AD-A119 605 VIRGINIA UNIV CHARLOTTESVILLE DEPT OF MECHANICAL AND--ETC F/G 20/11
OPTIMAL VIBRATION REDUCTION OVER A FREQUENCY RANGE.(U)
1982 W D PILKEY, L KITIS, B P WANG N00014-75-C-0374
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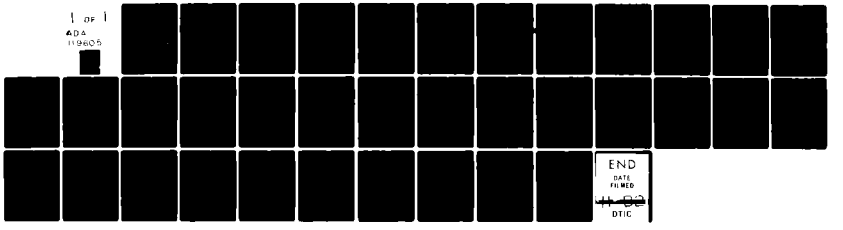

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER A-82-1	2. GOVT ACCESSION NO. AD-A119605	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) OPTIMAL VIBRATION REDUCTION OVER A FREQUENCY RANGE		5. TYPE OF REPORT & PERIOD COVERED technical report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) W. D. Pilkey L. Kitis B. P. Wang		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C0374
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mechanical & Aerospace Engineering University of Virginia Charlottesville, VA 22901		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217		12. REPORT DATE
		13. NUMBER OF PAGES 18
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

unlimited

DTIC ELECTE SEP 27 1982 S H D

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

optimal vibration
harmonic excitation
weight minimization
vibration absorber
damping

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is a review of optimal vibration reduction techniques for systems subject to harmonic excitation over a frequency range. Only passive means of control are considered. The objective functions used for optimization are restricted to those which relate directly to some measure of frequency response. Other common optimization goals such as weight minimization with response constraints are not included.

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Optimal Vibration Reduction

Over a Frequency Range

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OPTIMAL VIBRATION REDUCTION OVER A FREQUENCY RANGE

Abstract. This is a review of optimal vibration reduction techniques for systems subject to harmonic excitation over a frequency range. Only passive means of control are considered. The objective functions used for optimization are restricted to those which relate directly to some measure of frequency response. Other common optimization goals such as weight minimization with response constraints are not included in this survey.

CONVENTIONAL DYNAMIC VIBRATION ABSORBER

The simplest device used to attenuate the steady-state vibration of a mechanical system over a frequency range is the conventional dynamic absorber. In the classical analysis, the main system is modelled as a mass resonating on a spring (Figure 1). The absorber is tuned to resonate such that the motion if its mass m_2 becomes relatively large and the motion of the main mass m_1 is minimized. The first analysis of the absorber is usually attributed to Ormondroyd and Den Hartog [1]. A detailed discussion of optimal tuning and damping parameters is given in Den Hartog's book [2]. To write the equations of motion in a dimensionless form, the following symbols are introduced

$$
\begin{aligned}
x_{st} &= F/k_1 && \text{= static displacement of main mass produced by a force F}\\
\mu &= m_2/m_1 && \text{= mass ratio = absorber mass/main mass}\\
\Omega_n &= (k_1/m_1)^{1/2} && \text{= uncoupled natural frequency of main system}\\
\omega_n &= (k_2/m_2)^{1/2} && \text{= uncoupled natural frequency of damper system}
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
f &= \omega_n/\Omega_n && \text{= ratio of natural frequencies}\\
g &= \omega/\Omega_n && \text{= ratio of the exciting frequency to the uncoupled natural frequency of main system}\\
C_c &= 2m_2\Omega_n && \text{= critical damping}\\
\xi &= c_2/C_c && \text{= damping ratio}\\
z_i &= x_i/x_{st},\ (i = 1,2) && \text{= displacement ratio}
\end{aligned}
$$

With this notation the equations of motion in the frequency domain may be written

$$
\begin{aligned}
(1 - g^2)z_1 + \mu f^2(z_1 - z_2) + j2\mu\xi g(z_1 - z_2) &= 1\\
-g^2 z_2 + f^2(z_2 - z_1) + j2g\xi(z_2 - z_1) &= 0
\end{aligned}
\tag{2}
$$

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which gives

$$z_1 = \frac{A + j\xi B}{C + j\xi D} \tag{3}$$

where

$$A = f^2 - g^2$$
$$B = 2g$$
$$C = -\mu f^2 g^2 + (g^2 - 1)(g^2 - f^2) \tag{4}$$
$$D = 2g(1 - g^2 - \mu g^2)$$

If A/C = B/D, then z_1 becomes equal to A/C which is independent of damping ξ. For fixed μ and f, the condition A/C = B/D yields a quadratic equation in g^2 the two roots of which are the frequencies g_1, g_2 where the amplitude $|z_1|$ of the main mass is independent of damping. The points $(g_1, |z_1(g_1)|)$ and $(g_2, |z_1(g_2)|)$ on the frequency response curve $|z_1(g)|$ are called fixed points or invariant points. Den Hartog optimizes absorber performance by first choosing $f = \omega_n / \omega_n$ (tuning) so that the fixed points are adjusted to equal height and then finding the value of ξ (optimum damping) to make the frequency response curve pass through one of the fixed points with a horizontal tangent. His result for optimum tuning is

$$f = \frac{1}{1 + \mu} \tag{5}$$

To find optimum damping for fixed μ, substitute equation (5) into equation (3), find the derivative of $|z_1|$ with respect to g, then evaluate at one of the fixed points, say g_1, and equate the result to zero. From the equation thus obtained, the damping ratio may be found as

$$\xi^2 = \frac{\mu(3 - \sqrt{\mu(\mu + 2)})}{8(1 + \mu)^3} \tag{6}$$

On the other hand, if the fixed point chosen is the one at g_2, the resulting damping ratio is given by

$$\xi^2 = \frac{\mu(3 + \sqrt{\mu/(\mu + 2)})}{8(1 + \mu)^3} \tag{7}$$

Den Hartog recommends the use of the average value

$$\xi^2 = \frac{3\mu}{8(1+\mu)^3} \tag{8}$$

EXTENSIONS OF THE CONVENTIONAL DYNAMIC ABSORBER

Lewis [3] extended this optimization procedure to multiple degree-of-freedom, discrete, undamped systems to which a conventional viscous vibration absorber is attached. For such a system with N masses, Lewis proves the existence of 2N-2 fixed points. Thus, if the designer selects a resonant peak of primary interest, the adjacent invariant points in the vicinity of this peak can be adjusted as described above to obtain optimum tuning and damping. As a particular case, Lewis analyzes the two-mass system shown in Figure 2 and shows that the invariant points occur at the frequencies

$$\omega_{a,b}^2 = \frac{K(M_1 + M_2 + m)}{M_1(M_2 + m)} \pm \frac{K\sqrt{2M_1M_2m(M_1 + M_2 + m)}}{2M_1M_2(M_2 + m)} \tag{9}$$

with optimum tuning given by

$$\frac{k}{K} = \frac{m(2M_1M_2 + 2M_2^2 + 2M_2m - M_1m)}{2M_1(M_2 + m)^2} \tag{10}$$

and optimum damping by

$$c^2 = \frac{K(8M_1M_2 + 8M_2^2 + 8M_2m - M_1m)m^3}{4(M_2 + m)^3(M_1 + M_2 + m)} \tag{11}$$

Snowdon [4] modified the conventional absorber by adding a spring in series with the damper and showed that the resulting device, called a three-element dynamic absorber, active through a greater frequency range, reduced transmissibility (absolute value of the ratio of main mass displacement to the prescribed displacement of a foundation attached to the main mass by a spring) more effectively at the center of this range. In the same paper, Snowdon introduced dual absorbers to create a pronounced trough in the transmissibility curve while avoiding the two large compensating resonant peaks that appear in the transmissibility curve of the single absorber. Snowdon attaches these absorbers to an undamped single degree-of-freedom system and provides design information for them in graphical form. Two of his curves have been reproduced here to show the changes in transmissibility (Figures 3 and 4).

Randall et. al. [5] provided optimum absorber design curves for primary systems with damping, (Figure 5). They proved that once damping is introduced into the main system, invariant points no longer exist. Defining a performance measure G by

$$G = \max_{\omega} |x_1(\omega)| \tag{12}$$

they used a numerical search to compute tuning and damping parameters that minimize G. Soom and Lee [6] solved the same problem by nonlinear programming and investigated the possibility of using nonlinear springs to improve broad band response. They also tried several different objective functions, for example

$$\begin{aligned} G_1 &= \sum_{\omega} (|x_1(\omega) - 1|)^2 \quad \text{for } \omega \text{ such that } |x_1(\omega)| > 1 \\ G_2 &= \max_{\omega} (\omega |x_1(\omega)|) \\ G_3 &= \sum_{\omega} |x_1(\omega)|^2 \\ G_4 &= \sum_{\omega} \omega |x_1(\omega)|^2 \end{aligned} \tag{13}$$

Nonlinearities have also been treated by Roberson [7] and Arnold [8]. Roberson considered the system shown in Figure 6 where the spring between the masses M_1 and M_2 is nonlinear. Its load-deflection curve is the sum of a linear and a cubic term. Since the absorber is undamped in this case, its effectiveness is limited to a small frequency range. Roberson's synthesis criterion is the maximization of this suppression band. Arnold [8] studied the same system as Roberson but set c = 0. He provides frquency response curves of the system for hardening and softening coupling springs as well as for a softening spring designed according to Roberson's optimum system-parameter specifications. In both [7] and [8] the restoring force in the nonlinear spring as a function of spring extension is taken of the form

$$R(x) = c(x \pm \mu^2 x^3) \tag{14}$$

where the plus sign indicates a hardening spring characteristic and the minus sign indicates a softening one.

Kwak et. al. [9] and Haug and Arora [10] used a steepest descent algorithm to solve the classical absorber problem over a finite frequency range. Their objective function is

$$G = \max_{a \le \omega \le b} |z_1(\omega)| \tag{15}$$

where z_1 is as defined in equation (1) and the frequency interval of interest is [a,b]. They minimize G subject to the constraints

$$\max_{a<\omega<b} \left| \frac{x_2 - x_1}{x_1} \right| < Q_{max} \tag{16}$$

$$f_{MIN} < f < f_{MAX} \tag{17}$$

$$\zeta_{MIN} < \zeta < \zeta_{MAX} \tag{18}$$

where the notation is as in equation (1). They showed that for certain finite frequency intervals, designs superior to Den Hartog's infinite frequency range optimum may be found.

SYSTEMS WITH CONTINUOUS MEMBERS

Plunkett [11] proved the existence of invariant points for undamped continuous systems to which a single discrete damper has been connected and exploited this property to determine optimum damping for these systems. His approach is a generalization of the methods of Den Hartog [1,2] and Lewis [3]. He considers the vibration velocity at one point of a linear system resulting from a sinusoidal force at another point. Let the force applied at a point 1 be F_1 and v_1 be the velocity at the same point. Suppose a damper with damping constant c is applied across points with relative velocity v_2 and let the unknown vibration velocity at the point of interest be v_3. Plunkett shows that

$$\frac{v_3}{F_1} = jb_1 \frac{1 + jb_2c}{1 + jb_3c} \tag{19}$$

and notes that equation (19) has the same form as Den Hartog's equation (3). Thus when $b_2/b_3 = 1$, the ratio v_3/F_1 is independent of c and has the invariant points property described above in conjunction with the conventional dynamic vibration absorber. The value of c that gives a zero slope of the amplitude $|v_3/F_1|$ with respect to frequency ω at the invariant value of $|v_3/F_1|$ is an optimum value for c. Henney and Raney [12] applied a similar technique to optimize damping for vibrating uniform beams and studied the sensitivity of maximum displacement response to deviation from optimum damping.

Snowdon [13] ,[14] described how absorbers could reduce the force transmitted to the terminations of undamped cantilever beams at the resonant frequencies of beam vibration. The systems under consideration are shown in Figure 7. The force transmissibility T across the beam is defined as follows

$$T = |F_T/F_o| \tag{20}$$

where F_T is the force transmitted to the fixed end and F_o is the applied force. When a single absorber is used, as in Figures 7a and 7b, essentially the same procedure as that used for the conventional absorber is applicable. The fixed points are located by comparing transmissibility curves obtained for zero and infinite damping. The frequency ratio is then chosen so that the fixed points lie on transmissibility curve actually take the equal values of transmissibility at the fixed points. When two absorbers are applied simultaneously to the beam, Figure 7c, Snowdon's approach is to assign each absorber the values of optimum tuning and damping that were determined when the absorber was attached, individually, to its present position on the beam. Optimum supression of both the first and the second beam resonances results from this procedure.

Jacquot and Foster [15] considered an undamped single-degree-of-freedom system equipped with a damped cantilever beam serving as a vibration absorber. The authors developed an approximate system dynamics by an assumed modes approach retaining only one mode of the beam. Since the resulting equations were of the form obtained for the conventional absorber problem, the same tuning and damping approach was applicable. Snowdon [16] analyzed platelike absorbers attached either to a lumped mass-spring system or to a plate with small internal damping and presented optimum design parameters in graphical form.

Jacquot [17] appended a discrete absorber with structural or viscous damping to a beam, Figure 8. Using an assumed mode approach as in [15], he derived an approximate frequency response given by

$$y(x,\lambda) = \frac{\epsilon_i \varphi_i(x)}{EI\beta_i^4 L} \frac{T^2 - \lambda^2}{\lambda^4 - \lambda^2(1 + T^2(1 + \mu\varphi_i^2(a)) + T^2} \tag{21}$$

where

$$\lambda = (\omega/\beta_1^2)\sqrt{\rho A/EI}$$

$$T^2 = (k/m)(\rho A/EI\beta_1^4) \tag{22}$$

$$\mu = m/\rho AL$$

and β_i, ρ_i are the beam mode shapes and eigenvalues satisfying

$$\frac{d^4\rho_i}{dx^4} = \beta_i\rho_i \tag{23}$$

Because the form of equation (21) is the same as that found in the conventional absorber problem with the exception that μ is replaced by $\mu\rho^2(a)$, Jacquot was able to tune and damp to flatten the fundamental resonance response using Den Hartog's procedure.

Warburton and Ayorinde [18] extended Jacquot's representative mode method to plates and shells by defining appropriate effective mass and stiffness for such elastic bodies. The accuracy of this single-mode approximation was favorably affected if adjacent resonant frequencies were well separated from the natural frequency for which the absorber was being tuned and adversely affected as the absorber size increased. In a companion paper [19], the authors considered cylindrical shells as examples of dynamically complex structures, for which the ratio of adjacent natural frequencies tends toward unity. They showed that as dynamic complexity increased optimum absorber parameters deviated from those calculated for an equivalent single-degree-of-freedom system.

MULTIPARAMETER DESIGN BY NONLINEAR PROGRAMMING

McMunn and Plunkett [20] developed a computational method to optimize multiple dampers for large mechanical systems. Damping was defined to be optimum if the maximum response over a range of excitation frequencies was minimized. The response function of interest, for example

$$f(\omega,\underline{c}) = \left| \frac{x_i}{P_j} \right|^2 \tag{24}$$

where x_i is the displacement of the ith mass and P_j is the force on the jth mass, is first maximized with respect to excitation frequency ω using

$$\frac{\partial f}{\partial \omega} = 0 \qquad \frac{\partial^2 f}{\partial \omega^2} < 0 \tag{25}$$

and then minimized with respect to the vector $\underline{c}$ of damping values

$$\min_{\underline{c}} \quad f(\omega,\underline{c}) \tag{26}$$

In fact, the minimization over $\underline{c}$ also considers values of f at $\omega = \Omega_1$, and $\omega = \Omega_2$ if a finite frequency interval $[\Omega_1,\Omega_2]$ is being considered. Two multiple-degree-of-freedom, multiple-damper discrete systems and a column with complex modulus damping were studied as examples using this approach. A similar approach was adopted by Ng and Cunniff [21] who designed a three degree-of-freedom isolation system and verified their results by

experimental tests. In their formulation, the authors define a primitive function

$$|\chi|^2 = (P^{-1}\underline{f})(P^{-1}f)^* \tag{27}$$

where * denotes complex conjugate transpose,

$$P = -\omega^2 M + j\omega C + K \tag{28}$$

and $\underline{f}$ is the force vector. Their first step is to minimize over damping variables the primitive function maximized over frequency

$$\min_{c'} \max_{\omega} |\chi'|^2 \tag{29}$$

They repeat this procedure until an optimum is reached

$$|\chi|^2_{opt} = \min_{c^n} \max_{c^n} |\underline{x}^n|^2 \tag{30}$$

Dale and Cohen [22] extended the method of McMunn and plunkett to continuous systems whose steady-state equations of motion could be reduced to a set of ordinary differential equations containing spatial coordinates as independent variables. Both dissipative and nondissipative design parameters were included.

Lunden [23] presented a nonlinear programming solution based on a sequential unconstrained minimization technique to the problem of determining a continuous damping distribution which minimizes the maximum response of a vibrating beam over a specified frequency interval. In a second paper [24], the author applied the same approach to vibrating frames. In these references, the maximum response F in the frequency interval studied is written as

$$F(\eta^d, \eta^s) = \max_{a \leq \omega \leq b} f(\eta^d, \eta^s, \omega) \tag{31}$$

where η^d denotes distributed structural damping and η^s denotes the structural damping constants for discrete springs in the system. An exact displacement method is used with hysteretic damping introduced by the loss factor η^d giving a complex bending stiffness $EI(1 + j\eta^d)$. The 4 x 4 stiffness matrix for a beam element then takes the form

$$\underline{F} = EI(1 + j\eta^d)K\,\underline{x} \tag{32}$$

where K is a 4 x 4 matrix of transcendental functions of frequency.

Kitis [25] utilizes structural reanalysis and modal techniques with nonlinear programming to make tractable problems in which the systems under consideration contain a large number of degrees of freedom. The repetitive computations of response required in the nonlinear programming portion of the optimal design are carried out using efficient reanalysis methods or condensed eigenproblem solutions so that computation time in the structural analysis phase of the design is reduced.

IMPEDANCE MATCHING

The progressive wave solution to the wave equation for continuous chain-like systems has been used to minimize vibratory response over a frequency range by means of "impedance matching". To illustrate the basic idea of this technique consider the rod shown in Figure 9, [14]. Designating the displacement at any point x by ξ, the wave equation may be written as

$$\frac{\partial^2 \xi}{\partial x^2} + n^2 \xi = 0 \tag{33}$$

where

$$n^2 = \rho \frac{\omega^2}{E} \tag{34}$$

If the end of the rod at x = 0 is subjected to a sinusoidally varying force F_o, then the driving point impendance Z_o is given by

$$Z_o = \frac{F_o}{(\partial E/\partial t)_{x=0}} = \frac{F_o}{j\omega E(x=0)} \tag{35}$$

Snowdon [14] shows that the impedance Z_o defined in equation (35), may be written as

$$Z_o = A\sqrt{\rho E}\ \ \frac{1 - Re^{j\varphi}}{1 + Re^{j\varphi}} \tag{36}$$

where $Re^{j\varphi}$ describes the relative magnitude of and the phase difference between the incident and the reflected waves. The characteristic impedance Z_{ch} is defined as the impedance of an infinitely long rod in which reflections do not occur, the value of Z_{ch} is obtained from equation (36) by equating R to zero

$$Z_{ch} = A\sqrt{\rho E} \tag{37}$$

A matched condition will occur if a damper of damping constant Z_{ch} is attached to the rod at x = 0. This attachment causes the ratio R of reflected to incident wave to be zero so that no vibration response buildup due to reflected waves is possible. This idea has been applied to shafts on supports [26]. Here the dynamic response is expressed in progressive wave form like electrical response waves in transmisison line theory. Waveforms for a uniform shaft flexibly supported on two rotational and translational mass-spring-damper units at the ends and one such unit in the interior are obtained. The terminating impedance is made equal to the characteristic impedance of the shaft to obtain a matched condition.

LIMITING PERFORMANCE METHODS

A limiting performance approach was applied in Refs. [27] and [28] to the optimal design of vibratory systems over a frequency range. In this approach, instead of fixing the design configuration at the outset, those parts of the system to be designed are replaced by control forces. Then, for a selected cost function and design constraint the absolute optimal performance of the system is computed by solving an optimization problem in which the control forces are unknowns. The solution is called the limiting performance of the system. After the limiting performance characteristics have been found, the designer can choose a prospective configuration for the part of the system to be designed and apply parameter identification techniques to find optimum design variable values so that the designed system responds as closely as possible to the limiting performance response. The limiting performance characteristics are found by linear programming and parameter identification can be accomplished by such curve fitting techniques as least squares. This two-stage procedure has been demonstrated for the optimal design of rotor suspension systems in [27].

BOOKS AND MONOGRAPHS

The book by Haug and Arora [10] contains considerable material applicable to frequency response shaping, although specialized aspects are not treated in detail. Two other useful references are the monograph by Sevin and Pilkey [28] and the book by Snowdon [14]. Sevin and Pilkey present an introduction to the subject and a summary of the state-of-the art up to 1971. Snowdon's book contains a wealth of information on vibration absorbers and reduction of beam vibrations.

ACKNOWLEDGMENT

This work was supported by the Office of Naval Research, Arlington, Virginia.

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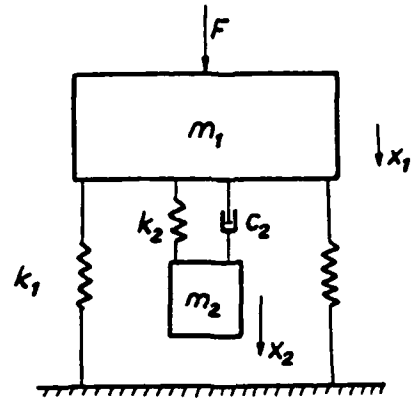


Figure 1. A Mass-Spring System with a Conventional Dynamic Vibration Absorber

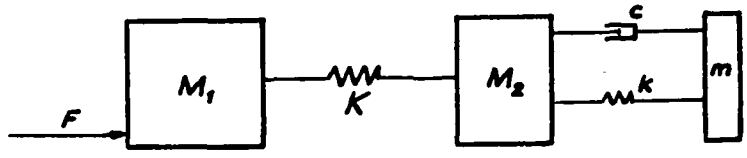
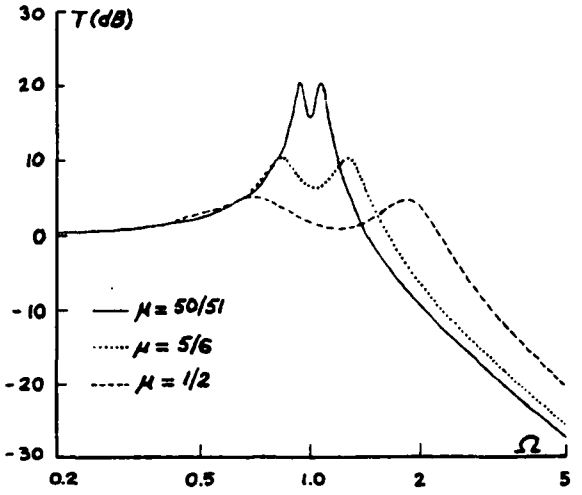


Figure 2. Two-Mass System with Tuned Damper



$\Omega = \frac{\omega}{\omega_o}$, $\omega_o^2 = \frac{K_1}{M_1+M_2}$, K_1 = spring constant of main system, M_1 = mass of main system, M_2 = mass of absorber, ω = excitation frequency.

Figure 3. Transmissibility of the Three-Element Dynamic Absorber

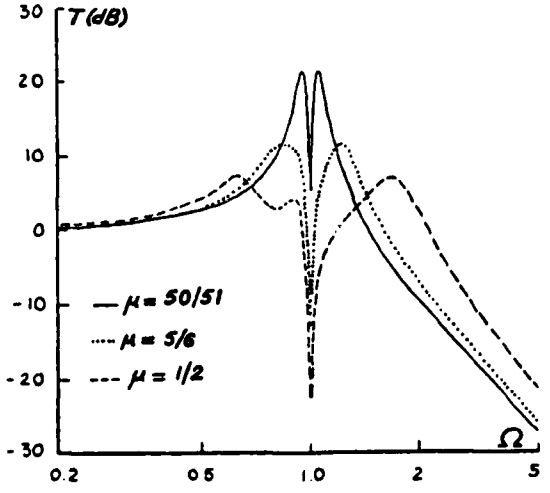


Figure 4. Transmissibility of the Dual Absorber

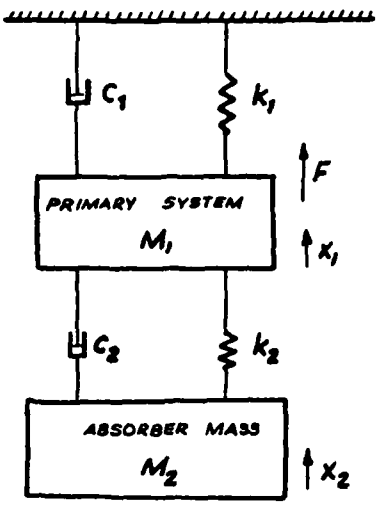


Figure 5. Vibration Absorber for a Damped System

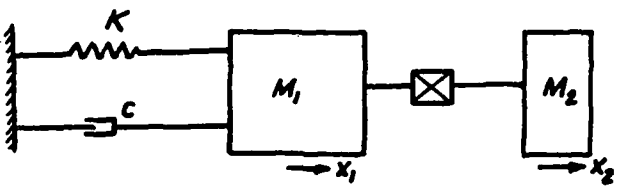
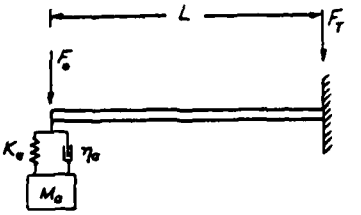
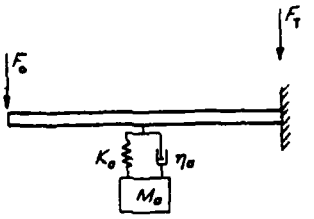


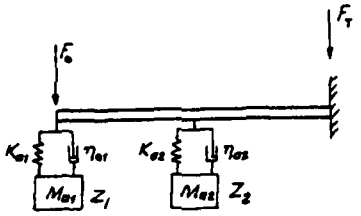
Figure 6. Vibration Absorber with Nonlinear Spring



a. Attached at end



b. Attached at midpoint



c. Attached at the end and midpoint

K_a = absorber spring constant

η_a = absorber damping constant

M_a = absorber mass

Z = absorber impedance

Figure 7. Dynamic Absorbers Attached to a Cantilever Beam Excited by a Sinusoidally Varying Force at its Free End

Optimal Vibration Reduction

Over a Frequency Range

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OPTIMAL VIBRATION REDUCTION OVER A FREQUENCY RANGE

Abstract. This is a review of optimal vibration reduction techniques for systems subject to harmonic excitation over a frequency range. Only passive means of control are considered. The objective functions used for optimization are restricted to those which relate directly to some measure of frequency response. Other common optimization goals such as weight minimization with response constraints are not included in this survey.

CONVENTIONAL DYNAMIC VIBRATION ABSORBER

The simplest device used to attenuate the steady-state vibration of a mechanical system over a frequency range is the conventional dynamic absorber. In the classical analysis,the main system is modelled as a mass resonating on a spring (Figure 1). The absorber is tuned to resonate such that the motion if its mass m_2 becomes relatively large and the motion of the main mass m_1 is minimized. The first analysis of the absorber is usually attributed to Ormondroyd and Den Hartog [1]. A detailed discussion of optimal tuning and damping parameters is given in Den Hartog's book [2]. To write the equations of motion in a dimensionless form, the following symbols are introduced

$$
\begin{aligned}
x_{st} &= F/k_1 && \text{= static displacement of main mass produced by a force F}\\
\mu &= m_2/m_1 && \text{= mass ratio = absorber mass/main mass}\\
\Omega_n &= (k_1/m_1)^{1/2} && \text{= uncoupled natural frequency of main system}\\
\omega_n &= (k_2/m_2)^{1/2} && \text{= uncoupled natural frequency of damper system}
\end{aligned}
\qquad (1)
$$

$$
\begin{aligned}
f &= \omega_n/\Omega_n && \text{= ratio of natural frequencies}\\
g &= \omega/\Omega_n && \text{= ratio of the exciting frequency to the uncoupled natural frequency of main system}\\
C_c &= 2m_2\Omega_n && \text{= critical damping}\\
\xi &= c_2/C_c && \text{= damping ratio}\\
z_i &= x_i/x_{st},\ (i = 1,2) && \text{= displacement ratio}
\end{aligned}
$$

With this notation the equations of motion in the frequency domain may be written

$$
\begin{aligned}
(1 - g^2)z_1 + \mu f^2(z_1 - z_2) + j2\mu\xi g(z_1 - z_2) &= 1\\
-g^2 z_2 + f^2(z_2 - z_1) + j2g\xi(z_2 - z_1) &= 0
\end{aligned}
\qquad (2)
$$

which gives

$$z_1 = \frac{A + j\xi B}{C + j\xi D} \tag{3}$$

where

$$\begin{aligned} A &= f^2 - g^2 \\ B &= 2g \\ C &= -\mu f^2 g^2 + (g^2 - 1)(g^2 - f^2) \\ D &= 2g(1 - g^2 - \mu g^2) \end{aligned} \tag{4}$$

If A/C = B/D, then z_1 becomes equal to A/C which is independent of damping ξ. For fixed μ and f, the condition A/C = B/D yields a quadratic equation in g^2 the two roots of which are the frequencies g_1, g_2 where the amplitude $|z_1|$ of the main mass is independent of damping. The points $(g_1, |z_1(g_1)|)$ and $(g_2, |z_1(g_2)|)$ on the frequency response curve $|z_1(g)|$ are called fixed points or invariant points. Den Hartog optimizes absorber performance by first choosing $f = \omega_n / \omega_n$ (tuning) so that the fixed points are adjusted to equal height and then finding the value of ξ (optimum damping) to make the frequency response curve pass through one of the fixed points with a horizontal tangent. His result for optimum tuning is

$$f = \frac{1}{1 + \mu} \tag{5}$$

To find optimum damping for fixed μ, substitute equation (5) into equation (3), find the derivative of $|z_1|$ with respect to g, then evaluate at one of the fixed points, say g_1, and equate the result to zero. From the equation thus obtained, the damping ratio may be found as

$$\xi^2 = \frac{\mu(3 - \sqrt{\mu(\mu + 2)})}{8(1 + \mu)^3} \tag{6}$$

On the other hand, if the fixed point chosen is the one at g_2, the resulting damping ratio is given by

$$\xi^2 = \frac{\mu(3 + \sqrt{\mu/(\mu + 2)})}{8(1 + \mu)^3} \tag{7}$$

Den Hartog recommends the use of the average value

$$\zeta^2 = \frac{3\mu}{8(1+\mu)^3} \tag{8}$$

EXTENSIONS OF THE CONVENTIONAL DYNAMIC ABSORBER

Lewis [3] extended this optimization procedure to multiple degree-of-freedom, discrete, undamped systems to which a conventional viscous vibration absorber is attached. For such a system with N masses, Lewis proves the existence of 2N-2 fixed points. Thus, if the designer selects a resonant peak of primary interest, the adjacent invariant points in the vicinity of this peak can be adjusted as described above to obtain optimum tuning and damping. As a particular case, Lewis analyzes the two-mass system shown in Figure 2 and shows that the invariant points occur at the frequencies

$$\omega^2_{a,b} = \frac{K(M_1 + M_2 + m)}{M_1(M_2 + m)} \pm \frac{K\sqrt{2M_1M_2m(M_1 + M_2 + m)}}{2M_1M_2(M_2 + m)} \tag{9}$$

with optimum tuning given by

$$\frac{k}{K} = \frac{m(2M_1M_2 + 2M_2^2 + 2M_2m - M_1m)}{2M_1(M_2 + m)^2} \tag{10}$$

and optimum damping by

$$c^2 = \frac{K(8M_1M_2 + 8M_2^2 + 8M_2m - M_1m)m^3}{4(M_2 + m)^3(M_1 + M_2 + m)} \tag{11}$$

Snowdon [4] modified the conventional absorber by adding a spring in series with the damper and showed that the resulting device, called a three-element dynamic absorber, active through a greater frequency range, reduced transmissibility (absolute value of the ratio of main mass displacement to the prescribed displacement of a foundation attached to the main mass by a spring) more effectively at the center of this range. In the same paper, Snowdon introduced dual absorbers to create a pronounced trough in the transmissibility curve while avoiding the two large compensating resonant peaks that appear in the transmissibility curve of the single absorber. Snowdon attaches these absorbers to an undamped single degree-of-freedom system and provides design information for them in graphical form. Two of his curves have been reproduced here to show the changes in transmissibility (Figures 3 and 4).

Randall et. al. [5] provided optimum absorber design curves for primary systems with damping, (Figure 5). They proved that once damping is introduced into the main system, invariant points no longer exist. Defining a performance measure G by

$$G = \max_{\omega} |x_1(\omega)| \tag{12}$$

they used a numerical search to compute tuning and damping parameters that minimize G. Soom and Lee [6] solved the same problem by nonlinear programming and investigated the possibility of using nonlinear springs to improve broad band response. They also tried several different objective functions, for example

$$\begin{aligned} G_1 &= \sum_{\omega}(|x_1(\omega) - 1|)^2 \quad \text{for } \omega \text{ such that } |x_1(\omega)| > 1 \\ G_2 &= \max_{\omega}(\omega|x_1(\omega)|) \\ G_3 &= \sum_{\omega}|x_1(\omega)|^2 \\ G_4 &= \sum_{\omega} \omega|x_1(\omega)|^2 \end{aligned} \tag{13}$$

Nonlinearities have also been treated by Roberson [7] and Arnold [8]. Roberson considered the system shown in Figure 6 where the spring between the masses M_1 and M_2 is nonlinear. Its load-deflection curve is the sum of a linear and a cubic term. Since the absorber is undamped in this case, its effectiveness is limited to a small frequency range. Roberson's synthesis criterion is the maximization of this suppression band. Arnold [8] studied the same system as Roberson but set c = 0. He provides frquency response curves of the system for hardening and softening coupling springs as well as for a softening spring designed according to Roberson's optimum system-parameter specifications. In both [7] and [8] the restoring force in the nonlinear spring as a function of spring extension is taken of the form

$$R(x) = c(x \pm \mu^2 x^3) \tag{14}$$

where the plus sign indicates a hardening spring characteristic and the minus sign indicates a softening one.

Kwak et. al. [9] and Haug and Arora [10] used a steepest descent algorithm to solve the classical absorber problem over a finite frequency range. Their objective function is

$$G = \max_{a<\omega<b} |z_1(\omega)| \tag{15}$$

where z_1 is as defined in equation (1) and the frequency interval of interest is [a,b]. They minimize G subject to the constraints

$$\max_{a<\omega<b} \left| \frac{x_2 - x_1}{x_1} \right| < Q_{max} \tag{16}$$

$$f_{MIN} < f < f_{MAX} \tag{17}$$

$$\zeta_{MIN} < \zeta < \zeta_{MAX} \tag{18}$$

where the notation is as in equation (1). They showed that for certain finite frequency intervals, designs superior to Den Hartog's infinite frequency range optimum may be found.

SYSTEMS WITH CONTINUOUS MEMBERS

Plunkett [11] proved the existence of invariant points for undamped continuous systems to which a single discrete damper has been connected and exploited this property to determine optimum damping for these systems. His approach is a generalization of the methods of Den Hartog [1,2] and Lewis [3]. He considers the vibration velocity at one point of a linear system resulting from a sinusoidal force at another point. Let the force applied at a point 1 be F_1 and v_1 be the velocity at the same point. Suppose a damper with damping constant c is applied across points with relative velocity v_2 and let the unknown vibration velocity at the point of interest be v_3. Plunkett shows that

$$\frac{v_3}{F_1} = jb_1 \frac{1 + jb_2c}{1 + jb_3c} \tag{19}$$

and notes that equation (19) has the same form as Den Hartog's equation (3). Thus when $b_2/b_3 = 1$, the ratio v_3/F_1 is independent of c and has the invariant points property described above in conjunction with the conventional dynamic vibration absorber. The value of c that gives a zero slope of the amplitude $|v_3/F_1|$ with respect to frequency ω at the invariant value of $|v_3/F_1|$ is an optimum value for c. Henney and Raney [12] applied a similar technique to optimize damping for vibrating uniform beams and studied the sensitivity of maximum displacement response to deviation from optimum damping.

Snowdon [13] ,[14] described how absorbers could reduce the force transmitted to the terminations of undamped cantilever beams at the resonant frequencies of beam vibration. The systems under consideration are shown in Figure 7. The force transmissibility T across the beam is defined as follows

$$T = |F_T/F_o| \tag{20}$$

where F_T is the force transmitted to the fixed end and F_o is the applied force. When a single absorber is used, as in Figures 7a and 7b, essentially the same procedure as that used for the conventional absorber is applicable. The fixed points are located by comparing transmissibility curves obtained for zero and infinite damping. The frequency ratio is then chosen so that the fixed points lie on transmissibility curve actually take the equal values of transmissibility at the fixed points. When two absorbers are applied simultaneously to the beam, Figure 7c, Snowdon's approach is to assign each absorber the values of optimum tuning and damping that were determined when the absorber was attached, individually, to its present position on the beam. Optimum supression of both the first and the second beam resonances results from this procedure.

Jacquot and Foster [15] considered an undamped single-degree-of-freedom system equipped with a damped cantilever beam serving as a vibration absorber. The authors developed an approximate system dynamics by an assumed modes approach retaining only one mode of the beam. Since the resulting equations were of the form obtained for the conventional absorber problem, the same tuning and damping approach was applicable. Snowdon [16] analyzed platelike absorbers attached either to a lumped mass-spring system or to a plate with small internal damping and presented optimum design parameters in graphical form.

Jacquot [17] appended a discrete absorber with structural or viscous damping to a beam, Figure 8. Using an assumed mode approach as in [15], he derived an approximate frequency response given by

$$\gamma(x,\lambda) = \frac{\epsilon_i \varphi_i(x)}{EI\beta_i^4 L} \frac{T^2 - \lambda^2}{\lambda^4 - \lambda^2(1 + T^2(1 + \mu\varphi_i^2(a)) + T^2} \tag{21}$$

where

$$\lambda = (\omega/\beta_1^2)\sqrt{\rho A/EI}$$

$$T^2 = (k/m)(\rho A/EI\beta_1^4) \tag{22}$$

$$\mu = m/\rho AL$$

and β_i, ρ_i are the beam mode shapes and eigenvalues satisfying

$$\frac{d^4\rho_i}{dx^4} = \beta_i\rho_i \tag{23}$$

Because the form of equation (21) is the same as that found in the conventional absorber problem with the exception that μ is replaced by $\mu\rho^2(a)$, Jacquot was able to tune and damp to flatten the fundamental resonance response using Den Hartog's procedure.

Warburton and Ayorinde [18] extended Jacquot's representative mode method to plates and shells by defining appropriate effective mass and stiffness for such elastic bodies. The accuracy of this single-mode approximation was favorably affected if adjacent resonant frequencies were well separated from the natural frequency for which the absorber was being tuned and adversely affected as the absorber size increased. In a companion paper [19], the authors considered cylindrical shells as examples of dynamically complex structures, for which the ratio of adjacent natural frequencies tends toward unity. They showed that as dynamic complexity increased optimum absorber parameters deviated from those calculated for an equivalent single-degree-of-freedom system.

MULTIPARAMETER DESIGN BY NONLINEAR PROGRAMMING

McMunn and Plunkett [20] developed a computational method to optimize multiple dampers for large mechanical systems. Damping was defined to be optimum if the maximum response over a range of excitation frequencies was minimized. The response function of interest, for example

$$f(\omega, \underline{c}) = \left| \frac{x_i}{P_j} \right|^2 \tag{24}$$

where x_i is the displacement of the ith mass and P_j is the force on the jth mass, is first maximized with respect to excitation frequency ω using

$$\frac{\partial f}{\partial \omega} = 0 \qquad \frac{\partial^2 f}{\partial \omega^2} < 0 \tag{25}$$

and then minimized with respect to the vector $\underline{c}$ of damping values

$$\min_{\underline{c}} \quad f(\omega, \underline{c}) \tag{26}$$

In fact, the minimization over $\underline{c}$ also considers values of f at $\omega = \Omega_1$, and $\omega = \Omega_2$ if a finite frequency interval $[\Omega_1, \Omega_2]$ is being considered. Two multiple-degree-of-freedom, multiple-damper discrete systems and a column with complex modulus damping were studied as examples using this approach. A similar approach was adopted by Ng and Cunniff [21] who designed a three degree-of-freedom isolation system and verified their results by

experimental tests. In their formulation, the authors define a primitive function

$$|\underline{x}|^2 = (P^{-1}\underline{f})(P^{-1}\underline{f})^* \tag{27}$$

where * denotes complex conjugate transpose,

$$P = -\omega^2 M + j\omega C + K \tag{28}$$

and $\underline{f}$ is the force vector. Their first step is to minimize over damping variables the primitive function maximized over frequency

$$\min_{c} \max_{\omega} |\underline{x}'|^2 \tag{29}$$

They repeat this procedure until an optimum is reached

$$|\underline{x}|^2_{opt} = \min_{c^n} \max_{c^n} |\underline{x}^n|^2 \tag{30}$$

Dale and Cohen [22] extended the method of McMunn and plunkett to continuous systems whose steady-state equations of motion could be reduced to a set of ordinary differential equations containing spatial coordinates as independent variables. Both dissipative and nondissipative design parameters were included.

Lunden [23] presented a nonlinear programming solution based on a sequential unconstrained minimization technique to the problem of determining a continuous damping distribution which minimizes the maximum response of a vibrating beam over a specified frequency interval. In a second paper [24], the author applied the same approach to vibrating frames. In these references, the maximum response F in the frequency interval studied is written as

$$F(\eta^d, \eta^s) = \max_{a \le \omega \le b} f(\eta^d, \eta^s, \omega) \tag{31}$$

where η^d denotes distributed structural damping and η^s denotes the structural damping constants for discrete springs in the system. An exact displacement method is used with hysteretic damping introduced by the loss factor η^d giving a complex bending stiffness $EI(1 + j\eta^d)$. The 4 x 4 stiffness matrix for a beam element then takes the form

$$\underline{F} = EI(1 + j\eta^d)K\,\underline{x} \tag{32}$$

where K is a 4 x 4 matrix of transcendental functions of frequency.

Kitis [25] utilizes structural reanalysis and modal techniques with nonlinear programming to make tractable problems in which the systems under consideration contain a large number of degrees of freedom. The repetitive computations of response required in the nonlinear programming portion of the optimal design are carried out using efficient reanalysis methods or condensed eigenproblem solutions so that computation time in the structural analysis phase of the design is reduced.

IMPEDANCE MATCHING

The progressive wave solution to the wave equation for continuous chain-like systems has been used to minimize vibratory response over a frequency range by means of "impedance matching". To illustrate the basic idea of this technique consider the rod shown in Figure 9, [14]. Designating the displacement at any point x by ξ, the wave equation may be written as

$$\frac{\partial^2 \xi}{\partial x^2} + n^2 \xi = 0 \tag{33}$$

where

$$n^2 = \rho \frac{\omega^2}{E} \tag{34}$$

If the end of the rod at x = 0 is subjected to a sinusoidally varying force F_o, then the driving point impendance Z_o is given by

$$Z_o = \frac{F_o}{(\partial E/\partial t)_{x=0}} = \frac{F_o}{j\omega E(x=0)} \tag{35}$$

Snowdon [14] shows that the impedance Z_o defined in equation (35), may be written as

$$Z_o = A\sqrt{\rho E}\ \ \frac{1 - Re^{j\varphi}}{1 + Re^{j\varphi}} \tag{36}$$

where $Re^{j\varphi}$ describes the relative magnitude of and the phase difference between the incident and the reflected waves. The characteristic impedance Z_{ch} is defined as the impedance of an infinitely long rod in which reflections do not occur, the value of Z_{ch} is obtained from equation (36) by equating R to zero

$$Z_{ch} = A\sqrt{\rho E} \tag{37}$$

A matched condition will occur if a damper of damping constant Z_{ch} is attached to the rod at x = 0. This attachment causes the ratio R of reflected to incident wave to be zero so that no vibration response buildup due to reflected waves is possible. This idea has been applied to shafts on supports [26]. Here the dynamic response is expressed in progressive wave form like electrical response waves in transmisison line theory. Waveforms for a uniform shaft flexibly supported on two rotational and translational mass-spring-damper units at the ends and one such unit in the interior are obtained. The terminating impedance is made equal to the characteristic impedance of the shaft to obtain a matched condition.

LIMITING PERFORMANCE METHODS

A limiting performance approach was applied in Refs. [27] and [28] to the optimal design of vibratory systems over a frequency range. In this approach, instead of fixing the design configuration at the outset, those parts of the system to be designed are replaced by control forces. Then, for a selected cost function and design constraint the absolute optimal performance of the system is computed by solving an optimization problem in which the control forces are unknowns. The solution is called the limiting performance of the system. After the limiting performance characteristics have been found, the designer can choose a prospective configuration for the part of the system to be designed and apply parameter identification techniques to find optimum design variable values so that the designed system responds as closely as possible to the limiting performance response. The limiting performance characteristics are found by linear programming and parameter identification can be accomplished by such curve fitting techniques as least squares. This two-stage procedure has been demonstrated for the optimal design of rotor suspension systems in [27].

BOOKS AND MONOGRAPHS

The book by Haug and Arora [10] contains considerable material applicable to frequency response shaping, although specialized aspects are not treated in detail. Two other useful references are the monograph by Sevin and Pilkey [28] and the book by Snowdon [14]. Sevin and Pilkey present an introduction to the subject and a summary of the state-of-the art up to 1971. Snowdon's book contains a wealth of information on vibration absorbers and reduction of beam vibrations.

ACKNOWLEDGMENT

This work was supported by the Office of Naval Research, Arlington, Virginia.

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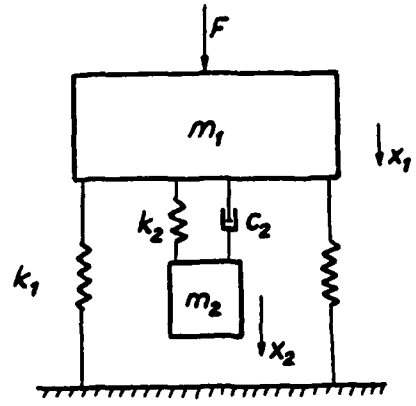


Figure 1. A Mass-Spring System with a Conventional Dynamic Vibration Absorber

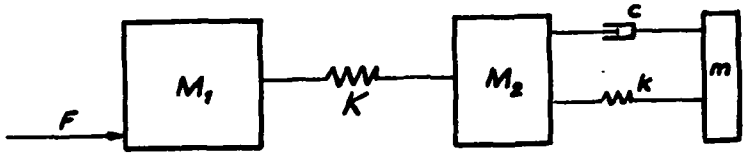
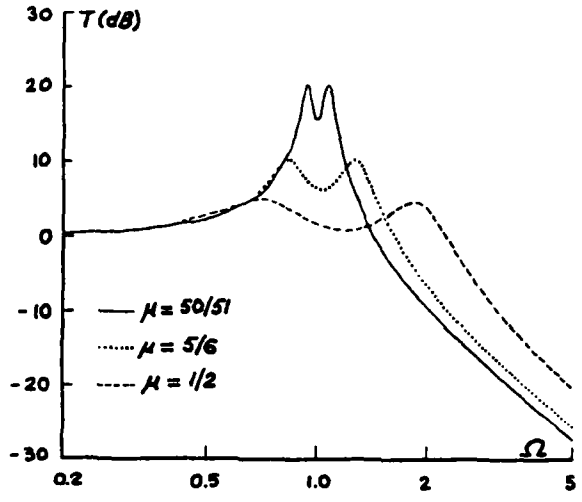


Figure 2. Two-Mass System with Tuned Damper



$\Omega = \frac{\omega}{\omega_o}$, $\omega_o^2 = \frac{K_1}{M_1+M_2}$, K_1 = spring constant of main system, M_1 = mass of main system, M_2 = mass of absorber, ω = excitation frequency.

Figure 3. Transmissibility of the Three-Element Dynamic Absorber

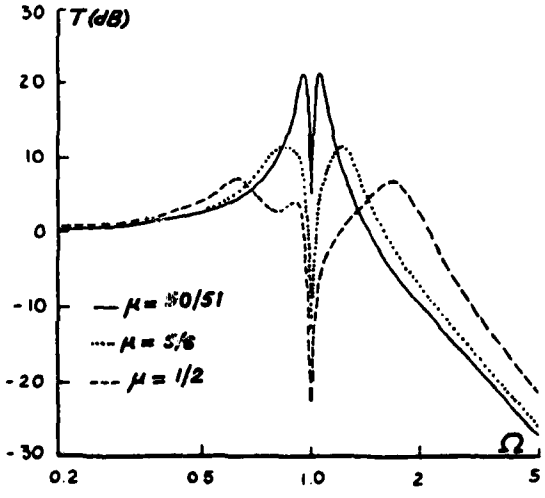


Figure 4. Transmissibility of the Dual Absorber

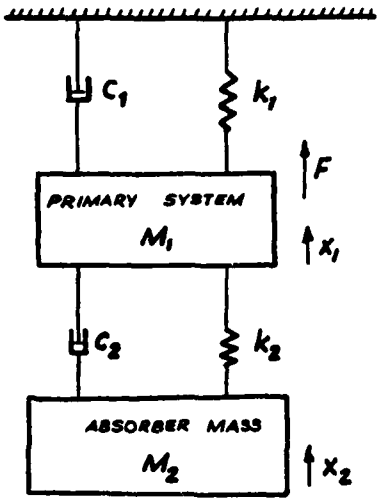


Figure 5. Vibration Absorber for a Damped System

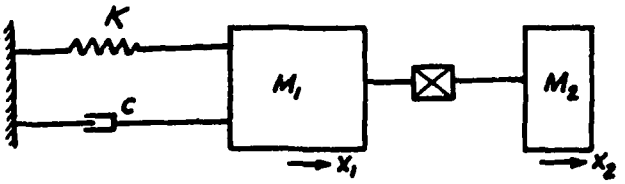
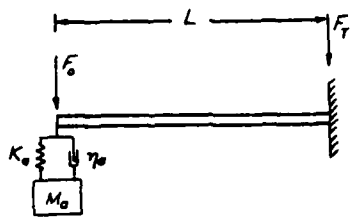
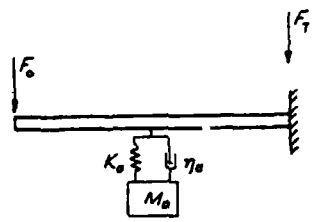


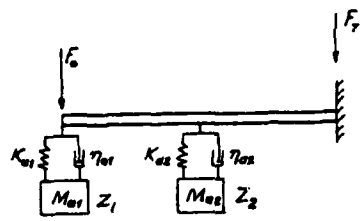
Figure 6. Vibration Absorber with Nonlinear Spring



a. Attached at end



b. Attached at midpoint



c. Attached at the end and midpoint

K_a = absorber spring constant

η_a = absorber damping constant

M_a = absorber mass

Z = absorber impedance

Figure 7. Dynamic Absorbers Attached to a Cantilever Beam Excited by a Sinusoidally Varying Force at its Free End

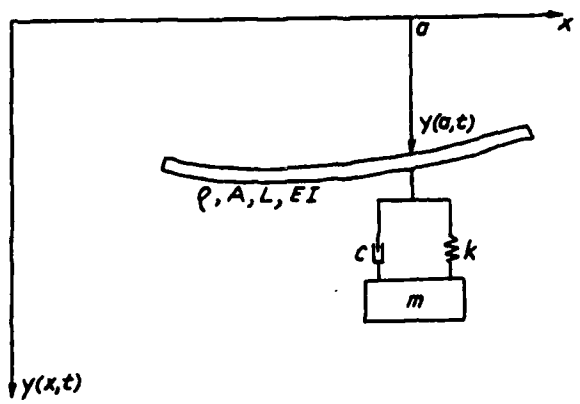


Figure 8. Euler-Bernoulli Beam with Dynamic Vibration Absorber

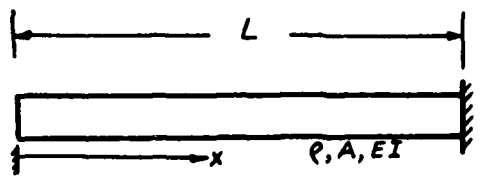


Figure 9. Rod Under Longitudinal Vibration

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