism of separation was seized by K. E. Stewartson and the aim of the present account is to outline the main ideas of the theory so far developed under his leadership.

To understand a little better how he succeeded in gaining a grip on the problem, it may be noted that the bubble thickness (Figure 4) is not observed to vary strongly over most of its length. This indicates that the rate of change of displacement thickness, $d\delta/dx_1$, also remains of the order of

Fig. 4



magnitude typical of weak interaction, except perhaps at a bubble end. As expected from weak-interaction experience, on the other hand, computation of the solutions of (6) along the front part of the airfoil surface runs into the terminal singularity as the observed, front bubble end with its reversal of flow direction is approached. That would seem to indicate a growth of x-derivatives of velocity or pressure to levels beyond those which a weak interaction can accommodate. In the magnifying glass of the Prandtl Lim, this would look like a singularity, and its resolution might require a stretching transformation also of x. These plausible interpretations of the experimental and theoretical oracles suggest the following three notions.

(I) The displacement thickness remains of order $Re^{-1/2}$ throughout.

(II) The neighborhood of a bubble end is a region of velocity or pressure changes that are stronger than weak interaction can describe, but not strong enough to affect the external-flow limit. The possibility needs consideration that those changes might penetrate farther into the fluid than does a weak interaction, but not to distances independent of Re (which the external-flow Lim would see).

(III) The region of those stronger changes near a bubble end appears quite short also in the direction along the surface, by comparison to the weak-interaction boundary layer and to the bubble (Figure 4). That exceptional region, in which the apparent singularity is resolved, may plausibly need to be thought of as shrinking to a point as $Re \to \infty$.

One of the aims of the present account is to show that these three notions can serve as the essential basis of the theory to be outlined and may therefore be considered to embody a definition of what will, for brevity, be called mild interaction. To clarify appeal to the components of this definition in the following discussion, they will be referred to briefly as

(I) Mass-flow bound,

(II) Penetration,

(III) Localization.

To simplify the presentation, it will be very helpful to restrict its scope by three working principles complementary to the main definition. Attention will be confined to the front end of the bubble, which is easier to think about because it involves a process growing out of a standard, unseparated, weak-interaction boundary layer. This identifies an asymptotic, initial condition for the mild interaction, which complements the definition and will be referred to as

(IV) Upstream condition.

Next, the notion of Lim (Section 2) admits a very large class of stretching transformations. It has been demonstrated in another context [Meyer and Wilson 1973] how it can be treated systematically to sort out uniquely those transformations of primary significance for a problem at hand. Such an approach, however, is a research tool for settling technical controversy and would lengthen the present discussion so as to obscure the matters of real interest. As a working principle, an a-priori restriction of the stretching functions X, Y, σ and π of Re in (3) to powers of Re will therefore be adopted. It should be remarked immediately that the advanced theory of mild interaction bristles with logarithms of Re, but for the present purposes, powers will turn out to be adequate.

This still leaves a large class of Lim's and not all of them can plausibly be of equal importance. Many will have to be discarded as inconsistent with the governing equations and other conditions of the problem that have been identified already. Others will be merely limits of other

Lim's that offer more complex and useful information and insight. These aspects of the Lim notion have been explored by Lagerstrom, Kaplun and others [Eckhaus 1979] and it would again be unprofitable for present purposes to enter upon them at length. Instead, the working principle will be adopted that the discussion is to be firmly directed towards a description of the mild interaction by the <u>minimum number</u> of Lim's. The technical term for them is 'significant limits' [Eckhaus 1979].

5. Two Decks

To apply now the notions formulated in Sections 2 and 4, begin by noting that the localization condition (III) implies a stretching function

$$X = Re^{-\alpha}, \qquad \alpha > 0$$

in (3), if x be measured from a suitable point in the mild interaction region.

The upstream condition (IV) requires consideration of Prandtl's Lim, for which $Y = \sigma = Re^{-1/2}$, by (5). The transformation for this "main deck" $\widetilde{\text{Lim}}$ is therefore

$$\begin{aligned} \tilde{x} &= xRe^{\alpha}, \qquad \tilde{y} = yRe^{1/2} \\ \psi(x,y) &= \tilde{\delta}\tilde{\psi}(\tilde{x},\tilde{y}) \qquad\qquad (10) \\ p(x,y) &= p_e(x) + \tilde{\pi}(Re)\tilde{p}(\tilde{x},\tilde{y}) \end{aligned}$$

with $\tilde{\delta} = Re^{-1/2}$ and $p_e(x)$, $u_e(x)$ used as in Section 3 to denote the known, surface values of pressure and tangential velocity of the external flow. Since $x \to 0$ as $Re \to \infty$ at all $\tilde{x}$, only the values $p_e(0)$ and $u_e(0)$ actually enter into the argument and these will be denoted briefly by p_e and u_e.

Since (10) fails to account for the penetration condition (II), consideration must also be given to another transformation that can bring greater distances from the body surface into view, and this is called the "upper deck" $\widehat{\text{Lim}}$. At such greater distances, the upstream condition (IV) implies that the streamlines came from the external flow because the mass-flow bound (I) excludes a displacement of streamlines near the body surface by much more than $\delta = 0(Re^{-1/2})$. By Kelvin's theorem, therefore, the flow in the upper deck remains an irrotational one, which has no distinguished direction of influence, and accordingly, $\widehat{\text{Lim}}$ cannot normally involve a different stretch for x and y and must have a transformation

$$\hat{x} = xRe^{\alpha}, \qquad \hat{y} = yRe^{\alpha}$$

(11) $$\psi(x,y) = u_e y + \hat{\sigma}(Re)\hat{\Psi}(\hat{x},\hat{y})$$

$$p(x,y) = p_e + \hat{\pi}(Re)\hat{p}(\hat{x},\hat{y})$$

with

(12) $$0 < \alpha < 1/2$$

in order that the upper deck do penetrate significantly beyond the main deck. The condition (I) that the mild interaction has a mass-flow deficiency $u_e y - \psi$ of the same order as that of a weak interaction (Section 3) implies

(13) $$\hat{\sigma} = \tilde{\sigma} = Re^{-1/2} \quad .$$

Further progress depends on direct use of the Navier-Stokes equations. If D^* abbreviates the convective derivative for general, stretched variables,

(14) $$D^* = u^* \frac{\partial}{\partial x^*} + v^* \frac{\partial}{\partial y^*} = \frac{\partial \psi^*}{\partial y^*}\frac{\partial}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial}{\partial y^*} \quad ,$$

then substitution of (10) into (2a) yields, by (13),

(15a) $$\tilde{D}\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} + \tilde{\pi}\frac{\partial \tilde{p}}{\partial \tilde{x}} = -Re^{-\alpha}p_e' - Re^{-\alpha}\left(\frac{\partial^3 \tilde{\Psi}}{\partial \tilde{y}^3} + Re^{2\alpha-1}\frac{\partial^3 \tilde{\Psi}}{\partial \tilde{x}^2 \partial \tilde{y}}\right)$$

$$\to 0 \quad \text{as} \quad Re \to \infty$$

by the Lim-definition of Section 2 and by (12). Similarly, substitution of (10) into (2b) and of (11) into (2a) and (2b) yields

(15b) $$-\tilde{D}\,\partial\tilde{\Psi}/\partial\tilde{x} + \tilde{\pi}\,Re^{1-2\alpha}\,\partial\tilde{p}/\partial\tilde{y} \to 0 \quad ,$$

(16a) $$\hat{D}\,\partial\hat{\Psi}/\partial\hat{y} + \hat{\pi}\,Re^{1-2\alpha}\,\partial\hat{p}/\partial\hat{x} \to 0 \quad ,$$

(16b) $$-\hat{D}\,\partial\hat{\Psi}/\partial\hat{x} + \hat{\pi}\,Re^{1-2\alpha}\,\partial\hat{p}/\partial\hat{y} \to 0 \quad \text{as} \quad Re \to \infty \quad .$$

Mainly through the localization condition (III), the enhanced streamwise changes in the mild interaction are thus seen to overshadow the viscous shear in both decks so that both are governed by inviscid limit equations!

To learn more about $\bar{\pi}$ and $\hat{\pi}$ in (15) and (16), we may now turn to a survey of alternatives typical of Lim-arguments. If $\hat{\pi}\,Re^{1-2\alpha} \to 0$ as $Re \to \infty$, (16a,b) would imply mere convection of a velocity perturbation already present upstream, and by (IV), could describe only the constant, weak-interaction correction to u_e and $v_e = 0$. With such a choice of stretching function, therefore, the transformation (11) cannot give nontrivial information on the velocity perturbation of the upper deck. If $\hat{\pi}\,Re^{1-2\alpha} \to \infty$, on the other hand, (16a,b) would imply $\hat{p} \equiv$ const, which could describe no more than the weak-interaction correction of p_e and therefore, would make the transformation (11) equally uninformative. The only stretch giving nontrivial information can therefore be $\hat{\pi} = Re^{2\alpha-1}$.

Similar information on $\bar{\pi}$ may now be deduced by appeal to consistency. As a preliminary remark, recall that the limit equations (16a,b) are an inviscid, Euler system and, by the upstream condition (IV) and Kelvin's theorem, must describe a potential-flow perturbation. This could be determined uniquely by specification of the 'surface' pressure distribution, $\hat{p}(\hat{x},0)$, together with the condition that the perturbation decays to zero as $\hat{x}^2 + \hat{y}^2 \to \infty$. Accordingly, $\hat{p}(\hat{x},0) \not\equiv 0$, for otherwise the perturbation would vanish identically and there would be no upper deck and no mild interaction. By (10) and (11), however, consistency of the representation of pressure as a continuous function requires

$$\hat{\pi} \lim_{\hat{y}\to 0} \hat{p}(\hat{x},\hat{y}) = \bar{\pi} \lim_{\tilde{y}\to\infty} \tilde{p}(\tilde{x},\tilde{y}) \quad ,$$

unless another Lim intervenes between the two decks. By the working principle that we look for the minimal number of Lim's, the argument against such intervention is omitted. Since $\hat{p}(\hat{x},0) \not\equiv 0$ and the Lim-definition requires $\tilde{p}$ to be bounded on compacts, at least, it follows that

(17) $$\tilde{\pi} = \hat{\pi} = Re^{2\alpha-1} \quad .$$

By (12) the limit equations (15), (16) for the upper and main decks, respectively, now become

(18a) $$\hat{D}\partial\hat{\psi}/\partial\hat{y} + \partial\hat{p}/\partial\hat{x} \to 0 \quad ,$$

(18b) $$-\hat{D}\partial\hat{\psi}/\partial\hat{x} + \partial\hat{p}/\partial\hat{y} \to 0 \quad ,$$

(19a) $$\tilde{D}\partial\tilde{\psi}/\partial\tilde{y} \qquad \to 0 \quad ,$$

(19b) $$-\tilde{D}\partial\tilde{\psi}/\partial\tilde{x} + \partial\tilde{p}/\partial\tilde{y} \to 0$$

as $Re \to \infty$, for all $\hat{x}$ and $\tilde{x}$ and for all $\hat{y}$ and $\tilde{y} > 0$.

The plausible suspicion that inviscid equations cannot adequately describe a boundary-layer process can be raised into more plastic relief by some striking conclusions following readily from (19a,b). From (19a), $\partial\tilde{\psi}/\partial\tilde{y} = \tilde{u}$ is merely convected through the main deck from the weak interaction upstream, so that it is a function only of $\tilde{\psi}$,

(20) $$\tilde{u} = \partial\tilde{\psi}/\partial\tilde{y} = f(\tilde{\psi}) \quad .$$

By another interpretation of (14), it is also seen from (19a) that the 'streamline slope', defined as

(21) $$-(\partial\tilde{\psi}/\partial\tilde{x})/(\partial\tilde{\psi}/\partial\tilde{y}) = \theta \quad ,$$

is independent of $\tilde{y}$. The same follows for $\partial\theta/\partial\tilde{x}$, and by (19b),

$$\partial\theta/\partial\tilde{x} = -(\partial\tilde{\psi}/\partial\tilde{y})^{-2}\,\partial\tilde{p}/\partial\tilde{y} \quad ,$$

so that $\partial\tilde{p}/\partial\tilde{y}$ is also independent of $\tilde{y}$. By (10), (11) and (17), however, consistency of the pressure representation requires

$$\lim_{\tilde{y}\to\infty} \partial\tilde{p}/\partial\tilde{y} = Re^{\alpha-1/2} \lim_{\hat{y}\to 0} \partial\hat{p}/\partial\hat{y} \to 0$$

by (12). Accordingly, $\partial\tilde{p}/\partial\tilde{y} \equiv 0$ and

(22) $$\theta \text{ and } \tilde{p} \text{ are independent of } \tilde{y} \quad ,$$

the <u>main deck merely transmits</u> the perturbations of <u>pressure and streamline slope</u> from the upper deck towards the body surface.

Now, the Navier-Stokes equations (2) express conservation of momentum by a balance of pressure, momentum and viscous stress. By their boundary condition at the body surface, the momentum near it must be arbitrarily small, and nontrivial pressure changes in a mild interaction can, by (22), be balanced near the body surface only by another deck of enhanced shear stress.

Hindsight now permits us to make a thumbnail sketch of the proof of this Section, which may help to pull its threads together and also to illuminate both the detailed reasoning behind the mild-interaction premises (I) - (III) and the possibilities of reducing those premises further. The short extent of a mild interaction prevents viscous shear from having a substantial, direct influence on the part of the bubble-end process visible in the classical Prandtl limit. As a result, his limit describes no more than a short stretch of weak interaction, unless significant perturbations extend beyond his boundary-layer thickness. Nontrivial perturbations extending beyond it, on the other hand, make the process visible in Prandtl's limit one of mere transmission, and momentum conservation close to the body surface is not then possible without enhanced, viscous shear there.

6. Lower Deck

Shear stresses substantially exceeding those of a weak interaction (Section 3) require $Re^{-1}\partial^2\psi/\partial y^2 \to \infty$ as $Re \to \infty$ and the third deck exhibiting them must be a $\bar{\text{Lim}}$ with transformation of form

$$
(23) \qquad \begin{aligned} &\bar{x} = x\, Re^{\alpha}, \qquad \bar{y} = y\, Re^{\beta}, \qquad \beta > \frac{1}{2} \\ &\psi(x,y) = \bar{\sigma}(Re)\bar{\psi}(\bar{x},\bar{y}), \qquad p(x,y) = p_e + \bar{\pi}\,\bar{p}(\bar{x},\bar{y}) \quad , \end{aligned}
$$

with again

$$\bar{\pi} = Re^{2\alpha-1}$$

for consistency with the transmitted pressure. Since $\beta > \frac{1}{2}$, (10) shows that (23) resolves the limit $\tilde{y} \to 0$ of the main deck.

The mass-flow in this lower deck depends on the particular nature of the weak interaction from which the mild interaction develops. The Reader will have grown impatient to see the issues resolved in one case, without enumeration of alternatives, and with attention already focused on the front end of the bubble (Figure 4), it will now be further restricted to the most common case of weak interaction with normal type of skin friction, $y_i^{-1}\partial\psi_i/\partial y_i \to \tau_i(x_i)$ as $y_i \to 0$. If $\tau_i(0) = \tau$, briefly, it follows from the upstream condition (IV) and from (10) and (12) that the main deck must also have

$$
(24) \qquad \lim_{\tilde{y}\to 0} (\tilde{y}^{-1}\partial\tilde{\psi}/\partial\tilde{y}) \to \tau \quad \text{as} \quad \tilde{x} \to -\infty \quad .
$$

In terms of $\tilde{\psi}$ and $\tilde{u} = \partial\tilde{\psi}/\partial\tilde{y}$, this relation is

$$\lim_{\tilde{\psi}\to 0} (\tilde{\psi}^{-1/2}\partial\tilde{\psi}/\partial\tilde{y}) = (2\tau)^{1/2} \quad ,$$

first as $\tilde{x} \to -\infty$, and then by (20), throughout the mild interaction. As $\tilde{y} \to 0$ and $\bar{y} \to \infty$, it follows from (10) and (23) by consistency that

$$\bar{\sigma}\, Re^{\beta}\, \partial\bar{\psi}/\partial\bar{y} \sim (2\tau\, \bar{\sigma}\, \bar{\psi}/\bar{\delta})^{1/2} \quad \text{as} \quad \bar{y} \to \infty$$

and then from (13) that

(25) $$\bar{\sigma} = Re^{-2\beta+1/2} \quad \text{and} \quad \partial\bar{\psi}/\partial\bar{y} - \tau\bar{y} \to A(\bar{x}) \quad \text{as} \quad \bar{y} \to \infty \; .$$

Substitution of (23) into (2a,b) now yields, by a computation analogous to that of Section 5,

(26) $$\bar{D}\,\frac{\partial\bar{\psi}}{\partial\bar{y}} + Re^{2(\alpha+\beta-1)}\,\frac{\partial\bar{p}}{\partial\bar{x}} - Re^{3\beta-\alpha-3/2}\,\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3} \to 0$$

$$\partial\bar{p}/\partial\bar{y} \to 0$$

as $Re \to \infty$. Just as in Section 5, the alternatives may now be explored to see fairly readily that positive or negative choices of $\alpha+\beta-1$ or $\beta - 1/2 - \alpha/3$ imply truncations of (26) that are either degenerations inconsistent with a mild interaction or are reduced forms of (26) describing only sublayers $\bar{y} \to 0$ or $\bar{y} \to \infty$ of (23). The choice

$$\alpha+\beta-1 = \beta - \frac{1}{2} - \alpha/3 = 0$$

is thus seen to yield the only significant $\overline{\text{Lim}}$ for the lower deck, and it implies

(27) $$\alpha = \frac{3}{8},\; \beta = \frac{5}{8},\; \bar{\sigma} = Re^{-3/4},\; \bar{\pi} = \tilde{\pi} = \hat{\pi} = Re^{-1/4} \quad ,$$

by (25) and (17).

All three stretching transformations (10), (11) and (23) have thus been determined from the Navier-Stokes equations, the definitions of the limit notion and the mild interaction, and the upstream condition (IV) with normal type of skin friction. Such a mild interaction must penetrate beyond the normal boundary layer thickness of order $Re^{-1/2}$ to distances of order $Re^{-3/8}$, it must involve three significant decks and must exhibit a much enhanced pressure perturbation of order $Re^{-1/4}$ (by contrast to that of order $Re^{-1/2}$ in a weak interaction).

From (26) and (14), the lower deck must be governed by the limit equations

(28a) $$\bar{u} = \partial\bar{\psi}/\partial\bar{y}, \qquad \bar{v} = -\partial\bar{\psi}/\partial\bar{x} \quad ,$$

(28b) $$\bar{u}\partial\bar{u}/\partial\bar{x} + \bar{v}\partial\bar{u}/\partial\bar{y} = -\partial\bar{p}/\partial\bar{x} + \partial^2\bar{u}/\partial\bar{y}^2 \quad ,$$

(28c) $$\partial\bar{p}/\partial\bar{y} = 0 \quad ,$$

which are of exactly the same form as Prandtl's equations (6a), (6b)! Their theory (Section 3) leaves no doubt that the physical body-surface condition of the Navier-Stokes equations,

(28d) $$\bar{u}(\bar{x},0) = \bar{v}(\bar{x},0) = \bar{\psi}(\bar{x},0) = 0$$

is appropriate to the lower deck equations and hence, there appears no call for further decks. The 'outer boundary condition' corresponding to (6d) is now (25), and since the lower deck in a weak interaction is merely the sublayer close to the body surface, the initial condition for (28) is (24) translated by (10) and (23) into its lower-deck representation,

(28e) $$\bar{u}(\bar{x},\bar{y}) \to \tau\,\bar{y} \quad \text{as} \quad \bar{x} \to -\infty \quad \text{for all} \quad \bar{y} > 0 \quad .$$

Comparison with (25) adds the information that $A(\bar{x}) \to 0$ as $\bar{x} \to -\infty$.

With demonstration of necessity and structure of the triple deck thus complete, it may be opportune to vent some qualms. Can powers as small as $Re^{1/8}$ plausibly distinguish orders of magnitude when the largest Reynolds numbers at which steady motion is practically realized do not much exceed 10^5? My own view is that the true asymptotic structure of a theory tends to emerge only at a late stage of development. It is reinforced when experimental results at even lower Re show agreement [Stewartson 1981] with theoretical distinctions based on even smaller powers of Re. The asymptotic notions in present use are not certain to be definitive, they are primarily a fruitful scheme of rational guidance to the discovery of candidates.

Another qualm is that the foregoing proof goes to show that the candidate discovered is the only one which can be consistent with the premises outlined in the early Sections. It does not address the question whether the fairly complicated system of equations (18), (19) and (28) with associated conditions is likely to be consistent, so that it can describe an actual candidate. In addition, (28e) sets an initial condition for the boundary-layer type equations (28) which is anomalous in the light of hydrodynamic stability theory: this 'boundary layer' is to start out with a 'profile' $\bar{u}(-\infty,\bar{y})$ of which every point is an inflection point.

A more urgent concern, however, arises from uniqueness. In Prandtl's equation (6b), dp_e/dx_i is a known forcing function, but no information concerning $\partial\bar{p}/\partial\bar{x}$ in (28b) is yet to hand, so that the lower-deck equations, at this point, are not a definite set of differential equations at all! Similarly, $u_e(x_i)$ is a known function in (6d), but the outer condition (25) for (28) contains an unknown function $A(\bar{x})$.

The upper-deck equations (18) are similarly indefinite. They describe a potential-flow perturbation, but no information is yet to hand on the shape of the 'body' past which this is a potential flow. Such a boundary condition for (18) as $\hat{y} \downarrow 0$ can arise only from the main deck, but that has been seen (Section 5) to be mainly a transmitting mechanism, so that the missing condition may be suspected to arise actually from the lower deck. The indeterminacies of upper and lower deck may, perhaps, be related?

7. Interaction

Apparently, a significant interaction between the decks implicit in the mild-interaction definition or limit notion remains to be uncovered. A natural starting point is the transmission statement (22) for the main deck. By (10), (11) and (13), the upper-deck representation for the main-deck streamline slope is

$$-\theta = \frac{\partial\tilde{\psi}/\partial\tilde{x}}{\partial\tilde{\psi}/\partial\tilde{y}} = \frac{\partial\hat{\psi}/\partial\hat{x} + \mathrm{Re}^{-2\alpha+\frac{1}{2}}\, u_e'\hat{y}}{u_e + \mathrm{Re}^{\alpha-\frac{1}{2}}\, \partial\hat{\psi}/\partial\hat{y}}$$

$$\to u_e^{-1}\partial\hat{\psi}/\partial\hat{x}$$

as $\mathrm{Re} \to \infty$, by (27). Consistency for the ratio v/u where main and upper deck merge therefore requires

$$\lim_{\hat{y}\to 0} \partial\hat{\psi}/\partial\hat{x} = -u_e \lim_{\tilde{y}\to\infty} \theta(\tilde{x},\tilde{y})$$

By (10) and (23), similarly,

$$-\theta = \mathrm{Re}^{-\beta+\frac{1}{2}}(\partial\bar{\psi}/\partial\bar{x})/(\partial\bar{\psi}/\partial\bar{y})$$

and some information on this ratio is obtainable from (25), which implies

$$\bar{\psi} \sim \frac{1}{2}\tau\,\bar{y}^2 - A(\bar{x})\bar{y} + B(\bar{x}) \quad \text{as} \quad \bar{y} \to \infty \quad ,$$

whence

$$(\partial\bar{\psi}/\partial\bar{x})/(\partial\bar{\psi}/\partial\bar{y}) \to A'(\bar{x})/\tau \quad \text{as} \quad \bar{y} \to \infty \quad ,$$

and by (27), consistency of v/u where main and lower deck merge requires

$$\theta(\tilde{x},0) = -\mathrm{Re}^{-1/8}A'(\bar{x})/\tau \quad .$$

This is a disappointing result because the transmission statement (22) for streamline slope only tells us now that

$$\lim_{\hat{y}\to 0} \partial\hat{\psi}/\partial\hat{x} \to 0 \quad \text{as} \quad \mathrm{Re} \to \infty$$

i.e., the potential-flow perturbation of the upper deck is a flow past a flat plate. It is not a good omen for the search for a non-vanishing upper deck.

The establishment of (27), however, has changed the basis for the proofs. While the limit arguments preceding (27) had to be based on comparison of powers of Re restricted only by inequalities, an opportunity now arises for re-examining the limits in the light of precise knowledge of those powers of Re. Indeed, a re-tracing of the argument leading from (15a) to the $\tilde{y}$-independence of (21) -- verbatim, except for the additonal fact that $\alpha = 3/8$ -- shows that also

$$\frac{\partial}{\partial \tilde{y}} (\mathrm{Re}^{1/8}\theta) = -\mathrm{Re}^{1/8}\left(\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\right)^{-2} \tilde{\delta} \frac{\partial \psi}{\partial \tilde{y}} \to 0 \quad \text{as} \quad \mathrm{Re} \to \infty$$

so that $\mathrm{Re}^{1/8}\theta$ is also transmitted unchanged across the main deck. The preceding calculation now supports the result

$$\lim_{\hat{y}\to 0} (\mathrm{Re}^{1/8}\partial\hat{\psi}/\partial\hat{x}) = u_e A'(\bar{x})/\tau \quad , \tag{29}$$

which expresses a mass-flow interaction between the lower and upper decks.

The upper deck perturbation is therefore a potential flow past a <u>permeable</u> flat plate $\hat{y} = 0$ through which passes an <u>outward volume-flow at a rate</u> $-\mathrm{Re}^{-1/8}u_e A'(\hat{x})/\tau$ per unit $\hat{x}$-distance (and unit span normal to the flow plane), where use has been made of the fact that $\bar{x} = \tilde{x} = \hat{x}$, by (10), (11) and (23). By standard potential theory, such a source distribution induces on the same surface $\hat{y} = 0$ an $\hat{x}$-component of velocity

$$\frac{\partial\hat{\psi}}{\partial\hat{y}} = \hat{u}(\hat{x},0) = -\mathrm{Re}^{-1/8}\frac{u_e}{\pi\tau}\int_{-\infty}^{\infty}\frac{A'(x_1)}{\hat{x}-x_1}\,dx_1 \quad , \tag{30}$$

where the integral denotes the Cauchy principal value, if it exists (and π is now used in its universal meaning).

To profit from the other half of the transmission statement (22), note that (23) and (27) cast Bernoulli's equation for the potential flow in the upper deck into the form

$$p + \frac{1}{2}(\partial\psi/\partial x)^2 + \frac{1}{2}(\partial\psi/\partial y)^2 = \text{const}$$

$$= p_e + \frac{1}{2}u_e^2 + Re^{-1/8}\, u_e \frac{\partial\hat{\psi}}{\partial\hat{y}} + Re^{-1/4}[\hat{p} + \frac{1}{2}(\frac{\partial\hat{\psi}}{\partial\hat{x}})^2 + \frac{1}{2}(\frac{\partial\hat{\psi}}{\partial\hat{y}})^2]$$

$$+ Re^{-1/2}u_e'\,\hat{y}(\partial\hat{\psi}/\partial\hat{x} + Re^{-1/4}u_e'\,\hat{y})$$

$$= p_e + \frac{1}{2}u_e^2$$

which is its value as $\hat{x} \to -\infty$. This is again of no direct help, but a re-examination of the transformation of (2) by (11), in the light now of (27), shows that (16) can be sharpened to

$$Re^{1/4}\,\hat{D}\begin{pmatrix}\partial\hat{\psi}/\partial\hat{y}\\ -\partial\hat{\psi}/\partial\hat{x}\end{pmatrix} + Re^{1/4}\begin{pmatrix}\partial\hat{p}/\partial\hat{x}\\ \partial\hat{p}/\partial\hat{y}\end{pmatrix} \to 0 \quad \text{as} \quad Re \to \infty \; .$$

If the classical Bernoulli calculation is applied to this, it supports the stronger statement

$$Re^{1/4}[p + \frac{1}{2}(\partial\psi/\partial x)^2 + \frac{1}{2}(\partial\psi/\partial y)^2 - p_e - \frac{1}{2}u_e^2] \to 0$$

and if (23) and (27) are used as before and note is taken of (29) and (30), the relation

$$\hat{p}(\hat{x},0) = -Re^{1/8}\, u_e\, \hat{u}(\hat{x},0)$$

of form familiar from linearized potential-perturbation theory results. This converts (30) into an equation for the pressure perturbation on the 'flat plate' $\hat{y} = 0$, and since the pressure stretch is the same for all decks, by (27), and the pressure perturbation is transmitted across the main deck, by (22), it is also an interaction condition on the lower-deck pressure,

(31) $$\bar{p}(\bar{x}) = \frac{u_e^2}{\pi\tau} \int_{-\infty}^{\infty} \frac{A'(x_1)}{\bar{x}-x_1}\, dx_1 \quad .$$

So, this Hilbert integral equation has to be added to (28a-e) and (25) to express a condition of mutual consistency betweer the three decks.

The initial similarity between Prandtl's equations (6) and the mild-interaction equations (28) is greatly reduced thereby, and the relevance of the weak-interaction knowledge (Section 3) for triple-deck theory is put in doubt. One of the main distinctions is that Prandtl's equations (6) are an inhomogeneous system forced by a known external flow, but the mild-interaction equations (25), (28), (31) are a homogeneous system and may therefore be suspected of admitting non-trivial solutions only for particular values of an as yet unidentified eigen-parameter. The combination of boundary-layer equations with a singular integral equation poses a novel mathematical challenge.

8. Successes and Failures

It will be recalled that the objective of the mild-interaction concept was to resolve local breakdowns of the weak-interaction principle and most of all, to elucidate the phenomenon of local separation characterized by reversal of flow directions and a region of recirculation of the fluid close to the body surface.

One of the simplest breakdowns of weak interaction occurs at the trailing edge of a flat plate placed edgewise in a uniform stream, where the boundary condition on the plane $y = 0$ changes suddenly from no-slip at the plate to no-shear at the wake-center. Here the triple deck has led to a rather convincing resolution [Stewartson 1981] of the weak-interaction singularity. Admittedly, it is not nearly as simple as the theory here discussed, since the analysis appears to require a 'staircase' of decks [Stewartson 1981], but the central role of the mild-interaction concept appears well assured. For instance, the predicted correction to Prandtl's flat-plate drag fits the experimental observation unbelievably well to Reynolds numbers as low as 10. The same analysis has been extended to wedge-shaped trailing edges of sufficiently small wedge-angle [Stewartson 1981]. While those are welcome results, however, they fall short of addressing the main issue because no separation occurs under these circumstances.

A much more resounding success has been achieved in the analysis of shock-boundary layer interaction in supersonic flight. Here the governing equations are much more complicated than (2) because the fluid motion is intrinsically coupled to thermodynamic processes, and an introduction to this branch of the theory would completely overload the present account with discussions of matters not central to the illumination of the triple-deck concept. If the restrictions be accepted, however, which support the

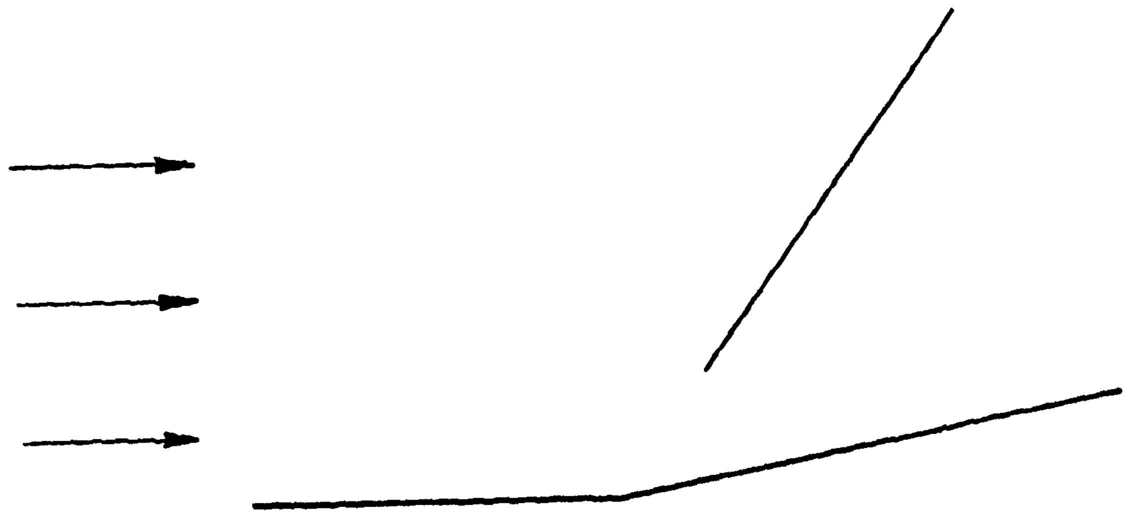
Illingworth-Stewartson transformation, then the compressible boundary-layer equations for weak interaction can be associated with a system of Prandtl equations (6) in a way explained briefly in the Appendix. This will make it easy to understand, at least in principle, why the theory set out in the preceding Sections furnishes an almost complete guide to parallel arguments extending it to compressible-flow problems. What emerges [Stewartson 1975, Messiter 1978] are the same triple-deck equations (25), (28) and this will be no surprise now, for they govern the lower deck in which the velocity must be small, in any case. There is one drastic change, however: the supersonic, potential-flow perturbation in the upper deck is governed by the wave equation, which replaces (30) by a strictly local relation leading to a much simpler interaction condition,

$$\bar{p}(\bar{x}) = -A'(\bar{x}) \tag{32}$$

in the place of (31). The supersonic triple-deck equations are therefore still a homogeneous system, by contrast with the compressible weak-interaction equations (Appendix), but they are a system only of differential equations.

There are several further, favorable circumstances in the supersonic case, and the simplest example for illustrating them may be the flow past a concave corner (Figure 5). For this, the wave equation and indeed, nonlinear inviscid supersonic theory [Courant and Friedrichs 1948], predict a very simple solution: two regions of uniform flow separated by a shock springing from the very corner and inclined at an angle related directly to the corner-angle. In reality, this can only be the asymptotic pattern far from the body surface because the effect of the surface boundary condition must be transmitted to the 'external' flow by a boundary-layer process, on which the precise location of the shock will depend, even though the ultimate shock-

Fig. 5



inclination does not. There is a sense in which this is a strong interaction, but clearly, the adjustment of the shock location is relatively easy to cope with.

The boundary-layer process, however, poses a more difficult problem when the corner is sharp (Figure 5). For if the radius of curvature of the metal surface is much smaller than the displacement thickness (Section 3), weak-interaction theory cannot serve and mild-interaction notions are the least that recourse is needed to.

Similar considerations apply to the more important example of shock-boundary layer interaction triggered by an incident shock (Figure 6). The external flow is then more complicated, but can still be synthesized from simple pieces readily adjustable to the boundary-layer process. A second, favorable feature, readily apparent also in the corner-flow, is that the whole process is of much more limited extent in the streamwise direction than a thin bubble (Figure 4).

The most important, favorable circumstance, however, was pointed out at an early stage by Oswatitsch and Wieghardt even before any equations relating to such interactions had ever been written down. The paradox was that the weak-interaction equations (Appendix) are parabolic for the boundary layer and hyperbolic, for the supersonic, external flow, whence mathematics proves that the corner (Figure 5) can have no influence on the flow upstream of it. Just that is observed, however, and to help explain it, they pointed out that the physical influences upon each other of the mechanisms of supersonic flow and (weak-interaction) boundary layer act in a sense favoring a self-contained, mutual interaction of these mechanisms. In first translating these physical considerations into formulae, Lighthill was able to give strong support to the plausibility of such a self-contained process indicating an eigensolution,

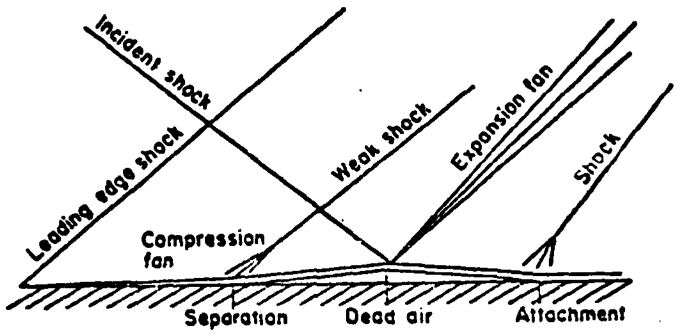


FIG. 1. Sketch of main properties of flow when a shock wave interacts with a laminar boundary layer.

Fig. 6. From K.Stewartson, Multistructured boundary layers on flat plates and related bodies, Adv.Appl.Mech. 14, p. 172, Academic Press, New York, 1974.

several properties of which he described. This gave a major impetus to Stewartson's formulation of the triple-deck concept and its application to shock-boundary layer interaction, which does involve separation.

It should be observed that the attendant reversal of flow direction near the body surface introduces a striking difficulty so far ignored in the present discussion. The boundary layer equations (6) or (28) are <u>parabolic</u> with a nonlinear diffusion coeficient related primarily to the tangential velocity component $u^* = \partial\psi^*/\partial y^*$ (as can be fully clarified by the Mises transformation given in the Appendix). In a weak interaction, $u^* > 0$ everywhere and the effective diffusion is positive. The reversal of flow direction inherent in separation, however, must make (28) describe a nonlinear 'heat-conduction' process that turns from forward to backward at a location which is obscure. Clearly, the standard coordinates parallel and normal to the body surface are then unsuitable variables, but discovery of a transformation resolving the difficulty at an affordable price does not seem imminent. Meanwhile, numerical experiments have had an understandable tendency to degenerate into a long brawl with unstable diffusion. ..

In addition, shock-boundary layer interaction is a much more complicated process than this account aims to discuss, involving at least two triple decks as well as other, distinct, mild-interaction processes. All the same, a combination of heuristic asymptotics with numerical experiments has, through the efforts of a number of investigators, led to a theoretical prediction of the detailed structure of supersonic shock-boundary layer interaction [Stewartson 1975, Messiter 1978], which displays a most encouraging measure of agreement with experimental observation. Similar success has been achieved with the help of triple decks in the theoretical description of supersonic flow past shallow corners (Figure 5) and even, past plate sections through which air is injected. All of them share the favorable features that the process is self-contained, that the external flow is readily mastered and adjusted, and that the interaction condition is (32). To learn more about the successful explanation of supersonic flow separation, the Reader may wish to turn to [Messiter 1978], [Stewartson 1981] and the literature there cited.

The subsonic bubble-end envisaged at the start of the present account as the main motivation for the mild-interaction definition has proven much more recalcitrant. When the qualitative influences upon each other of the mechanisms of potential flow and boundary layer are considered, a change of sign is found to occur at the sonic speed. As a result, the physical argument makes a self-contained, mutual interaction implausible. The analytical and numerical exploration failed to shake this prognosis and a consensus has developed [Stewartson 1974] that the triple-deck equations (25), (28), (31) do not possess a solution describing flow separation.

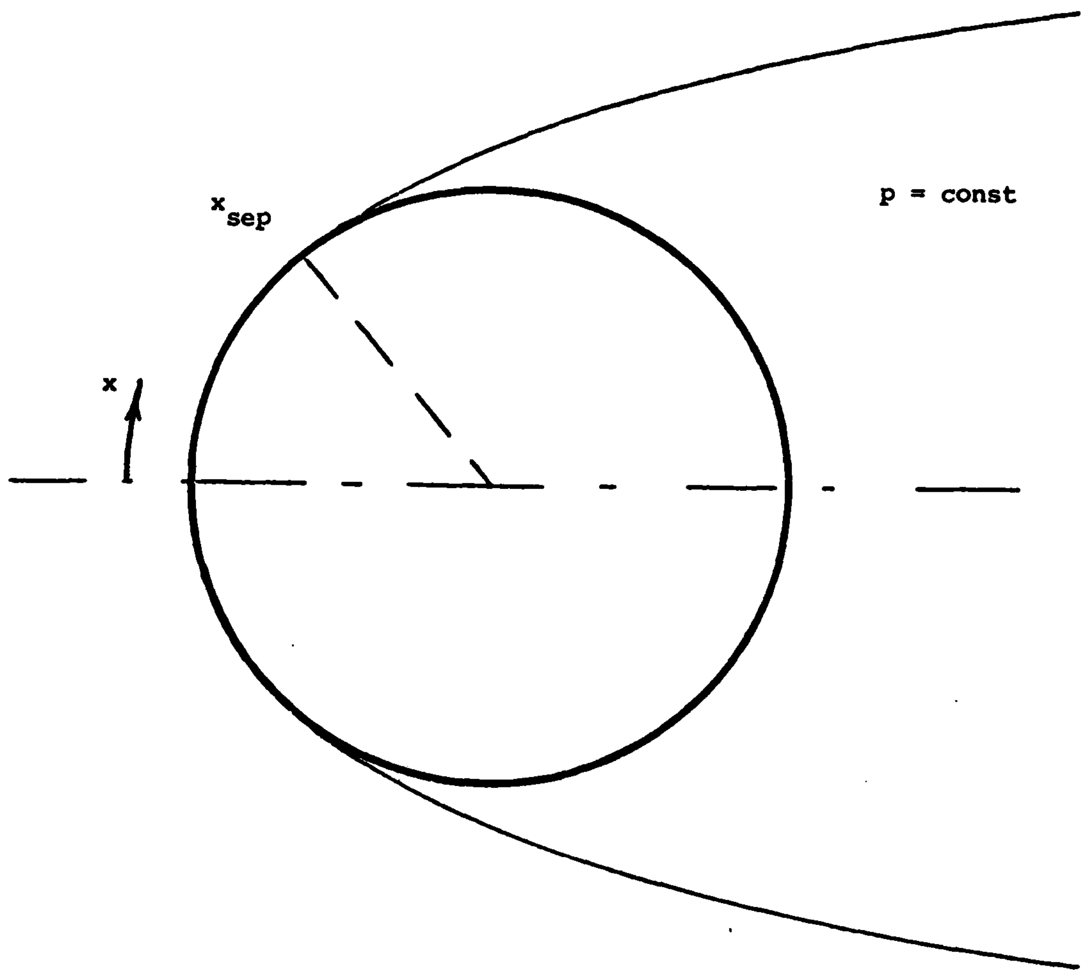
9. Sychev's Proposal

To break this impasse, Sychev suggested that some of the mathematical foundations of the theory be abandoned by relaxing the solution concept (Section 2). To describe his proposal better, it may help to recall first an old paradox of the inviscid break-away theory. To fix the ideas, flow past a cylinder is commonly observed (Figure 7) to break away so as to leave a wide region of slowly recirculating fluid as a wake. The first of all the potential flow models for this, due to Kirchhoff, neglects the motion in the wake so that the pressure there is a constant, p_w. That reduces the non-uniqueness of the potential problem to that of a family of definite problems dependent on the one parameter p_w. Wake models are still a matter of controversy, but Sychev's considerations involve them only in a local and conceptual sense. The Kirchhoff model may then play a useful role as a qualitative representation of the observation that the pressure on the cylinder surface varies much less rapidly with distance downstream from the break-away point than upstream.

For the discussion of the next equation, it is convenient to count the arclength x along the body surface from the nose stagnation point (Figure 7). The separation point, denoted by x_{sep}, marks a singularity of the potential flow because a streamline bifurcates here; the branch leaving the body surface is called separation streamline. It must leave the surface tangentially, because a stagnation point at x_{sep} is neither plausible nor observed, and potential theory shows the local character of the pressure distribution on the surface to be then described by [Brodetsky 1923]

$$(33) \qquad \begin{aligned} p &\sim p_w + k(x_{sep} - x)^{1/2} && \text{as} \quad x \uparrow x_{sep} \\ &= p_w && \text{for} \quad x > x_{sep} \end{aligned}$$

Fig. 7



with constant k. Potential theory also shows that p_w, x_{sep} and k are equivalent parameters standing in one-one relation, and for present purposes, k is more conveniently regarded as the family-parameter of the Kirchhoff model. Its value determines the potential solution uniquely and $k > 0$ is found to imply a separation-streamline curvature at x_{sep} exceeding the body surface curvature in the sense that the separation streamline enters the interior of the body. For this reason, positive values of k must be excluded for the Kirchhoff model.

Negative values, on the other hand, would imply an adverse pressure gradient of arbitrarily large magnitude <u>upstream</u> of x_{sep}, which a boundary layer cannot support without singularity and hence, the separation must occur before its defined position x_{sep} is reached. This contradiction is sharpened to a complete dilemma by noting that $k = 0$ would imply the absence of any singularity and the pressure on the body surface then continues to fall over an arclength interval extendng beyond x_{sep}, so that the weak-interaction solution also continues uniquely and there is no break-away of the type observed (Figure 7). This paradox inherent in the Kirchhoff model was one of the main reasons for its discreditation over many years.

Sychev suggested that it can be resolved by considering k to depend on the Reynolds number in such a way that

$$(34) \qquad k(Re) \to 0 \quad \text{as} \quad Re \to \infty \ .$$

The proposition is more startling than may be immediately evident. It shifts the singularity (33) from an external-flow property, as it is in the Kirchhoff model, to an internal, boundary-layer feature. An attention-getting *implication* is that, as $Re \to \infty$, separation is predicted to take place at a position where the pressure of the external flow is still <u>falling</u> in the streamwise direction. That is not only quite contrary to intuition, but also sharpens the negative implications for subsonic flow of the physical argument of Oswatitsch and Wieghardt (Section 8).

The relative weakness and local nature of the singularity (33), (34) suggest a possible connection with the notion of mild interaction. The singularity might then be a feature of the potential-flow upper deck. The next, natural question for Sychev was what Reynolds-number dependence of k would be consistent with this? That is readily deduced from (11) and (27), where the origin of local coordinates was taken at an appropriate point in the mild-interaction region, now seen to be x_{sep}. Accordingly, $x - x_{sep}$ is to be interpreted as just x and p_e, identified with p_w. In the upper-deck notation (11), the relation (33) then becomes

$$\begin{aligned} p_w + \hat{\pi}\hat{p} &\sim p_w + k|\hat{x}Re^{-\alpha}|^{1/2} && \text{as } \hat{x} \to -\infty \\ &\sim p_w && \text{as } \hat{x} \to \infty \end{aligned}$$

and by (27), since the respective stretches for x and $p-p_e$ are the same in all three decks,

$$\begin{aligned} k &\sim -\gamma\, Re^{-1/16} && \text{as } Re \to \infty \\ \bar{p} &\sim -\gamma\, |\bar{x}|^{1/2} && \text{as } \bar{x} \to -\infty \qquad (35) \\ &\sim 0 && \text{as } \bar{x} \to \infty \end{aligned}$$

with constant $\gamma > 0$, because positive values of k are excluded for the same reason as in the Kirchhoff paradox.

It may be worth remarking that (33) and (35) together express a condition of consistency between the triple-deck Lims and the ever-present, fourth deck of the external-flow Lim. What the observer looking at the external flow sees as a local limit (33) describing a singularity (at finite Re), must be seen by the observer looking at the upper deck as an asymptotic statement, but not neccessarily, as a more detailed condition. In Sychev's proposal, accordingly, the role of the triple deck is to resolve the apparent singularity of the external-flow Lim needed to make the total picture consistent with a locally Kirchhoff model of break-away.

However, (35) contradicts (31) because it makes that integral roundly divergent. Sychev proposes to avoid that difficulty simply by an interpretation of the integral in the finite-part sense. The more explicit meaning of (31) then becomes

$$\bar{p}(\bar{x}) + \gamma\, u_e^2 |\bar{x}|^{1/2} H(-\bar{x}) = \frac{u_e^2}{\pi\tau} \int_{-\infty}^{\infty} \frac{A'(s)+\gamma\tau\, s^{1/2} H(s)}{\bar{x}-s}\, ds \tag{36}$$

where $H(x)$ denotes the Heaviside unit step and the integral is again in the sense of Cauchy's principal value.

The introduction of the finite part involves, of course, a substantial shift of the solution concept into the realm of distributions and thereby mandates a revision from the ground up of the theory outlined in this account. A rational motivation of the Sychev form of mild-interaction theory, however, is not nearly as urgent a concern as the look forward, and in a striking role reversal, it was now Stewartson [1974] who called publicly for a mathematical existence proof! To appreciate the substantial and novel character of this challenge, it may help to collect here the set of equations (25), (28), (35), (36) arrived at:

$$\bar{u} = \partial\bar{\psi}/\partial\bar{y}, \qquad \bar{v} = -\partial\bar{\psi}/\partial\bar{x} \quad , \tag{37a}$$

$$\bar{u}\partial\bar{u}/\partial\bar{x} + \bar{v}\partial\bar{u}/\partial\bar{y} = -d\bar{p}/d\bar{x} + \partial^2\bar{u}/\partial\bar{y}^2 \quad , \tag{37b}$$

for all $\bar{x}$ and all $y > 0$,

$$\bar{u}(\bar{x},0) = \bar{v}(\bar{x},0) = \bar{\psi}(\bar{x},0) \quad \text{for all } \bar{x} \tag{37c}$$

$$\bar{u} - \tau\bar{y} \to A(\bar{x}) \quad \text{as } \bar{y} \to \infty, \quad \text{for all } \bar{x} \tag{37d}$$

with known constant τ,

(37e) $$\bar{u} \to \tau\bar{y} \quad \text{as} \quad \bar{x} \to -\infty, \quad \text{for all} \quad \bar{y} > 0$$

(37f) $$|\bar{x}|^{-1/2}\,\bar{p} \to -\gamma \quad \text{as} \quad \bar{x} \to -\infty$$

with unknown constant $\gamma > 0$,

(37g) $$\bar{p} \to 0 \quad \text{as} \quad \bar{x} \to \infty$$

and

(37h) $$\bar{p}(\bar{x}) + \gamma\, u_e^2 |\bar{x}|^{1/2} H(-\bar{x}) = \frac{u_e^2}{\pi\tau} \int_{-\infty}^{\infty} \frac{A'(s) + \gamma\tau\, s^{1/2} H(s)}{\bar{x}-s}\, ds$$

in the sense of Cauchy's principal value, with $H(x)$ denoting the Heaviside unit step. It is still a homogeneous system for the unknown functions $\bar{u}(\bar{x},\bar{y})$, $\bar{v}(\bar{x},\bar{y})$, $\bar{p}(\bar{x})$, $A(\bar{x})$, and the uppermost question is now clearly whether an eigenvalue γ exists for which these equations possess a nontrivial solution?

Sychev was content with showing that such a solution, if there is one, would have asymptotic properties, as $\bar{x} \to \infty$, that give a plausible description of the flow reversal and recirculation inherent in separation. The existence question was addressed by Smith [1977], and in the absence of a mathematical theory for anything resembling (38a-h), he attacked it numerically. It is not hard to appreciate that the task was very difficult and could not be accomplished with definitive completeness, but Smith [1977] did succeed in establishing very substantial evidence in favor of the conjecture that an eigenvalue of γ exists near 0.44.

Appendix: Illingworth-Stewartson Transformation

Interdisciplinary research is made appreciably more difficult by the custom that the literature of each branch of science takes a great deal for granted. Thus reviews of triple-deck theory pass quickly to the supersonic case where the main, early successes are found (Section 8). In so doing, they take a remarkable correlation between compressible and incompressible boundary layers for granted, and it may help non-aerodynamicists to gain ready access to the more advanced reviews of triple-deck theory, if a succinct exposition of that correlation be here appended.

(i) Boundary Layer Equations

The Navier-Stokes equations for steady, two-dimensional flow are

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0$$

$$\rho(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \frac{2}{3}\frac{\partial}{\partial x}[\mu(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})] + \frac{\partial}{\partial y}[\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})]$$

$$\rho(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}[\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})] + \frac{2}{3}\frac{\partial}{\partial y}[\mu(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x})]$$

$$\rho c_p(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \Phi + \frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda \frac{\partial T}{\partial y})$$

$$\Phi = \mu[2(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial v}{\partial y})^2 + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2 - \frac{2}{3}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2]$$

together with the equation of state of a perfect gas

$$p = (c_p - c_v)\rho T \quad .$$

Here c_p, c_v denote the specific heats, assumed constant, λ, the heat conduction coefficient, and T, the temperature. For a fairly full background of these equations, reference may be made, e.g., to [Meyer 1971].

Only weak interaction will be considered in this Appendix. For a boundary layer on a solid surface $y = 0$, a direct extension to these equations of the transformation (3), (5) reduces them to

(38a) $$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0$$

(38b) $$\rho(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) = - \frac{dp}{dx} + \frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y})$$

(38c) $$\rho c_p(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) = u \frac{dp}{dx} + \frac{\partial}{\partial y} (\lambda \frac{\partial T}{\partial y}) + \mu (\frac{\partial u}{\partial y})^2$$

(38d) $$(c_p - c_v)\rho T = p(x) = p_e(x)$$

as $Re = U L \rho_0/\mu_0 \to \infty$, where the asterisks and subscripts of Sections 2, 3 have now been omitted and U, L, ρ_0 and μ_0 denote reference values of velocity, length, density and viscosity, respectively, and the variables are nondimensional. The standard boundary conditions at a solid surface are

(38e) $u = v = 0$ and T or $\partial T/\partial y$ specified at $y = 0$ for all x

and the conditions of consistency with the external, inviscid flow are

(38f) $$u \to u_e(x),\ T \to T_e(x),\ p(x) = p_e(x) \quad \text{as} \quad y \to \infty \quad ,$$

where u_e, T_e, p_e denote again the body-surface values of the external flow. These are Prandtl's compressible boundary-layer equations.

As in the low-speed case (Section 3), a finite, Re-independent curvature of the body surface does not affect the equations to the order in Re considered, and x and y in (38a-f) can therefore again be thought of as curvilinear, orthogonal coordinates along, and normal to, the body surface.

(ii) Mises Transformation

Let

$$\rho u = \partial\psi/\partial y, \quad \rho v = \partial\psi/\partial x, \quad \psi(x,0) = 0$$

to satisfy (38a) and take as new, <u>independent</u> variables the stream function $\psi(x,y)$ and the distance $s(x,y)$ along streamlines. In the Lim under the stretch (3), (5),

$$s(x,y) = x$$

and for any differentiable function $f(\psi,s)$, the first derivatives become

$$\frac{\partial f}{\partial x} = -\rho v \frac{\partial f}{\partial \psi} + \frac{\partial f}{\partial s} , \qquad \frac{\partial f}{\partial y} = \rho u \frac{\partial f}{\partial \psi} .$$

Accordingly, the momentum equation (38b) is transformed into

$$\rho u \frac{\partial u}{\partial s} = - \frac{dp}{ds} + \rho u \frac{\partial}{\partial \psi} (\mu \rho u \frac{\partial u}{\partial \psi}) . \tag{39}$$

This form of the local momentum balance raises the roles of convection and nonlinear diffusion into clearer relief and is often useful in technical work.

(iii) Crocco Relation

In the absence of chemical reactions, the stagnation enthalpy, $c_p T + \frac{1}{2}(u^2+v^2)$, of inviscid flow is constant along streamlines and in particular, therefore, its body-surface value in the external flow,

$$c_p T_e + \frac{1}{2} u_e^2 = \text{const.} = I_e .$$

For gas-dynamical background, reference may be made, e.g., to [Meyer 1971]. If $I = c_p T + \frac{1}{2} u^2$ denotes the local value of the same quantity, then the usual combination of the energy equation (38c) plus u times the momentum equation (38b) reads

$$\begin{aligned} \rho(u \frac{\partial I}{\partial x} + v \frac{\partial I}{\partial y}) &= \frac{\partial}{\partial y} (\lambda \frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y}) \\ &= \frac{\partial}{\partial y} (\mu \frac{\partial I}{\partial y}) , \end{aligned} \tag{40}$$

if the Prandtl number $\mu c_p/\lambda = \sigma = 1$. This assumption, made in the following, is a common gas dynamical approximation because the range of σ is normally very small and not too far from unity; for air, e.g., $\sigma = 0.72$, approximately, over quite a wide range of thermodynamic conditions.

If there is also no heat transfer to the body surface, the boundary condition there is $\partial T/\partial y = 0$, and by (38e), also $\partial I/\partial y = 0$ there. An obvious solution of (40) satisfying this boundary condition and consistent with (38f) is

$$I \equiv \text{const.} = I_e .$$

Assuming uniqueness, this establishes a simple, direct dependence of local temperature on local velocity,

(41) $$c_p T = I_e - \frac{1}{2} u^2 \quad ,$$

which eliminats the need for further consideration of (38c).

(iv) Illingworth-Stewartson Transformation

If $\mu \propto T$, which also adopted now as a common gas dynamical approximation (for air, e.g., $\mu \propto T^{0.89}$ over quite a wide range of conditions), then by (38d), $\mu\rho$ is also constant across the boundary layer, i.e., $\mu\rho$ depends on x, but not on ψ. This clearly simplifies (30) a little, and if account is also taken of the Bernoulli relation, $\rho_e u_e du_e/ds = -dp/ds$, for the inviscid external flow, (39) takes the form

$$\frac{\partial u}{\partial s} = \frac{\rho_e u_e}{\rho u} \frac{du_e}{ds} + \mu_e \rho_e \frac{\partial}{\partial \psi} (u \frac{\partial u}{\partial \psi}) \quad .$$

If now u is referrd to the local unit u_e by setting

$$u(\psi,s) = u_e(s)\, u'(\psi,s) \quad ,$$

then $\partial u/\partial s = u_e \partial u'/\partial s + u' du_e/ds$ and the momentum equation becomes

(42a) $$\frac{\partial u'}{\partial s} + \frac{u'^2 - \rho_e/\rho}{u' u_e} \frac{du_e}{ds} = \mu_e \rho_e u_e \frac{\partial}{\partial \psi} (u' \frac{\partial u'}{\partial \psi})$$

with boundary conditions

(42b) $u'(\psi,s) \to 1$ as $\psi \to \infty$ for all s ,

(42c) $u'(0,s) = 0$ at the body surface, for all s .

This is a system involving only $u'(\psi,s)$ and functions of s because $\rho_e/\rho = T/T_e$, by (38d), and on account of the Crocco relation (41).

Now recall the equations (6) for the incompressible boundary layer, which are seen to be the special case of (38a,b,e,f) resulting for constant ρ and μ, say, ρ_i and μ_i respectively. If now the Mises transformation to independent variables ψ_i, s_i is applied to (6b) and if $u_i = u_{ei}u'_i$, then the resulting equation is the special case of (42) for $\rho = \rho_e = \rho_i$, $\mu_e = \mu_i$:

(43a) $$\frac{\partial u_i}{\partial s_i} + \frac{u_i'^2-1}{u_i' u_{ei}} \frac{du_{ei}}{ds_i} = \mu_i \rho_i u_{ei} \frac{\partial}{\partial \psi_i} \left(u_i' \frac{\partial u_i'}{\partial \psi_i}\right)$$

(43b) $u_i' \to 1$ as $\psi_i \to \infty$ for all s_i ,

(43c) $u_i'(0,s_i) = 0$ at the body surface, for all s_i .

Now (42a-c) and (43a-c) are seen to be the same equation system, if we identify

$$\psi = \psi_i \quad ,$$

(44) $$\mu_e \rho_e u_e ds = \mu_i \rho_i u_{ei} ds_i = d\xi, \text{ say,}$$

(45) $$\frac{u'^2 - \rho_e/\rho}{u'^2 - 1} u_e^{-1} \frac{du_e}{d\xi} = u_{ei}^{-1} \frac{du_{ei}}{ds_i} \quad .$$

That (45), like (44), refers only to the external flows can be deduced from Crocco's relation (41) as follows. By (38d),

$$\frac{\rho_e}{\rho} = \frac{c_p T}{c_p T_e} = \frac{I_e - \frac{1}{2}u^2}{c_p T_e} = \frac{c_p T_e + \frac{1}{2}u_e^2 - \frac{1}{2}u'^2 u_e^2}{c_p T_e} = 1 + \frac{u_e^2}{2c_p T_e}(1-u'^2)$$

so that $u'^2 - \rho_e/\rho = u'^2 - 1 - (1-u'^2)u_e^2/(2c_pT_e)$ and (45) becomes

(46) $$\left(1 + \frac{u_e^2}{2c_p T_e}\right) u_e^{-1} \frac{du_e}{d\xi} = u_{ei}^{-1} \frac{du_{ei}}{d\xi} \quad .$$

This completes proof of the Theorem discovered almost simultaneously by Illingworth and Stewartson:

Theorem. To any inviscid, compressible external flow without chemical reaction and with surface velocity distribution $u_e(x)$, we can associate an inviscid, incompressible, external flow of surface velocity distribution $u_{ei}(x_i)$ by

(47) $$\frac{du_{ei}/dx_i}{\mu_i\rho_i u_{ei}^2} = \left(1 + \frac{u_e^2}{2c_pT_e}\right)\frac{du_e/dx}{\mu_e\rho_e u_e^2}$$

and for $\sigma = 1$, $\mu \propto T$ and no heat transfer, the weak-interaction boundary layers are then related by

(48) $$u_i' = \frac{u_i(\psi,x_i)}{u_{ei}(x_i)} = u' = \frac{u(\psi,x)}{u_e(x)} \quad .$$

The theorem furnishes the solution of the compressible problem as soon as the solution of the associated incompressible problem has been obtained, though first in the Mises form $u = u(\psi,x)$. To complete the quasi-Cartesian form of the solution, a quadrature suffices:

Corollary 1. At fixed, corresponding x and x_i,

$$\frac{dy}{dy_i} = \frac{\rho_i u_i T}{\rho_e u T_e} \quad .$$

Proof. At fixed s and s_i, by definition, $d\psi = \rho u dy = \rho_i u_i dy_i$ so that $dy/dy_i = \rho_i u_i/(\rho u)$ and the Corollary follows from (38d).

It is worth remarking that the theorem can be simplified by reference to the local Mach number

$$M_e(x) = u_e(x)/a_e(x)$$

where $a_e(x)$ is the speed of sound defined by

(49) $$a_e^2 = \gamma p_e/\rho_e = (\gamma-1)c_pT_e$$

with now $\gamma = c_p/c_v$, by (38d). By (41) and (49),

$$c_p dT_e/dx = -u_e du_e/dx \quad ,$$

$$\frac{dM_e/dx}{M_e} = \frac{du_e/dx}{u_e} - \frac{dT_e/dx}{2T_e} = \left(1 + \frac{u_e^2}{2c_pT_e}\right)\frac{du_e/dx}{u_e}$$

so that (46) becomes simply

(50) $$u_{ei}/M_e = \text{const.} = a_0 \quad ,$$

say, and (44) completes the proof of

Corollary 2. (47) in the theorem may be replaced by

$$u_{ei}(x_i) = a_0 M_e(x) \quad ,$$

$$x_i = (\mu_i \rho_i a_0)^{-1} \int^x \mu_e(s)\rho_e(s)a_e(s)ds \quad .$$

Corollary 1 may also be simplified to

$$y = \frac{a_0 \rho_i}{a_e \rho_e} [y_i + \frac{\gamma-1}{2} \frac{u_{ei}^2}{a_0^2} \int_0^{y_i} (1 - u_i'^2) d\eta]$$

with integral taken at constant x because, at corresponding points (ψ, x_i) and (ψ, x), the theorem gives $u_i/u = u_{ei}/u_e = a_0/a_e$, by (50), and T is given in terms of u by (41).

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2391	2. GOVT ACCESSION NO. AD-A118 617	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Stewartson's Triple Deck		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. E. Meyer		8. CONTRACT OR GRANT NUMBER(s) MCS-8001960 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 2 - Physical Mathematics
11. CONTROLLING OFFICE NAME AND ADDRESS (see Item 18 below)		12. REPORT DATE June 1982
		13. NUMBER OF PAGES 54
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Boundary Layer, Flow Separation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An introduction to triple-deck theory for steady, two-dimensional boundary layers in low-speed flow is presented. It aims to clarify how the rational structure of the theory can rest on few premises and to make its new ideas and challenges more widely accessible.

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