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THE APPLICATION OF FUZZY-SET THEORY TO THE BURNTHROUGH RANGE EQ--ETC(U)
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The Application of Fuzzy-Set Theory to the Burnthrough Range Equation

F. GROSS, D. J. HANRAHAN, AND S. T. HOOD

ENEWS PROGRAM
Tactical Electronic Warfare Division

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Uncertainty of parameters in the burnthrough range equation is nonstatistical. Fuzzy-set theory may be a better means of treating such uncertainty than the usual personal-probability (degree-of-belief) approach. Analytical and computer methods are presented and exemplified for computation of the burnthrough range as a fuzzy quantity when some of the radar parameters are uncertain. The methods are generally applicable to any fuzzy equation where suitable substitutions can be made.

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CONTENTS

THE APPLICATION OF FUZZY-SET THEORY TO THE BURNTHROUGH RANGE EQUATION

INTRODUCTION

Electronic warfare (EW) analysis is fraught with uncertainty, because the tactics and systems of a potential enemy are never completely known. A continuing effort is made to reduce such uncertainty by intelligence collection, but there is always a residue. Furthermore the uncertainty is not statistical. If an EW problem is treated by probability, then personal or subjective probability, that is, the degree of belief [1], must be used. Some authorities hold this to be an improper use of mathematical probability [2]. An alternative approach is to use the theory of fuzzy sets [3], which is said to be particularly suitable for nonstatistical uncertainty, although agreement on this point is incomplete [4-8]. The purpose of this report is to demonstrate the application of fuzzy-set theory to an elementary EW problem and present methods that may be generally used with the extension principle of fuzzy-set theory.

THE BURNTHROUGH RANGE EQUATION

The burnthrough range equation for self-screening against a search radar may be written in the form [9]

$$R^2 = \frac{\sigma B_J E_R}{4\pi B_R E_J (\mathrm{SNR})_d} \left(\frac{\theta_w f_r}{\dot{\theta}_s} \right)^{0.8}, \tag{1}$$

where

σ	is the radar cross section of the target (square meters),
B_J	is the bandwidth of the jammer,
B_R	is the bandwidth of the radar receiver,
E_R	is the effective radiated power of the radar, $(\mathrm{ERP})_R$,
E_J	is the effective radiated power of the jammer, $(\mathrm{ERP})_J$,
θ_w	is the radar-antenna beamwidth,
$\dot{\theta}_s$	is the radar-antenna scan rate,
$(\mathrm{SNR})_d$	is the minimum integrated signal-to-noise ratio necessary for detection, and
f_r	is the radar pulse-rate frequency.

Generally, the values of some of the arguments in Eq. (1) are not precisely known. Uncertainty in the values of variables on the right-hand side of Eq. (1) is translated into a corresponding uncertainty in the value of R. We apply fuzzy-set theory and treat the uncertain variables as "fuzzy variables" that may be represented as a fuzzy set, each with a membership function. (The membership function for a fuzzy set expresses on a scale of 0 to 1 the degree of membership of each element of some universe of interest in the set.) Consider the case where two of the radar parameters, say E_R and B_R, are uncertain. Later we shall discuss the general case where any number of variables are uncertain. We rewrite Eq. (1) as

$$\mathbf{R}^2 = K^2 \frac{\mathbf{E}_R}{\mathbf{B}_R}, \tag{2}$$

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where K is a known constant and $\mathbf{R}$, $\mathbf{E}_R$, and $\mathbf{B}_R$ are the fuzzy variables. Membership functions for $\mathbf{E}_R$ and $\mathbf{B}_R$ are assumed given and are denoted respectively by $\mu_E(e)$ and $\mu_B(b)$. Then by Zadeh's extension principle [10a]. the membership function for $\mathbf{R}$ is

$$\mu_R(r) = \sup_{e,b} \min [\mu_E(e), \mu_B(b)] \tag{3}$$

subject to the condition

$$r = K\sqrt{e/b}. \tag{4}$$

A simple example of this case with piecewise linear functions for μ_E and μ_B is given in Appendix A along with an analytical solution.

MATHEMATICAL ASPECTS

We first present an analytical approach to the solution of "fuzzy equations" using the burnthrough range equation as an example.

Two Fuzzy Independent Variables

Normalizing r in Eqs. (3) and (4), we write

$$\mu_R(r) = \sup_{e,b} \min [\mu_E(e), \mu_B(b)] \tag{5}$$

subject to the condition

$$r^2 b = e. \tag{6}$$

Substituting for e in Eq. (5) we obtain

$$\mu_R(r) = \sup_{b} \min [\mu_E(r^2 b), \mu_B(b)]. \tag{7}$$

We let $[A,B]$ be the domain of $\mu_R(r)$. For each r in $[A,B]$, both $\nu_r(b) \triangleq \mu_E(r^2 b)$ and $\mu_B(b)$ have the same domain, say $[C,D]$. Furthermore, we assume that both functions $\nu_r(b)$ and $\mu_B(b)$ are analytic on $[C,D]$. It follows from the identity theorem [11] that for each r the function $\nu_r(b) - \mu_B(b)$ has at most finitely many zeros in the interval $[C,D]$. In other words, the functions $\nu_r(b)$ and $\mu_B(b)$ intersect at most finitely many times. We let these intersections take place at the values b_j, $j = 1, 2, \ldots, n$, $b_j \in [C,D]$. We now concentrate on a typical such interval, say $[b_{j-1}, b_j]$. Without any loss of generality, we assume that $\mu_B(b)$ lies below $\nu_r(b)$ for $b \in (b_{j-1}, b_j)$. Thus, typically, we have a situation like the one in Fig. 1.

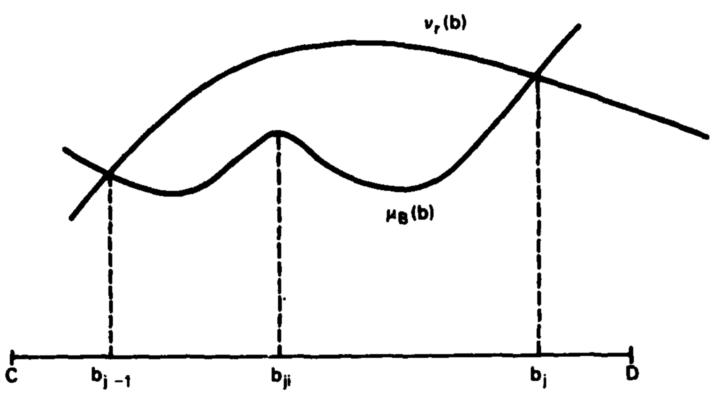

Fig. 1 — Typical interval between intersections of $\nu_r(b)$ and $\mu_B(b)$

The points b_{ji}, $i = 1, 2, \ldots, t$ in (b_{j-1}, b_j) at which $d\mu_B/db = 0$ and simultaneously $d^2\mu_B/db^2 < 0$ are the points at which $\mu_B(b)$ attains local maxima in (b_{j-1}, b_j). We need only examine the points $b_{j-1}, b_{j1}, b_{j2}, \ldots, b_{jt}, b_j$. The largest of the values $\mu_B(b_{j-1})$, $\mu_B(b_{ji})$ with $i = 1, 2, \ldots, t$, and $\mu_B(b_j)$ is equal to

$$\sup_{\substack{b \\ b \in [b_{j-1}, b_j]}} \min[\nu_r(b), \mu_B(b)].$$

We repeat this analysis for each of the intervals (b_{j-1}, b_j), $j = 1, 2, \ldots, n$, using C for b_0, and for the interval (b_n, D), exchanging μ_B and ν_r when $\nu_r(b)$ lies below $\mu_B(b)$. If we denote the largest such value (or a largest value) in $[b_{j-1}, b]$ by M_j and that in $[b_n, D]$ by M_d, then for the value of r used

$$\mu_R(r) = \max(M_d, \max_j M_j). \tag{8}$$

We then repeat the entire process for enough values of r to establish a curve for μ_R.

Thus far we have provided an analytical procedure to obtain the membership function for the burnthrough range as a fuzzy variable, provided that analytic membership functions for the effective radiated power and the bandwidth of the radar are given. The same procedure may be used for any function of two fuzzy variables when one of them can be expressed in terms of the other and the dependent variable, as in Eq. (6), and the given membership functions are analytic. The last condition is not true for the example of Appendix A, but local maxima are not involved there.

More Than Two Fuzzy Independent Variables

The only theorem that seems to deal with the more difficult case of more than two fuzzy variables is due to Baas and Kwakernaak [12,10b] and may be stated as follows:

Theorem: Let u_i, $i = 1, 2, \ldots, n$, be n piecewise continuously differentiable membership functions with [illegible] supports. Let g be a continuously differentiable mapping of $\mathbf{R}^n$ into $\mathbf{R}$ (the real line). At points where the respective derivatives exist, let $\mu'_i(x_i) = d\mu_i(x_i)/dx_i$ and $g_i(x) = \partial g(x_1, x_2, \ldots, x_n)/\partial x_i$. Suppose that the point $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in \mathbf{R}^n$ satisfies the following:

- $\mu'_i(\hat{x}_i)$ and $g_i(\hat{x})$, $i = 1, 2, \ldots, n$, exist and are nonzero.
- $\mu_1(\hat{x}_1) = \mu_2(\hat{x}_2) = \cdots = \mu_n(\hat{x}_n)$.
- $\mu'_i(\hat{x}_i)/g_i(\hat{x})$ has the same sign for each $i \in \{1, 2, \ldots, n\}$.

Then $\hat{x}$ is a strict relative maximum point of the mathematical programming problem

$$\text{maximize} \min_{i=1,2,\ldots,n} \mu_i(x_i)$$

subject to the condition

$$g(x_1, x_2, \ldots, x_n) = g(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) = g(\hat{x}).$$

Suppose that values for E_R, B_R, and f_r in Eq. (1) are uncertain. Then we define $F \triangleq (f_r)^{0.8}$ and write

$$\mathbf{R} = k\sqrt{\frac{\mathbf{E}_R\mathbf{F}}{\mathbf{B}_R}}, \tag{9}$$

where k is a constant and $\mathbf{R}$, $\mathbf{E}_R$, $\mathbf{B}_R$, and $\mathbf{F}$ are the fuzzy variables. A new membership function $\mu_F(f)$ for $\mathbf{F}$ is derived from a given membership function for f_r as a fuzzy variable. Then by the extension principle

$$\mu_R(r) = \sup_{e,b,f} \min\,[\mu_E(e),\ \mu_B(b),\ \mu_F(f)] \tag{10}$$

subject to the condition

$$r = k\sqrt{\frac{ef}{b}}. \tag{11}$$

Normalizing r in Eq. (9), we write

$$\mu_R(r) = \sup_{e,b,f} \min\,[\mu_E(e),\ \mu_B(b),\ \mu_F(f)] \tag{12}$$

subject to the condition

$$r^2 b = ef. \tag{13}$$

To apply the theorem of Baas and Kwakernaak, we must assume the following:

- Each of the membership functions $\mu_E(e)$, $\mu_B(b)$ and $\mu_F(f)$ is piecewise continuously differentiable with finite support;
- There exist $\hat{e}$, $\hat{b}$, and $\hat{f}$ such that $\mu'_E(\hat{e})$, $\mu'_B(\hat{b})$, and $\mu'_F(\hat{f})$ exist and are nonzero;
- $\mu_E(\hat{e}) = \mu_B(\hat{b}) = \mu_F(\hat{f})$;
- $\frac{\hat{e}}{\hat{f}}\mu'_E(\hat{e})$, $\frac{\hat{f}}{\hat{e}}\mu'_F(\hat{f})$, and $-\frac{\hat{b}^2}{\hat{e}\hat{f}}\mu'_B(\hat{b})$ have the same sign.

Under these four conditions it follows from the theorem of Baas and Kwakernaak that $\mu_R(r)$ is attained at $\hat{e}$, $\hat{b}$, and $\hat{f}$ for any given r. This result assures us that in principle under these four conditions we can find the desired value of $\mu_R(r)$, but it does not provide a procedure. Furthermore, the last two conditions are rather strong and often may not be satisfied. The theorem gives only sufficient conditions for a solution.

We now present a method which expands upon the previous one. This method not only assures the existence of a solution under reasonable hypotheses but also provides an analytical procedure for finding $\mu_R(r)$.

We let $T = \frac{E_R}{B_R}$, so that Eq. (9) becomes

$$\mathbf{R} = k\sqrt{\mathbf{FT}}. \tag{14}$$

By the procedure of the previous section we can determine the membership function $\mu_T(t)$ for $\mathbf{T}$; then Eq. (14) can be viewed as an equation with only two independent fuzzy variables, $\mathbf{F}$ and $\mathbf{T}$. Applying the procedure once more, we can determine the desired membership function $\mu_R(r)$ for $\mathbf{R}$.

It is important to demonstrate that we could just as well have chosen to make the substitution $T = E_R F$ and followed the same procedure. It is not obvious that the substitution chosen does not affect the final result, that is, the derived membership function $\mu_R(r)$. That $\mu_R(r)$ is independent of the choice follows from a theorem dealing with the general n-dimensional case, which we state and prove in Appendix B.

Thus we have provided an analytical method for obtaining the membership function for the burnthrough range, provided that analytic membership functions for the effective radiated power, the bandwidth, and the pulse rate frequency of the radar are given. More generally, the same procedure can be used recursively when any number of variables in Eq. (1) are fuzzy. More generally still, the method can be used for any fuzzy equation where appropriate substitutions can be made and the given membership functions are analytic.

If a single "best" value of a fuzzy variable is desired, it may be taken as the mean and called the (nonstatistical) expected value. Thus for burnthrough range **R**, the expected value is

$$E(\mathbf{R}) = \int_0^\infty r\mu_R(r)\,dr \Big/ \int_0^\infty \mu_R(r)\,dr. \tag{15}$$

This definition is not restricted to analytic membership functions.

DIGITAL COMPUTER SOLUTION

We now discuss an approach which is suitable for automatic computation. Equation (4) can be written in the parametric form

$$e = s,$$
$$b = \frac{K^2 s}{r^2} \tag{16}$$

where s is a parameter which generates all points (e, b) satisfying Eq. (4) with fixed K and r. Substituting for e and b in Eq. (5) gives

$$\mu_R(r) = \sup_s \min\left[\mu_E(s),\ \mu_B\left(\frac{K^2}{r^2}s\right)\right]. \tag{17}$$

This is a simple problem of maximization of a function of a single variable, for which standard numerical techniques are available. The functions μ_E and μ_B need not be continuous, much less analytic.

The method can be applied generally to any relation of the form

$$z = f(x, y) \tag{18}$$

for which a suitable parametric form

$$x = P_x(z, s),$$
$$y = P_y(z, s) \tag{19}$$

can be obtained. Functions P_x and P_y need not be analytic either, only calculable by an algorithm. The membership function for the dependent fuzzy variable is then given (in terms of membership functions for the independent fuzzy variables) by

$$\mu_Z(z) = \sup_s \min\{\mu_X[P_x(z, s)],\ \mu_Y[P_y(z, s)]\}, \tag{20}$$

which again only requires finding the maximum of a function of one variable.

Figure 2 illustrates the method with continuous functions μ_X and μ_Y. Loci of Eq. (18) or, alternatively, Eqs. (19), for two values of z are shown superimposed upon contours of

$$\mu_{X\times Y}(x, y) \triangleq \min[\mu_X(x),\ \mu_Y(y)]. \tag{21}$$

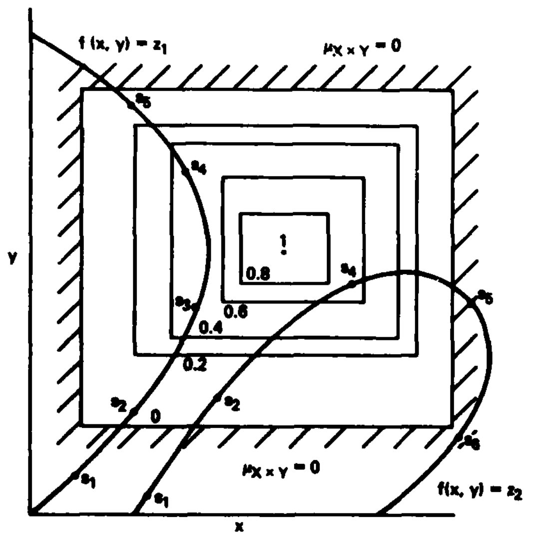

Fig. 2 — Illustration of the computer solution for $\mu_Z(z)$ when $z = f(x, y)$, given $\mu_X(x)$ and $\mu_Y(y)$. Rectangles represent contours of constant $\mu_{X\times Y}(x, y) = \min[\mu_X(x), \mu_Y(y)]$. One set of nested rectangles is obtained for convex μ_X and μ_Y; otherwise, more than one is obtained. Loci of $f(x, y)$ are shown for two values of z. For each value of z, the corresponding locus is traversed by varying parameter s, and $\mu_Z(z)$ is found as the maximum of $\mu_{X\times Y}$ on the locus.

Since μ_X and μ_Y are continuous functions, it follows from Eq. (21) that the loci of constant $\mu_{X\times Y}$ (contours) are rectangles (or degenerate rectangles, including points, rectangular areas or line segments) with their sides parallel to the x and y axes. For each value of z, the corresponding locus is traversed by varying s, and $\mu_Z(z)$ is found as the maximum of $\mu_{X\times Y}$ on the locus. A FORTRAN program for this procedure with the example of Appendix A is presented in Appendix C.

Equation (18) can be further generalized to the case of a function of n fuzzy variables:

$$z = f(x_1, x_2, \ldots, x_n). \tag{22}$$

A parametric representation of f now has the form

$$\begin{aligned} x_1 &= P_1(z, s_1, s_2, \ldots, s_{n-1}), \\ &\ldots, \\ x_n &= P_n(z, s_1, s_2, \ldots, s_{n-1}), \end{aligned} \tag{23}$$

where the $s_1, s_2, \ldots, s_{n-1}$ are independent parameters. The expression for μ_Z is then

$$\mu_Z(z) = \sup_{s_1, s_2, \ldots, s_{n-1}} \min\{\mu_1[P_1(z, s_1, s_2, \ldots, s_{n-1})], \mu_2[P_2(z, s_1, s_2, \ldots, s_{n-1})], \ldots, \mu_n[P_n(z, s_1, s_2, \ldots, s_{n-1})]\}, \tag{24}$$

where $\mu_i(x_i)$ is the membership function for the fuzzy variable $\mathbf{X}_i$. As the number of parameters increases, numerical efficiency becomes more important. In many situations the membership functions will be continuous and convex, allowing the use of efficient maximization algorithms.

A major problem in the application of fuzzy-set theory is the specification of membership functions. In the example of Appendix A, μ_E and μ_B are assumed given. A more general form of the program in Appendix C has been written which prompts an "expert" user to specify "most possible," "minimum possible," and "maximum possible" values of the fuzzy variables. These values are used to construct triangular membership functions. A further generalization would allow membership functions to be specified for any of the independent variables. Such a program might be termed a fuzzy calculator.

DISCUSSION

The analytical and computer methods presented have much wider application than the burnthrough equation and EW analysis. They may be used for any fuzzy equation involving any number of variables when suitable substitutions can be made and, for the analytic method, when the given membership functions are analytic. The last condition does not restrict the computer method, which is generally the most practical means of solution. The analytical method is most valuable for theoretical considerations.

The expected advantages of the fuzzy-set approach to analysis involving nonstatistical uncertainty are as follows:

- Membership functions (or agreement with suggested membership functions) for fuzzy variables should be easier to obtain from experts who tend to be unwilling to commit themselves to single numbers.
- The uncertainty is made explicit and carried through the analysis to appear as a fuzzy result. This is less misleading than the use of uncertain, single numbers leading to a sharp result.
- There is a built-in sensitivity analysis with simultaneous variation of uncertain quantities and weighting of their possible values (by means of their membership functions).

The above advantages are cited against the use of expected values with a personal-probability (degree-of-belief) approach. A more proper comparison would be against personal probability with distributions for, say, $\mathbf{E}_R$ and $\mathbf{B}_R$ as random variables that lead to a distribution for $\mathbf{R}$ as a random variable. But such an approach is not currently used. The relative merits of it and the fuzzy-set method need further study.

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Appendix A

EXAMPLE OF THE BURNTHROUGH EQUATION WITH TWO FUZZY INDEPENDENT VARIABLES

We assume the known parameters in Eq. (1) are

σ	$= 10^5$ m^2,
B_J	$= 10^7$ Hz,
E_J	$= 10^6$ W,
$(SNR)_d$	$= 10$,
θ_w	$\simeq \pi/180$ rad,
$\dot{\theta}_s$	$= \pi/5$ rad/s, and
f_r	$= 10^2$ Hz.

We let the membership functions for E_R and B_R as fuzzy variables be given by Figs. A1 and A2. These functions can be written

$$\mu_B(b) = \frac{1}{4\cdot 10^6}(b - 10^6),\ b \in [10^6,\ 5\cdot 10^6], \tag{A1a}$$

$$= \frac{1}{4}\left(5 - \frac{b}{5\cdot 10^6}\right),\ b \in [5\cdot 10^6,\ 10^7], \tag{A1b}$$

$$= \frac{3}{4}\left(3 - \frac{b}{5\cdot 10^6}\right),\ b \in [10^7,\ 1.5\cdot 10^7), \tag{A1c}$$

and

$$\mu_E(e) = \frac{1}{9}\left(\frac{e}{10^8} - 1\right),\ e \in [10^8, 10^9], \tag{A2a}$$

$$= \frac{1}{9}\left(10 - \frac{e}{10^9}\right),\ e \in [10^9, 10^{10}]. \tag{A2b}$$

It is convenient to make the substitutions $B_R = B_R^* \cdot 10^6$, $E_R = E_R^* \cdot 10^8$, and, correspondingly, $b = b^* \cdot 10^6$ with $e = e^* \cdot 10^8$. Then Eq. (2) becomes

$$\mathbf{R}^2 = (10K)^2 \frac{\mathbf{E}_R^*}{\mathbf{B}_R^*}, \tag{A3}$$

which we write as

$$\mathbf{R}^2 = (10K)^2 \mathbf{T}. \tag{A4}$$

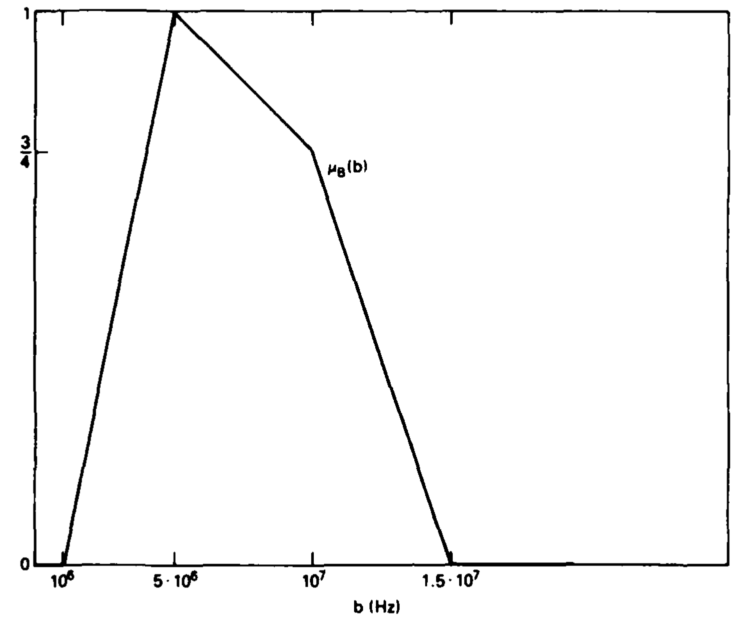

Fig. A1 — Membership function for B_R as a fuzzy variable

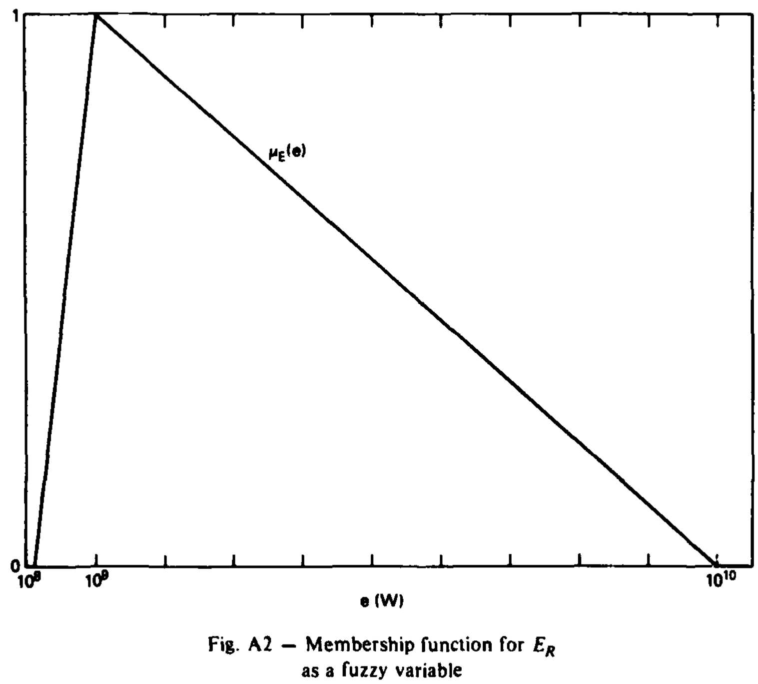

Fig. A2 — Membership function for E_R as a fuzzy variable

Our first task is to find the membership function $\mu_T(t)$ for **T** with $t \triangleq e^*/b^*$ (where $\triangleq$ means "defined as"), using the new membership functions:

$$\mu_B^*(b^*) = \frac{1}{4}(b^* - 1),\ b^* \in [1,5], \tag{A5a}$$

$$= \frac{1}{4}\left(5 - \frac{b^*}{5}\right),\ b^* \in [5,10], \tag{A5b}$$

$$= \frac{3}{4}\left(3 - \frac{b^*}{5}\right),\ b^* \in [10,15], \tag{A5c}$$

and

$$\mu_E^*(e^*) = \frac{1}{9}(e^* - 1),\ e^* \in [1,10], \tag{A6a}$$

$$= \frac{1}{9}\left(10 - \frac{e^*}{10}\right),\ e^* \in [10,100]. \tag{A6b}$$

According to the extension principle, the membership function for **T** is found from

$$\mu_T(t) = \sup_{b^*,e^*} \min\,[\mu_B^*(b^*), \mu_E^*(e^*)], \tag{A7}$$

subject to $t = e^*/b^*$, or simply

$$\mu_T(t) = \sup_{b^*} \min\,[\mu_B^*(b^*), \nu_t(b^*)], \tag{A8}$$

where $\nu_t(b^*) \triangleq \mu_E^*(tb^*)$.

Figure A3 is a plot of $\mu_B^*(b^*)$ and plots of $\mu_t(b^*)$ for $t = 0.5$, 1, 2, and 3 and indicates how a graphical solution might be obtained. In this simple case there are no local maxima, and solutions are found from intersections of $\mu_B^*(b^*)$ and $\nu_t(b^*)$. It is also clear from the figure which branches of the functions give a solution for different regions of the domain $\{b^*\}$.

Thus, analytically, we find

$$\mu_T(t) = \frac{5t - \frac{1}{3}}{\frac{20}{9}t + 3},\ t \in [0.0667, 0.775], \tag{A9a}$$

$$= \frac{25t - 1}{20t + 9},\ t \in [0.775, 2], \tag{A9b}$$

$$= \frac{100 - t}{90 + 4t},\ t \in [2, 100], \tag{A9c}$$

$$= 0,\ \text{otherwise}.$$

The desired membership function for the burnthrough range is then found, in view of (A4), from

$$\mu_R(r) = \mu_R(10K\sqrt{t}) = \mu_T(t). \tag{A10}$$

This is plotted in Fig. A4 as $\mu_{R'}(r')$.

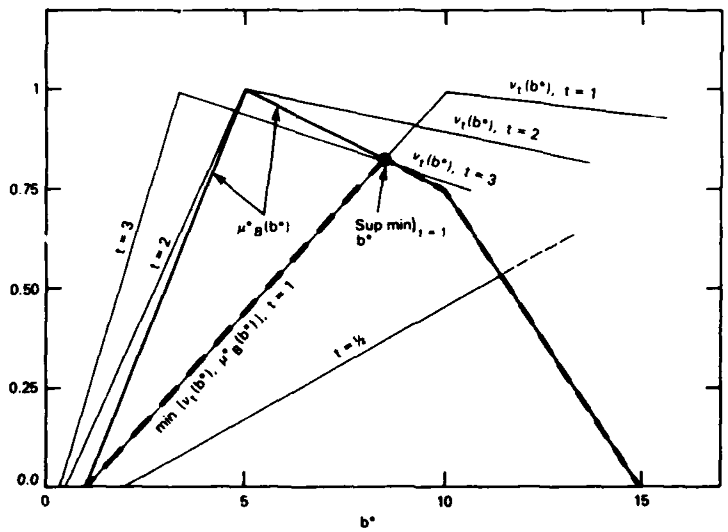

Fig. A3 — Intersections of $\mu^*_B(b^*)$ with $\nu_t(b^*)$ for various values of t

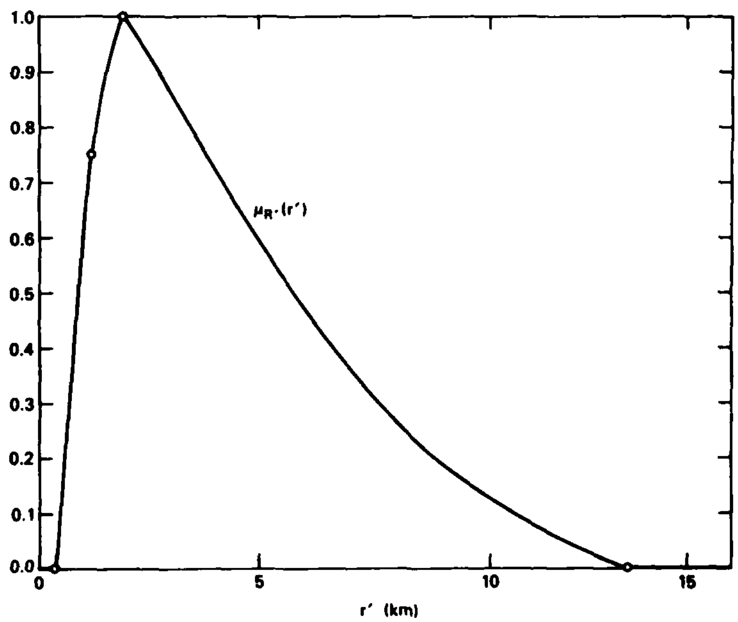

Fig. A4 — Resulting membership function for the burnthrough range

Appendix B

THEOREM THAT THE RESULT IS INDEPENDENT OF THE SUBSTITUTION USED

For a case of n fuzzy independent variables we use the extension principle in a more general form* than Eq. (10). We let X be a Cartesian product of universes, $X = X_1 \times \cdots \times X_n$, and let $A_1, \ldots, A_n$ be n fuzzy sets in $X_1, \ldots, X_n$ respectively. We let f be a mapping from X to a universe Y such that $y = f(x)$, $y \in Y$, $x \triangleq (x_1, \ldots, x_n)$, $x \in X$, and $x_i \in X_i$, $i = 1, 2, \ldots, n$. The extension principle allows us to induce from the n fuzzy sets A_i with membership functions $\mu_{A_i}(x_i)$ a fuzzy set B on Y through f such that the membership function for B is

$$\begin{aligned}\mu_B(y) &= \sup_{\substack{x \\ y=f(x)}} \min\,[\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)] \\ &= 0 \text{ if } f^{-1}(y) = \phi,\end{aligned} \qquad \text{(B1)}$$

where $f^{-1}(y)$ is the inverse image of y and ϕ is the empty set.

Theorem: Let h be a mapping from $X_{n-1} \times X_n$ to a universe Z and g be a mapping from $X_1 \times \cdots \times X_{n-2} \times Z$ to the universe Y such that for every $y \in Y$ the set of simultaneous solutions $\{x\}$ of $z = h(x_{n-1}, x_n)$ and $y = g(x_1, \ldots, x_{n-2}, z)$ is equal to the solution set of $y = f(x)$. Let C be a fuzzy set induced on Z thorough h by the fuzzy sets A_{n-1} and A_n. Then

$$\mu_B(y) = \sup_{\substack{x_1,\ldots,x_{n-2},z \\ y=g(x_1,\ldots,x_{n-2},z)}} \min[\mu_{A_1}(x_1), \ldots, \mu_{A_{n-2}}(x_{n-2}), \mu_C(z)]. \qquad \text{(B2)}$$

Proof: It suffices to prove the theorem for $n = 3$. The general case then follows by mathematical induction. Thus we want to prove that, for all $x = (x_1, x_2, x_3)$ in the solution set of $y = f(x)$,

$$\begin{aligned}&\sup_{\substack{x \\ y=f(x)}} \min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3)] = \\ &\sup_{\substack{x_1,z \\ y=g(x_1,z)}} \min\{\mu_{A_1}(x_1), \sup_{\substack{x_2,x_3 \\ z=h(x_2,x_3)}} \min[\mu_{A_2}(x_2), \mu_{A_3}(x_3)]\}.\end{aligned} \qquad \text{(B3)}$$

We let

$$S_i(y) = \{x : f(x) = y, \min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3)] = \mu_{A_i}(x_i)\}, \qquad \text{(B4)}$$

where $i = 1, 2, 3$. It is clear that $\bigcup_{i=1}^{3} S_i(y)$ is the solution set of $y = f(x)$. It suffices then to show that Eq. (B3) is valid for all $S_i(y)$.

---

*D. Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980, pp. 36-37.

We first evaluate the right-hand side of Eq. (B3) for $S_1(y)$. From definition (B4) we have for all $x \in S_1(y)$

$$\mu_{A_1}(x_1) \leqslant \min[\mu_{A_2}(x_2), \mu_{A_3}(x_3)] \leqslant \sup_{x_2, x_3} \min[\mu_{A_2}(x_2), \mu_{A_3}(x_3)]. \tag{B5}$$

Thus the right-hand side of Eq. (B5) becomes for $x \in S_1(y)$

$$\sup_{x_1, z} \min[\mu_{A_1}(x_1), \sup_{\substack{x_2, x_3 \\ z = h(x_2, x_3)}} \min[\mu_{A_2}(x_2), \mu_{A_3}(x_3)]] = \sup_{x_1} \mu_{A_1}(x_1). \tag{B6}$$

But for $x \in S_1(y)$ the left-hand side Eq. (B5) can be written, in view of Eq. (B4), as

$$\sup_{x} \min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3)] = \sup_{x_1} \mu_{A_1}(x_1). \tag{B7}$$

Thus Eq. (B5) is valid for $x \in S_1(y)$.

For $S_2(y)$ we have from Eq. (B4)

$$\mu_{A_2}(x_2) = \min[\mu_{A_2}(x_2), \mu_{A_3}(x_3)], \tag{B8}$$

so that in the right-hand side Eq. (B3) for $x \in S_2(y)$

$$\sup_{x_2, x_3} \min[\mu_{A_2}(x_2), \mu_{A_3}(x_3)] = \sup_{x_2} \mu_{A_2}(x_2). \tag{B9}$$

Also, since $\mu_{A_2}(x_2) \leqslant \mu_{A_1}(x_1)$ for $S_2(y)$, it follows that the right-hand side of Eq. (B3) becomes for $x \in S_2(y)$

$$\sup_{x_1, z} \min[\mu_{A_1}(x_1), \sup_{\substack{x_2 \\ z = h(x_2, x_3)}} \mu_{A_2}(x_2)] = \sup_{x_2} \mu_{A_2}(x_2). \tag{B10}$$

For $S_2(y)$ the left-hand side of Eq. (B3) becomes with (B4)

$$\sup_{x} \min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3)] = \sup \mu_{A_2}(x_2). \tag{B11}$$

Thus Eq. (B5) is valid for $x \in S_2(y)$ as well.

Similarly we can show that Eq. (B5) is valid for $x \in S_3(y)$ and, therefore, for all x in the solution set. The hypotheses of the theorem are sufficient; we can show that they are not necessary. Further, the hypotheses are satisfied with

$$f(x_1, \ldots, x_{n-1}, x_2) = g[x_1, \ldots, x_{n-2}, h(x_{n-1}, x_n)]. \tag{B12}$$

The problem of when f has the desired property is interesting and will be discussed in a subsequent report.

Appendix C
COMPUTER SOLUTION OF THE EXAMPLE IN APPENDIX A

The FORTRAN program listed in Table C1 calculates the membership function and average (mean) value of the burnthrough range for the example in Appendix A. To show the generality of the procedure, the parametric expressions for e and b and the functions μ_E and μ_B are implemented as FUNCTION subprograms. The subroutine LIMITS restricts the search to regions of the (e, b) plane where μ_E and μ_B are nonzero. Thus the program is applicable to any case where the effective radiated power and the bandwidth are the only uncertain radar parameters. Results are shown in Table C2 for the membership functions and known parameter values of Appendix A.

Table C1 — Computer program for the burn-through range when the radar's effective radiated power and bandwidth are uncertain

```
      PROGRAM FUZZ
C                                                                      *
C***********************************************************************
C     A PROGRAM FOR CALCULATING THE BURNTHROUGH RANGE FOR A HOSTILE    *
C     SEARCH RADAR AGAINST A SELF-SCREENING JAMMER.                    *
C                                                                      *
C***********************************************************************
C
C***********************************************************************
C                                                                      *
C     THE BURNTHROUGH RANGE, R, DEPENDS ON THE ERP, E, AND BANDWIDTH,*
C     B, OF THE RADAR THROUGH:  R**2=A*E/B, WHERE A DEPENDS ON KNOWN *
C     PROPERTIES OF THE JAMMER AND THE RADAR.                          *
C     THIS PROGRAM CALCULATES THE MEMBERSHIP FUNCTION FOR THE FUZZY    *
C     VARIABLE R WHICH  DEPENDS ON THE FUZZY VARIABLES E AND B.        *
C     FOR GIVEN R WE HAVE  E=PE(R,S), B=PB(R,S), WHERE S               *
C     IS A PARAMETER WHICH GENERATES ALL E,B PAIRS WHICH               *
C     YIELD R. E AND B HAVE MEMBERSHIP FUNCTIONS FMUE(E) AND FMUB(B). *
C                                                                      *
C***********************************************************************
C
      PARAMETER PI=3.14159
      COMMON SMIN,SMAX,RMIN,RMAX,A
      NR=100
      NS=1000
C
      TYPE *,'RADAR CROSS SECTION (SQ. M) ?'
      ACCEPT *,SIG
      TYPE *,'JAMMER BANDWIDTH (MHZ) ?'
      ACCEPT *,BJ
      TYPE *,'JAMMER ERP (WATT) ?'
      ACCEPT *,EJ
      TYPE *,'RADAR BEAMWIDTH (DEG) ?'
      ACCEPT *,W
      W=W*PI/180.
      TYPE *,'SCAN RATE (HZ) ?'
      ACCEPT *,SR
      SR=2.*PI*SR
      TYPE *,'MIN. SNR FOR DETECTION ?'
      ACCEPT *,SNR
      TYPE *,'PRF (PPS) ?'
      ACCEPT *,PRF
C
```

Table continues

Table C1 (continued)

```
         A=(SIG*BJ/(4.*PI*EJ*SNR))*(W*PRF/SR)**0.8
         A=A*1E8
 C
         CALL LIMITS(0.,0)
         SUM=0.
         WSUM=0.
         AVG=0.
 C
         DO 500 I=1,NR
          R=RMIN+(RMAX-RMIN)*FLOAT(I-1)/FLOAT(NR-1)
          CALL LIMITS(R,1)
         SUP=0.
         DO 400 J=1,NS
          S=SMIN+(SMAX-SMIN)*FLOAT(J-1)/FLOAT(NS-1)
          SUP=AMAX1(SUP,AMIN1(FMUE(PE(R,S)),FMUB(PB(R,S))))
400      CONTINUE
         R=R/1000.
         TYPE *,'R (KM) =',R,'MU(R) =',SUP
C
C  CALCULATE EXPECTED VALUE
C
         SUM=SUM+SUP*R
         WSUM=WSUM+SUP
C
500     CONTINUE
C
        IF (WSUM.GT.1E-6) AVG=SUM/WSUM
        TYPE *,'AVERAGE BURNTHROUGH RANGE (KM) =',AVG
C
        END
C
C$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
C
        SUBROUTINE LIMITS(R,I)
C
C***********************************************************************
C                                                                      *
C       THIS SUBROUTINE DETERMINES SEARCH LIMITS ON R AND S.           *
C       IT REQUIRES UPPER AND LOWER LIMITS ON E AND B                  *
C       OUTSIDE OF WHICH FMUE(E) AND FMUB(B) ARE ZERO,                 *
C       AND EXPRESSIONS FOR R AND S IN TERMS OF E AND B.               *
C                                                                      *
C***********************************************************************
C
        COMMON SMIN,SMAX,RMIN,RMAX,A
C
        DATA EMIN/1./,EMAX/100./,BMIN/1./,BMAX/15./
C
        IF (I.GT.0) GOTO 100
         RMIN=SQRT(A*EMIN/BMAX)
         RMAX=SQRT(A*EMAX/BMIN)
        RETURN
C
100     CONTINUE
         SMIN=AMIN1(EMIN,BMIN*R**2/A)
         SMAX=AMAX1(EMAX,BMAX*R**2/A)
        RETURN
C
        END
C
```

Table continues

Table C1 (continued)

```
      FUNCTION PE(R,S)
C
C*************************************************************************
C      THIS FUNCTION, TOGETHER WITH PB(R,S), DEFINES THE DEPENDENCE     *
C      OF R ON VARIABLES E AND B. THE RELATION IS    R**2=A*E/B.        *
C                                                                       *
C*************************************************************************
C
      COMMON SMIN,SMAX,RMIN,RMAX,A
C
      PE=S
      RETURN
      END
C
C$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
C
      FUNCTION PB(R,S)
C
C*************************************************************************
C      THIS FUNCTION, TOGETHER WITH PE(R,S), DEFINES THE DEPENDENCE     *
C      OF R ON VARIABLES E AND B. THE RELATION IS    R**2=A*E/B.        *
C                                                                       *
C*************************************************************************
C
      COMMON SMIN,SMAX,RMIN,RMAX,A
C
      PB=A*S
      IF (R.GT.1E-3) PB=PB/R**2
      RETURN
      END
C
C$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
C
      FUNCTION FMUE(E)
C
C*************************************************************************
C                                                                       *
C      THIS REPRESENTS THE MEMBERSHIP FUNCTION FOR FUZZY VARIABLE E.    *
C                                                                       *
C*************************************************************************
C
C  SIMPLE EXAMPLE
      FMUE=0.
      IF (E.GE.1..AND.E.LE.10.) FMUE=(E-1.)/9.
      IF (E.GT.10..AND.E.LE.100.) FMUE=(10.-E/10.)/9.
      RETURN
      END
C
C
      FUNCTION FMUB(B)
C
C*************************************************************************
C                                                                       *
C      THIS REPRESENTS THE MEMBERSHIP FUNCTION FOR FUZZY VARIABLE B.    *
C                                                                       *
C*************************************************************************
C
C  SIMPLE EXAMPLE
      FMUB=0.
      IF (B.GE.1..AND.B.LE.5.) FMUB=(B-1.)/4.
      IF (B.GT.5..AND.B.LE.10.) FMUB=(5.-B/5.)/4.
      IF (B.GT.10..AND.B.LE.15.) FMUB=(3.-B/5.)*3./4.
      RETURN
      END
C
```

Table C2 — Computer output for the program in Table C1 with the membership functions and known parameter values of Appendix A

```
$ RUN FUZZ
RADAR CROSS SECTION (SQ. M) ?
1.0E5
JAMMER BANDWIDTH (MHZ) ?
10.0
JAMMER ERP (WATT) ?
1.0E6
RADAR BEAM WIDTH (DEG) ?
1.0
SCAN RATE (HZ) ?
0.1
MIN. SNR FOR DETECTION ?
10.0
PRF (PPS) ?
100.0
R.(KM) =  0.3466001     MU(R) =  0.0000000E+00
R (KM) =  0.4786926     MU(R) =  9.1855764E-02
R (KM) =  0.6107852     MU(R) =  0.2004713
R (KM) =  0.7428777     MU(R) =  0.3220916
R (KM) =  0.8749703     MU(R) =  0.4459961
R (KM) =   1.007063     MU(R) =  0.5818230
R (KM) =   1.139155     MU(R) =  0.7087278
R (KM) =   1.271248     MU(R) =  0.7931786
R (KM) =   1.403341     MU(R) =  0.8506137
R (KM) =   1.535433     MU(R) =  0.9012267
R (KM) =   1.667526     MU(R) =  0.9414500
R (KM) =   1.799618     MU(R) =  0.9768122
R (KM) =   1.931711     MU(R) =  0.9953954
R (KM) =   2.063803     MU(R) =  0.9810811
R (KM) =   2.195896     MU(R) =  0.9656657
R (KM) =   2.327989     MU(R) =  0.9502503
R (KM) =   2.460081     MU(R) =  0.9337337
R (KM) =   2.592174     MU(R) =  0.9172173
R (KM) =   2.724266     MU(R) =  0.8995996
R (KM) =   2.856359     MU(R) =  0.8830831
R (KM) =   2.988451     MU(R) =  0.8651874
R (KM) =   3.120544     MU(R) =  0.8467467
R (KM) =   3.252636     MU(R) =  0.8291292
R (KM) =   3.384729     MU(R) =  0.8104104
R (KM) =   3.516821     MU(R) =  0.7926978
R (KM) =   3.648914     MU(R) =  0.7738354
R (KM) =   3.781007     MU(R) =  0.7561366
R (KM) =   3.913099     MU(R) =  0.7371097
R (KM) =   4.045192     MU(R) =  0.7195187
R (KM) =   4.177284     MU(R) =  0.7004953
R (KM) =   4.309377     MU(R) =  0.6831930
R (KM) =   4.441469     MU(R) =  0.6645205
R (KM) =   4.573562     MU(R) =  0.6475086
R (KM) =   4.705654     MU(R) =  0.6290241
R (KM) =   4.837747     MU(R) =  0.6128144
R (KM) =   4.969840     MU(R) =  0.5948097
R (KM) =   5.101932     MU(R) =  0.5781121
R (KM) =   5.234025     MU(R) =  0.5621302
R (KM) =   5.366117     MU(R) =  0.5451371
R (KM) =   5.498210     MU(R) =  0.5235189
```

Table continues

Table C2 (continued)

```
R (KM) =    5.630302     MU(R) =   0.5123141
R (KM) =    5.762395     MU(R) =   0.4966409
R (KM) =    5.894488     MU(R) =   0.4813002
R (KM) =    6.026580     MU(R) =   0.4665687
R (KM) =    6.158673     MU(R) =   0.4524058
R (KM) =    6.290765     MU(R) =   0.4387506
R (KM) =    6.422858     MU(R) =   0.4233998
R (KM) =    6.554951     MU(R) =   0.4097492
R (KM) =    6.687043     MU(R) =   0.3964981
R (KM) =    6.819135     MU(R) =   0.3828891
R (KM) =    6.951228     MU(R) =   0.3696639
R (KM) =    7.083320     MU(R) =   0.3581413
R (KM) =    7.215413     MU(R) =   0.3449887
R (KM) =    7.347506     MU(R) =   0.3319885
R (KM) =    7.479598     MU(R) =   0.3199065
R (KM) =    7.611691     MU(R) =   0.3084331
R (KM) =    7.743783     MU(R) =   0.2969701
R (KM) =    7.875876     MU(R) =   0.2860853
R (KM) =    8.007968     MU(R) =   0.2762794
R (KM) =    8.140061     MU(R) =   0.2663836
R (KM) =    8.272153     MU(R) =   0.2549562
R (KM) =    8.404246     MU(R) =   0.2448068
R (KM) =    8.536339     MU(R) =   0.2358705
R (KM) =    8.668430     MU(R) =   0.2248322
R (KM) =    8.800524     MU(R) =   0.2168122
R (KM) =    8.932616     MU(R) =   0.2074694
R (KM) =    9.064708     MU(R) =   0.1977779
R (KM) =    9.196801     MU(R) =   0.1901431
R (KM) =    9.328894     MU(R) =   0.1825137
R (KM) =    9.460986     MU(R) =   0.1730798
R (KM) =    9.593080     MU(R) =   0.1639993
R (KM) =    9.725171     MU(R) =   0.1559047
R (KM) =    9.857264     MU(R) =   0.1482931
R (KM) =    9.989355     MU(R) =   0.1406857
R (KM) =    10.12145     MU(R) =   0.1337029
R (KM) =    10.25354     MU(R) =   0.1277320
R (KM) =    10.38563     MU(R) =   0.1216348
R (KM) =    10.51773     MU(R) =   0.1140414
R (KM) =    10.64982     MU(R) =   0.1064510
R (KM) =    10.78191     MU(R) =   0.1000918
R (KM) =    10.91400     MU(R) =   9.5029056E-02
R (KM) =    11.04610     MU(R) =   8.7447226E-02
R (KM) =    11.17819     MU(R) =   8.2949318E-02
R (KM) =    11.31028     MU(R) =   7.6041251E-02
R (KM) =    11.44238     MU(R) =   7.0592560E-02
R (KM) =    11.57447     MU(R) =   6.4644873E-02
R (KM) =    11.70656     MU(R) =   6.0495801E-02
R (KM) =    11.83865     MU(R) =   5.3257197E-02
R (KM) =    11.97075     MU(R) =   4.9441695E-02
R (KM) =    12.10284     MU(R) =   4.3067295E-02
R (KM) =    12.23493     MU(R) =   3.8065791E-02
R (KM) =    12.36702     MU(R) =   3.4255862E-02
R (KM) =    12.49911     MU(R) =   3.0447721E-02
R (KM) =    12.63121     MU(R) =   2.2890359E-02
R (KM) =    12.76330     MU(R) =   1.9085437E-02
R (KM) =    12.89539     MU(R) =   1.5282005E-02
R (KM) =    13.02748     MU(R) =   1.1480063E-02
R (KM) =    13.15958     MU(R) =   7.6795518E-03
R (KM) =    13.29167     MU(R) =   3.8804412E-03
R (KM) =    13.42376     MU(R) =   0.0000000E+00
AVERAGE BURNTHROUGH RANGE (KM) =    4.430007
$
```

END

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