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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Signal detection problems with spherically exchangeable (Gaussian, conditionally Gaussian, and non-Gaussian) background noise are treated. Techniques for both historical data and cross-sectional data are developed using sufficient statistics and statistical noise. Extensions of the Kolmogorov-Smirnov statistic are employed. Each technique is illustrated by a numerical example.

SIGNAL DETECTION FOR SPHERICALLY EXCHANGEABLE (SE) STOCHASTIC PROCESSES

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1. Introduction and Summary

Consider signal detection in the following context. The background pure noise (PN) is produced by nature's choosing a parameter value and then generating a time series according to a (stochastic process) law with that parameter value. This process may be repeated N times. If there is signal present, what is generated is a (N + S), noise-plus-signal, time series. The statistical properties of the latter-type of time series are different from those of the former.

In the sequel, one will be concerned primarily with the case in which the parameter chosen is a variance and the time series subsequently generated is an i.i.d. zero-mean Gaussian time series with that variance. The variances are chosen by nature according to a positive cdf, corresponding to which will be a radial distribution. [See Appendix II].

The signal detection procedures developed below will be valid not only for the (conditional) Gaussian model, Ω(S-S-E), described above,

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but also for a wider family Ω(K-S-E) of pure noise (PN) stochastic processes. This is because for all S-E (spherically exchangeable) distributions, the set of observed radii $\{R_j\}$ are sufficient, and the set of "angles" are independent of the radii and have identical uniform distributions. [See Section 2.C. and Appendix II].

Five different signal detection models will be explicitly treated. The treatment of these models will indicate how a variety of related signal detection problems can be treated.

The paper is divided into ten sections. In Section 2, the basic S-E data models are given. Section 3 presents the five detection problems to be considered. Statistical preliminaries are presented for historical data and for cross-sectional data in Section 4. Detection procedures for the five problems are developed in Section 5 through 9. Open problems and concluding remarks are given in Section 10.

2. Data Structure of the Background Noise

It is assumed throughout that decisions are made on the basis of a finite number of observations. The actual time series being observed may be infinite, but one only uses a finite initial segment in the case of historical data; and a finite number of finite initial segments for cross-sectional data. In each case, the law, L, of the process determines the distribution of the initial segment, and vice versa. The two words are sometimes used interchangeably.

The data to be considered is of the form $\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N$, where

$\underset{\sim}{Z}_r = (Z_{r1}, \ldots, Z_{rk})$, and the probabilistic structures of interest are given below.

(2A) Normal Spherical Exchangeability Ω(N-S-E)

$\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_N$ are i.i.d. $N(\underset{\sim}{0}, \sigma^2 \tilde{I}_k)$, i.e., the k components of each Z-vector are i.i.d. $N(0,\sigma^2)$. This is the original spherical structure of interest. The family of laws of all such time series is Ω(N-S-E).

(2B) Schoenberg (1938) Spherical Exchangeability Ω(S-S-E).

$W_1, \ldots, W_N$ are i.i.d. $H(\cdot)$, with $H(0) = 0$, and, conditionally given $W_r = w_r$, $X_{r1}, \ldots, X_{rk}$ are i.i.d. $N(0, 1/w_r)$.

Definition 2.1. $H(\cdot)$ is called the mixing measure.

Here, one can say that each X_{rj}, has a multivariate distribution which is a variance-mixture of zero-mean normal random samples. If $H(\cdot)$ is a one-point (i.e, degenerate) cdf then one has case (2A) above. [Some relations between Ω(S-S-E) and Gaussian Markov processes is given in Appendix II.]. The family of laws of all such time series is Ω(S-S-E).

(2C) Kelker Spherical Exchangeability, Ω(K-S-E).

$\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N$ are i.i.d. $\tilde{F}(\cdot)$, where $\underset{\sim}{Z}_j = R_j \cdot \underset{\sim}{V}_j$ with (1) R_j and $\underset{\sim}{V}_j$ being independent; (2) $R_j \sim J(\cdot)$, where $J(0) = 0$; and (3) $\underset{\sim}{V}_j = (V_{j1}, \ldots, V_{jk})$ being uniformly distributed over the hypersphere $S_1^* = \{\underset{\sim}{v}: \sum_1^k v_j^2 = 1\}$, with $R_j^2 = \sum_{s=1}^k Z_{js}^2$ and $V_{js} = \frac{Z_{js}}{R_j}$.

Definition 2.2. $J(\cdot)$ is called the radial cdf. Some examples of radial distributions are given in Appendix II.

It is immediate that

Theorem 2.1. $\Omega(\text{N-S-E}) \subset \Omega(\text{S-S-E}) \subset \Omega(\text{K-S-E})$.

Schoenberg (1938) essentially proved that

Theorem 2.2. For infinite time series,

$$\Omega(\text{S-S-E}) = \Omega(\text{K-S-E}).$$

From Lord (1954); Kelker (1970); Bell, Avadhani and Woodroofe (1970); and Bell (1975, 1978), one has

Theorem 2.3. Let $\underset{\sim}{Z} = (Z_1, \ldots, Z_k)$ have a law L in $\Omega(\text{K-S-E})$. Then for $1 \leq m \leq k-1$, $[\sum_1^m Z_j^2][\sum_{m+1}^k Z_j^2]^{-1}(\frac{k-m}{m}) \sim F(m, k-m)$.

[This is Fisher's F-distribution, with m and $(k-m)$ degrees of freedom].

The salient point here is that there are non-Gaussian laws for which the F-distribution is valid.

These types of PN structures can be best illustrated by the following examples.

Example 2.1. $\Omega(\text{N-S-E})$. Let $Z_{r1}, \ldots, Z_{r9}$ be i.i.d. $N(0,5.4)$; $\underset{\sim}{Z}_r = (Z_{r1}, \ldots, Z_{r9})$; and $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_{50}$ be i.i.d. The $\underset{\sim}{Z}_r$ here are initial segments of zero-mean i.i d. Gaussian time series.

<u>Example 2.2</u>. Ω(S-S-E). Let $W_1, \ldots, W_{20}$ be i.i.d. Exp (2.5). Further, given $W_r = w_r$, let $Z_{r1}, \ldots, Z_{r8}$ be conditionally i.i.d. $N(0, 1/w_r)$. Then, the unconditional density of Z_r is

$$\tilde{f}(z_1, \ldots, z_8) = \int_0^\infty \left(\frac{w}{2\pi}\right)^4 \exp\left\{-\frac{w}{2}\sum_1^8 z_j^2\right\} (2.5) \exp\{-2.5w\}dw$$

$$= (120)\pi^{-4}\left[5.0 + \sum_1^8 z_j^2\right]^{-5} ; \text{ and } \underset{\sim}{Z}_1, \ldots \underset{\sim}{Z}_{20} \text{ are}$$

i.i.d. $\tilde{F}(\cdot, \ldots, \cdot)$. [See Table II.1 of Appendix II.].

In this example, each $R_j^2 = \sum_1^8 Z_{js}^2$, and the radial distribution $J(\cdot)$ satisfies

$$J(y) = P\{R_j \le y\} = \int_0^\infty F^*(yw^2) \; (2.5) \exp\{-2.5w\} \, dw,$$

where $F^*(\cdot)$ is the χ_8^2-cdf.

[Note: If $H(\cdot)$ were such that $P_H(\{5.2\}) = 1$, then $\tilde{f}(\underset{\sim}{z})$ would be $(10.4\pi)^{-4} \exp\left\{-\frac{1}{10.4}\sum_1^8 z_j^2\right\}$ and the law would be in Ω(N-S-E).]

<u>Example 2.3. Ω(K-S-E)</u>.

Let $\underset{\sim}{Z}_r = (Z_{r1}, \ldots, Z_{r5})$ and $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_{15}$ be i.i.d. $F_1(\cdot, \ldots, \cdot)$, with density of the form

$$\tilde{f}_1(z_1,\ldots,z_5) = \begin{cases} k_1^* \, \theta^{-1} \left(\sum_1^5 z_j^2\right)^{-2}, & 0 < \sum_1^5 z_j^2 < \theta^2 \\ 0, & \text{otherwise} \end{cases}$$

Then, the radial distribution, $J_1(\cdot)$ is $\mathcal{U}(0,\theta)$, i.e., $J_1'(y) = \theta^{-1}$, $0 < y < \theta$. [See Table II.1 of Appendix II.]

If the common density for the $\underset{\sim}{Z}$'s is of the form

$$\tilde{f}_2(z_1, \ldots, z_5) = k_2^* \left[\sum_1^5 z_j^2\right]^{-2} \exp\left\{-\lambda\sqrt{\sum_1^5 z_j^2}\right\},$$

then the radial distribution satisfies

$$J_2(y) = 1 - \exp\{-\lambda y\}, \quad y > 0.$$

[See Table II.1.].

Statistical detection procedures for these types of pure-noise data involve the observed radii, $R_1, \ldots, R_N$ which function as sufficient statistics; and the "angles" $\underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_N$ which are independent of the radii. [The $\underset{\sim}{V}$'s are not direction-angle vectors as in polar coordinates. The latter are discussed in Appendix II.]

For detection problems involving specific radial distributions one uses the $\{R_j\}$, while for detection problems involving the underlying spherical structure, the $\{\underset{\sim}{V}_j\}$ are employed.

The signal detection models to be treated in the sequel are given in the next section.

3. Some Signal Detection Problems

In each of the cases below, the PN (i.e., pure-noise) distribution or law, L, is described. Data reflecting a different stochastic process law would indicate that a signal is being received (in addition to the pure noise).

3.1. Detection Problems with Historical Data

For these problems, one assumes that the total data available is $\underset{\sim}{Z} = \{Z_r : 1 \leq r \leq k\}$. (In terms of the preceding section, one has $N = 1$).

The problems are as follows.

$$PN_1: \quad \mathcal{L}(\{\underset{\sim}{Z}\}) \in \Omega(\text{K-S-E})$$

This means the background pure noise is itself S-E.

$$PN_2: \quad \mathcal{L}(\{\underset{\sim}{Z} - a\}) \in \Omega(\text{K-S-E}) \quad \text{for some (unknown)} \quad a.$$

For this case, one hypothesizes that the pure noise is S-E about some unknown point a, possibly not zero. If "a" is known, subtraction leads to PN_1 above.

Certain other detection problems can only be handled when the data is cross-sectional, i.e., the data is of the form below.

$$\underset{\sim}{Z} = \begin{pmatrix} \underset{\sim}{Z}_1 \\ \vdots \\ \underset{\sim}{Z}_N \end{pmatrix} \quad \text{where} \quad \underset{\sim}{Z}_r = (Z_{r1}, \ldots, Z_{rk})$$

3.2. Detection Problems with Cross-Sectional Data.

$$PN_3: \quad \mathcal{L}(\underset{\sim}{Z}) = \mathcal{L}_0 \in \Omega(\text{K-S-E})$$

Here, one gives a specific pure-noise law. Any other stochastic behavior of the data indicates signal is present. $\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_n$ are i.i.d. and the associated i.i.d. radii $R_1, \ldots, R_n$ are used in the decision procedures.

$$PN_4: \quad L(\{\underset{\sim}{Z} - a\}) = L_0 \in \Omega(\text{K-S-E}) \quad \text{for some} \quad a.$$

This situation refers to a family of PN-laws indexed by "a", a nuisance parameter. The decision procedures will involve radii modified by "estimates" of "a".

The final detection model to be considered is a two-sample model. The observer observes two independent time series $\{X_r\}$ and $\{Y_r\}$, and the pure-noise situation entails their stochastic laws being equal. [See Bell (1964), Model II.]

The data here is

$$\underset{\sim}{Z} = \begin{pmatrix} \underset{\sim}{Z}_1 \\ \vdots \\ \underset{\sim}{Z}_N \end{pmatrix}$$

where

$$\underset{\sim}{Z}_r = \begin{cases} (X_{r1}, \ldots, X_{rk}), & \text{for } 1 \leq r \leq m \\ (Y_{r1}, \ldots, Y_{rk}) & \text{for } m + 1 \leq r \leq N \end{cases}$$

$$PN_5: \quad L(\{X_r\}) = L(\{Y_r\})$$

The mechanisms for handling these five detection problems (and some related problems) are given in the section below.

4. Statistical Preliminaries: Sufficient Statistics and Statistical Noise.

For developing the methodology it is easier to consider first $\Omega(S\text{-}S\text{-}E)$, since the family consists of laws involving conditional Gaussian distributions. Nuisance parameters, and M-S-S's (minimal sufficient statistics) can be handled in traditional ways. Then since the "angles" $\{\tilde{V}_j\}$ have the same behavior for $\Omega(N\text{-}S\text{-}E)$, $\Omega(S\text{-}S\text{-}E)$ and $\Omega(K\text{-}S\text{-}E)$, the decision procedures developed can be extended from $\Omega(S\text{-}S\text{-}E)$ to $\Omega(K\text{-}S\text{-}E)$.

One recalls that the above-mentioned extension is not necessary if the data vectors are initial segments of infinite S-E time series. For such cases (See Theorem 2.2) $\Omega(S\text{-}S\text{-}E) = \Omega(K\text{-}S\text{-}E)$.

To these ends, one now develops the mechanisms with $\Omega(S\text{-}S\text{-}E)$ in mind.

(4A) Basic Statistics

Let $\tilde{Z}$ be a data matrix with law L' in a family Ω' admitting a M-S-S (minimal sufficient statistic) $S(\tilde{Z})$.

Definition 4.1 Let $\delta(\tilde{Z}) = [S(\tilde{Z}), N(\tilde{Z})]$ where (1) $\delta(\cdot)$ is 1-1 a.e.; (2) $S(\tilde{Z})$ and $N(\tilde{Z})$ are independent. Then (a) $\delta(\cdot)$ is called the BDT (basic data transformation) for Ω'; and (b) $N(\tilde{Z})$ is called the M-S-N (maximal statistical noise) for Ω'.

[One should note that for any given family Ω', $S(\tilde{Z})$, $N(\tilde{Z})$ and $\delta(\cdot)$ need not be unique.].

These entities are used in constructing families of detection statistics with certain desirable properties.

(4B) Distribution-Free Statistics

Definition 4.2. (1) A statistic $T(\underset{\sim}{Z})$ is NPDF rt Ω', if there exists a cdf $Q(\cdot)$ such that $P\{T(\underset{\sim}{Z}) \leq y | L\} = Q(y)$ for all y and for all L in Ω'.

(2) A family of statistics $\{T^*(\underset{\sim}{Z},L): L \in \Omega'\}$ is PDF wrt Ω', if there exists a cdf $Q^*(\cdot)$ such that

$$P\{T^*(\underset{\sim}{Z},L) \leq y | L\} = Q^*(y) \text{ for all } y \text{ and for all}$$

L in Ω'.

A rule of thumb for the sequel is as follows.

4.A. For detection problems in which the PN-distribution involves a specific law, L_0 employ statistics of the form $T^*(\underset{\sim}{Z},L_0) = \psi^*[S(\underset{\sim}{Z}),L_0]$, This entails using the data only via the M-S-S. Such statistics are PDF wrt Ω'.

4.B. For detection problems which are concerned with the general structure of the PN-family, employ statistics of the form $T(\underset{\sim}{Z}) = \psi(N(\underset{\sim}{Z}))$. This entails using the data only through the M-S-N. Such statistics are NPDF wrt Ω'.

Example 4.1. Let $L_H \in \Omega(\text{S-S-E})$ and $\underset{\sim}{Z} = (Z_1, \ldots, Z_7)$. This means nature has chosen $W = w$, where $W \sim H(\cdot)$, and given $W = w$, $Z_1, \ldots Z_7$ are conditionally i.i.d. $N(0, \frac{1}{w})$. $R = (\sum_1^7 z_j^2)^{1/2}$ is the M-S-S for $H(\cdot)$, which determines and is determined by L (See Bell, et al, 1970).

$\underset{\sim}{V} = (\frac{Z_1}{R}, \ldots, \frac{Z_7}{R})$ is M-S-N; and

$\delta(\underset{\sim}{Z}) = (R, \underset{\sim}{V})$, where $S(\underset{\sim}{Z}) = R$ and $N(\underset{\sim}{Z}) = \underset{\sim}{V}$. Now, let

$$T_1(\underset{\sim}{Z}) = V_1 = \frac{Z_1}{R} \text{ and } T_2(\underset{\sim}{Z}, L_H) = J_L(R),$$

where $J_L(y) = P\{R \leq y | H\} = P\{R \leq y | L_H\}$, i.e., J_L is the radial cdf. Then, $T_1(\cdot)$ is NPDF wrt Ω(S-S-E) with cdf $Q(z) = \frac{1}{2} + \frac{1}{\pi}$ arc sin z, $|z| < \frac{\pi}{2}$. $T_2(\cdot,\cdot)$ is PDF wrt Ω(S-S-E).

Now for Ω(K-S-E), R is the M-S-S for the radial distribution, and $\underset{\sim}{V}$ is M-S-N. Therefore, T_1 is NPDF wrt Ω(K-S-E), and T_2 is PDF wrt Ω(K-S-E).

[Note: For analyses somewhat different from that of the sequel, one needs a polar coordinate model, angular distributions, and the relations between joint densities, radial distributions and characteristic functions. These are given in Lord (1954); Kelker (1969); Smith (1971); Bell and Smith (1970), and Ahmad (1975), and in Appendix II.].

(4C) Extraneous Statistical Noise (E-S-N)

One further statistical tool involves the use of E-S-N. The essentials of this method have been employed by several authors, e.g , Durbin (1961); and Bell and Doksum (1965).

Definition 4.3. Let $\underset{\sim}{Z} = (Z_1, \ldots, Z_k)$ be data governed by law L in Ω' with M-S-S, $S(\underset{\sim}{Z})$; M-S-N $N(\underset{\sim}{Z})$; and BDT, $\delta(\cdot)$. Let

$\underset{\sim}{Y} = (Y_1, \ldots, Y_k)$ be independent of $\underset{\sim}{Z}$ and be governed by L_1 in Ω'. Define $\underset{\sim}{Y}'$ to be $[Y_1', \ldots, Y_k'] = \delta^{-1}[S(\underset{\sim}{Y}), N(\underset{\sim}{Z})]$.

(1) $\underset{\sim}{Y}$ is called E-S-N (extaneous statistical noise); and

(2) $\underset{\sim}{Y}'$ is called R-S-N (randomized statistical noise).

Paralleling the proofs of the aforementioned authors, one can prove

Theorem 4.1. (Randomized Noise Theorem). $\underset{\sim}{Y}' \overset{d}{=} \underset{\sim}{Y}$.

This result illustrates a method of imposing a known (usually tractable). distribution on a problem in which the cdf of the data is unknown. It is particularly useful when the distribution of the M-S-N, $N(\underset{\sim}{Z})$, is relatively intractable.

Example 4.2. Let $\underset{\sim}{Z} = (Z_1, \ldots, Z_{20})$ be i.i.d. $N(0,u^*)$ where $u^* > 0$ is unknown. Let $\underset{\sim}{Y} = (Y_1, \ldots, Y_{20})$ be independent of $\underset{\sim}{Z}$ and i.i.d. $N(0,1)$. Then, the BDT is such that $\delta(\underset{\sim}{Z}) = [S(\underset{\sim}{Z}), N(\underset{\sim}{Z})]$, where $S(\underset{\sim}{Z}) = R = \sqrt{\sum_1^{12} Z_j^2}$, and $N(\underset{\sim}{Z}) = \underset{\sim}{V}$, with $V_j = \frac{Z_j}{R}$. $\underset{\sim}{V}$ is uniformly distributed over the hypersphere $S^*(1)$ [See Section 2.C]. $\delta^{-1}[S(\underset{\sim}{Y}), N(\underset{\sim}{Z})] = \frac{R^*}{R}(Z_1, \ldots, Z_{20}) = (Y_1', \ldots, Y_{20}') = \underset{\sim}{Y}'$, where $(R^*)^2 = \sum_1^{20} Y_j^2$. Then, $Y_1', \ldots, Y_{20}'$, are i.i.d. $N(0\ 1)$.

(4D) Nuisance Parameters and Kolmogorov-Smirnov Statistics

The next set of statistical tools of this section involve modified K-S (Kolmogorov-Smirnov)statistics. The original statistic of this class was (Kolmogorov (1933))

$$D(F_0; n) = \sup_z |F_n(z) - F_0(z)|, \text{ where}$$

$$F_n(z) = \frac{1}{n} \sum_1^n \varepsilon(z - Z_j), \text{ where } Z_1, \ldots, Z_n \text{ are i.i.d.}$$

$F_0(\cdot)$, continuous; and $\varepsilon(u) = 1$, if $u \geq 0$, 0, if $u < 0$.

For several of the problems of the sequel, $F_0(\cdot)$, is not known completely. It is known to be a member of a family of cdfs; and hence, there is a nuisance parameter involved. That is, $F_0(\cdot) \varepsilon \Omega'' = \{F(\cdot;\theta): \theta \varepsilon \Theta\}$.

For such situations, one uses extensions of the ideas of Lilliefors (1967, 1969), Srinivasan (1971) and Choi (1980).

<u>Definition 4.3</u>. Let $\underset{\sim}{Z}=(Z_1, \ldots, Z_n)$ be i.i.d. $F_0 \varepsilon \Omega'' = \{F(\cdot;\theta): \theta \varepsilon \Theta\}$, where Ω'' admits a M-S-S, $S(\underset{\sim}{Z})$, and a MLE, $\hat{\theta}$, of θ. Let $\hat{F}_n(y) = F(y;\hat{\theta})$ for all y; and $\tilde{F}_n(y) = E(F_n(y)|S(\underset{\sim}{Z}))$, for all y.

(1) $\hat{D}_n = \sup_y |F_n(y) - \hat{F}_n(y)|$, is the Lilliefors-type statistic for Ω''; and

(2) $\tilde{D}_n = \sup_y |F_n(y) - \tilde{F}_n(y)|$ is the Srinivasan-type statistic for Ω''.

<u>Example 4.3</u>. Let $\underset{\sim}{Z} = (Z_1, \ldots, Z_{15})$ be i.i.d. $N(0,\theta)$. Then, the M-S-S, $S(\underset{\sim}{Z}) = R$, and the M-S-N, $N(\underset{\sim}{Z}) = \underset{\sim}{V}$, where $R^2 = \sum_1^{15} Z_j^2$, $V_j = \frac{Z_j}{R}$, and $\underset{\sim}{V} = (V_1, \ldots, V_{15})$. Then, $\hat{F}_{15}(y) = \Phi(\frac{y\sqrt{15}}{R})$, since $\hat{\theta} = \frac{1}{15} R^2$.

Further $\tilde{F}_{15}(y) = E\{F_n(y)|R\} = P\{Z_1 \leq y|R\} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(\frac{y}{R})$ for $-R < y < R$. The Lilliefors-type statistic is, then,

$$\hat{D}_{15} = \sup_y |F_{15}(y) - \Phi(\frac{y\sqrt{15}}{R})| \text{ , and the Srinivasan-type}$$

statistic is

$$\tilde{D}_{15} = \sup_{-R<y<R} |F_{15}(y) - [\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(\frac{y}{R})]|$$

Tables for both of these statistics are given in Appendix I.

Example 4.4. Let $\underset{\sim}{Z} = (Z_1, \ldots, Z_9)$ be i i.d. $N(\underset{\sim}{\theta})$, where $\underset{\sim}{\theta} = (\mu, \sigma^2)$. Then $S(\underset{\sim}{Z}) = (\bar{Z}, S_Z)$, where $S_Z^2 = \frac{1}{9}\sum_1^9 (Z_i - \bar{Z})^2$; and $N(\underset{\sim}{Z}) = (\frac{Z_1 - \bar{Z}}{S_Z}, \ldots, \frac{Z_9 - \bar{Z}}{S_Z}) = (W_1, \ldots, W_9)$. Here $\hat{D}_9 = \sup_y |F_9(y) - \Phi(\frac{y - \bar{Z}}{S_Z})|$. This is the statistic studied by Lilliefors (1967).

$$\tilde{D}_9 = \sup |F_9(y) - \tilde{F}_9(y)|, \text{ where } \tilde{F}_9(\cdot),$$

is given by Srinivisan (1971), and is given explicitly in Example 6.2 of Appendix I.

What is of importance here is that each of the statistics $\hat{D}_{15}$, $\tilde{D}_{15}$ $\hat{D}_9$ and $\tilde{D}_9$ (of the preceding two examples) is a function of the data only through the relevant M-S-N. In fact, one can easily derive

(a) $\hat{D}_{15} = \sup_x |\frac{1}{15}\sum_1^{15} \varepsilon(x - V_j) - \Phi(X\sqrt{15})|$

(b) $\tilde{D}_{15} = \sup_{-1<x<1} |\frac{1}{15}\sum_1^{15} \varepsilon(x - V_j) - [\frac{1}{2} + \frac{1}{\pi}\tan^{-1}x]|$;

and

(c) $$\hat{D}_9 = \sup \left|\frac{1}{9} \sum_{1}^{9} \varepsilon\,(x - V_j) - \Phi(x)\right|$$

[The derivation for $\tilde{D}_9$ is not so straightforward.]

The final statistical tool of this section involves the use of Helmert matrices.

(4E) Helmert Matrices, M-S-S's and M-S-N.

Definition 4.4. An $(N \times N)$ square matrix $\tilde{H}_N = \{h_{ij}\}$ is called a Helmert matrix of order N if

$$h_{ij} = \begin{cases} \dfrac{1}{\sqrt{N}} & \text{for } i = N \\[2ex] \dfrac{1}{\sqrt{i(i-1)}} & \text{for } 1 \le i \le N-1,\ 1 < j \le i-1 \\[2ex] \dfrac{-(i-1)}{\sqrt{i(i-1)}} & \text{for } 1 < i \le N-1,\ j = i \\[2ex] 0 & \text{for } 1 \le i \le N-1,\ i+1 \le j \le N \end{cases}$$

It is clear that each $\tilde{H}_N$ is orthogonal, and that

Theorem 4.1. If $\underset{\sim}{Z} = (Z_1, \ldots, Z_N)$ is governed by a law L in Ω(K-S-E) and $\underset{\sim}{X} = \underset{\sim}{Z}\,\tilde{H}_N^T$, then $\underset{\sim}{X} \overset{d}{=} \underset{\sim}{Z}$.

The Helmert matrix can in some circumstances be used to construct the M-S-S and M-S-N.

Example 4.5. (Gaussian Random Sample). Let $\underset{\sim}{Z} = (Z_1, \ldots, Z_5)$ be i.i.d. $N(\mu, \sigma^2)$ and let $\underset{\sim}{Y} = (Y_1, \ldots, Y_5) = \underset{\sim}{Z}\,\underset{\sim}{H}_5^T$, where

$$\underset{\sim}{H}_5 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{-4}{\sqrt{20}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

Then, Y_1 Y_2, Y_3, Y_4 and Y_5 are independent, with $Y_j \sim N(0, \sigma^2)$ for $1 \leq j \leq 4$, and $Y_5 \sim N(\mu\sqrt{5}, \sigma^2)$. One has, then, that the M-S-S for (μ, σ^2) is $S(\underset{\sim}{Z}) = (Y_5, R^*)$, where $(R^*)^2 = \sum_1^4 y_j^2$ and the M-S-N is $N(\underset{\sim}{Z}) = (V_1, \ldots, V_4)$ where $V_j = \frac{Y_j}{R^*}$.

One can now treat the first detection problem.

5. S-E Background Noise.

PN: $L \in \Omega(\text{K-S-E})$ vs $N + S_1$ $L \notin \Omega(\text{K-S-E})$

As previously mentioned, the technique will be to develop the methodology for $\Omega(\text{S-S-E})$, and extend its validity to $\Omega(\text{K-S-E})$.

For $\Omega(\text{S-S-E})$, one historical "look" yields $\underset{\sim}{Z} = (Z_1, \ldots, Z_N)$ which are i.i d. $N(0, (w^*)^{-1})$, given $W = w^*$, where $W \sim H(\cdot)$, with $H(0) = 0$.

Then, one has

Theorem 5.1. Conditionally given $W = w^*$,

(a) the M-S-S is $S(\underset{\sim}{Z}) = R$;

(b) the M-S-N is $N(\underset{\sim}{Z}) = \underset{\sim}{V} = (\frac{Z_1}{R}, \ldots, \frac{Z_N}{R})$;

(c) the BDT is $\delta(\underset{\sim}{Z}) = (R, \underset{\sim}{V})$; and

(d) the MLE of w^* is $\frac{1}{N} R^2$

From these entities, one derives the decision rule based on the relevant statistics of *Section* 4.

Decision Rule 5.1: Decide $N + S_1$ iff

$$\hat{D}_N = \sup_y |F_N(y) - \Phi(\frac{y\sqrt{N}}{R})| > \hat{d}(\alpha, N).$$

[Critical values $\hat{d}(\alpha,N)$, are found in Table A.1 of Appendix I.]

Decision Rule 5.2. Decide $N + S_1$ iff

$$\tilde{D}_N = \sup_y |F_N(y) - [\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(\frac{y}{R})]| > \tilde{d}(\alpha, N).$$

[Values of $\tilde{d}(\alpha,N)$ are given in Table A.2 of Appendix I.]

Decision Rule 5.3. Decide $N + S_1$ iff

$$T = [\sum_1^m Z_j^2] [\sum_{m+1}^N Z_j^2]^{-1} (\frac{N - m}{m}) < f' \text{ or } > f''.$$

[The f'- and f''-values are found in a Fisher's F-table for m and (N - m) degrees of freedom].

Decision Rule 5.4. Decide $N + S_1$ if

$$D'_N = \sup_z \left| \frac{1}{N} \sum_1^N \varepsilon\,(z - Y'_j) - \Phi(z) \right| > d(\alpha,N), \quad \text{where } \underset{\sim}{Y} = (Y_1, \ldots, Y_N)$$

is E-S-N and $\underset{\sim}{Y}' = (Y_1, \ldots, Y_N)$ is R-S-N as in Section 4C. [The $d(\alpha,N)$-value are found in a standard one-sample K-S table].

[Note: If the data available is cross-sectional data, one might make the adaptations in decision rules suggested at the end of Section 6].

All of the decision rules are illustrated in Appendix I. Each example bears the same number as the decision rule which it illustrates, e.g., Example 5.4, illustrates Decision Rule 5.4.

It should be noted that each of the decision rules of this section involves the data solely via the M-S-N, $\underset{\sim}{V}$, and hence is NPDF. Also $\underset{\sim}{V}$ is the M-S-N for Ω(S-S-E) as well as for Ω(K-S-E). Hence, one has

Theorem 5 2. (1) Each of the statistics $\hat{D}_N$, $\tilde{D}_N$, T, D'_N can be written in the form $\psi[\underset{\sim}{V}]$, i.e. $\hat{D}_N = \psi_1[\underset{\sim}{V}]$; $\tilde{D}_N = \psi_2[\underset{\sim}{V}]$, $T = \psi_3[\underset{\sim}{V}]$ and $D'_n = \psi_4[\underset{\sim}{V}]$.

(2) Hence, each of these statistics is NPDF wrt Ω(K-S-E).

The next detection problem involves two nuisance parameters from the point of view of Ω(S-S E).

6. Background Noise S-E about an unknown Point, a.

PN_2: $L(\{\underset{\sim}{Z} - a\}) \in \Omega(\text{K-S-E})$ for some point a vs

$N + S_2$: $L(\{\underset{\sim}{Z} - a\}) \notin \Omega(\text{K-S-E})$ for any point a.

From the $\Omega(\text{S-S-E})$ point of view, the historical data is $\underset{\sim}{Z} = (Z_1, \ldots, Z_N)$, which are c.i.i.d. $N(a, (w^*)^{-1})$, given $W = w^*$.

Theorem 6.1. The basic statistics (conditionally) are then

(a) M-S-S: $S(\underset{\sim}{Z}) = (\overline{X}, S_X)$;

(b) M-S-N; $N(\underset{\sim}{X}) = (U_1, \ldots, U_N) = \underset{\sim}{U}$, where $U_j = \dfrac{X_j - \overline{X}}{S_X}$;

(c) BDT: $\delta(Z) = (\overline{X}, S_X, \underset{\sim}{U})$; and

(d) MLE of (a, w^*): $(\overline{X}, S_X^2)$.

Some important decision rules are then as follows:

Decision Rule 6.1. Decide $N + S_2$ iff

$$\hat{\hat{D}}_N = \sup_y |F_N(y) - \Phi(\frac{y - \overline{X}}{S_X})| > \hat{\hat{d}}(\alpha, N).$$

[The Lilliefors (1967) table yields critical values].

Decision Rule 6.2. Decide $N + S_2$ iff

$$\tilde{\tilde{D}}_N = \sup_y |F_N(y) - \tilde{F}_N(y)| > \tilde{\tilde{d}}(\alpha, n).$$

[Both $\tilde{F}_N(\cdot)$ and the critical values are given by Srinivasan (1970). See Example 6.2 of Appendix I.]

Decision Rule 6.3. Decide $N + S_2$ iff

$$D''_N = \sup_y \left| \frac{1}{N} \sum_1^N \varepsilon\, (y - Y'_j) - \Phi(y) \right| > d(\alpha,n)$$

where $Y'_j = \overline{Y} + S_Y(\frac{X_j - \overline{X}}{S_X})$, and $Y_1, \ldots, Y_N$ is E-S-N.

[The $d(\alpha,n)$ values are from the standard one-sample K-S table].

Some different procedures result from employing Helmert matrices here.

<u>Theorem 6.2</u>. Let $\underset{\sim}{Y} = (Y_1, \ldots, Y_N) = \underset{\sim}{Z}\, H_N^T$. Then conditionally, given $W = w^*$, one has

(a) $Y_1, \ldots, Y_N$ are independent;

(b) $Y_j \sim N(0, 1/w^*)$ for $1 \leq j \leq N - 1$;

(c) $Y_N \sim N(a\sqrt{N}, \frac{1}{w^*})$;

(d) (Y_N, R^*) is the M-S-S; where $R^* = [\sum_1^{N-1} Y_j^2]^{-1/2}$.

(e) $\underset{\sim}{V} = (V_1, \ldots, V_{N-1})$ is the M-S-N, with $V_j = \frac{Z_j}{R^*}$; and

(f) $(\frac{Y_N}{\sqrt{N}}, \frac{N}{(R^*)^2})$ is the MLE of (a, w^*).

The new decision rules are based on an F-ratio and the empirical cdf $F^*_{N-1}(\cdot)$, where $F^*_{N-1}(z) = \frac{1}{N-1} \sum_{j=1}^{N-1} \varepsilon\, (z - Y_j)$.

<u>Decision Rule 6.4</u>. Decide $N \rightarrow S_2$ iff

$$\hat{D}_{N-1} = \sup_z \left| F^*_{N-1}(z) - \Phi(\frac{z\sqrt{N-1}}{R^*}) \right| > \hat{d}(\alpha, N-1).$$

[See Decision Rule 5.1, for critical values].

<u>Decision Rule 6.5</u>. Decide $N \rightarrow S_2$ iff

$$\tilde{D}_{N-1} = \sup_z \left| F^*_{n-1}(z) - \left[\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{z}{R^*}\right)\right]\right| > \tilde{d}(\alpha, n-1).$$

[See Decision Rule 5.2 for critical values].

Decision Rule 6.6. Decide $N + S_2$ iff

$$D'_{N-1} = \sup_z \left| \frac{1}{N-1} \sum_1^{N-1} \varepsilon\,(z - X'_j) - \Phi(z)\right| > d(\alpha, n-1)$$

where $\underset{\sim}{X} = (X_1, \ldots, X_{N-1})$ is E-S-N and $X_j = \frac{R^{**}}{R^*} Y_j$ and $(R^{**})^2 = \sum_1^{N-1} X_j^2$. [See Decision Rule 5.4 for critical values].

Decision Rule 6.7. Decide $N + S_2$ iff

$$T' = \left[\sum_1^m Y_j^2\right] \left[\sum_{m+1}^{N-1} Y_j^2\right]^{-1} \left(\frac{N - m - 1}{m}\right) > f' \quad \text{or} \quad < f''$$

[See Decision Rule 5.3 for critical values].

One should note that in the event that cross-sectional data is available, some adjustments should be made in the decision rules of this section and the preceding section. One such adjustment is suggested by the following development, based on Bell (1975).

Let $\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N$ be i.i.d. initial k-segments of time series with common law L in Ω(K-S-E). Then one has

Theorem 6.3. (1) $P\{(\underset{\sim}{Z}_1, \underset{\sim}{Z}_2) = (Z_{11}, \ldots, Z_{1k}, Z_{21}, \ldots, Z_{2k})$ is S-E$\}$ $= 0$; and (2) $(\overline{Z}_{.1}, \ldots, \overline{Z}_{.k})$ is S-E, where $\overline{Z}_{.j} = \frac{1}{N} \sum_{r=1}^{N} Z_{rj}$.

This result suggests that the decision rules of Section 5 and 6 can be applied to the mean vectors of cross-sectional data.

Numerical examples illustrating all of these decision rules are given in Appendix I.

With so many decision rules being considered, it is natural to ask which rules are better. More specifically, one might ask how the decision rules based on the Z's compare with those based on the Y's of this section. No definitive answers are known at this time. [See Section 10].

One now turns to detection problems for which cross-sectional data is necessary and available.

7. Goodness-of-Fit Detection Problems

$$\underline{PN_3} \quad L = L_0 \in \Omega(\text{K-S-E}) \quad \text{vs} \quad \underline{N + S_3}: \quad L \neq L_0$$

It is convenient to write the data in matrix form

$$\underset{\sim}{Z} = \begin{pmatrix} Z_{11} & \cdots & Z_{1k} \\ \vdots & & \\ Z_{N1} & \cdots & Z_{Nk} \end{pmatrix}$$

As usual one views the problem first from the Ω(S-S-E) viewpoint. This means that $W_1, \ldots, W_N$ are i.i.d. $H(\cdot)$ with $H(0) = 0$; and $Z_{j1}, \ldots, Z_{jk}$ are c.i.i d. $N(0, [w_j^*]^{-1})$, given $W_j = w_j^*$; for $j = 1, 2, \ldots, N$.

One now defines $R_j^2 = \sum_{m=1}^{k} Z_{jm}^2$; $\underset{\sim}{V}_j = (V_{j1}, \ldots, V_{jk})$ where

$V_{jm} = \frac{Z_{jm}}{R_j}$; and $G_N(z) = \frac{1}{N} \sum_1^N \varepsilon\ (z - R_j)$.

It can be proved that

Theorem 7.1. Conditionally, given $W_j = w_j^*$ for $1 \leq j \leq N$,

(1) the M-S-S is $S(\underset{\sim}{Z}) = [R(1), \ldots, R(N)]$, the ordered radii;

(2) the M-S-N is $N(\underset{\sim}{Z}) = [R(R_1), \ldots R(R_N), \underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_N]$;

and

(3) $R_1, \ldots, R_N$ are i.i.d. $G_0(\cdot)$, where $G_0(z) = P\{R_1 \leq z | L_0\}$, i.e., $G_0(\cdot)$ is the radial cdf.

A pertinent decision rule is as follows.

Decision Rule 7.1. Decide $N + S_3$ iff

$$D_N = \sup_z |G_N(z) - G_0(z)| > d(\alpha, n).$$

[See Decision Rules 5.4 and 6.6 for critical values].

One aspect of the decision rule above worth knowing is the exact relationship between L_0, the law in Ω(S-S-E) and $H_0(\cdot)$, the mixing measure and cdf of the W's; and $\underset{\sim}{f}_0(\cdot, \ldots, \cdot)$ the joint density function of $\underset{\sim}{Z} = (Z_1, \ldots, Z_N)$. It can be proved that

Theorem 7.2. (1) $G_0(z) = \int_0^\infty N_k(wz^2)dH_0(z)$ where $N_k(\cdot)$ is the cpf of a X_k^2-distribution. (2) For fixed N, each of $G_0(\cdot)$, L_0, $H_0(\cdot)$ and $\underset{\sim}{f}_0(\cdot, \ldots \cdot)$ is determined by any other. [See, e.g., Kelker (1971), Bell, et al (1970), Bell (1975).].

This means that the detection problem $\underline{PN_3}$: $L = L_0$ is equivalent to $\underline{PN_3^*}$: $G = G_0$; and, hence, the decision rule is "completcly relevant."

A related detection problem is treated in the next section.

8. Goodness-of-Fit With a Nuisance Parameter.

PN_4 $L(\{Z_r - a\}) = L_0 \in \Omega(K\text{-}S\text{-}E)$ for some "a" (possibly non-zero)

vs

$N + S_4$: For each a $L(\{Z_r - a\}) \neq L_0$.

For the data matrix $\underset{\sim}{Z}$ as in Section 7, one forms $\underset{\sim}{Y} = \underset{\sim}{Z}\,\underset{\sim}{H}_N^T =$

$$\begin{pmatrix} Y_{11} & \cdots & Y_{1k} \\ \cdot & & \\ \cdot & & \\ Y_{N1} & \cdots & Y_{Nk} \end{pmatrix}$$

For constructing the M-S-S and M-S-N one needs

(a) $Y_j^* = Y_{jk}$ for $1 \leq j \leq N$;

(b) $(R_j^*)^2 = \sum_{s=1}^{k-1} Y_{js}^2$;

(c) $\underset{\sim}{Y}^* = [Y^*(1), \ldots, Y^*(N)]$, the ordered Y^*'s.

(d) $\underset{\sim}{R}^* = [R^*(1), \ldots, R^*(N)]$, the ordered R^*'s;

(e) $\underset{\sim}{V}_j = (V_{j1}, \ldots, V_{k,k-1})$, where $V_{js} = \dfrac{Y_{js}}{R_j^*}$.

One has then,

Theorem 8.1. Conditionally, given $W_j = w_j^*$ for $1 \leq j \leq N$,

(a) $S(\tilde{Z}) = S(\tilde{Y}) = [\underset{\sim}{Y}^*, \underset{\sim}{R}^*]$; and

(b) $N(\tilde{Z}) = N(\tilde{Y}) = [R(Y_1^*) \ldots, R(Y_N^*); R(R_1^*), \ldots, R(R_N^*); \underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_N]$

For the decision procedure, one needs the cdf of R_1^*. Call it $G_0^*(\cdot)$ where $G_0^*(z) = P\{R_1^* \leq z\}$.

Theorem 8 2. (a) $G_0^*(\cdot)$ determines and is determ :d by L_0

(b) $R_1^*, \ldots, R_N^*$ are i.i.d. G_0^*

The decision rule is then

Decision Rule 8.1. Decide $N + S_4$ iff

$$D_{N-1} = \sup_z \left| \frac{1}{N-1} \sum_{j=1}^{N-1} \varepsilon\, (z - R_j^*) - G_0^*(z) \right| > d(\alpha, N - 1).$$

This corresponds to a goodness-of-fit test for the modified radii $\{R_j^*\}$. One could have used any one of a number of other goodness-of-fit statistics, e.g., Cramer-von Mises. However, in this work, one uses consistently the Kolmogorov-Smirnov statistics.

The final detection problem to be considered here is a two-sample problem.

9. Signal Detection: A Two-sample Problem.

$$\underline{PN_5}: \; L_1 = L_2 \in \Omega(\text{K-S-E}) \qquad \text{vs} \qquad \underline{N + S_5}: \; L_1 \neq L_2$$

The data matrix here is

$$\underset{\sim}{Z} = \begin{pmatrix} X_{11} & \cdots & X_{1k} \\ \vdots & & \\ X_{m1} & \cdots & X_{mk} \\ Y_{11} & \cdots & Y_{1k} \\ \vdots & & \\ Y_{m1} & \cdots & Y_{nk} \end{pmatrix}$$

where $N = m + n.$

The relevant statistics are

$$R_j^2 = \begin{cases} \sum_{s=1}^{k} X_{js}^2 & \text{for } 1 \le j \le m; \text{ and} \\ \sum_{s=1}^{k} Y_{js}^2 & \text{for } m+1 \le j \le N. \end{cases}$$

One now has

<u>Decision Rule 9.1</u>. Decide $N + S_5$ iff $\sum_{1}^{m} R(R_j) \geq a_1$ or $\leq a_2$.

[The table here is that for the Mann Whitney-Wilcoxon Rank-Sum Statistic].

<u>Decision Rule 9.2</u>. Decide $N + S_5$ iff

$$D(m,n) = \sup_z \left| \frac{1}{m} \sum_{1}^{m} \varepsilon(z - R_j) - \frac{1}{n} \sum_{m+1}^{N} \varepsilon(z - R_j) \right| > d''(\alpha,m,n)$$

[The table here is that of the two sample K-S statistics].

Again a large number of nonparametric (NP) statistics are available for this detection problem. The two chosen above are representative and have some optimal properties

The preceding developments lead to some interesting observations and speculations.

10. Concluding Remarks and Open Problems.

(A) Parametric and NP Statistics

Signal detection problems related to SE time series utilize parametric as well as NP techniques. The sphericity property allows one to employ the F-statistic and other classical statistics; while the families of radial distributions are NP in character. Hence, one could have used a variety of goodness-of-fit and NP statistics. The relative utility of these other statistics is at this time an open question. [See (B) below].

(B) FDR and Power. The procedures given in the text are reasonable and commonly used. However very little precise information is available on FDR and power. A series of definitive studies, evaluating FDR for reasonable (N + S)-distributions, is needed. Some preliminary Monte Carlo results have been relatively costly and less than definitive. As of this writing, it is difficult to say which of the available procedures is better than which others. In particular, one asks how do the procedures based on the Z's compare with those based on the Y's in Section 6.

(C) Polynomial Drift. Techniques were presented above for time series S-E about 0 and about non-zero points "a". Of some interest would be background noise which is SE about some polynomial curve. This would be related to polynomial regression with SE (rather than Gaussian) errors and should be also related to some recent robustness studies.

(D) Guassian Markov Processes. It is developed in Appendix II, that S-E time series can be considered "imbedded" in certain mixtures of some Gaussian Markov processes. It is an open problem as to how the very rich literature on Gaussian Markov processes can be brought to bear on signal detection problems.

(E) Polar Coordinates. Some explicit detail about the polar coordinate aspects of S-E times series is given in Appendix II. The current interest in geometric probability may give some new insights into developing signal detection procedures different from the ones presented in this paper.

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APPENDIX I

1. Tables
2. Data Sets
3. Numerical Examples

(The tables were computed by S. M. Lee. He, A. Mason, and G. Muse did the computations for the numerical examples.)

TABLES

(See Section 5).

TABLE A.1. Critical Points for $\hat{D}_n$ (See Decision Rule 5.1).
[$\hat{D}_n$ is a modified Lilliefors-type K-S statistics].

Sample Size	.01	.05	.10	.15	.20	.25	30
2	0.8369	0.8202	0.7968	0.7697	0.7393	0.7062	0.6700
3	0.7975	0 7217	0.6566	0.5992	0.5507	0 5294	0.5107
4	0.7335	0.6210	0.5650	0.5289	0.4985	0.4701	0.4426
5	0.6603	0.5665	0.5074	0.4695	0.4403	0.4182	0.3969
6	0.6071	0.5140	0.4639	0.4289	0.4002	0.3790	0.3615
7	0.5663	0.4766	0.4288	0.39[illegible]	0.3752	0.3534	0.3355
8	0.5337	0.4497	0.4048	0 3739	0.3511	0.3322	0.3159
9	0.5048	0.4250	0.3823	0 3530	0.3318	0.3137	0.2981
10	0.4772	0.4005	0.3609	0.3351	0.3145	0.2978	0.2825
11	0.4659	0.3866	0.3458	0.3201	0.2999	0.2836	0.2695
12	0.4438	0.3709	0.3318	0.3069	0.2886	0.2725	0.2586
13	0.4278	0.3577	0 3198	0.2964	0.2774	0.2622	0.2491
14	0.4095	0.3411	0.3067	0.2838	0.2656	0.2513	0.2387
15	0.3973	0.3310	0.2974	0 2753	0.2589	0.2450	0.2328
16	0.3872	0.3215	0.2875	0.2674	0.2513	0.2374	0.2261
17	0.3769	0.3130	0.2819	0 2599	0.2439	0.2301	0 2186
18	0.3690	0.3024	0.2712	0.2511	0.2356	0.2226	0.2115
19	0.3567	0.2966	0.2655	0.2455	0.2311	0.2183	0.2078
20	0 3479	0.2896	0.2598	0.2408	0.2260	0.2137	0.2029
21	0.3369	0.2820	0.2521	0 2333	0.2193	0.2075	0.1973
22	0.3344	0.2781	0.2473	0.2284	0.2141	0.2023	0.1927
23	0.3189	0.2704	0.2415	0.2233	0 2097	0.1986	0.1888
24	0 3172	0.2627	0.2361	0.2183	0.2048	0.1938	0.1847
25	0.3082	0 2558	0.2306	0.2139	0.2015	0.1906	0.1809
26	0.3054	0.2551	0.2266	0 2095	0.1968	0.1863	0.1774
27	0.3016	0.2500	0.2244	0.2076	0.1943	0.1838	0.1750
28	0.2903	0.2444	0.2189	0.2031	0.1906	0.1803	0.1718
29	0.2897	0.2416	0.2169	0.2001	0 1879	0.1782	0.1689
30	0.2843	0.2356	0.2121	0.1975	0.[illegible]853	0.1748	0.1662

[Table computed by S. M. Lee, 20,000 repititions].

TABLE A.2. Critical Values of $\tilde{D}_n$ (See Decision Rule 5.2).

[$\tilde{D}_n$ is a modified Srinivasan-type K-S statistic].

Sample Size	.01	.05	.10	.15	.20	.25	.30	.35	.40
2	0.7451	0.7176	0.6907	0.6632	0.6380	0.6145	0.5917	0.5734	0.5484
3	0.6633	0.6104	0.5747	0.5435	0.5229	0.5019	0.4774	0.4591	0.4441
4	0.6025	0.5500	0.5203	0.4955	0.4767	0.4608	0.4398	0.4252	0.4117
5	0.5497	0.5123	0.4791	0.4590	0.4370	0.4238	0.4131	0.3996	0.3890
6	0.5326	0.4819	0.4529	0.4360	0.4239	0.4127	0.4020	0.3932	0.3816
7	0.5049	0.4707	0.4431	0.4271	0.4142	0.4050	0.3959	0.3858	0.3777
8	0.5011	0.4563	0.4344	0.4202	0.4100	0.4010	0.3917	0.3857	0.3804
9	0.4844	0.4478	0.4275	0.4177	0.4089	0.3985	0.3920	0.3862	0.3796
10	0.4725	0.4351	0.4166	0.4066	0.3996	0.3946	0.3889	0.3829	0.3791
11	0.4642	0.4372	0.4179	0.4085	0.4015	0.3944	0.3893	0.3842	0.3798
12	0.4626	0.4358	0.4194	0.4099	0.4033	0.3971	0.3919	0.3878	0.3837
13	0.4576	0.4360	0.4176	0.4091	0.4013	0.3950	0.3910	0.3870	0.3819
14	0.4491	0.4299	0.4153	0.4069	0.4003	0.3955	0.3912	0.3869	0.3843
15	0.4499	0.4284	0.4148	0.4069	0.4017	0.3969	0.3926	0.3881	0.3846
16	0.4472	0.4254	0.4133	0.4065	0.3995	0.3951	0.3912	0.3883	0.3855
17	0.4450	0.4247	0.4144	0.4088	0.4027	0.3976	0.3938	0.3909	0.3880
18	0.4390	0.4233	0.4107	0.4056	0.4012	0.3973	0.3934	0.3904	0.3878
19	0.4400	0.4216	0.4134	0.4074	0.4031	0.3982	0.3940	0.3907	0.3879
20	0.4427	0.4251	0.4148	0.4089	0.4088	0.4005	0.3975	0.3941	0.3912
21	0.4375	0.4220	0.4133	0.4081	0.4034	0.4001	0.3969	0.3943	0.3915
22	0.4377	0.4222	0.4140	0.4080	0.4044	0.4015	0.3977	0.3948	0.3923
23	0.4372	0.4237	0.4157	0.4103	0.4050	0.4024	0.3995	0.3972	0.3943
24	0.4348	0.4218	0.4138	0.4094	0.4058	0.4024	0.3998	0.3974	0.3951
25	0.4346	0.4220	0.4144	0.4099	0.4061	0.4031	0.4004	0.3980	0.3969
26	0.4359	0.4227	0.4152	0.4106	0.4074	0.4040	0.4012	0.3990	0.3962
27	0.4340	0.4249	0.4176	0.4120	0.4079	0.4050	0.4020	0.3998	0.3978
28	0.4348	0.4233	0.4151	0.4101	0.4075	0.4050	0.4026	0.4001	0.3978
29	0.4376	0.4239	0.4180	0.4128	0.4093	0.4066	0.4041	0.4015	0.3992
30	0.4345	0.4216	0.4160	0.4117	0.4087	0.4066	0.4040	0.4024	0.4007

Data Sets

(The data sets herein were generated as described in Section 1, and in Section 2.C. The actual laws of the underlying stochastic processes are in Ω(S-S-E).

Data Set A.

U(0,1)	F*	Distn	$X_{.1}$	$X_{.2}$	$X_{.3}$	$X_{.4}$	$X_{.5}$	$X_{.6}$	$X_{.7}$	$X_{.8}$	$X_{.9}$	$X_{.10}$
.71458	F_1	N(0,2)	-.466104	-1.933828	-.361703	.607492	.623703	2.219808	.968079	-2.095116	.199337	1.022671
.54211	F_1	N(0,2)	1.593956	1.763648	.029017	.092427	1.264293	.965183	-.525978	-2.974892	.061400	-3.402435
.75850	F_1	N(0,2)	-.071471	-1.246935	1.021264	-1.068502	-.197276	-1.467731	-.644328	-2.453513	1.505749	-1.955955
.03848	F_2	N(0,5)	3.085406	1.339974	2.898296	1.606342	.191939	1.833020	-1.146486	-.368396	.340160	1.750718
.51584	F_1	N(0,2)	.770832	2.270729	-.845985	1.639721	1.478145	-.986289	-.318554	-1.929726	-.167733	-1.029556
.43474	F_2	N(0,5)	-2.109770	.748547	1.543129	.918882	.326172	-1.332785	-2.063423	-3.803348	-1.231685	1.060268
		(ave)	.467142	.490356	.714003	.632727	.614496	.205193	-.621782	-2.270832	.117871	-.425715

(Six independent random samples of size 10 were generated. Each sample is from either a $F_1 = N(0,2)$ or $F_2 = N(0,5)$ populations. One first generates a random number U^* between 0 and 1; if U^* were greater than 0.5, $F^* = F_1$; if less than 0.5, $F^* = F_2$. Thus, one has a 50-50 mixture of $N(0,2)$ and $N(0,5)$ and $F = \frac{1}{2} F_1 + \frac{1}{2} F_2$.)

Data Set B.

(Ten independent random samples of size 6 from an Exp (0.5) population)

x_{1j}	x_{2j}	x_{3j}	x_{4j}	x_{5j}	x_{6j}	$\bar{x}$	$s_{\bar{x}}$
1.0784	1.5952	9.2919	1.7778	1.1270	0.2590	2.5215	3.3584
0.5370	0.9007	0.1444	1.5916	1.6744	1.5387	1.0644	0.6366
0.5882	1.6478	2.9246	4.0697	0.3525	2.1723	1.9591	1.4134
1.3147	2.9117	4.0288	1.0712	0.2252	1.6121	1.8606	1.3757
4.3221	2.0311	1.2146	0.1451	5.2428	1.8572	2.4688	1.9322
0.5331	7.4930	2.1394	9.8953	1.4496	0.4987	3.6681	4.0134
5.5134	1.8511	2.5914	3.7042	2.4613	0.8480	2.8282	1.6151
0.5144	2.6477	7.3539	1.5439	1.2565	1.5495	2.4776	2.4853
1.1833	0.3628	1.3510	0.2951	7.6608	5.6672	2.7473	3.1286
0.5530	0.2034	1.2320	0.7505	0.7187	5.2019	1.4432	1.8711

<u>Data Set C.</u>

(Ten independent random samples of size 6 from Exp (0.5) population).

Exp(5)	$X_{.1}$	$X_{.2}$	$X_{.3}$	$X_{.4}$	$X_{.5}$	$X_{.6}$	$X_{.7}$	$X_{.8}$	$X_{.9}$	$X_{.10}$
1	2.6403	1.5143	3.7287	2.9727	2.6965	.4378	5.0893	2.4227	1.6379	7.4846
2	.1862	7.3778	2.2325	.5703	.0527	1.5631	3.1559	7.1087	2.1588	4.0079
3	6.4833	1.0100	.1603	1.3867	2.1980	5.6003	.9843	6.4833	1.0933	.0149
4	2.7927	3.0798	3.3363	3.1414	2.4053	.0638	1.1351	1.6570	.1439	2.6980
5	1.2562	1.0740	3.1261	.8783	1.1263	2.0207	4.4420	.2774	1.2265	.1961
6	.7690	6.3259	8.338	3.6012	.4722	1.6717	2.0617	.6824	.0630	1.9675
(ave)	2.3555	3.3970	3.4870	2.0918	1.4918	1.8929	2.8114	3.1053	1.0539	2.7282
$(ave)^2$	5.5481	11.5394	12.1591	4.3755	2.2256	3.5831	7.9039	9.6426	1.1107	7.4429

[Note: The data here represents the same stochastic structure as in Data Set B.]

Data Set D.

Exp(.1)	Distn	$X_{.1}$	$X_{.2}$	$X_{.3}$	$X_{.4}$	$X_{.5}$	$X_{.6}$	$X_{.7}$	$X_{.8}$	$X_{.9}$	$X_{.10}$
23.9032	N(7,.0418)	6.9709	6.9879	7.0979	6.9961	7.0322	6.9174	6.9523	7.0079	7.0597	7.0263
4.6649	N(7,.2144)	6.9661	6.8182	7.2232	7.2502	7.0560	7.1289	7.0081	6.9929	7.2601	7.5083
3.8199	N(7,.2618)	6.8105	6.8879	6.2968	7.0456	6.7623	6.8660	6.8673	7.0181	6.1672	7.3398
3.3017	N(7,.3029)	6.8522	7.1212	7.1469	6.8540	6.9203	7.0984	6.7862	7.2329	7.4501	6.7519
.3968	N(7,2.2504)	3.4160	5.8129	9.1801	8.1291	6.2565	3.7941	11.2292	5.2786	1.9113	9.5053
21.3622	N(7,.0468)	6.9583	7.0599	7.0694	6.9825	6.9997	7.1097	7.0303	7.0102	7.0365	7.0890
	ave →	6.3290	6.7813	7.3357	7.2096	6.8378	6.4858	7.6456	6.7568	6.1475	7.5368
	$(\text{ave})^2$ →	40.0562	45.9860	53.8125	51.9783	48.7555	42.0656	58.4552	45.6543	37.7918	56.8034

(Six independent random samples of size 10 were generated from $N(7, 1/w)$ populations, where the w's were i.i.d. Exp (0,1).)

Numerical Examples for Section 5.

PN_1: $L \in \Omega(K\text{-}S\text{-}E)$ vs. $N + S_1$: $L \notin \Omega(K\text{-}S\text{-}E)$

5.1 Modified Lilliefors

5.2 Modified Srinivasan

5.3. F-test

5.4 E-S-N

The examples here are numbered corresponding to the decision rules of the text.

Example 5.1.

PN_1: $L \in \Omega$(K-S-E) vs. $N + S_1$: $L \in \Omega$(K-S-E)

Data:

1	2.38000E-2
2	9.20000E-2
3	-1.73000E-2
4	-8.65000E-2
5	1.07300E-1
6	1.73000E-2
7	-2.77500E-1
8	5.77000E-2
9	-1.36800E-1
10	2.03100E-1
11	-1.61000E-2
12	5.77000E-2
13	5.54000E-2
14	7.24000E-2
15	3.72000E-2
16	-1.01300E-1
17	-6.78000E-2
18	1.25300E-1
19	1.06800E-1

Decide $N + S_1$ iff

$$\hat{D}_N = \sup \left| F_N(y) - \Phi\left(\frac{y\sqrt{N}}{R}\right) \right| > \hat{d}(\alpha, N)$$

From Table A.1, one finds $d(.05,19) = 0.2966$. The obtained statistic value is $\hat{D}_{19} = 0.195$. Therefore, one decides PN.

Example 5.2.

$$PN_1 : \; L \in \Omega(\text{K-S-E}) \qquad \text{vs.} \qquad N + S_1 : \; L \notin \Omega(\text{K-S-E})$$

Data:

1	1.76000E2
2	1.88000E2
3	7.11000E2
4	7.32000E2
5	6.10000E2
6	1.00800E3
7	1.02400E3
8	1.19500E3
9	7.36000E2
10	1.19600E3
11	1.18100E3
12	1.04400E3
13	1.24500E3
14	1.50600E3
15	1.57600E3
16	1.34600E3
17	1.10300E3
18	1.29600E3
19	9.81000E2
20	1.44000E3

Decide $N + S_1$ iff

$$\tilde{D}_N = \sup_y \left| F_N(y) - \left[\frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{y}{R}\right)\right] \right| > \tilde{d}(\alpha, N)$$

From Table A.2, one finds $\tilde{d}(.05,20) = 0.4251$. The obtained statistic value is $\tilde{D}_{20} = 0.512$. Therefore, decide $N + S$.

Example 5.3.

PN_1: $L \in \Omega(K\text{-}S\text{-}E)$ vs. $N + S_1$: $L \notin \Omega(K\text{-}S\text{-}E)$

Data:

$Z = [6.9707, 6.9879, 7.0979, 6.9961$ $7.0322, 6.9174, 6.9523,$
$7.0079, 7.0597, 7.0263]$

F-statistic

$$T = \frac{\sum_{1}^{5} z_j^2}{\sum_{6}^{10} z_j^2} = \frac{246.202}{244.496} = 1.007$$

Decide $N + S_1$ iff $T > F_{(5,5,.975)} = 7.1464$

or $T < F_{(5,5,.025)} = .1399$

Therefore,

decide PN.

Example 5.4.

PN_1: $L \in \Omega(K\text{-}S\text{-}E)$ vs. $N + S_1$: $L \notin \Omega(K\text{-}S\text{-}E)$

Data:

$Z = [6.971, 6.988, 7.098, 6.996, 7.032, 6.917, 6.952,$
$7.008, 7.060, 7.026]$

E-S-N:

Generate $\underset{\sim}{Y} = (Y_1, \ldots, Y_{10})$ i.i.d. Φ independent of $\underset{\sim}{Z}$.

$$\underset{\sim}{Y} = [-0.511, -0.884, 0.307, -1.169, 1.146, -1.107, 0.140, -0.168, 1.698, 0.143]$$

$$R_{\underset{\sim}{Z}} = \left(\sum_1^{10} Z_j^2\right)^{1/2} = 22.152$$

$$R_{\underset{\sim}{Y}} = \left(\sum_1^{10} Y_j^2\right)^{1/2} = 2.827$$

R-S-N: $\underset{\sim}{V} = \frac{R_Y}{R_Z} \underset{\sim}{Z} = (.128)\underset{\sim}{Z}$

$$= [0.890, 0.894, 0.909, 0.895, 0.900, 0.885, 0.890, 0.897, 0.904, 0.899]$$

Test statistic:

$$D_N' = \sup_z \left|\frac{1}{10} \sum_1^{10} \varepsilon\,(z - V_j) - \Phi(z)\right|$$

$$= \max_k \left\{\max\left[\Phi(z) - \frac{k-1}{10}, \frac{k}{10} - \Phi(z)\right]\right\}$$

k	V(k)	$\Phi(V(k))$	$\Phi(V(k)) - \frac{k-1}{10}$	$\frac{k}{10} - \Phi(V(k))$
1	0.885	.8120	.8120	-
2	0.890	.8133	.7133	-
3	0.890	.8133	.6133	-
4	0.894	.8143	.5143	-
5	0.895	.8146	.4146	-
6	0.897	.8151	.3151	-
7	0.899	.8156	.2156	-
8	0.900	.8159	.1159	-
9	0.904	.8170	.0170	.0830
10	0.909	.8183	-	.1817

$D'_{10} = .8120 > .410 \quad (\alpha = .05)$

Decide N + S

<u>Numerical Examples for Section 6.</u>

$$PN_2: \quad L(\{\underset{\sim}{Z} - a\}) \in \Omega(K\text{-}S\text{-}E) \quad \text{vs.} \quad N + S_2: \quad L(\{\underset{\sim}{Z} - a\}) \notin \Omega(K\text{-}S\text{-}E)$$

6.1 Lilliefors

6.2 Srinivasan

6.3 E-S-N

6.4 Helmert → modified Lilliefors

6.5 Helmert → modified Srinivasan

6.6 Helmert → E-S-N

6.7 Helmert → F-Test

Example 6.1.

PN_2: $L(\{\underset{\sim}{Z} - a\}) \varepsilon\ \Omega(K\text{-}S\text{-}E)$ for some a

vs. $N + S_2$: $L(\{\underset{\sim}{Z} - a\}) \notin \Omega(K\text{-}S\text{-}E)$ for any a

Data Set D was used:

1st case: historical data ("first look")

2nd case: cross-sectional data (means)

Lilliefors (1967)

1. Compute $\overline{X} = \frac{\sum_1^N X_j}{N}$ and $S_X = \sqrt{\frac{\sum_1^N (X_j - \overline{X})^2}{N - 1}}$

2. Determine order statistics $X(1), X(2), \ldots, X(n)$.

3. Compute Lilliefors statistic

$$\hat{\hat{D}}_N = \sup_z \left|F_N(z) - \Phi\left(\frac{z - \overline{X}}{S_X}\right)\right|$$

$$= \max_k \left\{\max \left[\Phi\left(\frac{z - \overline{X}}{S_X}\right) - \frac{k - 1}{N},\quad \frac{k}{N} - \Phi\left(\frac{z - \overline{X}}{S_X}\right)\right]\right\}$$

4. Decide $N + S$ iff $\hat{\hat{D}}_N > \hat{\hat{d}}(\alpha,N)$ where $\hat{\hat{d}}(\alpha,N)$ is obtained from Lilliefors' table.

Case 1: Lilliefors method (historical data)

Data Set D (1st "look")

$$X_1 = 6.3290 \qquad S^2_{X_1} = 1.7008 \qquad S_{X_1} = 1.3041$$

k	Z(k)	$W=(\frac{Z-\overline{X}}{S_X})$	$\Phi(w)$	$\Phi(w)-\frac{k-1}{6}$	$\frac{k}{6}-\Phi(w)$
1	3.4160	-2.2337	.0127	.0127	.1540
2	6.8105	.3692	.6440	.4773	-
3	6.8522	.4012	.6600	.3267	-
4	6.9583	.4826	.6855	.1855	-
5	6.9661	.4885	.6870	.0203	.1463
6	6.9707	.4921	.6895	-	.3105

$\hat{\hat{D}}_6 = .4773 > .319 \quad (\alpha = .05)$ Decide N + S.

Case 2: Lilliefors method (cross-sectional data)

Data Set D

$$\mu_{\overline{X}} = 6.9006 \qquad S^2_{\overline{X}} = .2349 \qquad S_{\overline{X}} = .4846$$

$$\hat{\hat{D}}_N = \sup_z \left|F_N(z) - \Phi\left(\frac{Z - \overline{X}}{S_X}\right)\right|$$

$$= \max_k \left\{\max \left[\Phi\left(\frac{Z - \overline{X}}{S_X}\right) - \frac{k - 1}{10},\ \frac{k}{10} - \Phi\left(\frac{Z - \overline{X}}{S_X}\right)\right]\right\}$$

k	Z(k)	$W = \frac{Z-\overline{X}}{S_X}$	$\Phi(w)$	$\Phi(w) - \frac{k-1}{10}$	$\frac{k}{10} - \Phi(w)$
1	6.1475	-1.5664	.0585	.0585	.0415
2	6.3290	-1.1919	.1165	.0165	.0835
3	6.4858	- .8684	.3070	.1070	-
4	6.7568	- .3088	.3790	.0790	.0210
5	6.7813	- .2586	.3980	-	.1020
6	6.8378	- .1415	.4435	-	.1565
7	7.2096	- .6252	.7340	.1340	-
8	7.3357		.8120	.1120	-
9	7.5368	1.3005	.9035	.1035	-
10	7.6456	1.5249	.9365	.0365	.0635

$\hat{\hat{D}}_{10} = .1565 < .258 \ (\alpha = .05)$ Decide PN

Example 6.2.

Srinivasan (1970)

The Srinivasan statistic is

$$\tilde{\tilde{D}}_N = \sup_z |F_N(z) - \tilde{F}_0(z)|$$

where

$F_N(z)$ is the empirical distribution function

$\tilde{F}_0(z) = E[F_N(z)|M\text{-}S\text{-}S]$; and the

M-S-S is $S(\underset{\sim}{X}) = (\overline{X}, S_X^2)$. Hence

$$\tilde{F}_0(u;\mu,\sigma^2) = \int_{-\infty}^{u} f(x_1|\overline{X}, S_X^2)dx,$$

where

$$f(X_1 \mid \overline{X}, S_X^2) = \left(\frac{n}{n-1}\right)^{1/2} \frac{\Gamma(\frac{1}{2}[n-1])}{\pi\Gamma(\frac{1}{2}[n-2])} - \frac{1}{2}\left[1 - \frac{\overline{X} - X_1}{S_X} \cdot \frac{n}{n-1}\right]^{n-4/2}$$

$$\tilde{F}_0(u;\mu,\sigma^2) = 1 - \int_0^{V(u)} g(y)dy\,;$$

$$g(y) = \frac{\Gamma(n-2)}{\{\Gamma(\frac{1}{2}[n-2])\}^2}\, y^{n/2-2} \quad (1-y)^{n/2-2};$$

and

$$V(u) = \frac{1}{2}\left\{1 + \frac{\overline{X} - u}{S_X}\left(\frac{n}{n-1}\right)^{1/2}\right\}.$$

<u>Computation of Srinivasan statistic for $n = 10$</u>

Given a set of historical data: $X_1, X_2, \ldots, X_{10}$ one wishes to test

$$PN_2: \quad L(\{\underset{\sim}{X} - a\}) \in \Omega(K\text{-}S\text{-}E)$$

(i.e., S-E about some unknown a).

1. Compute order statistics $X(1), X(2), \ldots, X(10)$
2. Compute $\overline{X}$ and S_X.
3. Evaluate $V(u) = \frac{1}{2}\left\{1 + \left(\frac{n}{n-1}\right)^{1/2}\left(\frac{\overline{X} - u}{S_X}\right)\right\}$

   with $u = X(j)$ and $n = 10$ for $j = 1, \ldots, 10$,

   i.e., $V[X(j)] = \frac{1}{2}\left\{1 + \frac{\sqrt{10}}{3}\left[\frac{\overline{X} - X(j)}{S_X}\right]\right\}$ for $j = 1, \ldots, 10$
4. Evaluate

$$\tilde{F}_0[X(j);\mu,\sigma^2] = 1 - \{V[X(j)]\}^4 \; [35 - 84\, V[X(j)] + 70\, \{V[X(j)]\}^2 - 20\{V[X(j)]\}^3]$$

5. $\tilde{\tilde{D}}_n = \max\limits_{1 \le j \le 10} \left| \frac{j}{10} - \tilde{F}_0[X(j);\mu,\sigma^2] \right|$

j	X_j	$X(j)$	$X_j - \bar{X}$	$(X_j - \bar{X})^2$	$\frac{X - \bar{X}(j)}{S_X}$	$V[X(j)]$
1	-.4979	-.4979	-.2504	.0627	1.2743	1.1716
2	-.4105	-.4788	-.1630	.0266	1.1771	1.1204
3	-.0182	-.4105	.2293	.0526	.8295	.9372
4	.0388	-.3687	.2863	.0820	.6168	.8251
5	-.2179	-.3322	.0296	.0009	.4310	.7272
6	-.2353	-.2353	.0122	.0001	-.0621	.4673
7	.0454	-.2179	.2929	.0858	-.1506	.4206
8	-.4788	-.0182	-.2313	.0535	-1.1669	-.1150
9	-.3322	.0388	-.0847	.0072	-1.4570	-.2679
10	-.3687	.0454	-.1212	.0147	-1.4906	-.2856

$\bar{X} = -.2475$ $\qquad S_X^2 = .0386$

$S_X = .1965$

j	$[V(X(j))]^2$	$[V(X(j))]^3$	$[V(X(j))]^4$	$\tilde{F}_0[X(j)]$	$\frac{j}{10} - \tilde{F}_0[X(j)]$
1	1.3726	1.6082	1.8842	.0511	.0489
2	1.2553	1.4064	1.5758	.0082	.1918
3	.8783	.8232	.7715	.0031	.2969
4	.6808	.5617	.4635	.0203	.3797
5	.5288	.3846	.2797	.0940	.4060
6	.2184	.1020	.0477	.5709	.0291
7	.1769	.0744	.0313	.6693	.0307
8	.0132	-.0015	.0002	.9909	-
9	.0718	-.0192	.0052	.6728	.2272
10	.0816	-.0233	.0067	.5634	.4366

Decide N + S iff

$\tilde{D}_{10} > \tilde{d}(.05,10)=0.24$. $\tilde{D}_{10}= .4366$. Decide N + S.

Example 6.3.

E-S-N [Durbin (1961)]. The procedure is as follows.

1. Compute $\bar{X}$ and S_X.
2. Generate, by standard methods,

   E-S-N: $Y_1, Y_2, \ldots, Y_N$ idd Φ independent of $\underset{\sim}{X}$
3. let $Y_j' = \bar{Y} + \frac{S_Y}{S_X}(X_j - \bar{X})$
4. If $X_1, X_2, \ldots, X_N \in \Omega(\text{K-S-E})$ then $(Y_1', Y_2', \ldots, Y_N') \overset{d}{=} (Y_1, Y_2, \ldots, Y_N)$

   i.e., $Y_1', Y_2', \ldots, Y_N'$ are i.i.d. Φ.
5. Determine order statistics $Y'(1), Y'(2), \ldots, Y'(N)$

6. Use Kolmogorov-Smirnov test:

$$D''_N = \sup_z |F_N(z) - F_0(z)|$$

$$= \max_k \left[\max \left\{\Phi[Y'(k)] - \frac{k-1}{n}, \frac{k}{n} - \Phi[Y'(k)]\right\}\right]$$

7. Decide $N + S$ iff $D''_N > d(\alpha,N)$ where $d(\alpha,N)$ is value from Kolmogorov-Smirnov tables.

On applying the above procedure to Data Set D, one finds:

$\mu_{\overline{X}} = 6.9066$; $S^2_{\overline{X}} = .2349$; $S_{\overline{X}} = .4846$;

M-S-N: $N(\underset{\sim}{X}) = [-1.5664, -1.1919, -.8684, -.3088, -.2586, -.1415$
$.6252, .8854, 1.3005, 1.5249]$;

E-S-N: $\underset{\sim}{Y} = [.807, -1.306, .875, -.081, -1.056, 1.241, 1.126, -.397,$
$.244, 1.207]$; and

$\overline{Y} = .266$ $S^2_Y = .8019$ $S_Y = .8955$

$Y'_1 = .266 + (.8955)(-1.5664) = -1.1367$

$Y'_2 = -.8013$ $Y'_3 = -.5117$ $Y'_4 = -.0105$

$Y'_5 = .0344$ $Y'_6 = .1393$ $Y'_7 = .8259$

$Y'_8 = 1.0589$ $Y'_9 = 1.4306$ $Y'_{10} = 1.6315$.

The computations for the K-S test are as given below:

k	$Y'(k)$	$\Phi[Y'(k)]$	$\Phi[Y'(k)] - \frac{k-1}{10}$	$\frac{k}{10} - \Phi[Y'(k)]$
1	-1.1367	.1280	.1280	-
2	- .8013	.2080	.1080	-
3	- .5117	.3040	.1040	-
4	- .0105	.4940	.1940	-
5	.0344	.5135	.1135	-
6	.1393	.5550	.0550	.0450
7	.8259	.7955	.1955	-
8	1.0589	.8550	.1550	-
9	1.4306	.9238	.1238	-
10	1.6315	.9487	.0487	.0513

$D''_{10} = .1955 < .410 \ (\alpha = .05)$ Decide PN.

Examples 6.4 - 6.7

PN_2: $\mathcal{L}(\{\underset{\sim}{Z} - a\}) \in \Omega(K\text{-}S\text{-}E)$ for some a.

vs.

$N + S_2$: $\mathcal{L}(\{\underset{\sim}{Z} - a\}) \notin \Omega(K\text{-}S\text{-}E)$ for any a.

For these four examples one multiplies the original data vector by the Helmert matrix of appropriate dimension and reverts to the methods of Examples 5.1 - 5.4.

Original data: $\underset{\sim}{Z}$

1	5.33300E-1
2	-7.51000E-2
3	6.25200E-1
4	9.73000E-2
5	-4.60300E-1
6	-1.43000E-2
7	-2.29600E-1
8	-4.33000E-2
9	-1.74400E-1
10	-3.57700E-1
11	1.70900E-1
12	-4.59800E-1
13	6.54100E-1
14	5.53200E-1
15	1.18600E-1
16	-3.62000E-1
17	4.65700E-1
18	-4.56600E-1
19	5.39000E-1

The Data Vector, after being multiplied by the Helmert Matrix size 19 is: $\underset{\sim}{Y}$

4.30204E-1
-3.23414E-1
2.28486E-1
6.75717E-1
1.44580E-1
3.21522E-1
1.04178E-1
2.15479E-1
3.66623E-1
-1.72377E-1
4.46492E-1
-6.59488E-1
-5.13337E-1
-5.80276E-2
4.11059E-1
-4.16864E-1
5.03291E-1
-4.92980E-1
2.57909E-1

Example 6.4.

Decide N + S if

$$\hat{D}_{N-1} = \sup_{z} \left| F^{*}_{N-1}(z) - \Phi\left(\frac{Z\sqrt{N-1}}{R^{*}}\right) \right| > \hat{d}(\alpha, N-1)$$

From Table A.1, one finds $\hat{d}(.05, 18) = 0.3024$. The calculated value of the statistic is

$$\hat{D}_{18} = 0.214$$

Therefore, decide PN.

Example 6.5.

Decide N + S iff

$$\tilde{D}_{N-1} = \sup_z |F^*_{N-1}(z) + [\frac{1}{2} - \frac{1}{\pi}\tan^{-1}(\frac{Z}{R^*})]| > \tilde{d}(\alpha, N - 1)$$

From Table A.2, one finds $\tilde{d}(.05, 18) = 0.4233$. The calculated value of the statistic is $\tilde{D}_{18} = 0.373$. Therefore, decide PN.

Example 6.6.

1. Let $\underset{\sim}{Y} = [Y_1, \ldots, Y_{19}] = \underset{\sim}{Z} H^T_{19}$ where $\underset{\sim}{Z} = [Z_1, \ldots, Z_{19}]$ is the original data.
2. Generate $\underset{\sim}{X} = [X_1, \ldots, X_{18}]$ i.i.d. Φ independent of the original data.
3. Compute $R^* = (\sum_1^{18} Y_j^2)^{1/2}$ and $R^{**} = (\sum_1^{18} X_j^2)^{1/2}$
4. Generate $\underset{\sim}{X}' = [X'_1, \ldots, X'_{18}]$ where $X'_j = \frac{R^{**}}{R^*} Y_j$
5. Decide N + S iff

$$D'_{18} = \sup_z | \frac{1}{18} \sum_1^{18} (Z - X'_j) - \Phi(z)| > d(.05, 18)$$

The calculations yield:

E-S-N: $\underset{\sim}{X}$ = [.089, .233, .912, -1.10, -.515, 1.58, -1.41, -.245, 2.24, -.149, 1.639, -.798, -1.888, .344, .647, .718, -.125]

$$R^{**} = (\sum_1^{18} X_j^2)^{1/2} = 4.467$$

$$R^* = (\sum_1^{18} Y_j^2)^{1/2} = 1.793$$

R-S-N: $\underset{\sim}{X}' = [1.071, -.805, .568, 1.684, .361, .802, .259, .536, .914,$
$-.428, 1.111, -1.642, -1.278, -.144, 1.024, -1.039,$
$1.253, 1.228]$

k	X'(k)	Φ[X'(k)]	k/18	$\frac{k-1}{18}$	$\frac{k}{18}$ - Φ[X'(k)]	Φ[X'(k)] - $\frac{k-1}{18}$
1	-1.642	.045	.056	0	.011	.045
2	-1.278	.101	.111	.056	.010	.045
3	-1.039	.149	.167	.111	.018	.048
4	- .805	.210	.222	.167	.012	.043
5	- .428	.334	.278	.222	-	.112
6	- .144	.443	.333	.278	-	.165
7	.259	.602	.389	.333	-	.269
8	.361	.641	.444	.389	-	.252
9	.536	.704	.500	.444	-	.260
10	.568	.715	.556	.500	-	.215
11	.802	.789	.611	.556	-	.233
12	.914	.820	.667	.611	-	.209
13	1.024	.847	.722	.667	-	.180
14	1.071	.858	.778	.722	-	.136
15	1.111	.867	.833	.778	-	.089
16	1.228	.890	.889	.833	-	.057
17	1.253	.895	.944	.889	.049	.006
18	1.684	.954	1	.944	.046	.010

Decide N + S iff $D'_{18} > d(.05, 18) = .309$. $D'_{18} = .269$.

Decide PN.

Example 6.7.

Decide N + S iff

$$T' = \frac{\sum_{1}^{m} Y_j^2}{\sum_{m+1}^{N-1} Y_j^2} \left(\frac{N - m - 1}{m}\right) > F_{(N-m-1,m,1-\alpha/2)}$$

$$\text{or} \quad < F_{(N-m-1,m,\alpha/2)}$$

Critical values are: $(N = 19, \quad m = 9, \quad \alpha = .05)$

$$F_{(9,9,.025)} = 0.25$$

$$F_{(9,9,.975)} = 4.03$$

The obtained statistic value is

$$T' = \frac{0.96}{2.11} = 0.45$$

Therefore, decide PN.

Numerical Examples for Section 7.

PN_3: $L = L_0 \in \Omega(K\text{-}S\text{-}E)$ vs. $N + S_3$: $L \neq L_0$

Example 7.1.

Data: Data Set A (cross-sectional)

1. Compute $S_j^2 = \sum_{r=1}^{10} Z_{jr}^2$ for $1 \le j \le 6$

$S_1^2 = 1.6186$ $\quad S_2^2 = 2.8889$ $\quad S_3^2 = 1.4138$

$S_4^2 = 3.0323$ $\quad S_5^2 = 1.7226$ $\quad S_6^2 = 3.1484$

2. Decide $N + S_3$ iff

$$D_6 = \sup |G_\ell(z) - G_0(z)| > d(.05,\ 6)$$

where $G_0(z) = P[S_1 \le z \mid L_0] = \int_0^\infty F_{\chi_k^2}(wz^2)\,dJ_0 = [\frac{S^2(k)}{1+S^2(k)}]^5$.

Here $w = 1/\sigma^2$ and $J_0 = \mathrm{Exp}(5)$.

Kolmogorov-Smirnov test on S_j^2

k	$S^2(k)$	$[\frac{S^2(k)}{1+S^2(k)}]^5$	$[\frac{S^2(k)}{1+S^2(k)}]^5 - \frac{k-1}{6}$	$\frac{k}{6} - [\frac{S^2(k)}{1+S^2(k)}]^5$
1	1.4138	.0689	.0689	.0978
2	1.6186	.0902	-	.2431
3	1.7226	.1014	-	.3986
4	2.8889	.2262	-	.4405
5	3.0323	.2405	-	.5928
6	3.1484	.2518	-	.7482

$$D_6 = .7482 > D_{(6,.95)} = .521$$

Decide $N + S$.

Numerical Examples for Section 8.

PN_4: $\mathcal{L}(\{Z_r - a\}) = \mathcal{L}_0 \in \Omega(K\text{-}S\text{-}E)$ for some a vs.

$N + S_4$: $\mathcal{L}\{Z_r - a\} \neq \mathcal{L}_0$ for any a

Example 8.1.

Data: First five columns of data Set A

$$\tilde{Z} = \begin{pmatrix} Z_{11}, & \cdots, & Z_{15} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ Z_{61}, & \cdots, & Z_{65} \end{pmatrix}$$

1. Form new matrix

$$\tilde{Y} = \tilde{Z} H_5^T = \begin{pmatrix} Y_{11}, & \cdots, & Y_{15} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ Y_{61}, & \cdots, & Y_{65} \end{pmatrix}$$

$$= \begin{pmatrix} 1.037848 & -.684439 & -1.323319 & -1.039538 & -.684434 \\ -.119990 & 1.347044 & .897589 & -.352879 & 2.121287 \\ .831179 & -1.372096 & .839572 & -.128918 & -.698959 \\ 1.234207 & -.559795 & .723030 & 1.825137 & 4.079463 \\ -1.060587 & 1.932456 & -.786232 & -.464495 & 2.376244 \\ -2.021135 & -1.815677 & -.743263 & -.045593 & .638156 \end{pmatrix}$$

2. Compute $S_j^2 = \sum_{r=1}^{4} Y_{jr}^2$ for $1 \leq j \leq 6$

$S_1^2 = 4.377377$ $\quad S_2^2 = 2.759115$ $\quad S_3^2 = 3.295005$

$S_4^2 = 5.690535$ $\quad S_5^2 = 5.693147$ $\quad S_6^2 = 7.936188$

Decide $N + S_4$ iff

$$D_6 = \sup_z |G_6(z) - G_0(z)| > d(.05, 6)$$

$$G_0(z) = P[S_1 \leq z| \quad L_0 = [\frac{S^2(k)}{1 + S^2(k)}]^5$$

[The mixing measure here is Exp (5).]

Kolmogorov-Smirnov Test on S_j^2

k	$S^2(k)$	$[\frac{S^2(k)}{1+S^2(k)}]^5$	$[\frac{S^2(k)}{1+S^2(k)}]^5 - \frac{k-1}{6}$	$\frac{k}{6} - [\frac{S^2(k)}{1+S^2(k)}]^5$
1	2.759115	.213020	.213020	-
2	3.295005	.265743	.099076	.067590
3	4.377377	.357452	.024119	.142548
4	5.690535	.445104	-	.221563
5	5.693147	.445256	-	.388077
6	7.936188	.552457	-	.447543

$$D_6 = .448 < d(.05, 6) = .521$$

Decide PN

Numerical Example for Section 9.

$$PN_5: \quad L_1 = L_2 \in \Omega(\text{K-S-E}) \qquad \text{vs.} \quad N + S_5: \quad L_1 \neq L_2$$

9.1 Mann-Whitney-Wilcoxon Rank Sum Test

9.2 K-S two-sample test

Data sets B, C (cross-sectional data)

9.1. Mann-Whitney-Wilcoxon Rank Sum Test

Given two sets of cross-sectional data:

$$\tilde{C}_1 = \begin{pmatrix} X_{11}, & \ldots, & X_{1,10} \\ \vdots & & \\ X_{61}, & \ldots, & X_{6,10} \end{pmatrix} \qquad \tilde{C}_2 = \begin{pmatrix} Y_{11}, & \ldots, & Y_{1,10} \\ \vdots & & \\ Y_{61}, & \ldots, & Y_{6,10} \end{pmatrix}$$

Compute radii for each "look":

$$R_j = \left(\sum_{r=1}^{10} X_{jr}^2 \right)^{1/2}, \ 1 \le j \le m; \quad R_j = \left(\sum_{r=1}^{10} Y_j^2 \right)^{1/2} m + 1 \le j \le N,$$

where $m = n = 6$ and $N = 12$.

R_1	$= 7.4067$	(2)	R_7	$= 11.3978$	(8)
R_2	$= 9.2428$	(5)	R_8	$= 11.9761$	(4)
R_3	$= 13.4661$	(12)	R_9	$= 11.1978$	(7)
R_4	$= 11.7515$	(10)	R_{10}	$= 7.4447$	(3)
R_5	$= 10.0338$	(6)	R_{11}	$= 6.3221$	(1)
R_6	$= 8.7033$	(4)	R_{12}	$= 11.6063$	(9)

The Mann-Whitney U statistic is computed as follows: $(m = n = 6)$

$$U_1 = mn + \frac{m(m+1)}{2} - \sum_{1}^{m} R(R_j)$$

$$= (6)(6) + \frac{(6)(7)}{2} - \sum_{1}^{m} (2 + 5 + 12 + 10 + 6 + 4)$$

$$= 36 + 21 - 39 = 18$$

[Note: This statistic is equivalent to the statistic in Decision Rule 9.1].

$$U_2 = mn + \frac{n(n+1)}{2} - \sum_{m+1}^{N} R(R_j)$$

$$= 57 - (8 + 11 + 7 + 3 + 1 + 9) = 57 - 39 = 18$$

Decide N + S iff $\min\{U_1, U_2\} \leq U_{6,6,.025} = 5$

since $\min\{U_1, U_2\} = 18$

decide PN

9.2. Kolmogorov-Smirnov 2-sample test

Here $m = n = 60$, *and the decision rule is:*

Decide N + S iff $D_{(m,n)} > d_{m,n,.05} = .248$.

$D_{(60,60)} = .1667$. Decide PN.

APPENDIX II

(A) Gaussian Markov Processes

(B) Polar Coordinates

(C) Radial Distributions (Table II.1).

(A) Gaussian Markov Processes and SE Time Series

Consider the following two Gaussian Markov Processes.

(A) WL (Wiener-Levy) $\{W(t): t \geq 0\}$, where $W(t) = \mu(t) + V^{-1/2}W^*(t)$ with

(i) $\{W^*(t): t \geq 0\}$ being a Gaussian Process, satisfying

(ii $W^*(0) = 0$; and

(iii) $E[W^*(t)] \equiv 0$;

(iv) $\mathrm{Cov}[W^*(s), W^*(t)] = \min(s,t)$.

(B) O-U (Ornstein-Uhlenbeck) $\{U(t): t \geq 0\}$, where $V(t) = \mu(t) + V^{-1/2}U^*(t)$, with

(i) $\{U^*(t): t \geq 0\}$ being a Gaussian Process, satisfying

(ii) $U^*(0) = 0$;

(iii) $E[U^*(t)] \equiv 0$; and $\mathrm{Cov}[U^*(s), U^*(t)] = \exp\{-\gamma|s-t|\}$ for some $\gamma > 0$.

When V is a positive constant, the processes (A) and (B) above are the Wiener-Levy and Ornstern-Uhlenbeck processes, respectively.

Now let V be a positive random variable, i.e., one mixes the processes and form the random variables

a. $W_n = W(n\Delta)$;

b. $Y_n = W_n - W_{n-1}$

c. $V_n = V(n\Delta)$, and

d. $X_n = V_n - BV_{n-1}$, where $\Delta > 0$.

One can easily verify for the (mixed) W-L processes proposition

(1) The time series $\{Y_n: n \geq 1\}$ is S-E iff $\mu(t) \equiv 0$.

(2) The time series $\{Y_1 - \mu, Y_2, Y_3, \ldots,\}$ is S-E iff $\mu(t) \equiv \mu$.

(3) The time series $\{Y_n - B\Delta: n \geq 1\}$ is S-E iff $\mu(t) = Bt + \alpha$, for some α.

For the (mixed) O-U processes it is valid that

Theorem II.1

(1) $\{X: n \geq 1\}$ is S-E iff $B = \exp\{-\gamma\Delta\}$

(2) $\{V_n: n \geq 1\}$ is approximately S-E iff $\gamma\Delta$ is sufficiently large. [When $\gamma\Delta \geq 10$, one has approximately white noise.]

(B) Polar Coordinates, Direction Angles and Radial Distributions.

For an S-E time series, it is sometimes useful to formulate problems and solutions in polar coordinates. For an initial segment of length q, one has

$$
\begin{aligned}
X_1 &= R \sin \theta_1 \\
X_2 &= R \cos \theta_1 \sin \theta_2 \\
&\vdots \\
X_{q-1} &= R \cos \theta_1 \cos \theta_2 \ldots \cos \theta_{q-2} \cos \theta_{q-1} \\
X_q &= R \cos \theta_1 \cos \theta_2 \ldots \cos \theta_{q-2} \cos \theta_{q-1}
\end{aligned}
$$

where $|\theta_j| < \frac{\pi}{2}$ for $j = 1, 2, \ldots, q - 2$; and $|\theta_{q-1}| < \pi$; and $R^2 = \sum_1^q X_j^2$.

As previously implied (Section 2.C), R and $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_{q-1})$ are independent. Further, one can prove (e.g., Smith (1969))

Theorem II.2. The direction angles $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_{q-1})$ have joint density $\tilde{f}(\underset{\sim}{\theta}) = \frac{1}{2}\,\Gamma(\frac{q}{2})\,\pi^{-q/2}\,[\cos\theta_1]^{q-2}\,[\cos\theta_2]^{q-3} \ldots [\cos\theta_{q-2}]$.

The radial distributions, joint densities and characteristic functions (See Table A.II.1) are related by

Theorem II.3. For an initial q-segment, $\underset{\sim}{X} = (X_1, \ldots, X_q)$ of an S-E time series, (i) the characteristic function is of the form $\tilde{\phi}(t) = \psi(\sum_1^q t_j^2)$ (ii) the density (if it exists) has the form $\tilde{f}(x) = c_q\, \phi(\sum_1^q X_j^2)$, and (iii) R has (radial) density $2(r) = 2C_q\, \pi^{q/2}\, [\Gamma(\frac{q}{2})]^{-1}\, r^{q-1}\, \phi(r^2)$.

Several of the more important densities, $\tilde{f}(\cdot)$; characteristic functions, $\tilde{\phi}(\cdot)$; and radial distributions $g(\cdot)$ are given in Table II.1 below.

These are given for initial segments of length "q", in general, and for $q = 1, 2,$ and 3. Further a scale parameter "a" is included in all rows except the first row. The characteristic function is intractable in the uniform cases and has been omitted.

TABLE II.1 SOME MARGINAL DISTRIBUTIONS OF SPHERICALLY EXCHANGEABLE PROCESSES

DIM	FCN.	NORMAL	EXPONENTIAL	CAUCHY	UNIFORM
q	$\tilde{f}_1(\underset{\sim}{x})$	$(2\pi)^{-\frac{q}{2}}\exp(-\frac{1}{2}\Sigma x_j^2)$	$\left[\pi^{\frac{q-1}{2}}\,2^q\,\Gamma\left(\frac{q+1}{2}\right)\right]^{-1}\exp\left\{-\sqrt{\Sigma x_j^2}\right\}$	$\Gamma\left(\frac{q+1}{2}\right)\left[\pi(1+\Sigma x_j^2)\right]^{-\frac{q+1}{2}}$	$\frac{1}{2}q\Gamma(\frac{q}{2})\pi^{-\frac{q}{2}}$, $\Sigma x_j^2 \le 1$
	$\tilde{\phi}_1(\underset{\sim}{t})$	$\exp(-\frac{1}{2}\Sigma t_j^2)$	$[1+\Sigma t_j^2]^{-\frac{q+1}{2}}$	$\exp(-\sqrt{\Sigma t_j^2})$	
	$g_1(r)$	$\left[2^{\frac{q}{2}-1}\Gamma(\frac{q}{2})\right]^{-1} r^{q-1}\exp(-\frac{1}{2}r^2)$	$\Gamma(\frac{1}{2})\left[2^{q-1}\Gamma\left(\frac{q+1}{2}\right)\Gamma(\frac{q}{2})\right]^{-1} r^{q-1}e^{-r}$	$2\Gamma\left(\frac{q+1}{2}\right)r^{q-1}\left\{\Gamma(\frac{1}{2})\Gamma(\frac{q}{2})(1+r^2)^{\frac{q+1}{2}}\right\}^{-1}$	qr^{q-1}, $r \le 1$
q	$\tilde{f}_a(\underset{\sim}{x})$	$(2\pi a^2)^{-\frac{q}{2}}\exp\left\{-\frac{\Sigma x_j^2}{2a^2}\right\}$	$\left[\pi^{\frac{q-1}{2}}\,2^q a^q\,\Gamma\left(\frac{q+1}{2}\right)\right]^{-1}\exp\left\{-a^{-1}\sqrt{\Sigma x_j^2}\right\}$	$\Gamma\left(\frac{q+1}{2}\right)\left[\pi(a^2+\Sigma x_j^2)\right]^{-\frac{q+1}{2}}a^q$	$q\Gamma(\frac{q}{2})\left[2a^q\pi^{\frac{q}{2}}\right]^{-1}$, $\Sigma x_j^2 \le a^2$
	$\tilde{\phi}_a(\underset{\sim}{t})$	$\exp(-\frac{1}{2}a^2\Sigma t_j^2)$	$[1+a^2(\Sigma t_j^2)]^{-\frac{q+1}{2}}$	$\exp\left\{-a\sqrt{\Sigma t_j^2}\right\}$	
	$g_a(r)$	$\left[2^{\frac{q}{2}-1}\Gamma(\frac{q}{2})a^q\right]^{-1} r^{q-1}\exp\left\{-\frac{1}{2}\frac{r^2}{a^2}\right\}$	$\Gamma(\frac{1}{2})\left[2^{q-1}a^q\Gamma\left(\frac{q+1}{2}\right)\Gamma(\frac{q}{2})\right]^{-1} r^{q-1}e^{-\frac{r}{a}}$	$2\Gamma\left(\frac{q+1}{2}\right)r^{q-1}a^q\left\{\Gamma(\frac{1}{2})\Gamma(\frac{q}{2})(a^2+r^2)^{\frac{q+1}{2}}\right\}^{-1}$	$qr^{q-1}a^{-q}$, $r \le a$
1		$(2\pi a^2)^{-\frac{1}{2}}\exp\left\{-\frac{x^2}{2a^2}\right\}$	$(2a)^{-1}\exp\left\{-\frac{\lvert x\rvert}{a}\right\}$	$a[\pi(a^2+x^2)]^{-1}$	$(2a)^{-1}$, $\lvert x\rvert \le a$
		$\exp(-\frac{1}{2}a^2t^2)$	$[1+a^2t^2]^{-1}$	$\exp(-a\lvert t\rvert)$	$(ta)^{-1}\sin(ta)$
		$\frac{1}{a}\sqrt{\frac{2}{\pi}}\exp\left\{-\frac{x^2}{2a^2}\right\}$	$a^{-1}\exp(-\frac{1}{a}\lvert x\rvert)$	$2a[\pi(a^2+x^2)]^{-1}$	a^{-1}, $0 < x \le a$
2		$(2\pi a^2)^{-1}\exp\left\{-\frac{x_1^2+x_2^2}{2a^2}\right\}$	$(2\pi a^2)^{-1}\exp\left\{-\frac{1}{a}\sqrt{x_1^2+x_2^2}\right\}$	$(2\pi)^{-1}a^2(a^2+x_1^2+x_2^2)^{-\frac{3}{2}}$	$(\pi a^2)^{-1}$, $x_1^2+x_2^2 \le a^2$
		$\exp(-\frac{1}{2}a^2(t_1^2+t_2^2))$	$[1+a^2(t_1^2+t_2^2)]^{-\frac{3}{2}}$	$\exp\left\{-a\sqrt{t_1^2+t_2^2}\right\}$	
		$a^{-2}r\exp\left\{-\frac{r^2}{2a^2}\right\}$	$a^{-2}r\exp(-\frac{r}{a})$	$a^2r(a^2+r^2)^{-\frac{3}{2}}$	$2a^{-2}r$, $r \le a$
3		$(2\pi a^2)^{-\frac{3}{2}}\exp\left\{-\frac{\sum_1^3 x_j^2}{2a^2}\right\}$	$(8\pi a^3)^{-1}\exp\left\{-a^{-1}\sqrt{\sum_1^3 x_r^2}\right\}$	$a^3[\pi(a^2+\Sigma x_j^2)]^{-2}$	$3(4a^3)^{-1}$, $x_1^2+x_2^2+x_3^2 \le a^2$
		$\exp\left\{-\frac{1}{2}a^2\sum_1^3 t_j^2\right\}$	$\left[1+a^2\left(\sum_1^3 t_r^2\right)\right]^{-1}$	$\left\{-a\sqrt{\Sigma t_j^2}\right\}$	
		$\sqrt{\frac{2}{\pi}}\,a^{-3}r^2\exp\left\{-\frac{r^2}{2a^2}\right\}$	$(2a^3)^{-1}r^2\exp(-\frac{1}{a}r)$	$4r^2a^2[\pi(a^2+r^2)^2]^{-1}$	$3a^{-3}r^2$, $r \le a$

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