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ON THE EVALUATION OF CERTAIN MULTIVARIATE NORMAL PROBABILITIES

By

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CHAPTER I

INTRODUCTION AND SUMMARY

1.1. Statement of the Problem.

Suppose that X is an n-dimensional random vector that is normally distributed with mean zero and covariance matrix Σ, which is in fact a correlation matrix. For a given constant c, compute the probability of $A_k(c) = \{k$ of the n components of X exceed $c\}$: this is the problem that motivated the work contained herein. Denote the normal density with mean zero and correlation matrix Σ by $\phi(x;\Sigma)$; then the answer to this problem is just $\int_{A_k(c)} \phi(x;\Sigma)dx$. This n-dimensional integral is, however, quite useless in general. Rarely can it be expressed in terms of elementary functions; also, simulation and numerical integration in many dimensions are known to be prohibitively expensive and they do not provide general results. Our aim is to find approximations to the above integral that are easy to evaluate. In our search for these approximations, we have been able to extend or improve upon previously known results and we have discovered several interesting facts, which we also present.

1.2. Review of Earlier Work.

The problem of evaluating multivariate normal probabilities arises in many contexts. In fact whenever we wish to know the sampling properties of a test statistic or to make inferences when the population is (postulated to be) multivariate normal, we may expect to face this problem. Two examples of these applications are the following:

(i) The moments of certain nonparametric test statistics when the population is normal require orthant probabilities — see, e.g., Moran [18], who studies Spearman's rho.

(ii) The construction of simultaneous confidence intervals for the mean of a multivariate normal distribution often involves probabilities of rectangles — see, e.g., the books by Miller [17], and Tong [34]. In fact, Tong says that this topic is "perhaps a major source leading to the development of certain probability inequalities." (p. 165)

As we shall see later, when Σ has special structures such as equicorrelation or single latent factor, probability computations are fairly inexpensive, since they involve single integrals; however, efficient solutions have not been found in general. Many attempts to find efficient techniques have been made, and most of the methods proposed fall into one of several categories which we describe briefly.

(i) Reduction of dimensionality. In certain special cases, we can reduce the problem to that of evaluating a single or double integral, which is fairly inexpensive. For the general case, Plackett [20] uses a partial differential equation satisfied by $\phi(x;\Sigma)$ to provide a formula that expresses n-dimensional integrals in terms of lower dimensional integrals. His method is cumbersome, however, and he admits that "Whether (his) results offer a practical method of approach when n exceeds 4 is a moot point." (p. 358) His approach has much theoretical interest, though, for it elaborates upon the connection between the evaluation of orthant probabilities and computation of the volume of certain simplices on the (n-dimensional) unit sphere. The latter problem was extensively studied by

Schläfli [25] in the nineteenth century. Ruben [21,22,23] wrote several papers that have a similar bent. These methods are quite sophisticated mathematically, but they seem quite cumbersome in practice.

(ii) Asymptotic expansions. Another theoretically appealing approach is to write the probability (which is a function of ‡) in one of several infinite series expansion. Kendall [11] expanded bivariate normal probabilities in terms of tetrachoric functions and Moran [18] suggested the method for the quadrivariate case. They both found, however, that the series were useful only locally; that, in general, they converge very slowly. To overcome this problem, several authors (McFadden [13], Sondhi [29]) used a nonlinear series-to-series transformation suggested by Shanks [26] to accelerate the convergence in some special cases. To use Shanks' technique, we need to know the coefficients of several terms in the original series, and this points to yet another difficulty with asymptotic expansions: these coefficients are in general very hard to compute. In many cases, they involve complicated symmetric functions; in some examples that we tried, it can be shown that the coefficients are zonal polynomials (Takemura [33]), which are known only for the simplest cases. Thus, although a series expansion may be useful theoretically (in Chapter 2 we will provide a heuristic argument using it), the computational difficulties that it introduces usually render it useless in practice.

(iii) Probability Inequalities. The recent books by Tong [34] and Marshall and Olkin [16], and the expository article by Eaton [7] attest to the fact that the subject of probability inequalities in multivariate distributions is vast. We make no attempt to review the literature here. Perhaps the simplest inequality concerning the normal distribution is an elegant one first proved by Slepian [28]:

Lemma 1.1: Suppose X is normally distributed with mean zero and correlation Σ. Let R and T be correlation matrices. Then for all $a_1,\ldots,a_n$ we have $P_{\Sigma=R}(X_i > a_i, \forall i) \geq P_{\Sigma=T}(X_i > a_i, \forall i)$ whenever $r_{ij} \geq t_{ij}$ for all i,j.

Much work has been done to find more general classes of distributions (e.g. Das Gupta, et al. [3]) and other classes of regions (Sidak [27], Khatri [12]) for which similar inequalities hold. Our aim here is different. We take a closer look at Slepian's inequality in the normal case and "improve" upon it. The reason that we put quotes around improve is that our method does not always yield an inequality but it gives a much better approximation than does Slepian. This is a phenomenon that we shall see again when we study Mills' ratio: when we have an inequality, it is a bad approximation; when we have a good approximation, we no longer have the inequality that we desire.

(iv) Mills' Ratio. For certain problems, we need "simultaneous tail probabilities," $P\{X_i > a_i, \forall i\}$, where the a_i are large. In these cases, the multivariate Mills' ratio $R(a,\Sigma) = P_\Sigma(X_i > a_i, \forall i)/\phi(a,\Sigma)$ $(a = (a_1,\ldots,a_n)')$ becomes useful. Much work remains to be done on this topic. In an early paper, Savage [24] generalized the univariate case in a straightforward manner. Recently Steck [32] showed that Savage's methods are often inapplicable (because his conditions are too stringent). He then introduced three simple approximations to $R(a,\Sigma)$ in terms of the univariate Mills' ratio, and he showed some numerical examples. He was able to show that two of the approximations were indeed lower bounds to $R(a,\Sigma)$, but he did not study the third approximation, which appeared to be the best of the three (in the sense that it yielded the largest lower bound). After studying this topic, we have been able to give a partial answer to Steck's conjecture about the third approximation; we have also been able to "fine tune" his method to yield some improvements.

Finally there have also been several methods introduced (e.g. Henery [10], McFadden [14]) that take advantage of some special structure in $\ddagger$. They do not immediately fall into any of the categories that we have listed above.

One feature that we have noticed in the earlier literature is that some authors get very accurate results (numbers that are correct to the third, fourth, or even fifth decimal place), using methods that are hard to apply — see, e.g. Steck [30], Plackett [20]. Our approach is different. Our methods are very simple to apply, but they typically yield approximations that are good to the second (and sometimes only first) decimal place. If we desire more accuracy, we must do more mathematics to find other methods, and can expect that they will be harder to apply.

1.3. Notation.

Let $A_k(c) = \{k$ components of X exceed $c\}$, and $p_k^{(n)} = p(k,n,c,\ddagger) = P_{\ddagger}(A_k(c))$. If $S_n = \Sigma_1^n I\{X_i > c\}$, then $p_k^{(n)} = P\{S_n = k\}$. Thus we may regard $(p_0^{(n)},\ldots,p_n^{(n)})$ as a probability distribution — we call it the exceedance distribution — on the integers $\{0,\ldots,n\}$. Furthermore, since X and $-X$ have the same distribution, we may assume without loss of generality that $c \geq 0$. Finally, let $\phi(x)$ and $\Phi(x)$ be the standard univariate normal density and distribution functions, respectively.

1.4. Summary of Present Work.

In Chapter 2, we introduce an approximation to the exceedance distribution for arbitrary $\ddagger$, and then refine it. We also provide numerical examples which indicate when our methods are good.

In Chapter 3, we prove a variance inequality and use it to modify Slepian's inequality for orthant probabilities. We then consider the case of large c via Mills' ratio: we use a new result for the one-dimensional Mills' ratio to prove an inequality for the multi-dimensional Mills' ratio.

The equicorrelation case plays a central role in our work. Much has been written about it because its symmetry makes it mathematically tractable. We have some interesting new results that comprise Chapter 4.

Chapter 5 has some miscellaneous results. One section contains some work on asymptotic expansions that well illustrate the difficulties encountered. The other section contains some combinatorial results for some special cases.

CHAPTER II

AN APPROXIMATION

2.1. Approximation by Equicorrelation Case.

Notice that the event $A_k(c)$ is invariant under permutations of the coordinates of X. Thus

$$P_{\Sigma}(X \varepsilon A_k(c)) = P_{\Sigma}(\pi X \varepsilon A_k(x)) = P_{\pi\Sigma\pi'}(X \varepsilon A_k(c)) \; ,$$

where π is a permutation matrix. With this in mind, let $Y^{(i)}$ be distributed as $N(0,\pi_i\Sigma\pi_i')$ for $i = 1,\ldots,n!$. Also, let $\{Y^{(i)}\}_1^{n!}$ be independent. Let N be uniformly distributed on $\{1,\ldots,n!\}$ independently of X and $\{Y^{(i)}\}_1^{n!}$.

The exceedance distribution for $Y^{(N)}$ is the same as that for X. This is because

$$P_{\Sigma}(A_k(c)|N=i) = \int_{A_k(c)} \phi_{\pi_i\Sigma\pi_i'}(x)dx$$

and

$$P_{\Sigma}(A_k(c)) = E\,P_{\Sigma}(A_k(c)|N=i) = \int_{A_k(c)} \frac{1}{n!}\sum_{i=1}^{n!} \phi_{\pi_i\Sigma\pi_i'}(x)dx$$

$$= \int_{A_k(c)} m(x;\Sigma)dx \; ,$$

where $m(x;\Sigma) = \frac{1}{n!}\Sigma\, \pi_i\Sigma\pi_i$. Here $m(x;\Sigma)$ is not a normal density function; it is rather a mixture of normals. But if a random vector M has density $m(x;\Sigma)$, the $EM = EE(M|N) = 0$ and $EMM' = EE(MM'|N) = \Sigma_{\bar{\rho}}$, where

$$\Sigma_{\bar{\rho}} = \frac{1}{n!} \sum_{1}^{n!} \pi_i \Sigma \pi_i'$$

is an equicorrelation matrix with parameter $\bar{\rho}$ which is the average of the off-diagonal terms of Σ. Our approach here is to simply replace the non-normal density $m(x;\Sigma)$ by a normal density with the same first two moments: $\phi(x;\Sigma_{\bar{\rho}})$.

2.2. The Equicorrelation Case.

Of course, the above argument would be fruitless if the equi[illegible]tion case were not easily computable. That it is easy is proved in

Theorem 1. If $X \sim N(0,\Sigma)$, where Σ is an equicorrelation matrix with $\rho \geq 0$, then

$$P(A_k(c)) = \binom{n}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{k} \Phi\left(\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{n-k} \phi(t)dt .$$

Proof. Let $Z_0, Z_1, \ldots, Z_n$ be iid $N(0,1)$ variates. Let $Y_i = \sqrt{1-\rho}\, Z_i - \sqrt{\rho}\, Z_0$, $i = 1,\ldots,n$. Then $Y = (Y_1,\ldots,Y_n)'$ has the same distribution as X. So if we note that the X_i's are exchangeable and condition on the main effect Z_0, we get

$$P(A_k(c)) = \binom{n}{k} P(X_1 > c,\ldots,X_k > c, X_{k+1} \leq c,\ldots,X_n \leq c)$$

$$= \binom{n}{k} \int_{-\infty}^{\infty} P\left(Z_1 > \frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}},\ldots,Z_k > \frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}, Z_{k+1} \leq \frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}},\ldots,Z_n \leq \frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)\phi(t)dt$$

$$= \binom{n}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{k} \Phi\left(\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{n-k} \phi(t)dt .$$

QED

We pause now to take a closer look at this result. The expressions for $p_k^{(n)}$ are easily evaluated because we have a one-dimensional integral involving the well-tabulated functions Φ and ϕ. Gaussian quadrature as described in Abramowitz and Stegun [1] yields satisfactory results.

In the theorem above, we require $\rho \geq 0$. In order to generate exchangeable sequences with negative correlation, we may take $X_i = \sqrt{1+\rho}\, Z_i + \sqrt{-\rho}\, Z_0$, where $Z_1,\ldots,Z_n$ are iid $N(0,1)$ but now $\mathrm{corr}(Z_0,Z_i) = -\sqrt{-\rho/(1-\rho)}$. Unfortunately, the method of conditioning does not yield any useful results, for $Z_1,\ldots,Z_n$ are not conditionally independent. Recall that $\rho \geq -\frac{1}{n-1}$, so that for large n, $\ddagger$ will be very close to the identity. We shall see later that $\{p_k^{(n)}\}$ are very smooth functions of the components of $\ddagger$, so that for $-\frac{1}{n-1} < \rho \leq 0$, the exceedance distribution will be approximately a binomial.

But the expression

$$p_k^{(n)} = \binom{n}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^k \left(\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{n-k} \phi(t)dt$$

is formally correct, even for $\rho < 0$, because of the following argument that is due to Steck [22]. If we define $\Phi(z)$ for $z = x+iy$ by integrating along the line $\{t+iy: -\infty < t \leq x\}$, then

$$\Phi(z) = e^{y^2/2} \int_{-\infty}^{x} e^{-ity}\phi(t)dt = E(e^{-iyX}\, I\{X \leq x\})/Ee^{-iyX}$$

where $X \sim N(0,1)$. Steck then presents an argument to show that

$$P(X_1 \leq c,\ldots,X_n \leq c) = \int_{-\infty}^{\infty} \Phi\left(\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^n \phi(t)dt \quad \text{for } \rho < 0;$$

evidently the integral of the imaginary part is zero. A simple computation shows that we still have $\Phi(-z) = 1-\Phi(z)$. Thus, an inclusion-exclusion argument yields the formula for $p_k^{(n)}$. Steck remarks, though, that it "does not appear feasible to isolate the real part of the integrand," and so provides another formula for $F_n(c|\rho) \equiv P(X_1 \leq c_1, \ldots, X_n \leq c)$:

$$F_n(c|\rho) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} F_k(\alpha_n c|\rho_n) F_{n-k}(c|\rho) ,$$

where $\alpha_j^2 = \frac{1-\rho}{(1+(j-1)\rho)(1+(j-2)\rho)}$ and $\rho_n = \frac{-\rho}{1+(n-2)\rho}$. For the special case $c = 0$, he uses a recursion formula due to David [4] to get

$$F_n(0|\rho) = \frac{1}{2} - \sum_{j=1}^{[\frac{n}{2}]} \binom{n}{2j} F_k(0|\frac{\rho}{1+(2j-2)\rho}) F_{n-k}(0|\frac{\rho}{1+2j\rho}) .$$

The last formula has the advantage that there are only $[\frac{n}{2}]$ terms in the sum and each term is positive. Later, we shall present some Taylor series approximations to $F_n(c|\rho)$ that are quite easy to apply. For the special case $n = 2$, though, $p_k^{(2)}$ can be evaluated exactly. We can generate a negatively correlated bivariate normal $(X_1, X_2)'$ thus: $X_1 = \sqrt{1+\rho}\, Z_1 + \sqrt{-\rho} Z_0$ and $X_2 = \sqrt{1+\rho}\, Z_2 - \sqrt{-\rho}\, Z_0$, where Z_0, Z_1, Z_2 are independent standard normal variates. Then it is easy to see that

$$P(X_1 > c,\ X_2 > c) = \int_{-\infty}^{\infty} \Phi(-\frac{c-t\sqrt{-\rho}}{\sqrt{1+\rho}})\ \Phi(-\frac{c+t\sqrt{-\rho}}{\sqrt{1+\rho}})\ \phi(t)dt ,$$

$$P(X_1 \leq c,\ X_2 \leq c) = \int_{-\infty}^{\infty} \Phi(\frac{c-t\sqrt{-\rho}}{\sqrt{1+\rho}})\ \Phi(\frac{c+t\sqrt{-\rho}}{\sqrt{1-\rho}})\ \phi(t)dt , \quad \text{and}$$

$$p_1^{(2)} = 2\int_{-\infty}^{\infty} \Phi(-\frac{c-t\sqrt{-\rho}}{\sqrt{1+\rho}})\ \Phi(\frac{c+t\sqrt{-\rho}}{\sqrt{1+\rho}})\ \Phi(t)dt .$$

Because of the symmetry in the equicorrelation case, there are many interesting results that are analytically tractable. We will return to them later, and now say more about our approximation technique.

2.3. Local Argument for Approximation.

We will see later that approximating $m(x;\Sigma)$ by $\phi(x;\Sigma_{\bar{\rho}})$ does not always work well. In fact, as the dispersion among the $\{\rho_{ij}\}$, $\sum_{i<j}(\rho_{ij}-\bar{\rho})^2$, gets bigger, the approximation often gets worse. It may be tempting to use some other index of the center of the $\{\rho_{ij}\}$, e.g. the median, but the following argument supports $\bar{\rho}$, at least locally. We first need Plackett's identity, which is proved in [20]:

Lemma 2: If $\Sigma = (\rho_{ij})$ is a correlation matrix, then

$$\frac{\partial}{\partial\rho_{ij}}\phi(x;\Sigma) = \frac{\partial^2}{\partial x_i \partial x_j}\phi(x;\Sigma) .$$

Typically, we are interested in probabilities as a function of Σ, so we may expect to differentiate with respect to ρ_{ij}. However, the normal density contains the terms $|\Sigma|^{-1/2}$ and $x'\Sigma^{-1}x$, which are quite hard to handle. Plackett's identity simplifies matters by allowing us, instead, to differentiate the quadratic form $x'\Sigma^{-1}x$ with respect to x. We will use it many times in the sequel.

Now suppose that for a given Σ we want to expand $p_k^{(n)}$ about some equicorrelation Σ_ρ. Consider $T(\theta) = (1-\theta)\Sigma_\rho + \theta\Sigma$. Write $p_k^{(n)}(\theta) = \int_{A_k(c)} \phi(x;T(\theta))dx$ and expand in the Taylor series:

$$p_k^{(n)}(\theta) = p_k^{(n)}(0) + \theta p_k^{(n)\prime}(0) + \frac{\theta^2}{2}p_k^{(n)\prime\prime}(0) + \cdots .$$

Plackett's identity and the chain rule gives

$$p_k^{(n)\,'}(0) = \sum_{i<j} (\rho_{ij}-\rho) \int_{A_k(c)} x_i x_j \; \phi(x;\ddagger_\rho)dx$$

$$= \{\sum_{i<j} (\rho_{ij}-\rho)\} \int_{A_k(c)} x_1 x_2 \; \phi(x;\ddagger_\rho)dx \; ,$$

where the second equality is due to exchangeability. Thus, the first derivative vanishes when we choose $\rho = \bar{\rho} = \sum_{i<j} \rho_{ij}/\binom{n}{2}$. Locally, then, if $\ddagger - \ddagger_{\bar{\rho}}$ is of order θ, the difference in the respective exceedance probabilities is of order θ^2. The argument given above applies locally; as θ approaches 1, the higher order terms potentially play a large role. Also, the Taylor series is of theoretical value only. Computation of the higher order coefficients seems feasible only when $\bar{\rho} = 0$; even in that case, the computations are quite hard. We show some details and give some numerical results in section 5.1.

2.4. <u>Interpretations of Approximation and Extensions.</u>

There are at least three other ways of interpreting the $\ddagger_{\bar{\rho}}$ approximation that we introduced. Only one of them has proved fruitful; we briefly mention the other two and then delve into the fruitful one. (i) $\ddagger_{\bar{\rho}}$ is the average of $\ddagger$ over the permutation group: $\ddagger_{\bar{\rho}} = \frac{1}{n!} \sum_\pi \pi \ddagger \pi'$. Our problem does not seem to be invariant under other subgroups of the orthogonal group, $\mathcal{O}_n$. Averaging $\ddagger$ over $\mathcal{O}_n$ or the group of sign changes yields the identity. (ii) We earlier approximated $m(x;\ddagger)$ by $\phi(x;\ddagger_{\bar{\rho}})$ by fitting the first two moments. It may be worthwhile to try an Edgeworth expansion by considering fourth moments also (the third moments are zero). We have not been able to carry this idea far. Now we turn to an idea that yields nice results.

$\Sigma_{\bar\rho}$ is the projection of Σ onto the set of equicorrelation matrices under the norm $\|A-B\| = \{\sum_{i,j}(a_{ij}-b_{ij})^2\}^{1/2}$. But there is a larger class of correlation matrices that yield one-dimensional integrals for $p_k^{(n)}$. If we consider $X_i = \sqrt{1-\alpha_i^2}\, Z_i - \alpha_i Z_0$, with $|\alpha_i| < 1$, then X has correlation matrix $\Sigma = (\sigma_{ij})$, $\sigma_{ij} = \alpha_i\alpha_j$, $i \neq j$. It is called a "single-factor matrix" in one context of factor analysis. In this case, we have

$$p_k^{(n)} = \sum \int_{-\infty}^{\infty} \prod_{j=1}^{k} \Phi\left(-\frac{c+\alpha_{i_j} t}{\sqrt{1-\alpha_{i_j}^2}}\right) \prod_{j=k+1}^{n} \Phi\left(\frac{c+\alpha_{i_j} t}{\sqrt{1-\alpha_{i_j}^2}}\right) \phi(t)dt$$

where the summation is taken over the $\binom{n}{k}$ terms corresponding to the subevents that make up $A_k(c)$. $p_k^{(n)}$ is still given in terms of a one-dimensional integral, but because we no longer have exchangeability, we cannot avoid the task of enumerating the $\binom{n}{k}$ subevents that comprise $A_k(c)$.

The set of $n \times n$ single-factor matrices, $\mathcal{F}$, contains the set of equicorrelation matrices, $\mathcal{E}$. $\mathcal{F}$ is much larger than $\mathcal{E}$: $\mathcal{F}$ is described by n parameters, $\mathcal{E}$ by just one, and the set of all correlation matrices, $\mathcal{S}$, by $\binom{n}{2}$ parameters. If we picture $\mathcal{S}$ by plotting $(\rho_{12},\rho_{13},\ldots,\rho_{n-1,n})$ in $\mathbb{R}^{\binom{n}{2}}$, $\mathcal{F}$ and $\mathcal{E}$ are lower dimensional subsets when $n \geq 4$. But when $n = 3$, $n = \binom{n}{2}$, and $\mathcal{F}$ is an open set in $\mathcal{S}$. In fact, in the region $\rho_{ij} > 0$, $\mathcal{F}$ is characterized by $\rho_{12}\rho_{13}\rho_{23} < \min(\rho_{12}^2,\rho_{13}^2,\rho_{23}^2)$.

The projection of Σ onto $\mathcal{F}$ is closer to Σ than $\Sigma_{\bar\rho}$ is. Finding the projection onto $\mathcal{F}$ is hard, for it requires minimizing $\sum_{i,j}(\rho_{ij}-\alpha_i\alpha_j)^2$ over $\{(\alpha_1,\ldots,\alpha_n)\colon |\alpha_i| < 1, \forall i\}$, which is a restricted nonlinear least squares problem. It would be pointless to replace one hard problem (multi-dimensional integration) by another hard one, so we tried a simple (ad hoc) method that gave good numerical results. This alternative is to use a standard maximum likelihood fitting procedure (as provided by BMDP or SPSS) to get the loadings $\alpha_1,\ldots,\alpha_n$.

This last suggestion is based on practical considerations only. We motivated the $\Sigma_{\bar{\rho}}$ approximation by replacing a mixture of normals by a normal with the same first two moments. There does not seem to be any such "statistical" justification for the single-factor model approximation. We also showed that $\Sigma_{\bar{\rho}}$ was locally optimal among all equicorrelation matrices by requiring that the first derivative in a certain Taylor expansion vanish. If we use a Taylor expansion about an arbitrary single-factor matrix, G, the first derivative becomes

$$\sum_{i<j} (\rho_{ij}-\alpha_i\alpha_j) \int_{A_k(c)} x_i x_j \; \phi(x;F)dx \; ;$$

the integral is not independent of i and j, and it depends upon all the α_i's. It is impractical to set this expression equal to zero and then solve for $\alpha_1,\ldots,\alpha_n$. What we are left with, then, is that the single-factor matrix given by the maximum likelihood fitting procedure is closer to Σ than $\Sigma_{\bar{\rho}}$ is, so it ought to give better approximations than $\Sigma_{\bar{\rho}}$. It does not always, but as we see from the numerical examples it can give substantial improvement over $\Sigma_{\bar{\rho}}$.

2.5. Discussion of Numerical Examples.

Roughly speaking, the $\Sigma_{\bar{\rho}}$ approximation does well when the variance among the $\{\rho_{ij}\}$, $\Sigma(\rho_{ij}-\bar{\rho})^2$, is not too large; we have no convenient analytic expression to say what "too large" means. When $\Sigma(\rho_{ij}-\bar{\rho})^2$ is large, the single-factor matrix approximation improves upon $\Sigma_{\bar{\rho}}$ considerably. We have not found any examples in which the single-factor approximation does poorly.

Appendix 1 contains the data that we used to make the above claims. The first part of this appendix deals with Toeplitz forms: $\rho_{ij} = \rho^{i-j}$ for $i \geq j$. Several investigators indicated that such covariance structure is of wide applicability, so we studied this case and a modification of it extensively. The $\Sigma_{\bar{\rho}}$ approximation did surprisingly well; we did not check the single-factor approximation here. For each example, we indicate the value of the parameter ρ, the constant c, and the number of non-zero "super-diagonals," labelled diag. Thus if $n = 4$ and diag $= 3$,

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 \\ & 1 & \rho & \rho^2 \\ & & 1 & \rho \\ & & & 1 \end{pmatrix},$$

whereas if $n = 4$ and diag $= 2$,

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho^2 & 0 \\ & 1 & \rho & \rho^2 \\ & & 1 & \rho \\ & & & 1 \end{pmatrix}.$$

The column labelled "phat" contains the simulated values of the exceedance probabilities; that labelled "error" contains the estimated standard errors for the simulation value; and the column labelled "equi" contains the $\Sigma_{\bar{\rho}}$ approximation. It is clear from the data that the $\Sigma_{\bar{\rho}}$ approximation is excellent when all super-diagonals are non-zero. When some of the super-diagonals are zero, the approximation gets worse; but as we can see from the case $n = 10$, diag $= 2$, $\rho = .5$, the approximation is still good.

The second part of Appendix 1 provides a direct comparison of the $\Sigma_{\bar{\rho}}$ and single-factor approximations. We selected correlation matrices from the books by Anderson [2], Morrison [19], and Mardia, Kent, and Bibby [15], used BMDP's factor analysis package to get the maximum likelihood loadings, computed the two approximations, and compared them to the simulation results. Sometimes, the BMDP routine would return an α_i value of 1, which is not appropriate for our purposes (since we must divide by $\sqrt{1-\alpha_i^2}$ to compute $p_k^{(n)}$); we replaced such values by .95. In the appendix, we write Σ, the fitted α_i's, and the constant c. The columns labelled "phat", "error", and "equi" are as above, and that labelled "m.l.fit" contains the single-factor approximation. The row labelled "distances" contains the distance from the simulated vector $p = (p_0^{(n)},\ldots,p_n^{(n)})$ and the approximation $q = (q_0^{(n)},\ldots,q_n^{(n)})$: distance $= \{\sum_k (p_k^{(n)}-q_k^{(n)})^2\}^{1/2}$. Of course, other distance functions may also be appropriate, but our choice here seems quite reasonable.

It is evident that the single-factor approximation is often excellent, and is never a bad approximation. The $\Sigma_{\bar{\rho}}$ approximation can do very badly because the exceedance distributions for equicorrelation matrices are usually u-shaped or bell shaped (we prove this for $c = 0$ in Section 4.2), whereas for arbitrary $\ddagger$, the exceedance distribution may have more than one mode for $1 \leq k \leq n-1$. Such cases arise when $\ddagger$ is far from $\ddagger_{\bar{\rho}}$.

CHAPTER III

A CLOSER LOOK AT ORTHANT PROBABILITIES

3.1. A Variance Inequality.

We begin with a simple but useful lemma.

Lemma 3.1: Let X be a vector which has a bivariate normal distribution with mean zero and covariance matrix $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. For each fixed $c \geq 0$, $f(\rho) = P_\rho\{X_1 > c, X_2 > c\}$ is convex for $\rho \varepsilon (0,1)$. For each fixed $c > \sqrt{2} - 1$, f is convex in $(-1,1)$. For $c = 0$, f is convex in $(0,1)$ and concave in $(-1,0)$.

Proof. Let $\eta(x;\Sigma) = \frac{\partial^2}{\partial x_1 \partial x_2} \phi(x;\Sigma)$. By Plackett's identity, we have

$$f''(\rho) = \frac{\partial^2}{\partial \rho^2} \int_c^\infty \int_c^\infty \phi(x;\Sigma)dx = \int_c^\infty \int_c^\infty \frac{\partial^2}{\partial \rho^2} \phi(x;\Sigma)dx$$

$$= \int_c^\infty \int_c^\infty \frac{\partial^2}{\partial x_1 \partial x_2} \eta(x;\Sigma)dx = \eta((c,c);\Sigma)$$

$$= \frac{\phi(c)}{1-\rho^2} \{\rho + c^2 \frac{1+\rho}{1-\rho}\} .$$

The interchange of differentiation and integration above is easily justified. The sufficient conditions for $f'' > 0$ and $f'' < 0$ mentioned in the lemma are easily verified by checking the sign of $(\rho + c^2 \frac{1+\rho}{1-\rho})$. QED

Now let

$$h(\{\rho_{ij}\}) = \text{var}_\Sigma S_n = n\Phi(-c) - n^2\Phi(-c)^2 + \sum_{i \neq j} P_{\rho_{ij}}(X_i > c, X_j > c) ;$$

the variance of the exceedance distribution can be written in terms of bivariate quadrant probabilities (similarly, the k^{th} moment of the exceedance distribution is expressible in terms of k-dimensional orthant probabilities). Using Lemma 3.1, it is easy to see that the Hessian of h is positive definite whenever (i) $\rho_{ij} > 0$ $\forall i,j$ or (ii) $c > \sqrt{2} - 1$, and negative definite whenever $c = 0$ and $\rho_{ij} < 0$ $\forall i,j$. With this in mind, we have

Theorem 3.1: If (i) $\rho_{ij} > 0$ $\forall i,j$ or (ii) $c > \sqrt{2} - 1$, then $\mathrm{var}_{\ddagger} S_n > \mathrm{var}_{\ddagger_{\bar{\rho}}} S_n$. If $c = 0$ and $\rho_{ij} < 0$ $\forall i,j$, then $\mathrm{var}_{\ddagger} S_n < \mathrm{var}_{\ddagger_{\bar{\rho}}} S_n$.

Note. The theorem remains true if we appropriately replace the strict inequalities by weak ones.

Proof. The function h defined above is a symmetric (strictly) convex function whenever (i) or (ii) hold; hence it is Schur convex (see Marshall and Olkin [16], pp. 64-68). Since $(\rho_{12},\ldots,\rho_{n-1,n})$ majorizes $(\bar{\rho},\bar{\rho},\ldots,\bar{\rho})$, we have $h(\{\rho_{ij}\}) \geq h(\bar{\rho},\bar{\rho},\ldots,\bar{\rho})$. The proof for $c = 0$, $\rho_{ij} < 0$ is analogous. QED

3.2. An Application of the Variance Inequality.

Consider for the moment the case $c = 0$, and $\rho_{ij} > 0$, $\forall i,j$. We know that the mean of the exceedance distribution is $n/2$ and by Theorem 3.1 above we know that the exceedance distribution is more concentrated under $\ddagger_{\bar{\rho}}$ than under $\ddagger$. It is tempting to conjecture, then, that

$$P_{\ddagger}(X_1 > 0,\ldots,X_n > 0) \geq P_{\ddagger_{\bar{\rho}}}(X_1 > 0,\ldots,X_n > 0) .$$

Indeed, for $n = 3$, it is well known that

$$P_{\ddagger}(X_1 > 0, X_2 > 0, X_3 > 0) = \frac{1}{2} - \frac{1}{4\pi}(\cos^{-1}\rho_{12} + \cos^{-1}\rho_{13} + \cos^{-1}\rho_{23}) ,$$

which is a convex function of $(\rho_{12}, \rho_{13}, \rho_{23})$; thus, the inequality above does hold. Such a result is not true in general, however, as is seen by the following example:

$$\Sigma = \begin{Bmatrix} 1.0 & 0.5 & 0.5 & 0.0 \\ & 1.0 & 0.5 & 0.0 \\ & & 1.0 & 0.0 \\ & & & 1.0 \end{Bmatrix} , \quad \bar{\rho} = .25 \ , \ c = 0$$

$P_\Sigma(X_i > 0, \forall i) = .1250$, whereas the corresponding probability for $\Sigma_{\bar{\rho}}$ is .1265. (Of course, the fact that we have some zero off-diagonal terms in Σ is not the problem; we could just change them by some small ε). What does seem to be true, however, is that $p_0^{(n)} + p_1^{(n)}$ under Σ exceeds the corresponding sum under $\Sigma_{\bar{\rho}}$, although we have not found conditions for that.

We have also found in our numerical examples that although the inequality $P_\Sigma(X_i > 0, \forall i) \geq P_{\Sigma_{\bar{\rho}}}(X_i > 0, \forall i)$ does <u>not</u> hold, $P_{\Sigma_{\bar{\rho}}}(X_i > 0, \forall i)$ provides a much better approximation to $P_\Sigma(X_i > 0, \forall i)$ than the method suggested by Tong [34] and Steck [30] — their method uses Slepian's inequality. Tong suggests bounding the desired probability above and below by using equi-correlation matrices with $\rho_{max} = \max_{i,j} \rho_{ij}$ and $\rho_{min} = \min \rho_{ij}$, respectively, but these bounds are often useless, as we see from the following table. We took the first three examples from the second part of Appendix 1.

True Probability	Our Approximation	Upper Bound	Lower Bound
.216	.214	.272	.164
.134	.135	.228	.106
.059	.058	.105	.022
.194	.173	.290	.057
.056	.049	.174	.033

Notice, especially, that even when the $\Sigma_{\bar{\rho}}$ approximation is bad for the whole exceedance distribution, it does quite well in approximating $p_n^{(n)}$.

Of course, when c is so large that $p_0^{(n)}$ exceeds $p_k^{(n)}$ for $k = 1,\ldots,n$, then the $\Sigma_{\bar{\rho}}$ approximation will serve as an "upper bound" to the true $p_0^{(n)}$ and a "lower bound" for $p_n^{(n)}$; we use quotes here because although we do not always have a proper bound, we generally have a good approximation.

We now briefly mention an application of these ideas to the construction of convervative simultaneous confidence sets. Suppose that we observe $X^{(1)},\ldots,X^{(m)}$ iid $N(\mu,\Sigma)$ with Σ known, and we wish to get one-sided confidence bounds for μ. That is, we consider

$$R = \{t \in \mathbb{R}^n : t_i \leq \bar{x}_i + d_i,\ i=1,\ldots,n\} .$$

Now, if $\sigma_{ij} \geq 0$, we get $P_\Sigma(\mu \in R) \geq P_I(\mu \in R)$. Tong [34] suggests that if we choose $d_i \equiv d$, where d satisfies $\Phi(\sqrt{m}\, d) = (\gamma)^{1/n}$, the confidence is at least γ. He also suggests that if $\sigma_{ij} \geq \rho > 0$, then "a less conservative solution can be obtained by letting d satisfy

$$P_{\Sigma_\rho}\left(\bigcap_1^n \{X_i \leq \sqrt{n}\, d\}\right) = \gamma,"$$

where Σ_ρ is an equicorrelation matrix with correlation ρ. The discussion above indicates that we get a much more accurate estimate for the confidence coefficient for the confidence regions if we use $\bar{\rho} = \frac{1}{\binom{n}{2}} \sum_{i<j} \rho_{ij}$ instead of $\rho = \min\{\rho_{ij}\}$. We do not always get a conservative approximation for the probability. However, we do in general get a much better approximation than Tong's suggestion (which is an application of Slepian's inequality) yields.

3.3. The Multivariate Mills' Ratio.

The discussion of $p_n^{(n)}$ for large (positive) values of c naturally suggests the multidimensional analog of Mills' ratio. We first consider the univariate Mills' ratio from the viewpoint of exponential families (we have not been able to find this treatment in the literature) and prove a lemma that will be useful later.

Let $R(x) = \frac{\Phi(-x)}{\phi(x)}$ for $x > 0$. R is called Mills' ratio. A well known enveloping series for R is $R(x) = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{3.5}{x^7} + \cdots$ (see Feller [8]). To see that R can be used to define an exponential family, note that

$$R(x) = \int_0^\infty e^{-xt-t^2/2}\, dt = \int_0^\infty e^{-xt}\, \nu(dt)$$

where $\nu(dt) = e^{-t^2/2}\, dt$ is a measure on $[0,\infty)$. If $S(x) = \log R(x)$, then

$$(*) \qquad 1 = \int_0^\infty e^{-xt-S(x)}\, \nu(dt) = \int_0^\infty P_x(dt)$$

defines an exponential family through ν with $\mathbb{R}$ as the natural parameter space (here $-x$ takes the place of the symbol θ in the usual literature). Let W be any random variable with distribution $P_x(dt)$.

Differentiating the relation $(*)$ with respect to x several times (interchanging $\frac{d}{dx}$ and $\int_0^\infty$ are easily justified), we get

(i) R is a strictly convex decreasing function with $R(0) = \sqrt{\frac{\pi}{2}}$.

(ii) $E_x W = -S'(x) = -\frac{R'(x)}{R(x)} \equiv \mu(x) > 0$ (μ refers to mean.)

(iii) $-\mu'(x) = S''(x) = \mathrm{var}_x W > 0 \Rightarrow \mu'(x) < 0$.

(iv) $\mu''(x) = E_x(W-\mu(x))^3$.

When we try to prove certain inequalities for the multivariate Mills' ratio, we will need the signs of μ, μ' and μ''. The sign of μ'' is very hard to determine by using the standard approximations for R, but the following observation makes the problem easy. Notice that μ'' is the third central moment of W. The third central moment of any random variable is proportional to a standard measure of skewness; furthermore, the distribution of W is the same as that of a $N(-x,1)$ variate conditioned on the event that it is positive. It is clear that W has a positively skewed density, and hence $\mu'' > 0$. This fact is in fact a special case of the following lemma.

Lemma 3.2: Suppose X has density $s(x)$ that is symmetric, unimodal, and continuous with finite absolute third moment. Then, for c in the interior of the support of $s(x)$, $f_c(x) = s(x)/\int_c^\infty s(t)dt$, $x \geq c$, the conditional density of X given that $X > c$, has positive third central moment.

Proof. Suppose that Y has density f. Let $EY = \mu > 0$. It is easy to see that $c < \mu$, so that we may define $Z = (\frac{Y}{\mu} - 1)/(1 - \frac{c}{\mu})$. $Z \varepsilon [1,\infty)$, and the density of Z, $g(z)$, has the same shape as f does. Also, $EZ = 0$; thus, it suffices to show that $EZ^3 \geq 0$. When $c \geq 0$, the mode of g is at -1; when $c < 0$, the mode of g is at $-\frac{\mu}{\mu-c} \varepsilon (-1,0)$. Now

$$EZ^3 = E(Z^3-Z) = \int_{-1}^{\infty} (z^3-z)g(z)dz$$

$$= \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty} = \int_{0}^{1} (z^3-z)(g(z)-g(-z))dz + \int_{1}^{\infty} (z^3-z)g(z)dz \ .$$

The second term above is obviously positive. The first term is also positive because $z^3-z < 0$ and $g(z) < g(-z)$, since the mode of g is strictly negative. Thus $EZ^3 \geq 0$.

Note: It is in fact true that the support of g must include points in $(1,\infty)$. Also, the conditions of the lemma are not very general, but they are enough for our purposes.

Let $T = \Sigma^{-1}$. The multivariate Mills' ratio is

$$R(c,T) = \frac{P_{\Sigma}(X_1 > c,\ldots,X_n > c)}{\phi(c,c,\ldots,c;\Sigma)} \ .$$

Steck considers the more general $\{X_i > c_i, \forall i\}$; we specialize to $c_i \equiv c$. Most of our results apply to the general case too. Savage [24] suggests one and Steck [30] suggests three lower bounds to $R(c,T)$. Savage shows that

$$R(c,T) = \int_{u \geq 0} \exp(-cu'Te - \frac{1}{2} u'Tu)du \ ,$$

where $e = (1,\ldots,1)'$. Write $\Delta = cTe$ and assume $\Delta_i > 0$, he shows that

$$R(c,T) \geq (1 - \sum_{1}^{n} t_{ii}/\Delta_i^2 - \sum_{i<j} t_{ij}/\Delta_i\Delta_j)/\prod_{1}^{n} \Delta_i \ .$$

Steck shows in some numerical studies, however, that Savage's condition,

$\Delta_i > 0$, is too restrictive and that even when it applies the resulting lower bound is usually bad.

All of Steck's suggestions involve a likelihood ratio argument. By an appropriate change of variables, he writes $R(c,T) = Eg(V)$, where V is a random vector whose joint density is some product of univariate Mills' ratio densities: $\exp(-xt-t^2/2)/R(x)$. Then he uses the approximation $Eg(V) \approx g(EF)$. When g is convex, the approximation is a lower bound; in two cases, Steck was able to prove that this was indeed so. They are:

$$\text{(i)} \quad R(c,T) \geq \hat{R}_1 \equiv \Pi_1^n \, (R(z_i)/\sqrt{t_{ii}} \, \exp(-M'(Q-I)M/2) \; .$$

$$\text{(ii)} \quad R(c,T) \geq \hat{R}_3 = (\tfrac{\pi}{2})^{n/2} \, \Pi_1^n \, 1/\sqrt{t_{ii}} \, \exp(-\sqrt{\tfrac{2}{\pi}} \, \Sigma z_{ii} - \tfrac{2}{\pi} \sum_{i<j} q_{ij}) \; .$$

However, in numerical examples, it turned out that his other approximation, $\hat{R}_2$, yielded the best (largest) lower bound, so he conjectured, but did not prove, that $\hat{R}_2$ was indeed a lower bound.

Our work consists of providing a sufficient condition for $\hat{R}_2$ to be a lower bound. Using

$$R(c,T) = \int_{u\geq 0} \exp(-cu'Te - \tfrac{1}{2}\, u'Tu)du \; ,$$

and making the change of variables $u_i = v_i/\sqrt{t_{ii}}$, we have

$$R(c,T) = (\prod_1^n t_{ii}^{-1/2}) \int_{v\geq 0} \exp(-\tfrac{1}{2}\, v'(Q-I)v) \prod_1^n e^{-z_i v_i - v_i^2/2} dv_i$$

where $Q = (t_{ij}/\sqrt{t_{ii}t_{jj}})$ and $z_i = \Delta_i/\sqrt{t_{ii}}$. Now $\frac{1}{2}\, v'(Q-I)v = \sum_{i<j} v_i v_j q_{ij}$;

we combine exponents, we get

$$\left(\prod_1^n t_{ii}^{-1/2}\right)\exp\left(-v_1\{z_1+\sum_2^n v_jq_{1j}\}-v_1^2/2\right)\exp\left(-v_2\{z_2+\sum_3^n v_jq_{2j}\}-v_2^2/2\right)\cdots$$

$$\exp\left(-v_{n-1}\{z_{n-1}+v_nq_{n-1,n}\}-v_{n-1}^2/2\right)\exp\left(-v_nz_n-v_n^2/2\right) .$$

Now recall that $\exp(-xt-\frac{t^2}{2})/R(x)$ is a density. Thus multiplying and dividing by a product of such factors, we can write

$$R(c,T) = \left(\prod_1^n t_{ii}^{-1/2}\right)E\left[R(z_1+\sum_2^n V_jq_{1j})R(z_2+\sum_3^n V_jq_{2j})\cdots R(z_{n-1}+V_nq_{n-1,n})R(z_n)\right] ,$$

where $(V_1,\ldots,V_n)'$ has the density

$$f(v_1,\ldots,v_n) = \frac{\exp(-v_1\{z_1+\sum_2^n v_jq_{1j}\}-v_1^2/2}{R(z_1+\sum_2^n v_jq_{1j})}\cdots\frac{\exp(-v_nz_n-v_n^2/2)}{R(z_n)} .$$

$\hat{R}_2$ is then just gotten by replacing V_i in the expression above by $\mu(z_i+\sum_{i+1}^n V_jq_{2j})$ beginning with V_2 and proceeding with $V_3,\ldots,V_n$. Then $\hat{R}_2 = \prod_1^n R(w_i)/\sqrt{t_{ii}}$, where $w_n = z_n$, and $w_i = z_i+\sum_{k=i+1}^n q_{ik}\mu(w_i)$, for $i = n-1,n-2,\ldots,1$. Having defined $\hat{R}_2$, we now study it more closely.

Define $h(v_2,\ldots,v_n) = R(z_1+\sum_2^n v_2q_{1j})\cdots R(z_{n-1}+q_{n-1,n}v_n)R(z_n)$, so that $R(c,T) = (\prod t_{ii}^{-1/2})E\{h(V_2,\ldots,V_n)\}$.

Lemma 3.3: h is a convex function.

Proof. It is well known that h is convex whenever log h is convex. Using the notation that we developed earlier, we will show that the Hessian of log h is non-negative definite. For the case $n = 4$ it is easy to show that the Hessian is a weighted sum of the following three non-negative matrices:

$$\begin{pmatrix} q_{12}^2 & q_{12}q_{13} & q_{12}q_{14} \\ & q_{13}^2 & q_{13}q_{14} \\ & & q_{14}^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_{23}^2 & q_{23}q_{24} \\ 0 & & q_{24}^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_{34}^2 \end{pmatrix}$$

with positive weights $-\mu'(z_1 + \Sigma_2^4 v_j q_{1j})$, $-\mu'(z_2 + \Sigma_3^4 v_j q_{2j})$, and $-\mu'(z_3+q_{34}v_4)$, respectively. Thus, the Hessian is positive definite. The proof for other n is analogous. QED

Thus, by Jensen's inequality, we have that

$$R(c,T) = (\prod_1^n t_{ii}^{-1/2})Eh(V_2,\ldots,V_n) \geq (\prod_1^n t_{ii}^{-1/2})h(EV_2,\ldots,EV_n) .$$

Unfortunately, the right hand side above is not $\hat{R}_2$. To see the difference consider the case $n = 3$. Here $EV_3 = \mu(z_3)$ and $EV_2 = EE(V_2|V_3) = E\mu(z_2+q_{23}V_3) \geq \mu(z_2+q_{23}\mu(z_3))$. We get the last inequality here because by the earlier lemma on skewness, $\mu'' > 0$, so that μ is convex. Thus

$$R(c,T) \geq \Pi t_{ii}^{-1/2}\; h(EV_2,EV_3)$$

$$= \Pi t_{ii}^{-1/2}\; R(z_1+q_{12}E\mu(z_2+q_{23}V_3)+q_{13}\mu(z_3))R(z_2+q_{23}\mu(z_3))R(z_3) .$$

Since R is a strictly decreasing function, we can say that if $q_{12} \leq 0$, then $R(c,T) \geq \hat{R}_2$. But $q_{12} \leq 0$ if and only if $t_{12} \leq 0$. Since T can be replaced by $\pi T \pi'$ where π is any permutation matrix, we have in the three-dimensional case that if Σ^{-1} contains at least one negative off-diagonal term. Applying the same technique for larger n, we have

<u>Theorem 3.2</u>: If Σ^{-1} has at most $(n-1)$ positive off-diagonal terms, then $R(c,T) \geq \hat{R}_2$.

We omit the proof, since it is analogous to the discussion above.

CHAPTER IV

RESULTS FOR THE EQUICORRELATION CASE

Recall that when Σ is an equicorrelation matrix with $\rho \geq 0$,

$$p_k^{(n)} = \binom{n}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^k \Phi\left(\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{n-k} \phi(t)dt \ .$$

For $c = 0$, we have

$$p_n^{(n)} = \int_{-\infty}^{\infty} \Phi\left(t\sqrt{\frac{\rho}{1-\rho}}\right)^n \phi(t)dt \ ;$$

it is well known that $p_2^{(2)} = \frac{1}{2} - \frac{1}{2\pi}\cos^{-1}\rho$ and $p_3^{(3)} = \frac{1}{2} - \frac{3}{4\pi}\cos^{-1}\rho \ (-\frac{1}{2} \leq \rho \leq 1)$, for $n \geq 4$, however, $p_n^{(n)}$ cannot be expressed in terms of elementary functions (David [6], p. 34). In fact, we show in Theorem 4.4 that for $n \geq 4$ $p_n^{(n)}$ is everywhere convex in ρ, but that for $n \leq 3$, it is convex for $\rho > 0$ and concave for $\rho \leq 0$.

4.1. The Exceedance Distribution.

When $c = 0$ and $\rho = 1/2$. we have $p_k^{(n)} = 1/(n+1)$, independent of k. This is most easy to see if we write $X_i = \sqrt{1/2}\, Z_i - \sqrt{1/2}\, Z_0$ and notice that $p_k^{(n)}$ is just the probability that Z_0 is the $(n-k+1)$st order statistic in the iid sample $(Z_0,\ldots,Z_n)$. It is clear now that the Z_i need not be normal for the exceedance distribution to be uniform. This led us to consider other cases for which the exceedance distribution could be computed by combinatorial methods; we discuss these in Chapter 5.

Notice also that we can write

$$p_k^{(n)} = \binom{n}{k} \int_{-\infty}^{\infty} \Phi\left(- \frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{k} \Phi\left(\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{n-k} \phi(t)dt$$

$$= \binom{n}{k} \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} \int_{-\infty}^{\infty} \Phi\left(\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right)^{n-k+i} \phi(t)dt$$

$$= \binom{n}{k} \sum_{i=0}^{k} (-1)^{i} \binom{k}{i} P\{X_{n-k+i,n-k+i} \le c\}$$

where $X_{i,n}$ is the ith order statistic in a sample of size n. Thus if we know $p_0^{(k)}$ for $k = 0,1,\ldots,n$, then we can compute $p_k^{(j)}$ for $k = 0,1,\ldots,j$ and $j = 1,\ldots,n$. This is similar to some recursion relations given in David [6]. It can be argued similarly that $p_k^{(n)}$ is expressible in terms of $p_0^{(j)}$ for $j = 1,2,\ldots,n$ (for arbitrary ‡) by using the inclusion-exclusion formula. However, this does not turn out to be very useful computationally because repeated additions and subtractions of the estimates of $p_0^{(j)}$ typically accumulates too much roundoff error.

Earlier, we generalized the equicorrelation case to the single-factor case. We can generalize in another direction by writing the iid $N(0,1)$ variates $(Z_1,\ldots,Z_n)$ as $(Z_1,\ldots,Z_{n_1},Z_{n_1+1},\ldots,Z_{n_1+n_2},\ldots,Z_n)$ where $n = \Sigma_1^b n_i$. We add a separate main effect to each of the b blocks to get correlation matrices that look like this:

$$\begin{pmatrix} \begin{matrix} 1 & \rho_1 \\ & \ddots \\ & & 1 \end{matrix} & & & \bigcirc \\ & \begin{matrix} 1 & \rho_2 \\ & \ddots \\ & & 1 \end{matrix} & & \\ & & \ddots & \\ \bigcirc & & & \begin{matrix} 1 & \rho_b \\ & \ddots \\ & & 1 \end{matrix} \end{pmatrix}$$

In this case we have

$$p_k^{(n)} = \sum \prod_1^b \binom{n_i}{k_i} \int_{-\infty}^{\infty} \Phi\left(-\frac{c+t\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\right)^{k_i} \Phi\left(\frac{c+t\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\right)^{n_i-k_i} \phi(t)dt ,$$

where the summation is over $\{(k_1,\ldots,k_b): \Sigma_1^b k_i = k$ and $0 \leq k_i \leq n_i\}$. This case and the factor model case have been very important in our work. They provide a large class of correlation matrices for which we can efficiently (i.e., without simulation) compute the exceedance probabilities. In fact, we conjectured the variance inequality of Chapter 3 after looking at many specific examples.

Recall that $S_n = \Sigma_1^n I\{X_i > c\}$. For large n we have

<u>Theorem 4.1</u>:

$$P\{\frac{S_n}{n} \leq x\} \to \Phi\left(\frac{c+\sqrt{1-\rho}\ \Phi^{-1}(x)}{\sqrt{\rho}}\right) , \quad 0 \leq x \leq 1 ,$$

so that

$$\Phi^{-1}(\frac{S_n}{n}) \overset{d}{\to} N(\frac{c}{\sqrt{1-\rho}}, \frac{\rho}{1-\rho}) .$$

<u>Proof</u>. The strong law of large numbers says

$$\frac{1}{n}\sum_{1}^{n} I\{X_i > \frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\} \to \Phi\left(-\frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\right) \quad \text{a.e.}$$

Thus,

$$\begin{aligned}
P(\frac{S_n}{n} \le x) &= \int P(\frac{S_n}{n} \le x \mid Z_0 = t)\phi(t)dt \\
&= \int P(\frac{1}{n}\sum_{1}^{n} I\{Z_i > \frac{c+t\sqrt{\rho}}{\sqrt{1-\rho}}\} \le x)\phi(t)dt \\
&\to \int I\{1-\Phi(\frac{t\sqrt{\rho}+c}{\sqrt{1-\rho}}) \le x\}\phi(t)dt \\
&= P\{1-\Phi(\frac{c+Z_0\sqrt{\rho}}{\sqrt{1-\rho}}) \le x\} \\
&= P\{Z_0 > -\frac{c+\sqrt{1-\rho}\;\Phi^{-1}(x)}{\sqrt{\rho}}\} \\
&= \Phi(\frac{c+\sqrt{1-\rho}\;\Phi^{-1}(x)}{\sqrt{\rho}}) \; . \qquad \text{QED}
\end{aligned}$$

We thus have a two-parameter family of distributions on $[0,1]$: $F_{a,b}(x) = \Phi(a+b\Phi^{-1}(x))$, $a \in \mathbb{R}$, $b > 0$. The density is $f_{a,b}(x) = b\phi(a+b\Phi^{-1}(x))/\phi(\Phi^{-1}(x))$. We now state a few elementary facts about this family.

(i) $a = 0$, $b = 1 \Rightarrow$ we have a uniform distribution, which is consistent with the discrete uniform result for $c = 0$, $\rho = 1/2$.

(ii) $b = 1$, $a > 0$ $(a < 0) \Rightarrow$ the density decreases (increases) from ∞ to 0 (0 to ∞) as x goes from 0 to 1.

(iii) $b > 1 \Rightarrow$ the density is bounded with $f(0) = f(1) = 0$, and it is unimodal with mode at $x = \Phi(ab/(b^2-1))$.

(iv) $b < 1 \Rightarrow f(0) = f(1) = \infty$, and the density is u-shaped with a minimum at $x = \Phi(ab/(b^2-1))$.

These properties are quite similar to those of the beta distribution. Now if L has distribution $F_{a,b}$, then its moments are

$$EL^n = n \int_0^1 t^{n-1}(1-F_{a,b}(t))dt$$

$$= 1 - nP(Z_1 \leq Z_0, Z_2 \leq Z_0, \ldots, Z_{n-1} \leq Z_0, Z_n \leq a+bZ_0) \ .$$

So, in particular, $EL = 1-\Phi(\frac{a}{\sqrt{1+b^2}})$ ($= \Phi(-c)$ in the theorem above).

There seem to be no closed-form expresssions for the higher moments.

It is of interest to know when the approximation to the exceedance distribution by the limit distribution $\Phi(a+\Phi^{-1}(b))$ (with appropriate a,b) is good. Numerical examples with and without continuity correction indicate the following.

(i) $n \geq 15$ often yields good approximations.

(ii) The approximation with continuity correction is better than without it for k not too close to n.

(ii) Both approximations are quite bad for $p_0^{(n)}$ and $p_1^{(n)}$ when c is large. Typically, the approximation for $p_0^{(n)}$ is too small and the approximation for $p_1^{(n)}$ is too large.

Table 1 below gives some examples. The exact probabilities are compared with approximations with and without continuity correction. The values of c, ρ, and n are indicated.

Table 1

c = 1.280 rho = 0.250

i	exact prob.	with cont. corr.	without cont. corr.
0	0.314	0.202	0.000
1	0.229	0.325	0.386
2	0.154	0.188	0.247
3	0.103	0.112	0.145
4	0.068	0.068	0.087
5	0.046	0.042	0.053
6	0.030	0.026	0.033
7	0.020	0.016	0.020
8	0.013	0.010	0.012
9	0.009	0.006	0.007
10	0.005	0.003	0.004
11	0.003	0.002	0.002
12	0.002	0.001	0.001
13	0.001	0.001	0.001
14	0.001	0.000	0.000
15	0.000	0.000	0.000
16	0.000	0.000	0.000
17	0.000	0.000	0.000
18	0.000	0.000	0.000
19	0.000	0.000	0.000
20	0.000	0.000	0.000

c = 0.000 rho = 0.333

i	exact prob.	with cont. corr.	without cont. corr.
0	0.044	0.010	0.000
1	0.074	0.061	0.035
2	0.095	0.099	0.082
3	0.109	0.123	0.112
4	0.118	0.137	0.131
5	0.121	0.141	0.140
6	0.118	0.137	0.140
7	0.109	0.123	0.131
8	0.095	0.099	0.112
9	0.074	0.061	0.082
10	0.044	0.010	0.036

c = 0.000 rho = 0.600

i	exact prob.	with cont. corr.	without cont. corr.
0	0.149	0.105	0.000
1	0.110	0.129	0.174
2	0.100	0.110	0.117
3	0.095	0.104	0.106
4	0.092	0.102	0.103
5	0.095	0.104	0.103
6	0.100	0.110	0.106
7	0.110	0.129	0.117
8	0.149	0.105	0.174

c = 0.850 rho = 0.400

i	exact prob.	with cont. corr.	without cont. corr.
0	0.260	0.183	0.000
1	0.177	0.227	0.310
2	0.130	0.152	0.183
3	0.100	0.111	0.129
4	0.078	0.084	0.096
5	0.064	0.064	0.073
6	0.050	0.050	0.056
7	0.038	0.038	0.043
8	0.029	0.029	0.033
9	0.022	0.022	0.025
10	0.016	0.016	0.018
11	0.011	0.011	0.013
12	0.007	0.007	0.009
13	0.004	0.004	0.005
14	0.002	0.002	0.003
15	0.000	0.000	0.001

We saw above that for $c = 0$, and $\rho = 0$, the exceedance distribution is binomial; for $\rho = 1/2$, it is uniform; and for $\rho = 1$, it has point masses of $1/2$ each at 0 and n. As ρ tends from 0 to 1, we go smoothly from a bell-shape to a u-shape; the proof of this intuitively obvious fact is surprisingly hard. The following lemma makes this fact precise. Since the proof does not require the assumption of normality, we state it in some generality.

Lemma 4.1: Suppose Φ is any distribution function with $\Phi(-x) = 1-\Phi(x)$. Let $\Phi' = \phi$ be such that $L(x,\xi) = \frac{1}{\xi}\ \phi(\frac{x}{\xi})/\phi(x)$ is decreasing in $|x|$ for $0 < \xi \leq 1$ and increasing in $|x|$ for $1 \leq \xi < \infty$. Then for $0 \leq \rho \leq 1/2$ the exceedance distribution satisfies

$$p_0^{(n)} \leq p_1^{(n)} \leq \cdots \leq p_{[n/2]}^{(n)} \geq \cdots \geq p_n^{(n)} \ ;$$

for $\frac{1}{2} \leq \rho \leq 1$, the inequalities above are reversed.

Proof. The condition on $L(x,\xi)$ is a monotone likelihood ratio type of restriction. It is satisfied by the normal and Cauchy, for example. We first show that for $0 \leq \rho \leq 1/2$, $p_1 \geq p_0$ and that for $2 \leq k \leq \frac{n+1}{2}$, $p_k \_\ p_{k-1}$. Now, for $\xi = \sqrt{\rho/(1-\rho)}$,

$$p_1 - p_0 = \binom{n}{1} \int_{-\infty}^{\infty} \Phi(-\xi t)\Phi(\xi t)^{n-1}\phi(t)dt - \int_{-\infty}^{\infty} \Phi(\xi t)^n \phi(t)dt$$

$$= \int_{-\infty}^{\infty} (n\Phi(t)^{n-1} - (n+1)\Phi(t)^n) L(t,\xi)\phi(t)dt$$

$$= \int_{0}^{\infty} (f(\Phi(t)) + f(\Phi(-t))) L(t,\xi)\phi(t)dt$$

where $f(y) = n y^{n-1}(n+1) y^n$. f has roots at 0 and $\frac{n}{n+1}$; also, $g(y) = f(y) + f(1-y)$ is strictly decreasing for $\frac{n}{n+1} \leq y \leq 1$. g is strictly positive in $[\frac{1}{2}, \frac{n}{n+1}]$ and $g(1) = -1$, so there is a unique $y^* \varepsilon (\frac{n}{n+1}, 1)$ such that $g(y^*) = 0$. Now suppose $\Phi(t^*) = y^*$. Then

$$p_1 - p_0 = \int_0^\infty g(\Phi(t))L(t,\xi)\phi(t)dt$$

$$= \int_0^{t^*} g(\Phi(t))L(t,\xi)\phi(t)dt + \int_{t^*}^\infty g(\Phi(t))L(t,\xi)\phi(t)dt$$

$$\geq L(t^*,\xi)\{\int_0^{t^*} g(\Phi(t))\phi(t)dt + \int_{t^*}^\infty g(\Phi(t))\phi(t)dt$$

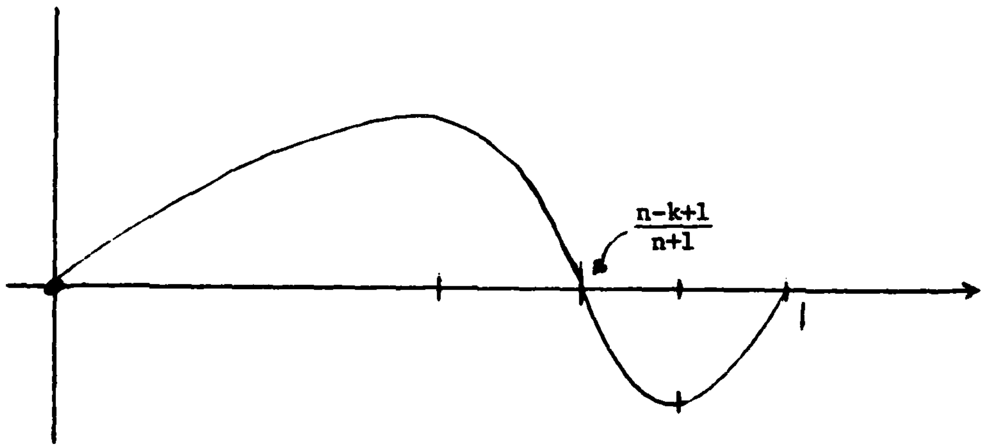
$$= L(t^*,\xi) \int_0^\infty g(\Phi(t))\phi(t)dt = 0 .$$

The above inequality $\geq$ is for $0 \leq \rho \leq \frac{1}{2}$; for $\frac{1}{2} \leq \rho \leq 1$ we get $\leq$. We get the last equality above because for $\rho = \frac{1}{2}$ (so that $\xi = 1$), $p_1 = p_0$. We are thus done for p_1 and p_0. Notice that we have used the likelihood ratio ($L(t,\xi)$ is one)technique that is so common in random walk theory — we transfer our attention from $\xi \neq 1$ to $\xi = 1$, where matters are simple since then $p_0 = \cdots = p_n = \frac{1}{n+1}$. The proof for $p_{k+1} \geq p_k$, for $k \geq 2$, $2k-1 \leq n$, $0 \leq \rho \leq \frac{1}{2}$ is similar, but the function g is no longer as simple as above. Now

$$p_k - p_{k-1} = \binom{n}{k} \int_{-\infty}^\infty \Phi(-\xi t)^k \Phi(\xi t)^{n-k}\phi(t)dt - \binom{n}{k-1} \int_{-\infty}^\infty \Phi(-\xi t)^{k-1}\Phi(\xi t)^{n-k}\phi(t)dt$$

$$= \binom{n}{k} \int_0^\infty g(\Phi(t))L(t,\xi)\phi(t)dt$$

where $f(y) = (1-y)^{k-1} y^{n-k}(1 - \frac{n+1}{n-k+1} y)$ and $g(y) = f(y) + f(1-y)$ is defined for $y \in [\frac{1}{2},1]$. f has roots at 0, 1 and $\frac{n-k+1}{n+1}$; f' has critical points at $\frac{n-k+1}{n+1} \pm \frac{\sqrt{k(1-\frac{k-1}{n})}}{n+1}$. Thus, f looks roughly like this:



We will be done if we can show that g has a unique root y^* in $[\frac{1}{2}, 1)$, that g is positive in $[\frac{1}{2}, y^*)$ and negative otherwise; for then we can use the same proof as above. Now g is strictly positive in $[\frac{1}{2}, \frac{n-k+1}{n+1}]$ and strictly decreasing in $[\frac{n-k+1}{n+1}, \frac{n-k+1}{n+1} + \frac{\sqrt{k(1-\frac{k-1}{n})}}{n+1}]$.

We will be done if we can show that for $y \in [\frac{n-k+1}{n+1} + \frac{\sqrt{k(1-\frac{k-1}{n})}}{n+1}, 1]$, $g(y) \leq 0$. For $y = 1$, this is trivial; so suppose $y \neq 1$. Note that $g(y) \leq 0 \iff$

$$(1-y)^{n-k}(1 - \frac{n+1}{n-k+1} y) + y^{k-1}(1-y)^{n-k}(1 - \frac{n+1}{n-k+1} (1-y)) \leq 0$$

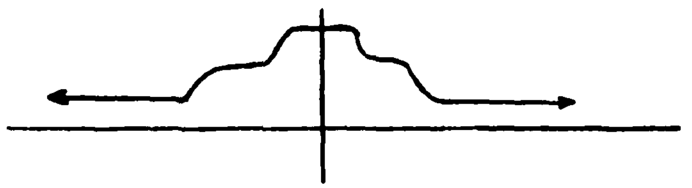
$$\iff (\frac{y}{1-y})^{n-2k+1} \geq 1 + \frac{n-2k+1}{(n+1)y - (n-k+1)} .$$

Since $n \geq 2k-1$, it suffices to check $g(y) \leq 0$ for $y \in [\frac{n-k+1}{n+1} + \frac{\sqrt{k/2}}{n+1}, 1)$.

In this range, $(\frac{y}{1-y})^{n-2k+1} \geq (\frac{n-k+1+\sqrt{k/2}}{k-\sqrt{k/2}})^{n-2k+1}$, since this function is strictly decreasing in y. So now it is enough to show that

$1 + \frac{n-2k+1}{\sqrt{k/2}} \leq (\frac{n-k+1+\sqrt{k/2}}{k-\sqrt{k/2}})^{n-2k+1}$ for $k = 2,3,\ldots$ and $n \geq 2k-1$. The inequality is trivial for $n = 2k-1$, $2k$, and $2k+1$. Finally, if we let $n-2k+1 = j$, we note that the functions $1 + \frac{x}{\sqrt{k/2}}$ and $(\frac{x+k+\sqrt{k/2}}{k-\sqrt{k/2}})^x$ are to be compared. It is rather easy to see that the latter dominates the former for all $x \geq 0$. Thus we have our desired inequality so that g has a unique root in $[\frac{1}{2}, 1)$, and we are done. QED

The condition on $L(t,\xi) = \frac{1}{\xi}\,\phi(\frac{t}{\xi})/\phi(t)$ above eliminates densities that look like this:



It is a somewhat puzzling condition. We suspect that the result is true without it, but have not been able to show that.

Now we come to an interesting result that is fairly easy to prove given the lemma above.

Theorem 4.2: Let $p(\rho) = (p_0^{(n)}(\rho),\ldots,p_n^{(n)}(\rho))$. If $0 \leq \rho_1 < \rho_2 \leq \frac{1}{2}$, then $p(\rho_1) \succ p(\rho_2)$; if $\frac{1}{2} \leq \rho_1 < \rho_2 \leq 1$, then $p(\rho_1) \prec p(\rho_2)$. Here, $x \prec y$ means that y majorizes x in the sense of Marshall and Olkin [16].

<u>Proof</u>. Assume that $n = 2m+1$ is odd; the proof for even n is similar. Here $p_i^{(n)}(\rho) = p_{2m+1-i}^{(n)}(\rho)$ since $c = 0$; also,

$$p_0^{(n)}(\rho) \leq p_1^{(n)}(\rho) \leq \cdots \leq p_m^{(n)}(\rho) = p_{m+1}^{(n)}(\rho) \geq \cdots \geq p_{2m+1}^{(n)}(\rho)$$

for $0 \leq \rho \leq \frac{1}{2}$ and

$$p_0^{(n)}(\rho) \geq p_1^{(n)}(\rho) \geq \cdots \geq p_m^{(n)}(\rho) = p_{m+1}^{(n)}(\rho) \leq \cdots \leq p_{2m+1}^{(n)}(\rho) .$$

Thus, it suffices to show that $p_m^{(n)}(\rho)$, $2p_m^{(n)}(\rho)$, $2p_m^{(n)}(\rho) + p_{m-1}^{(n)}(\rho)$, $2p_m^{(n)}(\rho) + 2p_{m-1}^{(n)}(\rho)$, $2p_m^{(n)}(\rho) + 2p_{m-1}^{(n)}(\rho) + p_{m-2}^{(n)}(\rho), \ldots$ etc. all decrease in ρ and that $p_o^{(n)}(\rho)$, $2p_0^{(n)}(\rho)$, $2p_0^{(n)}(\rho) + p_1^{(n)}(\rho), \ldots$ all decrease in ρ. Since $\Sigma_0^m\ p_i^{(n)}(\rho) \equiv 1/2$, it suffices to show that $p_m^{(n)}(\rho)$, $2p_m^{(n)}(\rho)$, $2p_m^{(n)}(\rho) + p_{m-1}^{(n)}(\rho), \ldots$ etc. are decreasing. To do this, we just show that all the derivatives are negative; the details are a bit messy but straightforward. Let $\xi = \sqrt{\rho/(1-\rho)}$ (ξ is a strictly increasing function of ρ). So, for $j < m$

$$f(\xi) = \sum_{i=0}^{j} p_{m-i}(\rho) = \sum_{j=0}^{j} \binom{2m+1}{m-i} \int_{-\infty}^{\infty} \Phi(-\xi t)^{m-i}\ \Phi(\xi t)^{m+i+1}\ \phi(t)dt$$

$$= \binom{2m+1}{m} \sum_{0}^{j} \frac{m!(m+1)!}{(m-i)!(m+i+1)!} \int_{-\infty}^{\infty} \Phi(-\xi t)^{m-i}\ \Phi(\xi t)^{m+i+1}\ \phi(t)dt$$

and

$$f'(\xi) = \binom{2m+1}{m} \int_{-\infty}^{\infty} \Big\{\sum_{0}^{m} \frac{m!(m+1)!}{(m-i)!(m+i)!} \; \Phi(-\xi t)^{m-i}\, \Phi(\xi t)^{m+i}$$

$$- \sum_{0}^{j} \frac{m!(m+1)!}{(m-i)!(m+i)!} \Phi(-\xi t)^{m-i-1}\, \Phi(\xi t)^{m+i+1}\Big\} \; \phi(t) t \phi(\xi t) dt$$

$$= \binom{2m+1}{m} m!(m+1)! \int_{0}^{\infty} \Big\{\sum_{0}^{j} \frac{\Phi(-\xi t)^{m-i}\Phi(\xi t)^{m+i} - \Phi(\xi t)^{m-i}\Phi(-\xi t)^{m+i}}{(m-i)!(m+i)!}$$

$$- \sum_{1}^{j+1} \frac{\Phi(-\xi t)^{m-k}\Phi(\xi t)^{m+k} - \Phi(\xi t)^{m-k}\Phi(-\xi t)^{m+k}}{(m-k)!(m+k)!}\Big\} \; t\, \phi(t)\, \phi(\xi t) dt$$

$$= - \binom{2m+1}{m} \frac{m!(m+1)!}{(m+j+1)!(m-j-1)!} \int_{0}^{\infty} (\Phi(-\xi t)\Phi(\xi t))^{m-j-1} \{\Phi(\xi t)^{2j+2} - \Phi(-\xi t)^{2j+2}\} \; t\phi(t)\phi(\xi t) dt$$

$$< 0 \,.$$

In $f'(\xi)$ above, the second equality is derived thus: if h is an odd function then

$$\int_{-\infty}^{\infty} g(x)h(x)dx = \int_{0}^{\infty} (g(x)-g(-x))h(x)dx \;;$$

above $t\phi(t)\phi(\xi t)$ plays the role of h. We then get a collapsing sum which yields the third equality there. We thus have that

$$\sum_{0}^{j} p_{m-i}(\rho) \quad (\text{and so} \quad 2\sum_{0}^{j} p_{m-i}(\rho))$$

is decreasing in ρ for $j < m$; the case $j = m$ is trivial for

$$\sum_{0}^{m} p_{m-i}(\rho) \equiv \frac{1}{2} \,.$$

Similar but tedious algebra shows the result for

$$2 \sum_{0}^{j} p_{m-i}(\rho) + p_{m-j-1}(\rho) .$$

So our proof for $0 \leq \rho \leq \frac{1}{2}$ is complete. The proof for $\frac{1}{2} \leq \rho \leq 1$ follows from the fact that

$$\sum_{0}^{m} p_{m-i}(\rho) \equiv \frac{1}{2} .$$

QED

Notice that the proof is heavily dependent upon the symmetry induced by letting $c = 0$; this tells us that the mode of the exceedance distribution is at $[n/2]$. An annoying feature of trying to prove majorization results here seems to be that we must know the location of the mode.

We have not yet extended this result to $c \neq 0$. Numerical evidence tells us that we do not have majorization for all c; but $p(\rho,c)$ (viewed now as a function of ρ and c) is a smooth function, so we should get a majorization for some $c \neq 0$. In particular, we expect to find an interval of c's for which majorization still holds.

4.2. Orthant Probabilities for the Equicorrelation Case.

In the last chapter, we studied $P(X > c, Y > c)$ in connection with the exceedance distribution. Here, we study the analogous function for $n \geq 3$.*

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* We arrived at results in this section when we tried to answer a query by Professor Olkin.

<u>Theorem 4.3</u>: For all n, and each fixed $c \geq 0$, $f_c(\rho) = P_\rho(X_1 > c,\ldots,X_n > c)$ is a strictly convex function of ρ in $(0,1)$. For each fixed $c > \sqrt{2}-1$ and all n, $f_c(\rho)$ is strictly convex for all $\rho \in (-\frac{1}{n-1}, 1)$.

<u>Proof</u>. The key fact involved in this proof is Plackett's identity:

$$\frac{\partial}{\partial \rho_{ij}} \phi(x;\Sigma) = \frac{\partial^2}{\partial x_i \partial x_j} \phi(x;\Sigma) .$$

We write $\phi_\Sigma(x)$ for $\phi(x;\Sigma)$. Now $f_c(\rho) = \int_c^\infty \phi_{\Sigma_\rho}(x)(dx)_n$, where $(dx)_n = \Pi_1^n dx_i$. Using the chain rule and Plackett's identity, we get

$$f_c'(\rho) = \int_c^\infty \frac{d}{d\rho} \phi_{\Sigma_\rho}(x)(dx)_n = \sum_{i<j} \int_c^\infty \frac{\partial}{\partial \rho_{ij}} \phi_\Sigma(x) \frac{\partial \rho_{ij}}{\partial \rho} (dx)_n$$

$$= \sum_{i<j} \int_c^\infty \frac{\partial^2}{\partial x_i \partial x_j} \phi_{\Sigma_\rho}(x)(dx)_n$$

$$= \frac{n(n-1)}{2} \int_c^\infty \phi_{\Sigma_\rho}(x_1,\ldots,x_{n-2},c,c)(dx)_{n-2}$$

$$= \frac{n(n-1)}{2} \phi_{\Sigma_\rho}(c,c) \int_c^\infty \frac{\phi_{\Sigma_\rho}(x_1,\ldots,x_{n-2},c,c)}{\phi_{\Sigma_\rho}(c,c)} (dx)_{n-2} .$$

Henceforth, a zero-dimensional integral is defined to be 1: that is, $\int \phi_\Sigma(x)(dx)_0 = 1$. Now the integrand is the conditional density of $X_1,\ldots,X_{n-2}$ given $X_{n-1} = X_n = c$. Now $(X_1,\ldots,X_{n-2}|X_{n-1} = X_n = c)$ has the same distribution as $\frac{2\rho c}{1+\rho} e + \sqrt{\frac{(1-\rho)(1+2\rho)}{(1+\rho)}} X$, where $e' = (1,\ldots,1)$, $J = ee'$, $\tilde{\Sigma}_\rho = \frac{1+\rho}{1+2\rho} I + \frac{\rho}{1+2\rho} J$ and X is a $N(0,\tilde{\Sigma}_\rho)$ variate. Notice that if $-\frac{1}{n-1} < \rho < 1$, then $-\frac{1}{n-3} < \frac{\rho}{1+2\rho} < \frac{1}{3}$. Thus

$$f_c'(\rho) = \binom{n}{2} \phi_{\Sigma_\rho}(c,c) \int_{ca(\rho)}^{\infty} \phi_{\tilde{\Sigma}_\rho}(x)(dx)_{n-2}$$

where $a(\rho) = \sqrt{\dfrac{(1+\rho)(1+2\rho)}{(1-\rho)}}$.

Next,

$$f_c''(\rho) = \binom{n}{2} \frac{d}{d\rho} \phi_{\Sigma_\rho}(c,c) \int_{ca(\rho)}^{\infty} \phi_{\tilde{\Sigma}_\rho}(x)(dx)_{n-2}$$

$$= \binom{n}{2} \phi_{\tilde{\Sigma}_\rho}(c,c) \; \{(\frac{c^2}{(1+\rho)^2} + \frac{\rho}{1-\rho^2}) \int_{ca(\rho)}^{\infty} \phi_{\tilde{\Sigma}_\rho}(x)(dx)_{n-2}$$

$$+ \frac{d}{d\rho} \int_{ca(\rho)}^{\infty} \phi_{\tilde{\Sigma}_\rho}(x)(dx)_{n-2}\} \quad .$$

To evaluate the last term above, we use a simple extension of Leibnitz's rule. Recall that

$$\frac{d}{d\rho} \int_{u(\rho)}^{\infty} h(\rho,x)dx = \int_{u(\rho)}^{\infty} \frac{\partial}{\partial\rho} h(\rho,x)dx - h(\rho,u(\rho))u'(\rho) \; .$$

We can use induction to show that

$$\frac{d}{d\rho} \int_{u(\rho)}^{\infty} \cdots \int_{u(\rho)}^{\infty} h(\rho,x_1,\ldots,x_n)dx_1,\ldots,dx_n =$$

$$\int_{u(\rho)}^{\infty} \cdots \int_{u(\rho)}^{\infty} \frac{\partial}{\partial\rho} h(\rho,x_1,\ldots,x_n)dx_1,\ldots,dx_n -$$

$$\sum_{i=1}^{n} u'(\rho) \int_{u(\rho)}^{\infty} \cdots \int_{u(\rho)}^{\infty} h(\rho,x_1,\ldots,u(\rho),\ldots,x_n)(dx)_{n-1},$$

where $u(\rho)$ replaces x_i in the last integral. Thus

$$f_c''(\rho) = \binom{n}{2}\phi_{\ddagger_\rho}(c,c)\ \{[(\frac{c}{1+\rho})^2 + (\frac{\rho}{1-\rho^2})] \int_{ca(\rho)}^{\infty} \phi_{\tilde{\ddagger}_\rho}(x)(dx)_{n-2}$$

$$+ \frac{1}{(1+2\rho)^2}\binom{n-2}{2} \int_{ca(\rho)}^{\infty} \phi_{\tilde{\ddagger}_\rho}(x_1,\ldots,x_{n-4},ca(\rho),ca(\rho))(dx)_{n-4}$$

$$- c(n-2)a'(\rho) \int_{ca(\rho)}^{\infty} \phi_{\tilde{\ddagger}_\rho}(x_1,\ldots,x_{n-3},ca(\rho))(dx)_{n-3}\} \ .$$

The last two integrals are proportional to certain conditional probabilities, so

$$f_c''(\rho) = \binom{n}{2}\phi_{\tilde{\ddagger}_\rho}(c,c)\{((\frac{c}{1+\rho})^2 + \frac{\rho}{1-\rho^2}) \int_{ca(\rho)}^{\infty} \phi_{\tilde{\ddagger}_\rho}(x)(dx)_{n-2}$$

$$+ \frac{1}{(1-2\rho)^2}\binom{n-2}{2}\phi_{\tilde{\ddagger}_\rho}(ca(\rho),ca(\rho)) \int_{cb(\rho)}^{\infty} \phi_{\ddagger^*_\rho}(x)(dx)_{n-4}$$

$$+ (n-2)\frac{a'(\rho)}{a(\rho)}\ \phi'(ca(\rho)) \int_{cd(\rho)}^{\infty} \phi_{\ddagger^{**}_\rho}(x)(dx)_{n-3}\}$$

$$= \binom{n}{2}\ \phi_{\ddagger_\rho}(c,c)\ \{I + II + III\}$$

where $\ddagger^*_\rho = (1 - \frac{\rho}{1+4\rho})I + (\frac{\rho}{1+4\rho})J$, $\tilde{\ddagger}_\rho = (1 - \frac{\rho}{1+2\rho})I + (\frac{\rho}{1+2\rho})J$, $\ddagger^{**}_\rho = (1 - \frac{\rho}{1+3\rho})I + (\frac{\rho}{1+3\rho})J$ are of the appropriate dimension; $b(\rho)$, $d(\rho)$ are rather messy and not important for our purposes, so we omit them; and $a(\rho) = \sqrt{\frac{(1+\rho)(1+2\rho)}{(1-\rho)}}$. Notice that for $\rho > -\frac{1}{2}$, $a(\rho) > 0$ and $\frac{a'(\rho)}{a(\rho)} = -(\frac{1}{1-\rho^2} + \frac{1}{1+2\rho}) < 0$. Notice further that II is always positive; III is positive for $c > 0$ and $\rho > -\frac{1}{2}$; and I is positive for either (i) $\rho \geq 0$ or (ii) $|c| > \sqrt{2}-1$ and all ρ. Thus we have some simple sufficient conditions for $f_c''(\rho) > 0$, and the theorem is proved. QED

We have not been able to describe what happens when $\rho \varepsilon (-\frac{1}{n-1}, 0)$ and $0 < c < \sqrt{2} - 1$. For $c = 0$, recall that $p_2^{(2)} = \frac{1}{2} - \frac{1}{2\pi} \cos^{-1}\rho$ and $p_3^{(3)} = \frac{1}{2} - \frac{3}{4\pi} \cos^{-1}\rho$, and that for $n \geq 4$, $p_n^{(n)}$ cannot be expressed in terms of elementary functions. The following theorem, whose proof is rather delicate, sheds some more light on this fact.

<u>Theorem 4.4</u>: For $n = 2$ and 3, $f_0(\rho) = P_\rho(X_1 > 0, \ldots, X_n > 0)$ is strictly concave for $\rho \varepsilon (-\frac{1}{n-1}, 0)$ and strictly convex for $\rho \varepsilon (0,1)$. For $n \geq 4$, $f_0(\rho)$ is strictly convex for all $\rho \varepsilon (-\frac{1}{n-1}, 1)$.

<u>Proof</u>. For $n = 2$ and 3, $f_0''(\rho)$ is $\frac{1}{2\pi} \frac{\rho}{(1-\rho^2)^{3/2}}$ and $\frac{3}{4\pi} \frac{\rho}{(1-\rho^2)^{3/2}}$, respectively, which are strictly negative for $\rho < 0$. For $n = 4$, we have

$$f_0''(\rho) = \frac{3}{\pi} \frac{1}{\sqrt{1-\rho^2}} \left\{ \frac{1}{(1+2\rho)\sqrt{(1+\rho)(1+3\rho)}\; 2\pi} + \frac{\rho}{1-\rho^2} P_{\tilde{\rho}}(X > 0, Y > 0) \right\}$$

where $\text{corr}(X,Y) = \tilde{\rho} = \frac{\rho}{1+2\rho}$; for $-\frac{1}{3} < \rho \leq 0$, the first term is at least $\frac{1}{2\pi} \leq .159$, and the second term is, in absolute value, less than $\frac{3}{32} \simeq .094$. Thus $f'' > 0$ and for $n = 4$ f is strictly convex in its entire domain $(-\frac{1}{3}, 1)$. For $n = 5$, $f''(\rho) = \frac{5}{\pi\sqrt{1-\rho^2}} \{ \frac{\rho}{1-\rho^2} P_{\tilde{\rho}}(X_1 > 0, X_2 > 0, X_3 > 0)$ $\frac{3}{4\pi} \frac{1}{(1+2\rho)\sqrt{(1+\rho)(1+3\rho)}} \}$ where $(X_1, X_2, X_3)' \sim N(0, \Sigma_{\tilde{\rho}})$. For $-\frac{1}{4} < \rho \leq 0$, the second term exceeds $\frac{3}{4\pi} \simeq .238$, and the first term is, in absolute value, less than $\frac{1}{30} \simeq .033$. Thus f is again convex in its entire domain $(-\frac{1}{4}, 1)$. The case $n \geq 6$ is a bit more involved. Here,

$$f''(\rho) = \binom{n}{2}\frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\{\frac{\rho}{1-\rho^2} P\,(X_1 > 0,\ldots,X_{n-2} > 0) +$$

$$\binom{n-2}{2}\frac{1}{2\pi}\frac{1}{(1+2\rho)\sqrt{(1+\rho)(1+2\rho)}}\; P_{\rho*}(X_1 > 0,\ldots,X_{n-4} > 0) \quad ,$$

where $\tilde{\rho} = \frac{\rho}{1+2\rho}$ and $\rho^* = \rho/(1+4\rho)$. Notice that $\tilde{\rho} \in (-\frac{1}{n-3}, \frac{1}{3})$ and $\rho^* \in (-\frac{1}{n-5}, \frac{1}{5})$; we only need consider the case $\tilde{\rho} < 0$, $\rho^* < 0$ here. Now if $X \sim N(0,\ddagger_{\tilde{\rho}})$, then $X_1,\ldots,X_{n-4}|X_{n-3} = x$, $X_{n-2} = y$ has the same distribution as $\frac{(x+y)}{1+\tilde{\rho}}\, e + \sqrt{\frac{(1-\tilde{\rho})(1+2\tilde{\rho})}{1+\tilde{\rho}}}\, T$ where $T \sim N(0,\ddagger_{\rho*})$. Thus

$$P_\rho(X_1 > 0,\ldots,X_{n-2} > 0)$$

$$= \int_0^\infty\!\!\int P_{\tilde{\rho}}(X_1 > 0,\ldots,X_{n-4} > 0|X_{n-3}=3, X_{n-2}=y)\,\phi_{\ddagger_{\tilde{\rho}}}(x,y)\,dxdy$$

$$\leq \int_0^\infty\!\!\int P_{\rho*}(T_1 > 0,\ldots,T_{n-4} > 0)\,\phi_{\ddagger_{\tilde{\rho}}}(x,y)\,dxdy$$

(since $\tilde{\rho} < 0$, $1+\tilde{\rho} > 0, x+y > 0$)

$$= P_{\rho*}(T_1 > 0,\ldots,T_{n-4} > 0)\,P_{\tilde{\rho}}(X_1 > 0, X_2 > 0) \quad .$$

Thus, since $\rho < 0$,

$$f''(\rho) \geq \binom{n}{2}\frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\; P_{\rho*}(X_1 > 0,\ldots,X_{n-4} > 0)\;\{\frac{\rho}{1-\rho^2} P_{\tilde{\rho}}(X_1 > 0, X_2 > 0) +$$

$$\binom{n-2}{2}\frac{1}{2\pi}\frac{1}{(1+2\rho)\sqrt{(1+\rho)(1+3\rho)}}\} \quad .$$

For $\rho \in (-\frac{1}{n-1}, 0)$, the last term above is at least $\binom{n-2}{2}\frac{1}{2\pi}$; the first term, in absolute value, is less than $\frac{1}{3}\frac{n-1}{n(n-2)}$. It is easy to see that for $n \geq 4$, $\frac{1}{4}\frac{(n-2)}{n(n-2)} < \frac{1}{2\pi}\binom{n-2}{2}$. Thus $f''(\rho) > 0$ and f is strictly convex in $(-\frac{1}{n-1}, 1)$. QED

CHAPTER V

MISCELLANEOUS RESULTS

5.1. Series Expansions.

In Chapter 2, we used an asymptotic expansion about the equicorrelation case to show that the $\ddagger_{\bar{\rho}}$ approximation is locally optimal. In this section, we use an expansion about the identity to compute the exceedance probabilities. We will see that (i) the coefficients in the expansion are hard to compute, and (ii) the first two terms of the series are good only locally. It will also be clear that using the expansion about other equicorrelation matrices is much too cumbersome in practice. Earlier, we used Plackett's identity to get the first coefficient; here, differentiating directly is more convenient for describing the coefficients.

As before, consider

$$p_k^{(n)}(\theta) = p_k(\theta) = (2\pi)^{-n/2} \int_{A_k(c)} |\theta\ddagger+(1-\theta)I|^{-1/2} \exp(-\tfrac{1}{2}x'(\theta\ddagger+(1-\theta)I)^{-1}x)dx \ .$$

We would like to write

$$p_k(\theta) = p_k(0) + \theta p_k'(0) + \frac{\theta^2}{2} p_k''(0) + \cdots ; \quad k = 0,1,\ldots,n \ .$$

If we write

$$a(x) = x^{-1/2} \ , \ b(\theta) = |\theta\ddagger+\bar{\theta}I| \ , \ g(\theta) = a(b(\theta))$$

$$c(t) = e^{-t/2} \ , \ d(\theta) = x'(\theta\ddagger+\bar{\theta}I)^{-1}x \ , \ h(\theta) = c(d(\theta))$$

(we suppress the dependence of h and d on x), then Leibnitz's rule says the rth derivative of p_k is

$$(2\pi)^{-n/2} \sum_{i=0}^{r} g^{(i)}(\theta)h^{(r-i)}(\theta)dx \ .$$

The derivatives of the composition $g(\theta) = a(b(\theta))$ are given by the formula of Faá di Bruno:

$$g^{(i)}(\theta) = \sum \frac{i!}{m_1!\cdots m_i!} a^m(b(\theta)) \prod_{j=1}^{i} \left(\frac{b^{(j)}(\theta)}{j'}\right)^{m_j}$$

where the summation is taken over $m_j \geq 0$, $\Sigma\, m_j = m$, and $\Sigma\, jm_j = i$ — that is, over all partitions of i. From elementary calculus, we know that $a^{(m)}(1) = m'\binom{-1/2}{m}$. Also, if $\{\alpha_i\}_1^n$ are the eigenvalues of $(I-\ddagger)$, then

$$b(\theta) = \det(\theta\ddagger+\bar{\theta}I) = \prod_1^n (1-\theta\alpha_i) = 1-\theta u_1(\alpha)+\theta^2 u_2(\alpha) +\cdots+ (-1)^n\theta^n u_n(\alpha) \ ,$$

where $u_i(\alpha)$ is the ith elementary symmetric function of the eigenvalues $\{\alpha_i\}_1^n$:

$$u_i(\alpha) = \sum_{j_1<\cdots<j_i} \alpha_{j_1} \cdots \alpha_{j_i} \ .$$

Then $\frac{b^{(j)}(0)}{j!} = (-1)^j u_j(\alpha)$ and we get

$$g^{(i)}(0) = \sum \frac{i!(-1)^i}{m_1!\ldots m_i!} m!\binom{-1/2}{m} \prod_{j=1}^{n} u_j(\alpha)^{m_j} \ .$$

Also, $c^{(p)}(t) = (-\frac{1}{2})^p e^{-t/2} = (-\frac{1}{2})^p c(t)$ and $d^{(i)}(0) = i!x'(I-\ddagger)^i x$, so that

$$h^{(i)}(0) \sum \frac{i!}{m_1!\cdots m_i!} \left(-\frac{1}{2}\right)^m \prod_{j=1}^{i} (x'(I-\Sigma)^j x)^{m_j} .$$

The expressions above do not require Σ to be a correlation matrix. When Σ is a correlation matrix,

$$p_k' = \frac{1}{2} \sum_{i \neq j} \rho_{ij} \int_{A_k(c)} x_1 x_2 \, \phi(x;I)dx ,$$

$$p_k''(0) = \frac{1}{2} \sum_{i \neq j} \rho_{ij}^2 \, p_k(0) + \int_{A_k(c)} \left\{\left(\frac{x'(I-\Sigma)x}{2}\right)^2 - x'(I-\Sigma)x\right\}\phi(x;I)dx .$$

Higher order derivatives are considerably more cumbersome, and we omit them. Even if we had them, we would be faced with the task of expanding powers of the quadratic form $x'(I-\Sigma)^i x$, and then integrating over $A_k(c)$. Typically, the terms are proportional to $\int_{A_k(c)} (\prod_1^n x_i^{p_i})\phi(x;I)dx$. When the constant is zero we have a nice answer:

<u>Lemma 5.1</u>:

$$\int_{A_k(0)} \left(\prod_1^n x_i^{p_i}\right)\phi(x;I) = \frac{1}{2^n} E\left(\prod_1^n |x_i|^{p_i}\right) \sum_j (-1)^j \binom{p}{j}\binom{n-p}{n-k-j} ,$$

where p is the number of p_i's.

<u>Proof</u>. There are n x_i's, k of which are positive. Here is a picture of it:

$$\underbrace{- - - \cdots -}_{k \text{ positive}} \Big| \underbrace{- - - \cdots -}_{(n-k)\text{negative}} .$$

We can take j of the x_i's with odd exponents and put them in the $(n-k)$ "negative slots" above: there are $\binom{p}{j}$ ways of doing that. Of the remaining $(n-p)$ x_i's (which have even exponents) we choose $(n-k-j)$ of them to fill the remaining negative slots. It is clear that each such configuration contributes

$$(-1)^j \frac{1}{2^n} E \prod_1^n |X_i|^{p_i}$$

to the integral above, so we get the result by just adding up these contributions. QED

There does not seem to be a corresponding nice result for the integral above when $c \neq 0$, although we have enough expressions to evaluate $f_k'(0)$ and $f_k''(0)$. for $c \neq 0$,

$$\text{(i)} \quad \int_{A_k(c)} \left(\prod_1^b x_i\right)\phi_I(x)dx = \sum_i (-1)^i \binom{b}{i}\binom{n-b}{k-i}\phi(c)^b\Phi(-c)^{k-i}\Phi(c)^{n-k+i-b} ,$$

$$\text{(ii)} \quad \int_{A_k(c)} \left(\prod_1^b x_i^2\right)\phi_I(x)dx =$$

$$\sum_i \binom{b}{i}\binom{n-b}{k-i}\Phi(-c)^{k-i}\Phi(c)^{n-k+i-b}(c\phi(c)+\Phi(-c))^i(-c\phi(c)+\Phi(c))^{b-i} ,$$

and

$$\text{(iii)} \int_{A_k(c)} (x_1^2x_2x_3)\phi_I(x)dx = (-c\phi(c)+\Phi(-c)) \sum_i \binom{2}{i}\binom{n-3}{k-i}(-1)^i\phi(c)^2\Phi(c)^{n-k+i-3}\Phi(-c)^{k-i}$$

$$+ (c\phi(c)+\Phi(-c)) \sum_i (-1)^i\binom{2}{i}\binom{n-3}{k-i-1}\phi(c)^2\Phi(c)^{n-k-2+i}\Phi(-c)^{k-i-1} .$$

If we wanted to expand the probabilities about other equicorrelation matrices, the coefficients we would get are integrals of the terms (i), (ii), (iii) above.

Appendix 2 contains some examples of the use of expansions. For comparison, we took Σ to be an equicorrelation matrix with $\rho = .5$, and $c = 0$. We got the first column by numerical integration, and the second column by using the first two terms of the expansion. For $\theta = .02$, the approximation is excellent, as might be expected. However, for $\theta = .6$, it is quite bad - in fact, it can yield negative values!

5.2. Some Combinatorial Results.

Recall that for the equicorrelation case, with $\rho = 1/2$ and $c = 0$, the exceedance distribution is uniform and this result does not depend upon the fact that the X_i's are normal — it is essentially a combinatorial result. This led us to seek out other distribution-free results via combinatorics. The results in this section are quite limited in scope, but they are interesting nevertheless. Let $c = 0$ in this section.

Consider $(Z_0, Z_1, \ldots, Z_n)$ iid; if $X_i = \pm Z_{i-1} \pm Z_i$, $i = 1, \ldots, n$, then we can get $\{p_k^{(n)}\}$ by counting methods. For example, if $X_i = Z_i - Z_{i-1}$, we notice that $(Z_0, \ldots, Z_n)$ is a random permutation of the order statistics $(Z_{(0)}, Z_{(1)}, \ldots, Z_{(n)})$ to get

$$p_k^{(n)} = P(k \text{ of the } (Z_i - Z_{i-1}) \text{ are positive})$$

$$= P((Z_0,\ldots,Z_n) \text{ has } k \text{ positive successive differences})$$

$$= P(\text{a random permutation of } (Z_{(0)},\ldots,Z_{(n)}) \text{ has } k \text{ positive successive differences})$$

$$= P(\text{a random permutation of } \{0,\ldots,n\} \text{ has } k \text{ positive successive differences})$$

$= \{{n+1 \atop k}\}/(n+1)!$, where $\{{n \atop k}\}$ satisfies the recurrence relation $\{{n \atop k}\} = k\{{n-1 \atop k}\} + (n-k+1)\{{n-1 \atop k-1}\}$ with boundary conditions $\{{n \atop 0}\} = 1$, $\{{n \atop n}\} = 1$, $\{{n \atop n+1}\} = \{{n \atop n+2}\} = \cdots = 0$.

$\{{n \atop k}\}$ are called Eulerian numbers, which are tabulated in David-Kendall-Barton [5] for $n \leq 15$; for larger n, we can use the central limit theorem for m-dependent variates to say that this distribution is asymptotically $N(\frac{n-1}{2}, \frac{n+1}{12})$. The result, of course, is completely distribution free.

The next result requires that Z_i be symmetric about zero. Now suppose $X_i = Z_i + Z_{i-1}$, $i = 1,\ldots,n$. Here we want $P\{k \text{ of the } (Z_i + Z_{i-1})$ are positive$\}$. To compute this, notice that $Z_i + Z_{i-1}$ are arrived at by randomly permuting $|Z|_{(0)},\ldots,|Z|_{(n)}$ — where $|Z|_{(i)}$ is the i-th ordered absolute variate — and then independently attaching a plus or minus sign with probability 1/2 each. So to compute the probabilities, we may think of randomly permuting $\{0,1,\ldots,n\}$ and then randomly attach signs to get what we call a "signed permutation." The probability $p_k^{(n)}$ is the number, say $T_{n+1,k}$, of signed permutations that contain k positive successive sums, divided by $2^{n+1}(n+1)!$. The recursion relations for $T_{n,k}$ are more complicated than those for $\{{n \atop k}\}$:

$$T_{n+1,0} = (n+1)T_{n,0} + T_{n,1}$$

$$T_{n+1,1} = 2T_{n,0} + (n-1)\ T_{n,1} + 2T_{n,2}$$

$$T_{n+1,j} = (n-j+1)T_{n,j-2} + (j+1)T_{n,j-1} + (n-j+1)T_{n,j} + (j+1)T_{n,j+1}$$

$$\text{for } j = 2,\ldots,n-2$$

$$T_{n+1,n-1} = 2T_{n,n-3} + nT_{n,n-2} + 2T_{n,n-1}$$

$$T_{n+1,n} = T_{n,n-2} + (n+1)T_{n,n-1}\ ,$$

with the boundary conditions $T_{2,0} = T_{2,1} = 4$ and $T_{n,n} = T_{n+1,n+1} = \cdots = 0$. *It is interesting to note that* $p_0^{(n)} = a_n/(n+1)!$ where a_n is the nth Euler number — $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 16$, $a_5 = 61$, $a_6 = 272 \cdots$. Again the exceedance distribution is asymptotically $N(n/2,\ (n+1)/12)$.

It should be clear now that if we have a tri-diagonal $\ddagger$ with only $\pm\ 1/2$ on the super- and sub-diagonals, we can generate $\ddagger$ by appropriately choosing signs for $(\pm Z_0 \pm Z_1,\ldots,\pm Z_{n-1} \pm Z_n)$ and proceed as above.

Another situation for which combinatorial arguments are known are for $\ddagger$ with $\sigma_{ij} = i \wedge j \quad \forall i,j$. This $\ddagger$ is generated by partial sums of an iid $\{Z_i\}_1^\infty$ process. We then get the discrete arcsine distribution:

$p_k^{(n)} = \binom{2k}{k}\binom{2n-2k}{n-k}2^{-2n}$. A close look at the discussion in Feller [8] shows that this result is true for partial sums generated by exchangeable variates. That is, if $(Z_1,Z_2,\ldots)$ is exchangeable with $\text{cov}(Z_i,Z_j) = \rho \geq 0$, and $S_n = \Sigma_1^n\ Z_i$, then the covariance matrix of $(S_1,\ldots,S_n)$ also yields a discrete arcsine law.

Appendix I
Part 1.

rho=0.1 c=0.0 diag = 3

k	phat	error	equi
0	0.075	0.001	0.075
1	0.251	0.002	0.249
2	0.350	0.002	0.351
3	0.248	0.002	0.249
4	0.076	0.001	0.075

rho=0.1 c=0.0 diag = 2

k	phat	error	equi
0	0.075	0.001	0.075
1	0.251	0.002	0.249
2	0.350	0.002	0.351
3	0.248	0.002	0.249
4	0.076	0.001	0.075

rho=0.1 c=0.25 diag = 3

k	phat	error	equi
0	0.146	0.001	0.146
1	0.333	0.002	0.333
2	0.327	0.002	0.326
3	0.160	0.001	0.161
4	0.035	0.001	0.034

rho=0.1 c=0.25 diag = 2

k	phat	error	equi
0	0.146	0.001	0.146
1	0.333	0.002	0.333
2	0.327	0.002	0.326
3	0.160	0.001	0.161
4	0.035	0.001	0.034

rho=0.1 c=0.68 diag = 3

k	phat	error	equi
0	0.339	0.002	0.337
1	0.396	0.002	0.398
2	0.205	0.002	0.205
3	0.054	0.001	0.054
4	0.006	0.000	0.006

rho=0.1 c=0.68 diag = 2

k	phat	error	equi
0	0.339	0.002	0.337
1	0.396	0.002	0.398
2	0.205	0.002	0.205
3	0.054	0.001	0.054
4	0.006	0.000	0.006

rho=0 1 c=0.85 diag = 3

k	phat	error	equi
0	0.431	0.002	0.430
1	0.384	0.002	0.385
2	0.150	0.001	0.151
3	0.031	0.001	0.031
4	0.003	0.000	0.003

rho=0.1 c=0.85 diag = 2

k	phat	error	equi
0	0.431	0.002	0.430
1	0.384	0.002	0.385
2	0.150	0.001	0.151
3	0.031	0.001	0.031
4	0.003	0.000	0.003

rho=0.1 c=1.28 diag = 3

k	phat	error	equi
0	0.665	0.002	0.663
1	0.277	0.002	0.278
2	0.052	0.001	0.053
3	0.006	0.001	0.005
4	0.000	0.000	0.000

rho=0.1 c=1.28 diag = 2

k	phat	error	equi
0	0.665	0.002	0.663
1	0.277	0.002	0.278
2	0.052	0.001	0.053
3	0.006	0.001	0.005
4	0.000	0.000	0.000

rho=0.1 c=1.96 diag = 3

k	phat	error	equi
0	0.906	0.001	0.905
1	0.090	0.001	0.090
2	0.005	0.000	0.005
3	0.000	0.000	0.000
4	0.000	0.000	0.000

rho=0.1 c=1.96 diag = 2

k	phat	error	equi
0	0.906	0.001	0.905
1	0.090	0.001	0.090
2	0.005	0.000	0.005
3	0.000	0.000	0.000
4	0.000	0.000	0.000

rho=0.3 c=0.0 diag = 3

k	phat	error	equi
0	0.110	0.001	0.109
1	0.241	0.002	0.242
2	0.300	0.002	0.299
3	0.240	0.002	0.242
4	0.110	0.001	0.109

rho=0.3 c=0.25 diag = 3

k	phat	error	equi
0	0.188	0.002	0.187
1	0.301	0.002	0.304
2	0.285	0.002	0.281
3	0.169	0.001	0.170
4	0.058	0.001	0.057

rho=0.3 c=0.68 diag = 3

k	phat	error	equi
0	0.384	0.002	0.380
1	0.339	0.002	0.346
2	0.192	0.002	0.190
3	0.071	0.001	0.070
4	0.015	0.000	0.014

rho=0.3 c=0.85 diag = 3

k	phat	error	equi
0	0.471	0.002	0.468
1	0.328	0.002	0.332
2	0.149	0.001	0.147
3	0.044	0.001	0.045
4	0.008	0.000	0.007

rho=0.3 c=1.28 diag = 3

k	phat	error	equi
0	0.688	0.002	0.685
1	0.238	0.002	0.243
2	0.062	0.001	0.060
3	0.011	0.000	0.011
4	0.001	0.000	0.001

rho=0.3 c=1.96 diag = 3

k	phat	error	equi
0	0.910	0.001	0.908
1	0.082	0.001	0.084
2	0.008	0.000	0.007
3	0.001	0.000	0.001
4	0.000	0.000	0.000

rho=0.3 c=0.0 diag = 2

k	phat	error	equi
0	0.109	0.001	0.108
1	0.241	0.002	0.242
2	0.301	0.002	0.301
3	0.241	0.002	0.242
4	0.109	0.001	0.108

rho=0.3 c=0.25 diag = 2

k	phat	error	equi
0	0.186	0.002	0.186
1	0.302	0.002	0.305
2	0.286	0.002	0.283
3	0.169	0.001	0.170
4	0.057	0.001	0.056

rho=0.3 c=0.68 diag = 2

k	phat	error	equi
0	0.382	0.002	0.378
1	0.341	0.002	0.347
2	0.192	0.002	0.191
3	0.070	0.001	0.070
4	0.014	0.000	0.014

rho=0.3 c=0.85 diag = 2

k	phat	error	equi
0	0.470	0.002	0.467
1	0.329	0.002	0.334
2	0.149	0.001	0.148
3	0.044	0.001	0.044
4	0.007	0.000	0.007

rho=0.3 c=1.28 diag = 2

k	phat	error	equi
0	0.688	0.002	0.684
1	0.239	0.002	0.244
2	0.062	0.001	0.060
3	0.011	0.000	0.011
4	0.001	0.000	0.001

rho=0.3 c=1.96 diag = 2

k	phat	error	equi
0	0.910	0.001	0.908
1	0.082	0.001	0.084
2	0.008	0.000	0.007
3	0.001	0.000	0.001
4	0.000	0.000	0.000

rho=0.5 c=0.0 diag = 3

k	phat	error	equi
0	0.157	0.001	0.156
1	0.220	0.002	0.223
2	0.245	0.002	0.243
3	0.218	0.002	0.223
4	0.159	0.001	0.156

rho=0.5 c=0.25 diag = 3

k	phat	error	equi
0	0.243	0.002	0.242
1	0.260	0.002	0.265
2	0.234	0.002	0.231
3	0.169	0.001	0.170
4	0.093	0.001	0.092

rho=0.5 c=0.68 diag = 3

k	phat	error	equi
0	0.440	0.002	0.434
1	0.275	0.002	0.285
2	0.170	0.002	0.167
3	0.085	0.001	0.084
4	0.031	0.001	0.030

rho=0.5 c=0.85 diag = 3

k	phat	error	equi
0	0.522	0.002	0.518
1	0.263	0.002	0.271
2	0.138	0.001	0.135
3	0.059	0.001	0.059
4	0.019	0.001	0.018

rho=0.5 c=1.28 diag = 3

k	phat	error	equi
0	0.720	0.002	0.714
1	0.189	0.002	0.199
2	0.068	0.001	0.064
3	0.019	0.001	0.019
4	0.004	0.000	0.004

rho=0.5 c=1.96 diag = 3

k	phat	error	equi
0	0.918	0.001	0.915
1	0.068	0.001	0.072
2	0.012	0.000	0.011
3	0.002	0.000	0.002
4	0.000	0.000	0.000

rho=0.5 c=0.0 diag = 2

k	phat	error	equi
0	0.152	0.001	0.150
1	0.225	0.002	0.226
2	0.247	0.002	0.249
3	0.223	0.002	0.226
4	0.153	0.001	0.150

rho=0.5 c=0.25 diag = 2

k	phat	error	equi
0	0.237	0.002	0.235
1	0.266	0.002	0.270
2	0.237	0.002	0.237
3	0.172	0.001	0.171
4	0.088	0.001	0.088

rho=0.5 c=0.68 diag = 2

k	phat	error	equi
0	0.434	0.002	0.427
1	0.281	0.002	0.292
2	0.171	0.002	0.170
3	0.086	0.001	0.083
4	0.028	0.001	0.028

rho=0.5 c=0.85 diag = 2

k	phat	error	equi
0	0.517	0.002	0.511
1	0.268	0.002	0.278
2	0.139	0.001	0.137
3	0.059	0.001	0.057
4	0.017	0.001	0.017

rho=0.5 c=1.28 diag = 2

k	phat	error	equi
0	0.718	0.002	0.710
1	0.192	0.002	0.204
2	0.068	0.001	0.064
3	0.019	0.001	0.018
4	0.003	0.000	0.004

rho=0.5 c=1.96 diag = 2

k	phat	error	equi
0	0.918	0.001	0.914
1	0.069	0.001	0.074
2	0.011	0.000	0.010
3	0.002	0.000	0.002
4	0.000	0.000	0.000

rho=0.7 c=0.0 diag = 3

k	phat	error	equi
0	0.226	0.002	0.223
1	0.182	0.002	0.187
2	0.183	0.002	0.180
3	0.180	0.002	0.187
4	0.229	0.002	0.223

rho=0.7 c=0.25 diag = 3

k	phat	error	equi
0	0.321	0.002	0.317
1	0.202	0.002	0.209
2	0.175	0.002	0.173
3	0.151	0.001	0.154
4	0.151	0.001	0.147

rho=0.7 c=0.68 diag = 3

k	phat	error	equi
0	0.512	0.002	0.505
1	0.199	0.002	0.211
2	0.137	0.001	0.132
3	0.089	0.001	0.091
4	0.063	0.001	0.062

rho=0.7 c=0.85 diag = 3

k	phat	error	equi
0	0.587	0.002	0.581
1	0.187	0.002	0.198
2	0.116	0.001	0.111
3	0.069	0.001	0.069
4	0.042	0.001	0.041

rho=0.7 c=1.28 diag = 3

k	phat	error	equi
0	0.761	0.002	0.755
1	0.133	0.001	0.144
2	0.064	0.001	0.060
3	0.030	0.001	0.028
4	0.013	0.000	0.013

rho=0.7 c=1.96 diag = 3

k	phat	error	equi
0	0.928	0.001	0.926
1	0.051	0.001	0.055
2	0.015	0.000	0.014
3	0.005	0.000	0.004
4	0.001	0.000	0.001

rho=0.7 c=0.0 diag = 2

k	phat	error	equi
0	0.210	0.002	0.204
1	0.192	0.002	0.198
2	0.195	0.002	0.196
3	0.192	0.002	0.198
4	0.211	0.002	0.204

rho=0.7 c=0.25 diag = 2

k	phat	error	equi
0	0.305	0.002	0.296
1	0.213	0.002	0.225
2	0.185	0.002	0.188
3	0.164	0.001	0.160
4	0.134	0.001	0.132

rho=0.7 c=0.68 diag = 2

k	phat	error	equi
0	0.500	0.002	0.486
1	0.207	0.002	0.230
2	0.143	0.001	0.142
3	0.098	0.001	0.090
4	0.052	0.001	0.052

rho=0.7 c=0.85 diag = 2

k	phat	error	equi
0	0.578	0.002	0.564
1	0.193	0.002	0.217
2	0.120	0.001	0.118
3	0.076	0.001	0.067
4	0.033	0.001	0.034

rho=0.7 c=1.28 diag = 2

k	phat	error	equi
0	0.756	0.002	0.743
1	0.138	0.001	0.159
2	0.065	0.001	0.062
3	0.033	0.001	0.027
4	0.008	0.000	0.010

rho=0.7 c=1.96 diag = 2

k	phat	error	equi
0	0.928	0.001	0.923
1	0.052	0.001	0.060
2	0.015	0.000	0.013
3	0.005	0.000	0.004
4	0.000	0.000	0.001

rho=0.9 c=0.0 diag = 3

k	phat	error	equi
0	0.338	0.002	0.335
1	0.111	0.001	0.121
2	0.101	0.001	0.087
3	0.110	0.001	0.121
4	0.340	0.002	0.335

rho=0.9 c=0.25 diag = 3

k	phat	error	equi
0	0.438	0.002	0.437
1	0.114	0.001	0.112
2	0.099	0.001	0.103
3	0.099	0.001	0.105
4	0.250	0.002	0.243

rho=0.9 c=0.68 diag = 3

k	phat	error	equi
0	0.616	0.002	0.611
1	0.104	0.001	0.117
2	0.079	0.001	0.068
3	0.068	0.001	0.075
4	0.132	0.001	0.128

rho=0.9 c=0.85 diag = 3

k	phat	error	equi
0	0.680	0.002	0.680
1	0.096	0.001	0.097
2	0.069	0.001	0.066
3	0.058	0.001	0.054
4	0.097	0.001	0.092

rho=0.9 c=1.28 diag = 3

k	phat	error	equi
0	0.819	0.001	0.816
1	0.067	0.001	0.077
2	0.042	0.001	0.037
3	0.031	0.001	0.031
4	0.040	0.001	0.040

rho=0.9 c=1.96 diag = 3

k	phat	error	equi
0	0.948	0.001	0.946
1	0.025	0.001	0.029
2	0.012	0.001	0.011
3	0.008	0.000	0.007
4	0.007	0.000	0.007

rho=0.1 c=0.0 diag = 10

k	phat	error	equi
0	0.002	0.000	0.002
1	0.014	0.000	0.014
2	0.051	0.001	0.052
3	0.121	0.001	0.121
4	0.196	0.001	0.197
5	0.230	0.001	0.230
6	0.196	0.001	0.197
7	0.123	0.001	0.121
8	0.051	0.001	0.052
9	0.014	0.000	0.014
10	0.002	0.000	0.002

rho=0.1 c=0.25 diag = 10

k	phat	error	equi
0	0.009	0.000	0.009
1	0.048	0.001	0.048
2	0.126	0.001	0.126
3	0.205	0.001	0.207
4	0.237	0.001	0.235
5	0.190	0.001	0.193
6	0.118	0.001	0.116
7	0.051	0.001	0.050
8	0.015	0.000	0.015
9	0.003	0.000	0.003
10	0.000	0.000	0.000

rho=0.1 c=0.68 diag = 10

k	phat	error	equi
0	0.069	0.001	0.068
1	0.196	0.001	0.196
2	0.269	0.001	0.271
3	0.235	0.001	0.235
4	0.142	0.001	0.142
5	0.063	0.001	0.062
6	0.021	0.000	0.020
7	0.004	0.000	0.005
8	0.001	0.000	0.001
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.1 c=0.85 diag = 10

k	phat	error	equi
0	0.125	0.001	0.124
1	0.270	0.001	0.272
2	0.285	0.001	0.287
3	0.191	0.001	0.191
4	0.090	0.001	0.089
5	0.030	0.001	0.030
6	0.007	0.000	0.007
7	0.001	0.000	0.001
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.1 c=1.28 diag = 10

k	phat	error	equi
0	0.363	0.002	0.361
1	0.370	0.002	0.373
2	0.189	0.001	0.189
3	0.061	0.001	0.061
4	0.014	0.000	0.014
5	0.002	0.000	0.002
6	0.000	0.000	0.000
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.1 c=1.96 diag = 10

k	phat	error	equi
0	0.780	0.001	0.779
1	0.193	0.001	0.194
2	0.024	0.025	0.024
3	0.002	0.000	0.002
4	0.000	0.000	0.000
5	0.000	0.000	0.000
6	0.000	0.000	0.000
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.3 c=0.0 diag = 10

k	phat	error	equi
0	0.006	0.000	0.005
1	0.026	0.001	0.026
2	0.068	0.001	0.068
3	0.126	0.001	0.126
4	0.175	0.001	0.176
5	0.198	0.001	0.197
6	0.176	0.001	0.176
7	0.126	0.001	0.126
8	0.068	0.001	0.068
9	0.026	0.001	0.026
10	0.006	0.001	0.005

rho=0.3 c=0.25 diag = 10

k	phat	error	equi
0	0.019	0.000	0.008
1	0.068	0.001	0.047
2	0.134	0.001	0.125
3	0.187	0.001	0.208
4	0.200	0.001	0.237
5	0.171	0.001	0.194
6	0.118	0.001	0.115
7	0.065	0.001	0.049
8	0.027	0.001	0.014
9	0.007	0.000	0.003
10	0.001	0.000	0.000

rho=0.3 c=0.68 diag = 10

k	phat	error	equi
0	0.102	0.001	0.066
1	0.201	0.001	0.195
2	0.239	0.001	0.272
3	0.204	0.001	0.237
4	0.136	0.001	0.142
5	0.073	0.001	0.062
6	0.030	0.001	0.019
7	0.010	0.000	0.004
8	0.003	0.000	0.001
9	0.001	0.000	0.000
10	0.000	0.000	0.000

rho=0.3 c=0.85 diag = 10

k	phat	error	equi
0	0.165	0.001	0.121
1	0.260	0.001	0.272
2	0.247	0.001	0.289
3	0.172	0.001	0.192
4	0.095	0.001	0.088
5	0.041	0.001	0.029
6	0.014	0.000	0.007
7	0.004	0.000	0.001
8	0.001	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.3 c=1.28 diag = 10

k	phat	error	equi
0	0.401	0.002	0.359
1	0.328	0.001	0.375
2	0.173	0.001	0.189
3	0.069	0.001	0.061
4	0.022	0.000	0.014
5	0.005	0.000	0.002
6	0.001	0.000	0.000
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.3 c=1.96 diag = 10

k	phat	error	equi
0	0.790	0.001	0.787
1	0.173	0.001	0.180
2	0.032	0.001	0.028
3	0.005	0.000	0.004
4	0.000	0.000	0.000
5	0.000	0.000	0.000
6	0.000	0.000	0.000
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.5 c=0.0 diag = 10

k	phat	error	equi
0	0.017	0.000	0.015
1	0.047	0.001	0.046
2	0.082	0.001	0.085
3	0.122	0.001	0.123
4	0.151	0.001	0.150
5	0.159	0.001	0.160
6	0.150	0.001	0.150
7	0.123	0.001	0.123
8	0.084	0.001	0.085
9	0.047	0.001	0.046
10	0.018	0.000	0.015

rho=0.5 c=0.25 diag = 10

k	phat	error	equi
0	0.045	0.001	0.040
1	0.091	0.001	0.094
2	0.134	0.001	0.139
3	0.163	0.001	0.163
4	0.164	0.001	0.164
5	0.145	0.001	0.144
6	0.114	0.001	0.112
7	0.077	0.001	0.076
8	0.042	0.001	0.043
9	0.019	0.000	0.019
10	0.006	0.000	0.005

rho=0.5 c=0.68 diag = 10

k	phat	error	equi
0	0.158	0.001	0.143
1	0.198	0.001	0.211
2	0.201	0.001	0.208
3	0.167	0.001	0.168
4	0.124	0.001	0.119
5	0.077	0.001	0.075
6	0.044	0.001	0.042
7	0.021	0.000	0.022
8	0.008	0.000	0.008
9	0.002	0.000	0.003
10	0.000	0.000	0.000

rho=0.5 c=0.85 diag = 10

k	phat	error	equi
0	0.229	0.001	0.212
1	0.237	0.001	0.254
2	0.202	0.001	0.210
3	0.147	0.001	0.145
4	0.094	0.001	0.090
5	0.051	0.001	0.050
6	0.025	0.001	0.025
7	0.010	0.000	0.011
8	0.004	0.000	0.004
9	0.001	0.000	0.001
10	0.000	0.000	0.000

rho=0.5 c=1.28 diag = 10

k	phat	error	equi
0	0.464	0.002	0.444
1	0.265	0.001	0.292
2	0.148	0.001	0.149
3	0.073	0.001	0.068
4	0.032	0.001	0.029
5	0.012	0.000	0.012
6	0.004	0.000	0.004
7	0.001	0.000	0.001
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.5 c=1.96 diag = 10

k	phat	error	equi
0	0.812	0.001	0.802
1	0.141	0.001	0.157
2	0.036	0.001	0.032
3	0.009	0.000	0.007
4	0.002	0.000	0.002
5	0.000	0.000	0.000
6	0.000	0.000	0.000
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.7 c=0.0 diag = 10

k	phat	error	equi
0	0.056	0.001	0.049
1	0.072	0.001	0.076
2	0.089	0.001	0.095
3	0.107	0.001	0.108
4	0.116	0.001	0.115
5	0.117	0.001	0.117
6	0.117	0.001	0.115
7	0.105	0.001	0.108
8	0.092	0.001	0.095
9	0.072	0.001	0.076
10	0.057	0.001	0.049

rho=0.7 c=0.25 diag = 10

k	phat	error	equi
0	0.108	0.001	0.098
1	0.112	0.001	0.112
2	0.122	0.001	0.128
3	0.126	0.001	0.127
4	0.121	0.001	0.120
5	0.111	0.001	0.109
6	0.095	0.001	0.095
7	0.077	0.001	0.079
8	0.060	0.001	0.062
9	0.041	0.001	0.043
10	0.026	0.001	0.022

rho=0.7 c=0.68 diag = 10

k	phat	error	equi
0	0.261	0.001	0.231
1	0.174	0.001	0.199
2	0.147	0.001	0.159
3	0.123	0.001	0.124
4	0.097	0.001	0.095
5	0.072	0.001	0.070
6	0.052	0.001	0.050
7	0.037	0.001	0.035
8	0.021	0.000	0.022
9	0.012	0.000	0.012
10	0.005	0.000	0.004

rho=0.7 c=0.85 diag = 10

k	phat	error	equi
0	0.339	0.002	0.307
1	0.189	0.001	0.220
2	0.145	0.001	0.155
3	0.110	0.001	0.110
4	0.080	0.001	0.077
5	0.056	0.001	0.053
6	0.036	0.001	0.035
7	0.024	0.000	0.022
8	0.013	0.000	0.013
9	0.006	0.000	0.006
10	0.002	0.000	0.002

rho=0.7 c=1.28 diag = 10

k	phat	error	equi
0	0.561	0.002	0.528
1	0.184	0.001	0.224
2	0.108	0.001	0.113
3	0.065	0.001	0.062
4	0.038	0.001	0.034
5	0.022	0.000	0.019
6	0.012	0.000	0.010
7	0.006	0.000	0.005
8	0.002	0.000	0.003
9	0.001	0.000	0.001
10	0.000	0.000	0.000

rho=0.7 c=1.96 diag = 10

k	phat	error	equi
0	0.849	0.001	0.830
1	0.092	0.001	0.119
2	0.035	0.001	0.032
3	0.015	0.000	0.011
4	0.006	0.000	0.004
5	0.002	0.000	0.002
6	0.001	0.000	0.001
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.9 c=0.0 diag = 10

k	phat	error	equi
0	0.194	0.001	0.173
1	0.076	0.001	0.088
2	0.069	0.001	0.076
3	0.067	0.001	0.074
4	0.065	0.001	0.062
5	0.065	0.001	0.053
6	0.065	0.001	0.062
7	0.066	0.001	0.074
8	0.069	0.001	0.076
9	0.074	0.001	0.088
10	0.191	0.001	0.173

rho=0.9 c=0.25 diag = 10

k	phat	error	equi
0	0.281	0.001	0.252
1	0.091	0.001	0.116
2	0.076	0.001	0.086
3	0.071	0.001	0.063
4	0.064	0.001	0.060
5	0.062	0.001	0.067
6	0.060	0.001	0.064
7	0.057	0.001	0.054
8	0.056	0.001	0.057
9	0.060	0.001	0.072
10	0.123	0.001	0.110

rho=0.9 c=0.68 diag = 10

k	phat	error	equi
0	0.462	0.002	0.428
1	0.101	0.001	0.128
2	0.075	0.001	0.089
3	0.063	0.001	0.069
4	0.054	0.001	0.048
5	0.049	0.001	0.040
6	0.042	0.001	0.043
7	0.037	0.001	0.042
8	0.034	0.001	0.035
9	0.031	0.001	0.034
10	0.050	0.001	0.044

rho=0.9 c=0.85 diag = 10

k	phat	error	equi
0	0.537	0.002	0.502
1	0.097	0.001	0.134
2	0.072	0.001	0.083
3	0.058	0.001	0.052
4	0.048	0.001	0.042
5	0.040	0.001	0.042
6	0.035	0.001	0.037
7	0.030	0.001	0.028
8	0.026	0.001	0.024
9	0.024	0.001	0.026
10	0.033	0.001	0.029

rho=0.9 c=1.28 diag = 10

k	phat	error	equi
0	0.715	0.001	0.687
1	0.080	0.001	0.109
2	0.051	0.001	0.059
3	0.038	0.001	0.041
4	0.029	0.001	0.025
5	0.023	0.000	0.018
6	0.019	0.000	0.017
7	0.015	0.000	0.015
8	0.011	0.000	0.011
9	0.008	0.000	0.009
10	0.009	0.000	0.009

rho=0.9 c=1.96 diag = 10

k	phat	error	equi
0	0.905	0.001	0.893
1	0.038	0.001	0.053
2	0.020	0.000	0.022
3	0.013	0.000	0.012
4	0.008	0.000	0.006
5	0.006	0.000	0.004
6	0.004	0.000	0.004
7	0.002	0.000	0.003
8	0.002	0.000	0.002
9	0.001	0.000	0.001
10	0.001	0.000	0.001

rho=0.5 c=0.0 diag = 2

k	phat	error	equi
0	0.013	0.000	0.011
1	0.041	0.001	0.039
2	0.076	0.001	0.081
3	0.124	0.001	0.125
4	0.159	0.001	0.159
5	0.172	0.001	0.171
6	0.159	0.001	0.159
7	0.124	0.001	0.125
8	0.079	0.001	0.081
9	0.040	0.001	0.040
10	0.014	0.000	0.011

rho=0.5 c=0.25 diag = 2

k	phat	error	equi
0	0.038	0.001	0.031
1	0.084	0.001	0.086
2	0.131	0.001	0.139
3	0.171	0.001	0.172
4	0.175	0.001	0.175
5	0.155	0.001	0.153
6	0.117	0.001	0.115
7	0.074	0.001	0.073
8	0.037	0.001	0.038
9	0.015	0.000	0.015
10	0.004	0.000	0.003

rho=0.5 c=0.68 diag = 2

k	phat	error	equi
0	0.144	0.001	0.127
1	0.197	0.001	0.211
2	0.206	0.001	0.219
3	0.180	0.001	0.180
4	0.129	0.001	0.125
5	0.077	0.001	0.075
6	0.040	0.001	0.039
7	0.017	0.000	0.017
8	0.006	0.000	0.006
9	0.001	0.000	0.002
10	0.000	0.000	0.000

rho=0.5 c=0.85 diag = 2

k	phat	error	equi
0	0.216	0.001	0.194
1	0.237	0.001	0.259
2	0.212	0.001	0.223
3	0.157	0.001	0.154
4	0.096	0.001	0.091
5	0.049	0.001	0.047
6	0.021	0.000	0.021
7	0.008	0.000	0.008
8	0.003	0.000	0.003
9	0.001	0.000	0.001
10	0.000	0.000	0.000

rho=0.5 c=1.28 diag = 2

k	phat	error	equi
0	0.453	0.002	0.427
1	0.272	0.001	0.308
2	0.155	0.001	0.157
3	0.077	0.001	0.068
4	0.029	0.001	0.026
5	0.010	0.000	0.009
6	0.003	0.000	0.003
7	0.001	0.000	0.001
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

rho=0.5 c=1.96 diag = 2

k	phat	error	equi
0	0.809	0.001	0.797
1	0.144	0.001	0.165
2	0.036	0.001	0.031
3	0.009	0.000	0.006
4	0.002	0.000	0.001
5	0.000	0.000	0.000
6	0.000	0.000	0.000
7	0.000	0.000	0.000
8	0.000	0.000	0.000
9	0.000	0.000	0.000
10	0.000	0.000	0.000

Appendix I
Part 2.

Sigma

1.000	0.579	0.241
0.579	1.000	0.582
0.241	0.582	1.000

loadings 0.579 0.950 0.582

c = 0.0

k	phat	error	equi	m.l.fit
0	0.243	0.002	0.241	0.246
1	0.258	0.002	0.259	0.254
2	0.255	0.002	0.259	0.254
3	0.245	0.002	0.241	0.246
distances			0.006	0.005

c = 0.68

k	phat	error	equi	m.l.fit
0	0.534	0.002	0.532	0.537
1	0.252	0.002	0.259	0.249
2	0.145	0.002	0.142	0.142
3	0.069	0.001	0.068	0.070
distances			0.008	0.005

Sigma

1.000	0.717	-0.232
0.717	1.000	-0.471
-0.232	-0.471	1.000

loadings 0.717 0.950 -0.471

c = 0.0

k	phat	error	equi	m.l.fit
0	0.129	0.002	0.126	0.121
1	0.369	0.002	0.374	0.379
2	0.370	0.002	0.374	0.379
3	0.132	0.002	0.126	0.121
distances			0.009	0.020

c = 0.68

k	phat	error	equi	m.l.fit
0	0.452	0.002	0.426	0.328
1	0.361	0.002	0.419	0.254
2	0.173	0.002	0.139	0.152
3	0.014	0.001	0.016	0.010
distances			0.072	0.165

Sigma

1.000	0.600	0.636	0.472
0.600	1.000	0.383	0.458
0.636	0.383	1.000	0.705
0.472	0.458	0.705	1.000

loadings 0.729 0.561 0.863 0.774

c = 0.0

k	phat	error	equi	m.l.fit
0	0.216	0.002	0.214	0.210
1	0.185	0.002	0.192	0.198
2	0.198	0.002	0.188	0.184
3	0.184	0.002	0.192	0.198
4	0.218	0.002	0.214	0.210
distances			0.016	0.026

Sigma

1.000	0.585	0.177	0.197
0.585	1.000	0.209	0.217
0.177	0.209	1.000	0.291
0.197	0.217	0.291	1.000

loadings 0.728 0.796 0.279 0.294

c = 0.0

k	phat	error	equi	m.l.fit
0	0.134	0.002	0.135	0.128
1	0.231	0.002	0.232	0.240
2	0.271	0.002	0.267	0.264
3	0.228	0.002	0.232	0.240
4	0.137	0.002	0.135	0.128
distances			0.007	0.020

Sigma

1.000	0.735	0.711	0.704
0.735	1.000	0.693	0.709
0.711	0.693	1.000	0.839
0.704	0.709	0.839	1.000

loadings 0.803 0.797 0.904 0.908

c = 0.0

k	phat	error	equi	m.l.fit
0	0.283	0.002	0.283	0.282
1	0.147	0.002	0.150	0.154
2	0.137	0.002	0.132	0.128
3	0.145	0.002	0.150	0.154
4	0.287	0.002	0.283	0.282
distances			0.009	0.016

Sigma

1.000	0.885	0.905	0.883
0.885	1.000	0.826	0.769
0.905	0.826	1.000	0.923
0.883	0.769	0.923	1.000

loadings 0.950 0.871 0.963 0.938

c = 0.0

k	phat	error	equi	m.l.fit
0	0.347	0.002	0.349	0.354
1	0.106	0.001	0.114	0.114
2	0.090	0.001	0.074	0.064
3	0.103	0.001	0.114	0.114
4	0.354	0.002	0.349	0.354
distances			0.022	0.031

Sigma

1.000	0.735	0.711	0.704
0.735	1.000	0.693	0.709
0.711	0.693	1.000	0.839
0.704	0.709	0.839	1.000

loadings 0.803 0.797 0.904 0.908

c = 0.0

k	phat	error	equi	m.l.fit
0	0.283	0.002	0.283	0.282
1	0.147	0.002	0.150	0.154
2	0.137	0.002	0.132	0.128
3	0.145	0.002	0.150	0.154
4	0.287	0.002	0.283	0.282
distances			0.009	0.016

Sigma

1.000	0.720	-0.440	0.600
0.720	1.000	-0.130	0.680
-0.440	-0.130	1.000	-0.290
0.600	0.680	-0.290	1.000

loadings 0.839 0.856 -0.337 0.760

c = 0.0

k	phat	error	equi	m.l.fit
0	0.104	0.001	0.110	0.108
1	0.292	0.002	0.241	0.284
2	0.206	0.002	0.297	0.215
3	0.293	0.002	0.241	0.284
4	0.104	0.001	0.110	0.108
distances			0.117	0.016

Sigma

1.000	0.490	0.390	0.280	0.170
0.490	1.000	0.680	0.500	0.470
0.390	0.680	1.000	0.740	0.680
0.280	0.500	0.740	1.000	0.730
0.170	0.470	0.680	0.730	1.000

loadings 0.406 0.702 0.916 0.820 0.763

c = 0.0

k	phat	error	equi	m.l.fit
0	0.171	0.002	0.171	0.169
1	0.170	0.002	0.165	0.180
2	0.156	0.002	0.164	0.151
3	0.155	0.002	0.164	0.151
4	0.174	0.002	0.165	0.180
5	0.173	0.002	0.171	0.169
distances			0.015	0.014

Sigma

1.000	0.553	0.547	0.410	0.389
0.553	1.000	0.610	0.485	0.437
0.547	0.610	1.000	0.711	0.665
0.410	0.485	0.711	1.000	0.607
0.389	0.437	0.665	0.607	1.000

loadings 0.646 0.711 0.864 0.786 0.742

c = 0.0

k	phat	error	equi	m.l.fit
0	0.180	0.002	0.181	0.187
1	0.162	0.002	0.162	0.163
2	0.156	0.002	0.157	0.149
3	0.153	0.002	0.157	0.149
4	0.167	0.002	0.162	0.163
5	0.182	0.002	0.181	0.187
distances			0.007	0.013

Sigma

1.000	0.425	-0.055	-0.003	0.193
0.425	1.000	-0.042	0.049	0.069
-0.055	-0.042	1.000	0.747	0.169
-0.003	0.049	0.747	1.000	0.265
0.193	0.069	0.169	0.265	1.000

loadings -0.030 0.056 0.747 0.265 0.990

c = 0.68

k	phat	error	equi	m.l.fit
0	0.332	0.002	0.314	0.186
1	0.296	0.002	0.327	0.248
2	0.225	0.002	0.214	0.183
3	0.105	0.001	0.103	0.096
4	0.034	0.001	0.036	0.029
5	0.008	0.000	0.007	0.003
distances			0.037	0.160

Sigma

1.000	0.584	0.615	0.601	0.570	0.600
0.584	1.000	0.576	0.530	0.526	0.555
0.615	0.576	1.000	0.940	0.875	0.878
0.601	0.530	0.940	1.000	0.877	0.886
0.570	0.526	0.875	0.877	1.000	0.924
0.600	0.555	0.878	0.886	0.924	1.000

loadings 0.637 0.584 0.959 0.960 0.929 0.936

c = 0.0

k	phat	error	equi	m.l.fit
0	0.225	0.002	0.223	0.215
1	0.135	0.001	0.124	0.166
2	0.099	0.001	0.105	0.090
3	0.076	0.001	0.096	0.057
4	0.098	0.001	0.105	0.090
5	0.137	0.001	0.124	0.166
6	0.230	0.002	0.223	0.215
distances			0.029	0.051

Sigma

1.000	0.088	0.334	0.191	0.173	0.123
0.088	1.000	0.186	0.384	0.262	0.040
0.334	0.186	1.000	0.343	0.144	0.080
0.191	0.384	0.343	1.000	0.375	0.142
0.173	0.262	0.144	0.375	1.000	0.334
0.123	0.040	0.080	0.142	0.334	1.000

c = 0.0

k	phat	error	equi	m.l.fit
0	0.058	0.001	0.058	0.056
1	0.135	0.002	0.135	0.138
2	0.196	0.002	0.197	0.197
3	0.219	0.002	0.219	0.217
4	0.198	0.002	0.197	0.197
5	0.136	0.002	0.135	0.138
6	0.058	0.001	0.058	0.056
distances			0.002	0.006

Sigma

1.000	0.440	0.440	0.110	0.190	0.100	0.200	0.200	-0.240
0.440	1.000	0.340	0.250	0.320	0.400	0.490	0.160	-0.220
0.440	0.340	1.000	0.040	0.060	0.080	0.070	0.050	-0.070
0.110	0.250	0.040	1.000	0.180	0.120	0.230	0.020	-0.200
0.190	0.320	0.060	0.180	1.000	0.220	0.480	0.140	-0.080
0.100	0.400	0.080	0.120	0.220	1.000	0.400	-0.110	-0.300
0.200	0.490	0.070	0.230	0.480	0.400	1.000	0.150	-0.130
0.200	0.160	0.050	0.020	0.140	-0.110	0.150	1.000	-0.100
-0.240	-0.220	-0.070	-0.200	-0.080	-0.300	-0.130	-0.100	1.000

loadings 0.514 0.728 0.346 0.328 0.509 0.299 0.660 0.223 -0.311

c = 0.0

k	phat	error	equi	m.l.fit
0	0.008	0.000	0.012	0.009
1	0.049	0.001	0.047	0.049
2	0.111	0.001	0.099	0.107
3	0.156	0.001	0.154	0.155
4	0.176	0.002	0.188	0.180
5	0.178	0.002	0.188	0.180
6	0.156	0.001	0.154	0.155
7	0.110	0.001	0.099	0.107
8	0.049	0.001	0.047	0.049
9	0.008	0.000	0.012	0.009
distances			0.023	0.026

c = 0.68

k	phat	error	equi	m.l.fit
0	0.128	0.001	0.137	0.131
1	0.259	0.002	0.234	0.252
2	0.234	0.002	0.238	0.237
3	0.168	0.001	0.182	0.171
4	0.108	0.001	0.114	0.109
5	0.061	0.001	0.059	0.061
6	0.030	0.001	0.025	0.029
7	0.010	0.000	0.008	0.010
8	0.002	0.000	0.002	0.002
9	0.000	0.000	0.000	0.000
distances			0.031	0.009

Appendix II

n = 5

k	true prob.	approx from expansion
0	0.033	0.034
1	0.158	0.158
2	0.309	0.308
3	0.309	0.308
4	0.158	0.158
5	0.033	0.034

n = 10

k	true prob.	approx from expansion
0	0.001	0.001
1	0.011	0.012
2	0.048	0.048
3	0.119	0.120
4	0.201	0.201
5	0.239	0.238
6	0.201	0.201
7	0.119	0.120
8	0.048	0.048
9	0.011	0.012
10	0.001	0.001

$c = 0.0$

$(1-\theta)I+\theta\ddagger_{.5} = T(\theta)$

$\theta = .02$

n = 15

k	true prob.	approx from expansion
0	0.000	0.000
1	0.001	0.001
2	0.004	0.005
3	0.017	0.017
4	0.046	0.047
5	0.094	0.095
6	0.150	0.149
7	0.188	0.188
8	0.188	0.188
9	0.150	0.149
10	0.094	0.095
11	0.046	0.047
12	0.017	0.017
13	0.004	0.005
14	0.001	0.001
15	0.000	0.000

n = 5

k	true prob.	approx from expansion
0	0.521	0.761
1	0.272	0.063
2	0.128	0.092
3	0.055	0.167
4	0.019	0.090
5	0.005	0.011

n = 10

k	true prob.	approx from expansion
0	0.353	1.345
1	0.245	0.211
2	0.159	-0.950
3	0.101	-0.618
4	0.063	0.266
5	0.038	0.456
6	0.022	0.220
7	0.012	0.055
8	0.006	0.008
9	0.002	0.001
10	0.000	0.000

n = 15

k	true prob.	approx from expansion
0	0.266	1.905
1	0.210	1.236
2	0.154	-0.993
3	0.112	-2.106
4	0.081	-1.233
5	0.058	0.278
6	0.041	0.890
7	0.028	0.651
8	0.019	0.276
9	0.013	0.078
10	0.008	0.019
11	0.005	0.002
12	0.003	0.000
13	0.001	0.000
14	0.001	0.000
15	0.000	0.000

$c = 1.0$

$(1-\theta)I+\theta\ddagger_{.5} = T(\theta)$

$\theta = .6$

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ON THE EVALUATION OF CERTAIN MULTIVARIATE NORMAL PROBABILITIES

Report #322

Satish Iyengar

— Consider the following problem: if X is an n-dimensional normally distributed random vector with mean zero and covariance Σ, evaluate the probability, p_k, that k components of X exceed a given constant. We call $(p_0,\ldots,p_n)$ the exceedance distribution and study its behavior as Σ varies. The p_k's can be expressed as multidimensional integrals; these expressions, however, are not helpful, for simulation and numerical integration in high dimensions are very expensive.

When Σ is an equicorrelation matrix or has single-factor structure, the probabilities can be written as single integrals. In this dissertation, we propose some methods for approximating the above multidimensional integrals by such single integrals. Ample numerical evidence is given to show that the approximations are quite good. We also prove a theorem which gives conditions for the variance of the exceedance distribution to be greater than that of the approximation. We use this inequality to improve upon earlier approximations and inequalities.

A closely related problem is that of evaluating "vanishing" orthant probabilities: i.e., $P\{X_1 > a_1,\ldots,X_n > a_n\}$ for large a_i. A natural approach here is to use generalizations of Mills' ratio. Several authors have studied this approach and stated some conjectures. Studying Mills' ratio from the viewpoint of exponential families, we provide proofs for some of the conjectures and some extensions.

Finally, we state and prove some interesting special results for the equicorrelation case. This case lends itself to much analysis because of its symmetry. These results are new and shed some light on other results that are in the literature.

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