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Finite plane deformation of an elastic half-space due to a concentrated surface load.

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by

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Summary

In this paper the fully nonlinear equilibrium theory of homogeneous and isotropic, incompressible elastic solids is used to study the elastostatic field in plane strain on a half-plane deformed by a concentrated surface load. Under suitable restrictions on the form of the elastic potential at severe deformations, it is shown that, for materials which ultimately "harden" in simple shear, the displacement is bounded near the point of application of the load. This is not the case for materials which ultimately "soften" in shear. Estimates of the true stress tensor near the singular point are given.

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INTRODUCTION

Under certain circumstances, the description of the elastostatic field furnished by the classical linearized theory of elasticity may be inadequate, even when the applied loads are small. Such breakdowns in the linearized theory are ordinarily local in nature and are brought about, for example, by stress concentrations such as those induced by holes or cracks in the interior of the loaded solid. The most extreme examples of problems of this kind involve a singular point in the elastostatic field--the tip of a crack, for example--near which the displacement gradient is unbounded. Since the basic approximative assumption underlying the linear theory requires that this gradient be negligibly small in comparison with unity, it is hardly surprising that results based on this theory may be in error near such a singular point.

Problems involving large displacement gradients properly fall within the scope of the finite theory of elasticity. In recent years there have been several investigations within the framework of the finite theory of the local structure of the elastostatic field near a geometrically-induced singular point. Much of this work is summarized in the review articles [1,2], where references are given. In general, the analyses of singular problems reviewed in [1,2] are necessarily <u>local</u> in character; they reveal that the results from linear theory near the singular point are invariably incorrect quantitatively, and in some instances may be qualitatively misleading as well. Since it is often the field <u>near</u> the singular point which is of primary physical interest, analyses based on finite elasticity are of considerable significance.

In the present paper, a further singular problem in elastostatics is considered within the scope of the theory of finite elasticity. This is the plane strain problem of a concentrated uniform normal line force applied to an elastic body which, in the undeformed state, occupies a half-space. Here the singularity arises because of the character of the applied load, rather than from the geometry of the undeformed body, as is the case in most singular problems previously treated within the finite theory [1,2]. The present analysis aims at the asymptotic determination of the displacements and stresses near the point of application of the load. We deal with the fully nonlinear equilibrium theory for homogeneous, isotropic incompressible materials that possess an elastic potential. The only restriction on this potential is one which pertains to its asymptotic behavior at large deformations; it is this regime of deformation which dominates the local field near the singular point. Again, it is found that the structure of the stress and displacement fields near the singular point differs from that predicted by the linear theory.

The only previous works devoted to the effect of nonlinearity on the elastostatic field near the point of application of a concentrated force are those of Arutiunian [3] and Atkinson [4]. Both of these authors retain the assumption of infinitesimal displacement gradients appropriate to the linearized theory, but replace the constitutive law of the latter theory by a nonlinear one.

Section 1 contains a review of some prerequisites from the theory of finite plane elastostatics for homogeneous, isotropic, incompressible elastic solids. We also introduce in Section 1 the special class of such solids underlying the subsequent analysis. Section 2 is devoted to the formulation, analysis, and discussion of the concentrated force problem. Only the case of a tensile force is treated in detail.

1. PRELIMINARIES FROM PLANE FINITE ELASTOSTATICS

In this work we shall be concerned with the analysis, within the finite theory, of plane elastostatic fields in incompressible, homogeneous, and isotropic elastic materials in the absence of body forces.[1]

Consider an elastic body which--in the undeformed state--is an infinite cylinder, and let Π denote a plane open cross-section of this cylinder perpendicular to its generators. Let (x_1, x_2) be the coordinates of a generic point in Π relative to a fixed two-dimensional rectangular cartesian coordinate system in the plane of Π.

A plane deformation of the body is given by the transformation

$$y_\alpha = \hat{y}_\alpha(x_1, x_2) = x_\alpha + u_\alpha(x_1, x_2) \quad \text{on } \Pi \;, \quad \alpha = 1,2 \;, \tag{1.1}$$

where y_α are the components of the position vector $\underset{\sim}{y}$ of the particle in the deformed body whose position vector in the undeformed configuration is $\underset{\sim}{x}$; u_α are the components of the displacement vector $\underset{\sim}{u}$, all with respect to the rectangular coordinate system. The function $\hat{\underset{\sim}{y}}$ is required to be twice continuously differentiable on Π, and it is further required that the mapping $\underset{\sim}{x} \leftrightarrow \underset{\sim}{y}$ be one-to-one and that its inverse $\hat{\underset{\sim}{x}}$ have the same smoothness.

The deformation gradient tensor $\underset{\sim}{F}$ associated with $\underset{\sim}{y}$ has components

$$F_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta} \;. \tag{1.2}$$

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[1] For a discussion of the foundations of finite elasticity see Gurtin [5]. For further reading on plane finite elastostatics of incompressible materials, see [6].

Since the material is presumed to be incompressible, the deformation (1.1) must be locally volume-preserving, whence the Jacobian determinant of the mapping must satisfy

$$J = \det[F_{\alpha\beta}] = 1 \quad \text{on } \Pi \quad . \tag{1.3}$$

Define the right and left two-dimensional Cauchy-Green tensors $\underset{\sim}{C}$ and $\underset{\sim}{G}$, respectively, by

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \quad , \qquad \underset{\sim}{G} = \underset{\sim}{F}\,\underset{\sim}{F}^T \quad . \tag{1.4}$$

These deformation tensors have common fundamental scalar invariants given by

$$\begin{aligned} I_1 &= \operatorname{tr} \underset{\sim}{C} = F_{\alpha\beta}F_{\alpha\beta} = I \text{ , say,} \\ I_2 &= \det \underset{\sim}{C} = J^2 = 1 \quad . \end{aligned} \tag{1.5}$$[1]

The invariant I is found to obey

$$I \geq 2 \quad \text{on } \Pi \quad . \tag{1.6}$$

Moreover, $I = 2$ if and only if $\underset{\sim}{F} = \underset{\sim}{1}$, where $\underset{\sim}{1}$ is the two-dimensional unit tensor.

Let $\underset{\sim}{\tau}$ be the two-dimensional true (Cauchy) stress tensor regarded as a function of position on the deformation image Π^* of Π. Its components $\tau_{\alpha\beta}$ represent forces per unit deformed area. If $\underset{\sim}{\sigma}$ is the associated nominal (Piola) stress tensor field on Π, whose components $\sigma_{\alpha\beta}$

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[1]Repeated subscripts are summed over the range (1,2).

represent forces per unit undeformed area, one has

$$\underset{\sim}{\sigma} = \underset{\sim}{\tau}(\underset{\sim}{F}^T)^{-1} \quad . \tag{1.7}$$

For an equilibrium deformation in the absence of body forces, it is necessary that $\underset{\sim}{\tau}$ satisfy

$$\text{div}\, \underset{\sim}{\tau} = 0 \quad , \quad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T \quad \text{on } \Pi^* . \tag{1.8}$$

It follows from (1.7),(1.8) that

$$\text{div}\, \underset{\sim}{\sigma} = 0 \quad , \quad \underset{\sim}{\sigma}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{\sigma}^T \quad \text{on } \Pi \quad . \tag{1.9}$$

Suppose that Γ is a regular arc in Π which is mapped onto Γ^* in Π^* by the deformation (1.1), and denote by $\underset{\sim}{n}$ and $\underset{\sim}{n}^*$ unit normal vectors of Γ and Γ^*, respectively. The true traction vector $\underset{\sim}{t}$ and the associated nominal traction vector $\underset{\sim}{s}$ are given by

$$\left.\begin{aligned} \underset{\sim}{s} &= \underset{\sim}{\sigma}\underset{\sim}{n} \quad \text{on } \Gamma \ , \\ \underset{\sim}{t} &= \underset{\sim}{\tau}\underset{\sim}{n}^* \quad \text{on } \Gamma^* . \end{aligned}\right. \tag{1.10}$$

It can be shown that

$$\underset{\sim}{s} = \underset{\sim}{0} \text{ on } \Gamma \text{ if and only if } \underset{\sim}{t} = \underset{\sim}{0} \text{ on } \Gamma^* \quad . \tag{1.11}$$

Moreover, (1.11) continues to hold true for an arc Γ on the boundary of Π if the deformation and nominal stress field are suitably regular on the closure $\bar{\Pi}$ of Π. This important fact allows the boundary condition for a traction-free surface Γ^* in the deformed body to be specified on the known pre-image Γ of Γ^* in the undeformed body.

The mechanical response of the homogeneous, isotropic, incompressible material under consideration is governed by the strain energy density W per unit undeformed volume. For a plane deformation of the type described above, W depends only on the deformation invariant I:

$$W = W(I) \quad . \tag{1.12}$$

The stress-deformation relation is

$$\tau_{\alpha\beta} = 2W'(I)\, F_{\alpha\rho}F_{\beta\rho} - p\delta_{\alpha\beta} \quad \text{on } \Pi^*, \tag{1.13}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and the scalar field p is an arbitrary hydrostatic pressure whose presence is necessary because of the constraint of incompressibility. Because of the presence of p, the true stress tensor is not completely determined by the deformation for an incompressible material. From (1.13), (1.7) it follows that

$$\sigma_{\alpha\beta} = 2W'(I)F_{\alpha\beta} - p\varepsilon_{\beta\gamma}\varepsilon_{\alpha\rho}F_{\rho\gamma} \quad \text{on } \Pi, \tag{1.14}$$

provided $\varepsilon_{\alpha\beta}$ are the components of the two-dimensional alternator. In the foregoing, W' denotes the derivative of W with respect to I; we assume that W is twice continuously differentiable for $I \geq 2$. It is further assumed that W vanishes in the undeformed state, so that

$$W(2) = 0 \quad , \tag{1.15}$$

and that

$$W'(I) > 0 \quad , \quad I \geq 2 \; , \tag{1.16}$$

so that the Baker-Ericksen inequality is not violated.[1]

The linear theory of elastostatic plane strain is recovered from the finite deformation theory briefly described above by a systematic linearization with respect to the displacement gradients $u_{\alpha,\beta}$. Under this linearization, the distinction between true and nominal stresses disappears, and the constitutive law passes over into

$$\tau_{\alpha\beta} = \sigma_{\alpha\beta} = 2\mu\, \gamma_{\alpha\beta} - p\delta_{\alpha\beta} \quad , \tag{1.17}$$

where

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{1.18}$$

are the components of the infinitesimal strain tensor, and

$$\mu = 2W'(2) \tag{1.19}$$

is the infinitesimal shear modulus. The incompressibility condition $J = 1$ linearizes to

$$\gamma_{\alpha\alpha} = u_{\alpha,\alpha} = \operatorname{div} \underset{\sim}{u} = 0 \ . \tag{1.20}$$

The approximate form of W for infinitesimal deformations is found by linearization to be

$$W = \frac{\mu}{2}\, \gamma_{\alpha\beta}\, \gamma_{\alpha\beta} \ . \tag{1.21}$$

A deformation (1.1) of the form

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[1]See [7].

$$y_\alpha = B_{\alpha\beta}\, x_\beta \quad , \tag{1.22}$$

where the $B_{\alpha\beta}$ are constants satisfying

$$\det[B_{\alpha\beta}] = 1 \quad , \tag{1.23}$$

is a <u>homogeneous deformation</u> of the incompressible body. Two particular homogeneous deformations are of special interest: <u>uniaxial stress</u> and <u>simple shear</u>. For the former one takes

$$y_1 = \lambda x_1 \ , \qquad y_2 = \frac{1}{\lambda}\, x_2 \ , \qquad \lambda > 0 \ , \tag{1.24}$$

with λ constant. From (1.13), one then finds that $\tau_{12} = \tau_{21} = 0$, and, if p is chosen to be

$$p = 2W'(I)\, \lambda^{-2} \ , \tag{1.25}$$

where

$$I = \lambda^2 + \lambda^{-2} \quad , \tag{1.26}$$

one has

$$\tau_{22} = 0 \quad , \tag{1.27}$$

as well. The only nonvanishing stress component is then found from (1.13) to be

$$\tau_{11} = 2W'(I)(\lambda^2 - \lambda^{-2}) \ . \tag{1.28}$$

For simple shear, one has the homogeneous deformation

$$y_1 = x_1 + kx_2 \quad , \qquad y_2 = x_2 \ , \tag{1.29}$$

where the constant k is the amount of shear. From (1.13) one finds the relation between the true shear stress τ_{12} and the amount of shear k to be

$$\tau_{12} = 2W'(I)\,k \ , \tag{1.30}$$

where now

$$I = 2 + k^2 \ . \tag{1.31}$$

We shall assume throughout that W has the following property:

$$W(I) = AI^n + o(I^n) \quad \text{as} \quad I \to \infty \ , \tag{1.32}$$

where A and n are material constants satisfying

$$A > 0 \ , \qquad n > 1/2 \ . \tag{1.33}$$

For an incompressible material satisfying (1.32), one sees from (1.26),(1.28) that in extreme uniaxial stress ($\lambda \to \infty$), one has

$$\tau_{11} \sim 2nA\lambda^{2n} \ , \quad \lambda \to \infty \ . \tag{1.34}$$

For severe simple shear ($k \to \infty$), one obtains from (1.31),(1.30),(1.32)

$$\tau_{12} \sim 2nAk^{2n-1} \ , \quad k \to \infty \ . \tag{1.35}$$

Since $n > 1/2$, the stress response in uniaxial stress is, by (1.34), always asymptotically <u>hardening</u> as $\lambda \to \infty$, in the sense that $d\tau_{11}/d\lambda$ is increasing with increasing λ. In shear, the stress response of (1.35) is <u>hardening</u> as $k \to \infty$ for those materials with $n > 1$, softening for $\frac{1}{2} < n < 1$, and asymptotically linear if $n = 1$. The asymptotic forms of the stress response curves in uniaxial stress and simple shear for materials satisfying (1.32),(1.33) are shown in Figure 1.

If one were to permit $n < 1/2$ in (1.32) one would find that the field equations of the equilibrium theory would cease to be an elliptic system at sufficiently severe deformations; see [6].

Before proceeding to the specific problems to be discussed, it is useful to take note of an implication of the field equations (1.3), (1.9), (1.14). One can show that $\det \underset{\sim}{F} \equiv 1$ implies that

$$\varepsilon_{\beta\gamma}\, \varepsilon_{\alpha\rho}\, F_{\rho\gamma,\beta} \equiv 0 \quad \text{on } \Pi \,. \tag{1.36}$$

Substitution from (1.14) into the equilibrium equations (1.9) then gives, with the help of (1.36), the equation

$$[2W'(I)\, F_{\alpha\beta}]_{,\beta} = p_{,\beta}\, \varepsilon_{\beta\gamma}\, \varepsilon_{\alpha\rho}\, F_{\rho\gamma} \quad \text{on } \Pi \,. \tag{1.37}$$

If one multiplies (1.37) by $F_{\alpha\lambda}$, makes use of the fact that $\det \underset{\sim}{F} \equiv 1$ as well as of the definitions $(1.4)_1$,$(1.5)_1$, one finds that

$$\underset{\sim}{\nabla} p = 2W'(I)\, \underset{\sim}{F}^T \nabla^2 \underset{\sim}{y} + 2W''(I)\, \underset{\sim}{F}^T \underset{\sim}{F}\, \underset{\sim}{\nabla} I \quad \text{on } \Pi \,. \tag{1.38}$$

We will find this form of the equilibrium equations helpful in the sequel.

2. THE HALF-PLANE DEFORMED BY A CONCENTRATED FORCE

A. Formulation of the Problem

We consider the case in which the open cross-section Π of the undeformed body is the half-plane $x_1 > 0$, $-\infty < x_2 < \infty$, and we denote by $\overset{\circ}{\bar{\Pi}}$ the closure of Π with the origin deleted. Given the plane strain elastic potential $W(I)$ of the homogeneous, isotropic, incompressible material to be considered, we seek a deformation $y_\alpha = \hat{y}_\alpha(x_1,x_2)$ on $\overset{\circ}{\bar{\Pi}}$ such that the nominal stresses $\sigma_{\alpha\beta}$ generated by the deformation through (1.2)-(1.5) and (1.14) conform to the equation of equilibrium (1.9). We further assume that the free-surface conditions

$$\sigma_{11}(0,x_2) = \sigma_{21}(0,x_2) = 0 \quad , \quad |x_2| > 0 \tag{2.1}$$

hold and we require that, as $|\underset{\sim}{x}| \to \infty$, the true stress field should tend to zero:

$$\tau_{\alpha\beta}(x_1,x_2) \to 0 \quad \text{as} \quad |\underset{\sim}{x}| \to \infty \quad , \quad x_1 \geq 0 \, . \tag{2.2}$$

Further, we impose the requirement that

$$\sigma_{\alpha\beta} = O(r^{-1}) \ \text{ as } r \to 0, \ \text{ uniformly in } \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \, , \tag{2.3}$$

where r,θ are polar coordinates at the origin: $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. We next prescribe that

$$\int_{-\pi/2}^{\pi/2} \sigma_{\alpha\beta}\, n_\beta\, r\, d\theta = F\,\delta_{1\alpha} \quad , \qquad r > 0 \tag{2.4}$$

corresponding to a concentrated force of magnitude $|F|$ acting on the boundary of the deformed body in a direction parallel to the x_1-axis. We shall limit our attention to the case $F > 0$, so that the force is in the negative x_1-direction and is therefore tensile. In (2.4), $\underset{\sim}{n}$ is the unit vector in the radial direction. The compressive case could be treated by a similar analysis.

We finally assume the elastostatic field to be symmetric about the x_1-axis. This in particular rules out a concentrated moment at the origin; symmetry also implies that (2.4) holds automatically for $\alpha = 2$.

B. The Elastostatic Field near the Origin--Lowest Order Asymptotic Analysis

We now assume that the elastic potential $W(I)$ satisfies (1.32), (1.33), and we investigate the local structure of the field near the point of application of the force. We begin by making the Ansatz

$$y_\alpha = r^{m_\alpha} v_\alpha(\theta) + o(r^{m_\alpha - 1}) \quad \text{as } r \to 0, \text{ (no sum on } \alpha) \ , \tag{2.5}$$

uniformly for $-\pi/2 \leq \theta \leq \pi/2$, where m_1 and m_2 are constants restricted by

$$m_1 < 1 \quad , \quad m_2 > 1 \tag{2.6}$$ [1]

and neither of the unknown functions $v_\alpha(\theta) \in C^2([-\frac{\pi}{2}, \frac{\pi}{2}])$ vanish identically.[2] Moreover, in view of the prevailing symmetry one has

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[1] One can show systematically that (2.6) are necessary when the applied force is tensile. They correspond physically to the fact that, along the line of symmetry $\theta = 0$, the principal stretch λ_1 in the x_1-direction is large, while λ_2 is small. The hypothesis (2.6) must be altered when the load is compressive. Note that we do not assume $m_1 \geq 0$.

[2] We actually need the slightly stronger assumption that $v_1(\theta) \not\equiv 0$ and v_2 has at most a finite number of zeros in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$v_1(-\theta) = v_1(\theta) \quad , \qquad v_2(-\theta) = -v_2(\theta) \; . \tag{2.7}$$

It is assumed that (2.5) may be formally differentiated twice.

From (2.5) one obtains for the deformation gradient tensor $\underset{\sim}{F}$ the local asymptotic representation

$$F_{\alpha\beta} \sim f_{\alpha\beta}\, r^{m_\alpha - 1} \quad \text{as } r \to 0 \text{ (no sum on } \alpha) \; , \tag{2.8}$$

provided

$$f_{\alpha\beta} = m_\alpha v_\alpha(\theta)\, c_\beta(\theta) + \varepsilon_{\gamma\beta} c_\gamma(\theta)\, \dot{v}_\alpha(\theta) \quad \text{(no sum on } \alpha) \; . \tag{2.9}$$

Here the dot denotes differentiation with respect to θ and we have introduced the abbreviations

$$c_1(\theta) = \cos\theta \quad , \qquad c_2(\theta) = \sin\theta \; . \tag{2.10}$$

From (2.8),(2.9) there follows

$$J \equiv \det \underset{\sim}{F} = (m_1 v_1 \dot{v}_2 - m_2 v_2 \dot{v}_1)\, r^{m_1 + m_2 - 2} + o(r^{m_1 + m_2 - 2}) \; , \quad r \to 0 \tag{2.11}$$

Since incompressibility requires $J \equiv 1$, we must have

$$m_1 + m_2 - 2 \leq 0 \quad , \tag{2.12}$$

and either

$$m_1 v_1 \dot{v}_2 - m_2 v_2 \dot{v}_1 = 0 \qquad \text{if } m_1 + m_2 < 2 \; , \tag{2.13}$$

or

$$m_1 v_1 \dot{v}_2 - m_2 v_2 \dot{v}_1 = 1 \quad \text{if} \quad m_1 + m_2 = 2 \ . \tag{2.14}$$

From (1.5),(2.8),(2.9) we obtain

$$I \sim r^{2(m_1-1)} G(\theta) \quad \text{as} \quad r \to 0 \tag{2.15}$$

where

$$G(\theta) = \dot{v}_1^2(\theta) + m_1^2 \, v_1^2(\theta) \ . \tag{2.16}$$

In view of the assumption (2.6) concerning m_1, one has from (2.15) that $I \to \infty$ as $r \to 0$. The material assumption (1.32) then yields

$$\left.\begin{aligned} W(I) &\sim AG^n(\theta)\, r^{2n(m_1-1)} , \\ W'(I) &\sim nAG^{n-1}(\theta)\, r^{2(n-1)(m_1-1)} , \\ W''(I) &\sim n(n-1)\, AG^{n-2}(\theta)\, r^{2(n-2)(m_1-1)} , \end{aligned}\right\} \text{as } r \to 0 \ . \tag{2.17}$$

We now recall the field equations in the form (1.38); with the help of (2.17),(2.8),(2.9),(2.6),(2.15) we find from (1.38) that

$$\frac{\partial p}{\partial r} \sim 2n\, A\, m_1 v_1(\theta)\, Z(\theta)\, r^{2(m_1-1)n-1} , \tag{2.18}$$

and

$$\frac{1}{r}\frac{\partial p}{\partial \theta} \sim 2n\, A\, \dot{v}_1(\theta)\, Z(\theta)\, r^{2(m_1-1)n-1} , \tag{2.19}$$

as $r \to 0$, where

$$Z(\theta) = G^{n-2}(\theta)\ \{G(\theta)[\ddot{v}_1(\theta) + m_1^2\ v_1(\theta)]$$

$$+ (n-1)[\dot{G}(\theta)\ \dot{v}_1(\theta) + 2m_1(n-1)\ G(\theta)\ v_1(\theta)]\} . \qquad (2.20)$$

Compatibility of (2.18),(2.19) requires that v_1 and Z satisfy

$$m_1 v_1 \dot{Z} + [1 - (2n-1)(m_1-1)]\ \dot{v}_1 Z = 0 . \qquad (2.21)$$

Once (2.21) is fulfilled, one finds from either (2.18) or (2.19) that

$$p \sim \frac{m_1}{m_1-1} A\ v_1(\theta)\ Z(\theta)\ r^{2n(m_1-1)} \qquad \text{as} \quad r \to 0 . \qquad (2.22)$$

We next consider the boundary conditions (2.1). Because of (1.14) these are

$$\left.\begin{aligned} 2W'(I)F_{11} - pF_{22} &= 0 \\ 2W'(I)F_{21} + pF_{12} &= 0 \end{aligned}\right\} \quad \text{at } \theta = \pm\frac{\pi}{2}, \quad r > 0 . \qquad (2.23)$$

Multiplying the first of (2.23) by F_{11}, the second by F_{21}, adding the results, and using (1.3), we obtain

$$p = 2W'(I)(F_{11}^2 + F_{21}^2) \qquad \text{at} \quad \theta = \pm\frac{\pi}{2}, \quad r > 0 . \qquad (2.24)$$

On the other hand, eliminating p between the two equations (2.23) yields, in view of (1.16),

$$F_{11}F_{12} + F_{21}F_{22} = 0 \qquad \text{at} \quad \theta = \pm\frac{\pi}{2}, \quad r > 0 . \qquad (2.25)$$

Making use of (2.17),(2.8), and (2.9), we obtain from (2.24) the result

$$p \sim 2n\,A\,G^{n-1}(\theta)\,\dot{v}_1(\theta)\,r^{2n(m_1-1)} \quad \text{as} \quad r \to 0,\ \theta = \pm\frac{\pi}{2}\ . \tag{2.26}$$

Comparing (2.22) at $\theta = \pm\frac{\pi}{2}$ and (2.26) leads to

$$2nG^{n-1}(\pm\tfrac{\pi}{2})\,\dot{v}_1(\pm\tfrac{\pi}{2}) = \frac{m_1}{m_1-1}\,v_1(\pm\tfrac{\pi}{2})\,Z(\pm\tfrac{\pi}{2})\ . \tag{2.27}$$

The second boundary condition (2.25), with the help of (2.8),(2.9),(2.15) gives

$$\dot{v}_1(\pm\tfrac{\pi}{2})\ \ v_1(\pm\tfrac{\pi}{2}) = 0\ . \tag{2.28}$$

We now show that the two boundary conditions (2.27) and (2.28), together with the differential equation (2.21), imply that

$$\dot{v}_1(\pm\tfrac{\pi}{2}) = 0 \tag{2.29}$$

and

$$Z(\pm\tfrac{\pi}{2}) = 0\ . \tag{2.30}$$

We first establish (2.30) by showing that the hypothesis $Z(\pi/2) \neq 0$ leads to a contradiction. If $Z(\pi/2) \neq 0$, there is an interval $[\theta_0,\pi/2]$, $\theta_0 < \pi/2$, on which $Z(\theta)$ vanishes nowhere, by continuity. Equation (2.21) can then be integrated on $[\theta_0,\pi/2]$ to give

$$v_1(\theta) = C\left|Z(\theta)\right|^{\frac{m_1}{(2n-1)(m_1-1)-1}}\ , \qquad \theta_0 \le \theta \le \frac{\pi}{2}\ , \tag{2.31}$$

where C is a constant. If $C = 0$, then $v_1(\theta) \equiv 0$ on $[\theta_0,\pi/2]$, whence by (2.16),(2.20), $Z(\theta) \equiv 0$ on $[\theta_0,\pi/2]$, contradicting the hypothesis $Z(\pi/2) \neq 0$. Thus $C \neq 0$, and so by (2.31), $v_1(\theta) \neq 0$ for all

$\theta \in [\theta_0,\pi/2]$. In particular, $v_1(\pi/2) \neq 0$, so that by (2.28), $\dot{v}_1(\pi/2) = 0$. Thus (2.27) has been violated unless $m_1 = 0$. But if $m_1 = 0$ we conclude from (2.21) that $v_1 \equiv$ constant on $[\theta_0,\pi/2]$, which leads via (2.20),(2.16) to $Z \equiv 0$ on $[\theta_0,\pi/2]$, again contradicting the hypothesis. Thus indeed, $Z(\pi/2) = 0$, and, since $Z(\theta)$ is even, (2.30) holds. From (2.27) and (2.16) it then follows that (2.29) holds as well.

An argument similar to that just used to establish (2.30) can now be constructed to show that (2.21),(2.30) imply that

$$Z(\theta) \equiv 0 \quad \text{on } [-\pi/2,\pi/2] \tag{2.32}$$

From (2.32),(2.20),(2.29) we then obtain a nonlinear eigenvalue problem for $m_1, v_1(\theta)$:

$$[G^{n-1}(\theta)\ \dot{v}_1(\theta)]^{\cdot} + [m_1^2 + 2m_1(m_1-1)(n-1)]G^{n-1}(\theta)\ v_1(\theta) = 0\ , \qquad -\pi/2 \leq \theta \leq \pi/2 \tag{2.33}$$

$$\dot{v}_1(\pm \tfrac{\pi}{2}) = 0\ . \tag{2.34}$$

The differential equation (2.33) is identical with one which has arisen in the local analysis of the elastostostatic field near the tip of a crack; see [8],[9],[10], [11]

In view of (2.32), we have from (2.22) that

$$p = o(r^{2n(m_1-1)}) \quad \text{as} \quad r \to 0\ . \tag{2.35}$$

Suppose that $m_1 = 0$. Then (2.33),(2.34) imply that $v_1(\theta) \equiv$ constant and the leading term $r^{m_1}v_1(\theta)$ in the expansion of y_1 near $r = 0$ may be viewed as a rigid body translation parallel to the x_1-axis. Since the boundary value problem determines the elastostatic field at best to within an arbitrary translation of this kind, we shall discard the case $m_1 = 0$ and assume henceforth that, in addition to (2.6),

$$m_1 \neq 0 \quad , \tag{2.36}$$

also holds.

It is now possible to prove that (2.36),(2.6) and the assumptions made concerning v_1 and v_2 imply that (2.13) leads to the contradiction $v_1 \equiv 0$. Thus (2.14) must hold, and thus

$$m_1 + m_2 = 2 \; . \tag{2.37}$$

From (1.14) we have

$$\sigma_{1\beta} = 2W'(I)\, F_{1\beta} - p\, \varepsilon_{\beta\gamma}\, F_{2\gamma} \; . \tag{2.38}$$

Making use of (2.17),(2.8),(2.9),(2.35)-(2.38) we can show that the first term on the right in (2.38) dominates the second, and hence that

$$\sigma_{1\beta} \sim 2n\,A\,m_1 G^{n-1}(\theta)\; f_{1\beta}(\theta)\; r^{(2n-1)(m_1-1)} \quad \text{as} \quad r \to 0 \; . \tag{2.39}$$

From (2.3) we conclude that $(2n-1)(m_1-1) \geq -1$. In order to use (2.4) with $\alpha = 1$, we first observe that $n_\beta = c_\beta$ (see (2.10)), and from (2.39), (2.9) that

$$\int_{-\pi/2}^{\pi/2} \sigma_{1\beta} n_\beta \; r \, d\theta \sim 2nm_1 A \int_{-\pi/2}^{\pi/2} G^{n-1}(\theta) \; v_1(\theta) \; d\theta \; r^{(2n-1)(m_1-1)+1} \quad \text{as } r \to 0 \; . \qquad (2.40)$$

It follows that

$$m_1 = \frac{2(n-1)}{2n-1} < 1 \quad , \qquad n \neq 1 \; . \qquad (2.41)^1$$

From the boundary value problem (2.33),(2.34) with m_1 given by (2.41), one finds

$$v_1(\theta) = k_1 = \text{constant} \quad , \qquad -\pi/2 \leq \theta \leq \pi/2 \; , \qquad n \neq 1 \; . \qquad (2.42)$$

Since (2.37) holds, we have

$$m_2 = 2 - m_1 = \frac{2n}{2n-1} > 1 \quad , \qquad n \neq 1 \; . \qquad (2.43)$$

With the help of (2.42) and the fact that $v_2(\theta)$ is odd, we can now determine v_2 from (2.14) as

$$v_2(\theta) = \frac{1}{k_1 m_1} \theta = k_2 \theta \quad , \qquad -\pi/2 \leq \theta \leq \pi/2 \; , \qquad n \neq 1 \; . \qquad (2.44)$$

Finally, we return to (2.4) with $\alpha = 1$ to determine k_1 in terms of F. Using (2.40)-(2.42) and (2.16), we obtain

$$[(m_1 k_1)^2]^{n-1} \; (m_1 k_1) = \frac{F}{2n \, A\pi} \quad , \qquad n \neq 1 \quad , \qquad (2.45)$$

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[1]Recall that $m_1 = 0$ has been excluded.

and hence

$$k_1 = \frac{2n-1}{2(n-1)} \left(\frac{F}{2n A\pi}\right)^{\frac{1}{2n-1}} , \qquad n \neq 1 \quad . \tag{2.46}$$

Thus, for $n \neq 1$, we have determined the first terms (2.5) in the approximation to the deformation near $r = 0$ as follows:

$$\left.\begin{aligned} y_1 &\sim k_1 \, r^{\frac{2(n-1)}{2n-1}} , \\ y_2 &\sim \frac{2n-1}{2(n-1)} \frac{1}{k_1} r^{\frac{2n}{2n-1}} \theta , \end{aligned}\right\} \text{as } r \to 0, \ -\pi/2 \leq \theta \leq \pi/2 , \quad n \neq 1 , \tag{2.47}$$

with k_1 related to F through (2.46). The deformation image of the boundary $\theta = \pm\pi/2$ of the half-plane is then given in first approximation by

$$y_1 \sim \frac{2n-1}{n-1} \frac{1}{\pi} \left(\frac{F}{4nA}\right)^{1/n} |y_2|^{1-\frac{1}{n}} , \quad |y_2| \to 0 , \quad n \neq 1 . \tag{2.48}$$

We note that if $n > 1$, so that the material is asymptotically hardening in simple shear [see (1.35) and Figure 1], the displacement under the load is finite, while this is not the case for softening materials ($n < 1$). A sketch of the deformed surface based on (2.48) is given in Figure 2.

The case $n = 1$ (a material which is asymptotically linear in shear (Figure 1)) has been excluded in the results (2.47),(2.48). To treat this case, it is necessary to replace the Ansatz (2.5), (2.6) by

$$\left.\begin{aligned} y_1 &\sim (\log r) \, v_1(\theta) , \\ y_2 &\sim r^{m_2} \, v_2(\theta) , \qquad m_2 > 1 \end{aligned}\right\} \text{as } r \to 0 . \tag{2.49}$$

The special nature of the case $n=1$ arises because $m_1 = 0$ and $m_1 = 2(n-1)/(2n-1)$ are both eigenvalues (adjacent ones in fact) of the problem (2.33),(2.34). They are distinct as long as $n \neq 1$, but coalesce[1] as $n \to 1$. This coalescence may be used to motivate the form of the new Ansatz (2.49) for $n = 1$; we omit the details. One finds from (2.49) that $m_2 = 2$, $v_1(\theta) = k_1 =$ constant, $v_2(\theta) = (1/k_1)\theta$, where $k_1 = F/2A\pi$. The counterparts of (2.47) are

$$\left.\begin{aligned} y_1 &\sim \frac{F}{2A\pi}\log r \quad , \\ y_2 &\sim \frac{2A\pi}{F} r^2 \theta \quad , \end{aligned}\right\} \text{ as } r \to 0,\ -\pi/2 \le \theta \le \pi/2\ ,\quad n=1, \tag{2.50}$$

while the deformed boundary is now given approximately by

$$y_1 \sim \frac{F}{4A\pi}\ \log\left(\frac{F|y_2|}{A\pi^2}\right) \quad \text{as } |y_2| \to 0\ ,\quad n = 1\ . \tag{2.51}$$

The displacement is unbounded near the point of application of the load, as it is for the softening material ($n < 1$).

Although the nominal stresses σ_{11}, σ_{12} are fully determined to leading order as $r \to 0$ at this stage, the fact that, as yet, only the weak estimate (2.35) is available for the hydrostatic pressure p makes the asymptotic determination of σ_{22}, σ_{21} impossible without higher order considerations. In view of the relationship (1.7) between the nominal stresses $\sigma_{\alpha\beta}$ and the true stresses $\tau_{\alpha\beta}$, the full asymptotic determination

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[1] A similar but more complicated coalescence of eigenvalues arises in crack problems, see [8], [9].

of the latter must also await such higher order results.

C. Higher-Order Asymptotic Considerations

For the present, suppose that $n \neq 1$ and replace (2.5) by the two-term asymptotic representations

$$\left.\begin{aligned} y_1 &\sim k_1 r^{m_1} + w_1(\theta) r^{s_1} , \\ y_2 &\sim k_2 \theta r^{m_2} + w_2(\theta) r^{s_2} , \end{aligned}\right\} \text{ as } r \to 0 , \quad -\pi/2 \leq \theta \leq \pi/2 , \tag{2.52}$$

with the stipulation that

$$s_1 > m_1 , \quad s_2 > m_2 , \quad w_\alpha \in C^2([-\tfrac{\pi}{2}, \tfrac{\pi}{2}]) , \quad w_\alpha \not\equiv 0 \text{ on } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] , \tag{2.53}$$

and that w_1, w_2 have the respective parity of v_1 and v_2. Equations (1.2), (2.52) lead to the following representation for the components of the deformation gradient tensor:

$$F_{\alpha\beta} = f_{\alpha\beta}\, r^{m_\alpha - 1} + g_{\alpha\beta}\, r^{s_\alpha - 1} + o(r^{s_\alpha - 1}) , \text{ (no sum on } \alpha), \tag{2.54}$$

provided $f_{\alpha\beta}$ is given by (2.9) and

$$g_{\alpha\beta}(\theta) = s_\alpha c_\beta(\theta)\, w_\alpha(\theta) + \varepsilon_{\gamma\beta} c_\gamma(\theta)\, \dot{w}_\alpha(\theta) \quad \text{(no sum on } \alpha). \tag{2.55}$$

The asymptotic representation of the deformation invariant I depends on the value of s_1. One can show after some calculation that necessarily

$$m_1 < s_1 < 4 - 3m_1 . \tag{2.56}$$

Then the asymptotic representation for I is

$$I \sim m_1^2\, v_1^2\, r^{2(m_1-1)} + 2m_1 s_1 v_1 w_1\, r^{m_1+s_1-2} \quad . \tag{2.57}$$

Equations (2.17),(2.54),(2.55),(2.9),(2.57) and (1.38) yield

$$\left.\begin{aligned} \frac{\partial p}{\partial r} &\sim 2n\,A\,G_1^{2n-1}\; Y(\theta)\; r^{s_1-3} \quad , \\ \frac{1}{r}\frac{\partial p}{\partial \theta} &= o(r^{s_1-3}) \quad , \end{aligned}\right\} \; r \to 0 \;, \tag{2.58}$$

where

$$\left.\begin{aligned} Y(\theta) &= \ddot{w}_1 + \kappa w_1 \quad , \\ \kappa &= s_1[(2n-1)s_1 - 2(n-1)] \quad , \end{aligned}\right\} \tag{2.59}$$

and

$$G_1(\theta) = m_1 v_1(\theta) = m_1 k_1 \quad . \tag{2.60}$$

On the other hand, the boundary conditions (2.23) lead to

$$w_1(\pm\tfrac{\pi}{2}) = 0 \quad , \qquad p(r, \pm\tfrac{\pi}{2}) \sim o(r^{2(s_1-1)-m_1}) \; . \tag{2.61}$$

Integrate the first of (2.58) with respect to r to get

$$p \sim \frac{2n\,A\,G_1^{2n-1}}{s_1 - 2}\; Y(\theta)\; r^{s_1-2} \quad , \qquad (s_1 \neq 2) \quad . \tag{2.62}$$

Comparing (2.62) with the second of (2.61), one deduces that

$$Y(\pm\tfrac{\pi}{2}) = 0 \quad . \tag{2.63}$$

The compatability of (2.58),(2.62) together with (2.63) then gives

$$Y(\theta) \equiv 0 \qquad \text{on } [-\pi/2, \pi/2] \quad . \tag{2.64}$$

Equations (2.59) and the first of (2.61) imply

$$w_1(\theta) = B \cos \sqrt{\kappa}\,\theta \quad , \text{ B constant,} \quad -\pi/2 \leq \theta \leq \pi/2 \quad , \tag{2.65}$$

$$\sqrt{\kappa}\,\frac{\pi}{2} = j\pi \qquad , \; j = 0, \pm 1, \pm 2, \cdots \quad . \tag{2.66}$$

The second of (2.59) and (2.66) imply that

$$s_1 = \frac{2(n-1) \pm \sqrt{[2(n-1)]^2 + 16(2n-1)\,j^2}}{2(2n-1)} \; ; \quad j = \pm 1, \pm 2, \cdots \; . \tag{2.67}$$

One seeks the smallest value of s_1 satisfying (2.56). This occurs for $j = 1$, so that

$$s_1 = \frac{n-1}{2n-1} + \sqrt{\left(\frac{n-1}{2n-1}\right)^2 + \frac{4}{2n-1}} \quad . \tag{2.68}$$

The asymptotic results for the spatial coordinates deduced this far may be summarized as follows:

$$y_1 \sim k_1\, r^{\frac{2(n-1)}{2n-1}} + B \cos 2\theta\, r^{s_1} \quad , \quad n \neq 1 \quad , \tag{2.69}$$

$$y_2 \sim k_2\, \theta\, r^{\frac{2n}{2n-1}} \quad , \qquad n \neq 1 \quad , \tag{2.70}$$

$$p \sim o(r^{s_1 - 2}) \quad , \qquad n > 1/2 \quad . \tag{2.71}$$

The case $n = 1$ is treated at the end. Substituting from (2.54) into (1.3), using (2.11),(2.14) gives that

$$\begin{aligned} J \sim 1 + (k_2 s_1 w_1 - m_2 k_2 \dot{w}_1 \theta)\, r^{s_1 + m_2 - 2} \\ + m_1 k_1 \dot{w}_2\, r^{s_2 + m_1 - 2} + O(r^{s_1 + s_2 - 2}) \; . \end{aligned} \tag{2.72}$$

A simple analysis gives

$$s_2 = s_1 + m_2 - m_1 = s_1 + \frac{2}{2n-1} \quad , \tag{2.73}$$

and

$$\dot{w}_2 = \frac{m_2}{m_1}\frac{k_2}{k_1}\theta\,\dot{w}_1 - \frac{k_2 s_1}{k_1 m_1} w_1 \quad , \tag{2.74}$$

which on integration, using (2.65), yields

$$w_2 = \frac{B}{(m_1 k_1)^2}\left\{m_2\theta\cos 2\theta - \frac{1}{2}(m_2 + s_1)\sin 2\theta\right\} \quad . \tag{2.75}$$

For <u>n = 1</u>, we can similarly show that

$$y_1 \sim k_1 \log r + B\cos 2\theta\, r^2 , \qquad n = 1 \quad . \tag{2.76}$$

In an effort to find a strong estimate for the pressure field, we now assume the following three-term asymptotic representation for the deformation:

$$y_\alpha \sim v_\alpha r^{m_\alpha} + w_\alpha r^{s_\alpha} + z_\alpha r^{t_\alpha} \quad , \text{ (no sum on } \alpha), \tag{2.77}$$

with the stipulation that

$$t_1 > s_1 > m_1 \quad , \qquad t_2 > s_2 > m_2 \quad , \tag{2.78}$$

and z_1, z_2 are functions possessing derivatives of second order on $[-\pi/2, \pi/2]$, which fail to vanish identically and have the same parity as v_1 and v_2.

For n > 1, it can be shown that

$$z_1(\theta) = D\cos 2\sqrt{t_1}\,\theta + k_2^3(\mu_1\theta^2 + \mu_2) \quad , \tag{2.79}$$

and

$$t_1 = 2m_2 - m_1 = \frac{2(n+1)}{2n-1} \quad , \tag{2.80}$$

where

$$\left.\begin{aligned} D &= \frac{(m_2 + 2\mu_1)\ k_2^3 \frac{\pi}{2}}{2\sqrt{t_1}\ \sin \sqrt{t_1}\,\pi} \quad , \qquad 1 < \sqrt{t_1} < 2 \ , \\ \mu_1 &= \frac{1}{8t_1} [(2 - 3m_1)(m_2 - m_1) - 4m_2]\ m_2 \quad , \\ \mu_2 &= \frac{1}{4t_1} [\nu - 2(m_1 + \mu_1)] \quad , \\ \nu &= (2 - 3m_1)[1 - (m_2 - m_1)\ \frac{m_2}{2}\ (\frac{\pi}{2})^2] \quad , \end{aligned}\right\} \tag{2.81}$$

and we arrive at a strong estimate for the pressure field,

$$p \sim 2n\,A\,G_1^{2(n-1)}\ k_2^2\{(m_2 - m_1)\ \frac{m_2}{2}\ [\theta^2 - (\frac{\pi}{2})^2] + 1\}\ r^{\frac{2(2-n)}{2n-1}} \ , \quad n > 1 \ . \tag{2.82}$$

For <u>$n < 1$</u>, we can show that

$$z_1(\theta) = B^2(\hat{\mu}_1 \cos 4\theta - \hat{\mu}_2) \quad , \tag{2.83}$$

and

$$t_1 = 2s_1 - m_1 \quad , \tag{2.84}$$

where

$$\hat{\mu}_1 = -\frac{1}{4}\ (2n-1)t_1 s_1 k_2 \ , \qquad \hat{\mu}_2 = \frac{1}{2}\ (n-1)s_1 k_2 \quad , \tag{2.85}$$

but for the pressure field we have only the weak estimate

$$p \sim o(r^{2s_1 - m_1 - 2}) \quad . \tag{2.86}$$

At this stage we can determine z_2, t_2 from condition (1.3); however, it is not necessary to record the results here.

To find a strong estimate for the pressure field for $\underline{n < 1}$ we assume the deformation admits the representation

$$y_\alpha \sim v_\alpha r^{m_\alpha} + w_\alpha r^{s_\alpha} + z_\alpha r^{t_\alpha} + q_\alpha r^{\ell_\alpha} \quad , \text{ (no sum on } \alpha), \qquad (2.87)$$

with the stipulation that

$$\ell_1 > t_1 > s_1 > m_1 \quad , \qquad \ell_2 > t_2 > s_2 > m_2 \quad ,$$

$$q_\alpha \in C^2([-\pi/2, \pi/2]) \quad , \qquad q_\alpha \equiv 0 \quad \text{on } [-\pi/2, \pi/2] \quad .$$

We can now show that

$$\ell_1 = 2m_2 - m_1 \qquad , \qquad 7/12 < n < 1 \qquad , \qquad (2.88)$$

and that $q_1(\theta)$ is given by the value of $z_1(\theta)$ in (2.79). Thus, we can see a trade in dominance between the third and fourth term of (2.87), for $\alpha = 1$, as n passes through $n = 1$. Condition (1.3) is again used to determine ℓ_2 and q_2. For $7/12 < n < 1/2$, equation (2.82) is found to give a strong estimate for the pressure field. The value $n = 7/12$ is a transition point for the pressure field. A strong estimate for n in the range (1/2, 7/12) requires much further analysis.

At this point, we will record the results for $n = 1$.

$$\left.\begin{aligned} y_1 \sim\ & k_1 \log r + B \cos 2\theta\ r^2 + \frac{3}{8} k_2^3 \cos 4\theta\ r^4 \log r \\ & + [E \cos 4\theta - \frac{k_3^2}{16} (3\theta \sin 4\theta + 4\theta^2 + \pi^2 - \frac{5}{2})] r^4 \quad , \end{aligned}\right\}$$

$$\left.\begin{aligned} y_2 &\sim k_2\theta\, r \quad , \\ p &\sim A k_2^2 (4\theta^2 - \pi^2 + 2)\, r^2 \quad , \end{aligned}\right\} \quad (2.89)$$

where B and E are constants.

D. Summary of Results for Deformation and Stresses

The asymptotic results for the spatial coordinates and pressure field may be summarized as follows:

$$\left.\begin{aligned} y_1 &\sim k_1\, r^{\frac{2(n-1)}{2n-1}} + B\cos 2\theta\, r^{s_1} + B^2(\hat{\mu}_1 \cos 4\theta - \hat{\mu}_2) r^{2s_1 - m_1} \quad , \quad \tfrac{1}{2} < n \le \tfrac{7}{12} \, , \\ y_1 &\sim k_1\, r^{\frac{2(n-1)}{2n-1}} + B\cos 2\theta\, r^{s_1} + B^2(\hat{\mu}_1 \cos 4\theta - \hat{\mu}_2) r^{2s_1 - m_1} \\ &\qquad + [D\cos 2\sqrt{t_1}\,\theta + k_2^3(\mu_1\theta^2 + \mu_2)] r^{4-3m_1} \quad , \quad \tfrac{7}{12} < n < 1 \quad , \\ y_1 &\sim k_1^{\frac{2(n-1)}{2n-1}} + B\cos 2\theta\, r^{s_1} + [D\cos 2\sqrt{t_1}\,\theta + k_2^3(\mu_1\theta^2 + \mu_2)] r^{4-3m_1} \quad , \quad n > 1 \, , \\ y_2 &\sim k_2\,\theta\, r^{\frac{2n}{2n-1}} + w_2(\theta)\, r^{s_1 + \frac{2}{2n-1}} \quad , \quad n \neq 1 \\ p &\sim \frac{F}{\pi} k_2^3 \left\{\frac{2n}{(2n-1)^2}\left[\theta^2 - \left(\frac{\pi}{2}\right)^2\right] + 1\right\} r^{\frac{2(2-n)}{2n-1}} \quad , \quad \tfrac{7}{12} < n < \infty , \ n \neq 1 \, , \end{aligned}\right] \quad (2.90)$$

where k_1, k_2 are given in (2.44),(2.46); $w_2(\theta)$ in (2.75); s_1 and t_1 in (2.68),(2.80); D, μ_1, μ_2 in (2.81); and $\hat{\mu}_1, \hat{\mu}_2$ in (2.85). B, a constant, is left undetermined by the local analysis.

We now turn to the asymptotic determination of the actual stresses $\tau_{\alpha\beta}$. From (1.13),(2.17),(2.8),(2.9),(2.41)-(2.44), and (2.35), one finds

$$\tau_{11} \sim \frac{F}{\pi}(m_1k_1)\, r^{\frac{-2n}{2n-1}} \quad , \qquad n \neq 1 \quad , \tag{2.91}$$

$$\tau_{12} = \tau_{21} \sim \frac{F}{\pi}(m_2k_2)\, r^{\frac{-2(n-1)}{2n-1}}\,\theta,\ n \neq 1 \quad , \tag{2.92}$$

and using (2.82),

$$\tau_{22} \sim \frac{F}{\pi} m_2 k_2^3 \left\{\theta^2 + \frac{1}{2n-1}\left(\frac{\pi}{2}\right)^2\right\} r^{\frac{2(2-n)}{2n-1}} \quad , \quad \frac{7}{12} < n < \infty \ , \quad n \neq 1 \ . \tag{2.93}$$

The true stresses here are referred to the material polar coordinates (r,θ). The stress component τ_{11}, which is of primary physical interest, becomes unbounded at the origin for all admissible values of the hardening parameter, the singularity becoming more severe with decreasing values of n, and for the range of n under consideration is always stronger than that predicted by the linear theory. The other normal stress component τ_{22} remains bounded for $n \leq 2$, but for $n > 2$ it becomes unbounded, while the actual shearing stress τ_{12} is bounded for $n < 1$, and becomes unbounded at the origin for hardening materials $(n > 1)$. In both stresses τ_{22}, τ_{12}, the severity of the singularity increases with increasing values of n; however, the order of the singularity is less than that predicted by the linear theory for all allowable n.

For $n = 1$ we have, in summary, that

$$\left.\begin{aligned} y_1 &\sim k_1 \log r + B\cos 2\theta\, r^2 + \frac{3}{8} k_2^3 \cos 4\theta\ r^4 \log r \\ &\qquad + \left[E\cos 4\theta + \frac{k_2^3}{16}\left(6\theta \sin 4\theta + 4\theta^2 + \pi^2 - \frac{5}{2}\right)\right] \quad , \\ y_2 &\sim k_2 \theta\, r^2 + 2k_2^2 B\,(\theta\cos\theta - \sin\theta)\, r^4 \quad , \end{aligned}\right] \tag{2.94}$$

$$p \sim Ak_2^2(4\theta^2 - \pi^2 + 2)\ r^2 \quad .$$

Here, $k_1 = 1/k_2 = \frac{F}{2A\pi}$, while B and E are constant, undetermined by the local analysis. The components of the actual stress tensor are, using (1.13), (2.17), (2.8), (2.9), (2.50), and (2.35),

$$\tau_{11} \sim \frac{F}{\pi} k_1\ r^{-2} \quad , \qquad \tau_{22} \sim \frac{F}{\pi}\ 2k_2^3\{\theta^2 + (\tfrac{\pi}{2})^2\}\ r^2 \quad ,$$
$$\tau_{12} = \tau_{21} \sim \frac{F}{\pi}\ (2k_2)\ \theta \quad . \tag{2.95}$$

At this stage, a comparison of the results given in (2.94),(2.95) with those predicted by the classical linear theory reveals the presence of a log r singularity in the dominant term of the deformation in the x_1 direction in both cases. This is the only similarity! According to linear theory all the components of the stress tensor possess a 1/r singularity at the origin, but from (2.95) we see the stress component τ_{11} is more singular, while the other components are, in fact, bounded there.

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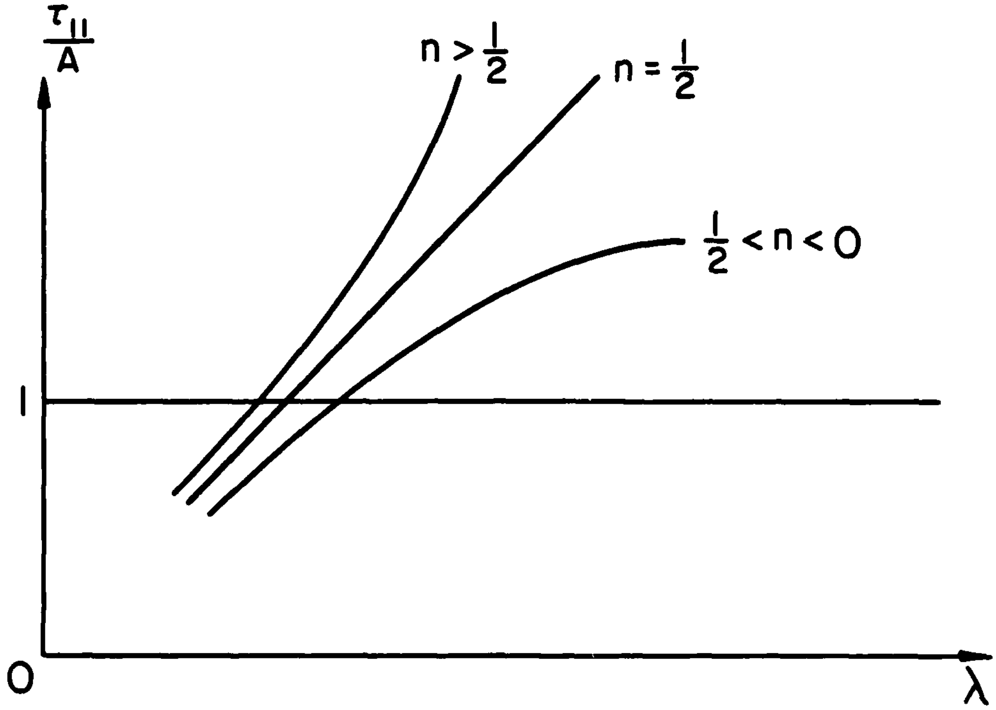
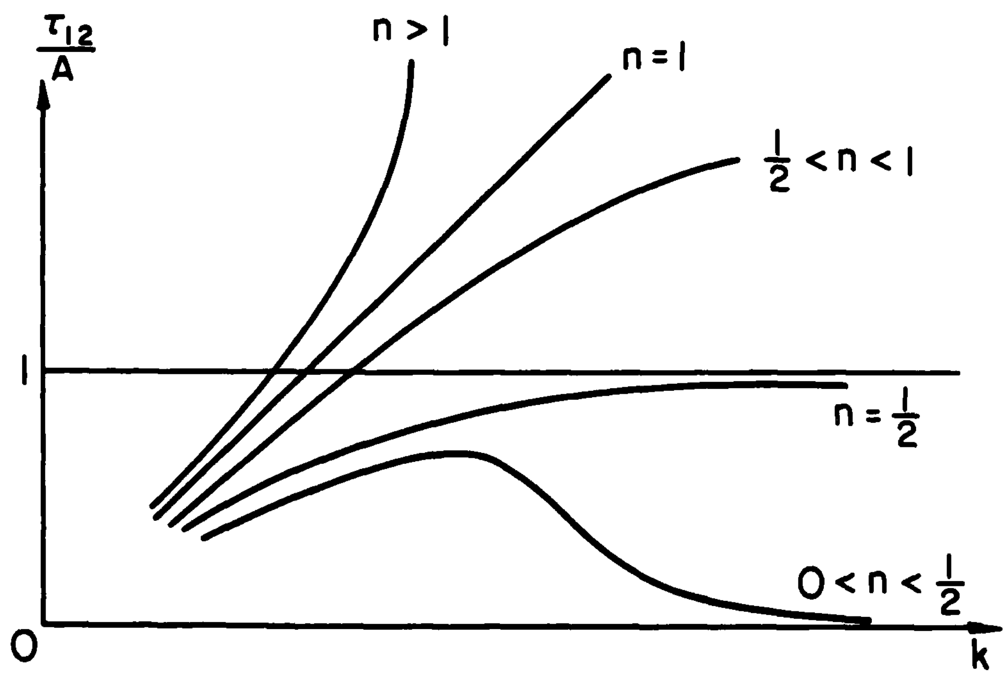
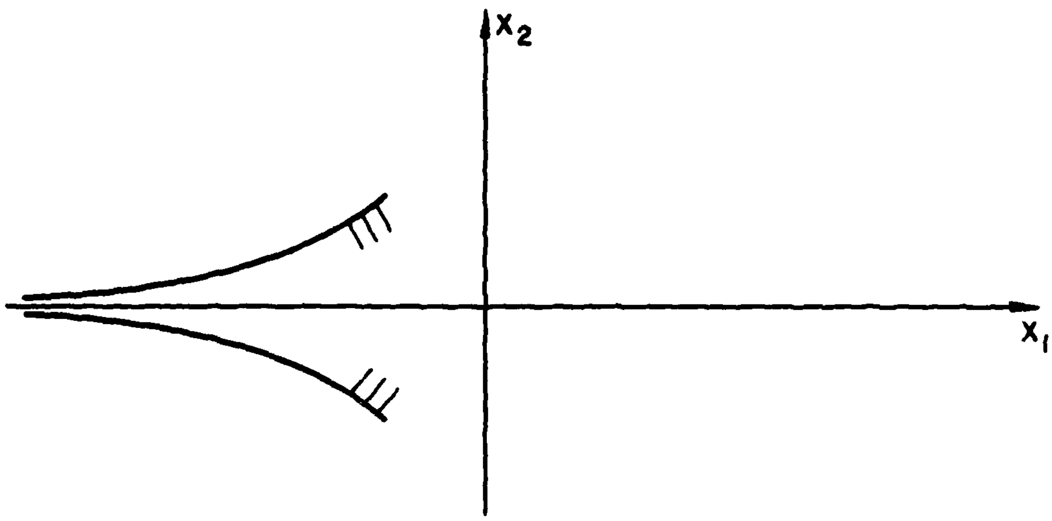


Power-law material response curves for extreme uniaxial stress.

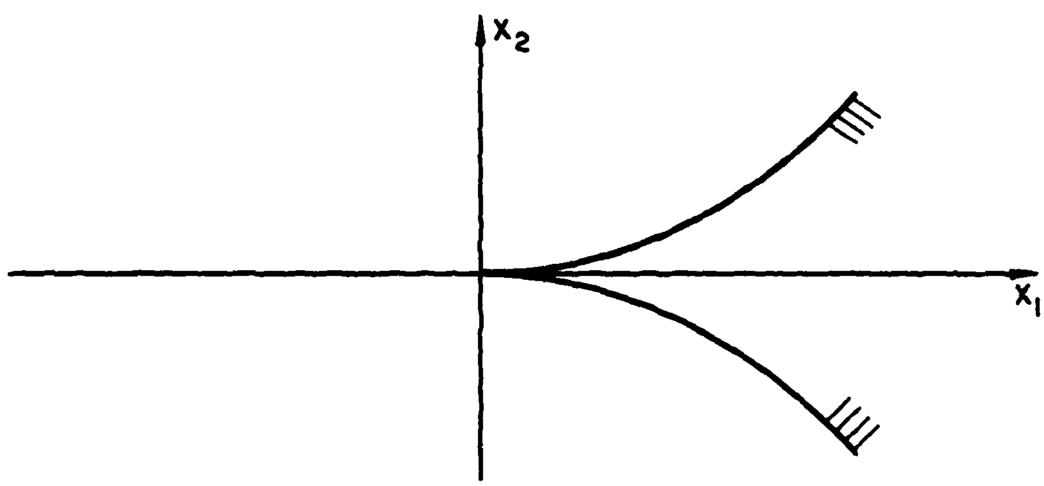


Power-law material response curves for severe simple shear.

FIGURE 1



$\frac{1}{2} < n \leq 1$ (SOFTENING MATERIAL)



$n > 1$ (HARDENING MATERIAL)

Local image of half-plane deformed by a tensile concentrated force.

FIGURE 2

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In this paper the fully non-linear equilibrium theory of homogeneous and isotropic, incompressible elastic solids is used to study the elastostatic field plane strain on a half-plane deformed by a concentrated surface load. Under suitable restrictions on the form of the elastic potential at severe deformations, it is shown that, for materials which ultimately "harden" in simple shear, the displacement is bounded near the point of application of the load. This is not the case for materials which ultimately "soften" in shear. Estimates of the true stress tensor near the singular point are given.

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