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DETERMINING THE PROBABILITY OF AT LEAST ONE SUCCESS IN TRIALS CONDUCTED ON THE LIGHTED PORTION OF A STAR SHAPED CURVE SUBJECT TO A POISSON SHADOWING PROCESS

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DETERMINING THE PROBABILITY OF AT LEAST ONE SUCCESS IN TRIALS CONDUCTED ON THE LIGHTED PORTION OF A STAR SHAPED CURVE SUBJECT TO A POISSON SHADOWING PROCESS

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Abstract

A star shaped curve, C, in the plane is subject to a Poisson shadowing process. According to this process, disks of random size appear at random locations in a region between a source of light, which is at the origin, and the curve C. These disks cast shadows on C. Trials are conducted along the lighted portion of C. Each trial requires a fixed length, ℓ, of C. The different trials are independent and have a fixed probability, p, of success. The number of trials conducted along C is a random variable, N, which depends on the random length of the lighted portion of C. The success probability is $P = 1 - E\{q^N\}$, where $q=1-p$. Lower and upper bounds for P are derived. A numerical example shows cases in which these bounds could be very close.

Key Words: *Poisson Shadowing Process, Random Fields, Measure of Visibility, Moments of Visibility, Success Probabilities*

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1. Introduction

Consicer a star-shaped curve in the plane, C , and a source of light at the origin, 0 . If there are no obstacles between the origin and the curve C , the whole curve is in the light or visible. Certain experiments (trials) can be conducted along the lighted portion of the curve. Each such trial requires a length ℓ of C , and the probability of its success is p , $0<p<1$. Let $L\{C\}$ be the total length of C . If C is completely visible, $N = [L\{C\}/\ell]$ trials can be conducted. ([a] designates the integer part of a.) Assume that all these trials are independent, having the same success probability, p , (Bernoulli trials). Thus, the probability of at least one success, when C is completely lighted, is $S = 1 - q^N$, where $q = 1 - p$. In reality, C may not be completely visible, due to shadows cast on it by objects (disks, say), which are randomly dispersed in a region between 0 and C . The centers of the disks follow a Poisson process and their diameters are of random size. Thus, the total number of trials that can be performed along the visible portion of C is the random variable

$N = \sum_{i=1}^{K} [X_i/\ell]$, where K is the random number of connected subsets (disjoint segments) of C , which are visible, and X_i is the length of the i-th such subset. The probability of at least one success, under this random shadowing, is $P = 1 - Q$, where $Q = E\{q^N\}$. The present study develops a method for determining upper and lower bounds for Q . This method is based on the methodology developed by Yadin and Zacks [1], for determining the moments of the total visible portion of C ,

$V\{C\} = \sum_{i=1}^{k} X_i$. We show that a lower bound for Q (an upper bound for P , respectively) is the value of the moment generating function (MGF) of $V\{C\}$, at the point $N = \ell nq/\ell$. The upper bound for Q (lower bound for P) can be obtained by considering $Q^* = E\{q^{N*}\}$, where

$N = \sum_{i=1}^{k} \left(\frac{X_i-\ell}{\ell}\right)_+$, and $a_+^* = \max(a,0)$. In section 2 we present the structure of the Poisson field of shadowing disks and review the main results of [1], which lead to the MGF of $V\{C\}$. In Section 3 we present lower and upper bounds for Q, which are based on the MGF of $V\{C\}$ and $V^*\{C\} = \sum_{i=1}^{K} (X_i-\ell)_+$.

Section 4 presents a numerical example for annular regions and a standard-uniform Poisson field of shadowing disks. It is shown, in the numerical example of Section 4, that the lower and upper bounds for Q developed in the present paper could be very close and effective. The present paper is very tightly linked with our previous paper [1], in which the general theory for the determination of the moments of visibility, $\mu_n = E\{V^n\{C\}\}$, is presented. As such it could be considered as an extension of [1] for an important class of applications.

2. The Poisson Field of Shadowing Obstacles, and the Moments of $V\{C\}$.

Consider a star-shaped curve, C , given by a function $r(s)$ on an interval $[s',s'']$, i.e.,

$$C = \{(\rho,\theta) \; ; \; \rho = r(\theta) \; , \; s' \le \theta \le s''\} \quad . \tag{2.1}$$

We further assume that shadows on C are cast by disks, which are randomly distributed within a region, C_1 , bounded by the curves
$U = \{(\rho,\theta) \; ; \; \rho = u(\theta) \; , \; s^* \le \theta \le s^{**}\}$ and
$W = \{(\rho,\theta) \; ; \; \rho = w(\theta) \; , \; s^* \le \theta \le s^{**}\}$. Each disk is characterized by a point (ρ,θ,y) , where (ρ,θ) are the polar coordinates of its center and y is its diameter. It is assumed that the centers are uniformly distributed between U and W and the diameters of the disks are i.i.d. random variables having a c.d.f. $G(y)$ concentrated on $[a,b]$ (the standard case). Moreover, $u(\theta)$, $w(\theta)$ and b are such that both the origin, 0 , is uncovered and the curve C is not intersected by any one of the random disks. For the precise conditions see [1]. It is further assumed that the number of disks whose centers fall within a subset C of C_1 has a Poisson distribution, with mean A , when A is the area of C .

A point $P = (r(s),s)$ is said to be visible, if the ray $\overline{OP}$ is not intersected by any shadowing disk.

The measure of visibility $V\{C\}$ is defined as

$$V\{C\} = \int_{s'}^{s''} I(s)\ell(s)ds \quad , \tag{2.2}$$

where $I(s)=1$ if the point $(r(s),s)$ is in the light (visible), and $I(s)=0$ otherwise. $\ell(s)ds = [r^2(s) + (r'(s))^2]^{1/2}ds$ is the infinitismal length of C at $(r(s),s)$. The moments of $V\{C\}$ where expressed in [1] in terms of the K-functions, $K_-(s,t)$ and $K_+(s,t)$. $\mu K_-(s,t)$ and $\mu K_+(s,t)$ are, respectively, the expected number of disks in C_1 , whose centers have orientation coordinates in $[s-t,s]$ ($[s,s+t]$, resp.), and which do not intersect the ray with orientation s . It is shown in [1] that these functions are given by

$$K_-(s,t) = \int_{s-t}^{s} \int_{u(\theta)}^{w(\theta)} G(y(\rho,s-\theta))\rho d\rho d\theta$$

and (2.3)

$$K_+(s,t) = \int_{s}^{s+t} \int_{u(\theta)}^{w(\theta)} G(y(\rho,\theta-s))\rho d\rho d\theta \quad ,$$

where

$$y(\rho,\theta-s) : y(\rho,s-\theta) = \begin{cases} 2\rho\sin|s-\theta| & , \text{ if } |s-\theta|<\pi/2 \\ 2\rho & , \text{ if } |s-\theta|\geq\pi/2 \end{cases} \tag{2.4}$$

is the maximal diameter of a disk centered at (ρ,θ), which does not intersect the ray with orientation s.

It is shown in [1] that the n-th moment of $V\{C\}$ is

$$\mu_n = n! \int_{s'\leq s_1\leq\ldots\leq s_n\leq s''} \cdots \int p(s_1,\ldots,s_n) \prod_{i=1}^{n} \ell(s_i)ds_i \quad , \tag{2.5}$$

where $p(s_1,\ldots,s_n)$ is the probability that n points on C, with orientation coordinates $s_1,\ldots,s_n$ are simultaneously visible. It is further shown that

$$p(s_1,\ldots,s_n) = \exp\left\{\nu\{C_1\}\right\} \exp\left\{\mu K_-(s_1,s_1-s^*) + \mu K_+(s_n,s^{**}-s_n) + \mu \sum_{i=1}^{n-1} \left(K_+(s_i, \frac{s_{i+1}-s_i}{2}) + K_-(s_{i+1}, \frac{s_{i+1}-s_i}{2})\right)\right\} \quad , \tag{2.6}$$

in which

$$\nu\{C_1\} = \frac{\mu}{2} \int_{s^*}^{s^{**}} \left(w^2(s)-u^2(s)\right)ds \quad . \tag{2.7}$$

Furthermore, let

$$\Psi_o(s) = \exp\{\mu K_-(s,s-s')\} \quad , \tag{2.8}$$

and define recursively, for $j \geq 1$

$$\Psi_j(s) = \int_{s'}^{s} \ell(y)\Psi_{j-1}(y) \exp\{\mu K_-(y,\frac{s-y}{2}) + \mu K_+(y,\frac{s-y}{2})\}dy \tag{2.9}$$

Then

$$\mu_n = n! \exp\left\{-\nu\{C_1\}\right\} \int_{s'}^{s''} \ell(s)\Psi_{n-1}(s) \exp\{\mu K_+(s,s^{**}-s)\}ds \quad . \tag{2.10}$$

3. The MGF of $V\{C\}$ and The Bounds for Q

In Section 1 we introduced the random variables $K, X_1, X_2, \ldots, X_K$, which are the number of lighted (visible) disjoint segments of C, and their length. Accordingly,

$V\{C\} = \sum_{i=1}^{K} X_i$. We also defined the random variable $N = \sum_{i=1}^{k} \left[X_i/\ell \right]$.

Thus $N \le V\{C\}/\ell$, with probability one. It follows, for every q, $0<q<1$, that

$$Q = E\{q^N\} \ge E\left\{q^{V\{C\}/\ell}\right\} = M_V(\ln q/\ell) \quad , \tag{3.1}$$

where $M_V(u)$ is the MGF of $V\{C\}$. Thus, $M_V(\ln g/\ell)$ is a lower bound for Q. Notice that, since $V(C) \le L(C) < \infty$, all the moments of $V(C)$ are bounded by powers of $L(C)$. Hence, the MGF of $V(C)$ can be expressed as

$$M_V(u) = \sum_{i=0}^{\infty} \frac{u^i}{i!} \mu_i \qquad , -\infty<u<\infty \quad . \tag{3.2}$$

For the derivation of an upper bound for Q, we consider the random variable $N^* = \sum_{i=1}^{k} \frac{X_i-\ell}{\ell}_+$. Since $N^* < N$ with probability one,

$$Q^* = E\left\{q^{N^*}\right\} \ge Q \quad . \tag{3.3}$$

In order to obtain Q^* we define a new visibility measure

$$V^*(C) = \sum_{i=1}^{K} (X_i-\ell)_+ \tag{3.4}$$

The moments of $V^*(C)$ can be obtained by the formulae presented in Section 2, in which the K-functions in (2.6) - (2.10) are modified in the following manner. Replace $K_-(s,t)$ and $K_+(s,t)$ by $K_-\left(s-\tau_1(s)\right.$, $\left.(t-\tau_1(s))_+\right)$ and $K_+\left(s+\tau_1(s)\right.$, $\left.(t-\tau_1(s))_+\right)$, respectively, where $\tau_1(s)$ should be determined so that

$$\int_{s-\tau_1(s)}^{s} \ell(s)ds = \ell/2 \tag{3.5}$$

and, similarly, $\tau_2(s)$ should satisfy the equation

$$\int_{s}^{s+\tau_2(s)} \ell(s)ds = \ell/2 \quad . \tag{3.6}$$

By definition, $K_{\pm}(s,0) = 0$ for all s .

More specifically, let μ_n^* $(n=1,2,\ldots)$ be the n-th moment of $V^*(C)$, which is given by

$$\mu_n^* = n! \int \underset{S}{\cdots} \int p^*(s_1,\ldots,s_n) \prod_{i=1}^{n} \ell(s_i)ds_i \quad , \tag{3.7}$$

where $\qquad S = \left\{s' \le s_1 \le \ldots \le s_n \le s''\right\}$.

The function $p^*(s_1,\ldots,s_n)$ is the probability that the union of n segments of C , each one of length ℓ , centered around the points $(r(s_1),s_i)$, $i=1,\ldots,n$, is completely visible. In other words, define the indicator function $I^*(s)$, which is equal to 1 if the segments of C of length ℓ , centered at $(r(s),s)$, is completely visible and is equal to 0 otherwise. Accordingly,

$$V^*(C) = \int_{s'}^{s''} I_{\ell}^*(s)\ell(s)ds \quad . \tag{3.8}$$

As explained above, $p^*(s_1,\ldots,s_n) = E\{\prod_{i=1}^{n} I^*(s_i)\}$. Following the theory developed in [1] , $p^*(s_1,\ldots,s_n)$ is given, as in (2.6), by

$$\begin{aligned} p^*(s_1,\ldots,s_n) = \exp\left\{-\nu\{C_1\}\right\} \exp\Bigg\{&\mu K_-\left(s_1-\tau_1(s_1)\ ,\ (s_1-\tau_1(s_1)-s)_+\right) \\ &+ \mu K_+\left(s_n+\tau_2(s_n)\ ,\ (s^{**}-s_n-\tau_2(s_n))_+\right)+ \\ \mu \sum_{i=1}^{n-1} &\left(K_+\left(s_i+\tau_2(s_i)\ ,\ \left(\frac{s_{i+1}-s_i}{2} - \tau_2(s_i)\right)_+\right)+\right. \\ &\left. K_-\left(s_i-\tau_1(s_i)\ ,\ \left(\frac{s_{i+1}-s_i}{2} - \tau_1(s_i)\right)_+\right)\right)\Bigg\} \quad . \end{aligned} \tag{3.9}$$

Finally, if $M_{V*}(u)$ denotes the MGF of $V^*(C)$, then $Q^* = M_{V*}(\ln q/\ell)$, which is the upper bound for Q . In the following section we illustrate these bounds in a special case.

4. Lower And Upper Bounds For Q in A Special Case

In the present section we exhibit the method of determining lower and upper bounds for the failure probability Q in the following special case. The curve C is an arc on a circle of radius r, centered at the origin, limited by rays having orientations s' and s'', $-\frac{\pi}{2} < s' \le s'' \le \frac{\pi}{2}$. The centers of the disks are distributed between U and W, where U and W are circles centered at the origin, with radii $0 < u < w < r$. Moreover, we assume that the centers of the disks are uniformly distributed within this annular region, and their random diameters, Y, are uniformly distributed between $[a,b]$ independently of their centers, where $0 < \frac{b}{2} \le u < w \le r - \frac{b}{2}$. This special case was previously studied in [1]. We have shown that in the present case, $K_-(s,t) = \hat{K}_+(s,t) \equiv \hat{K}(t)$. Explicit formulae for this function can be found in [1]. Notice that in the present case of C being a circular arc, $\tau_1(s) = \tau_2(s) = \ell/2r$ for all s. Accordingly, in the determination of Q^* we replace $\hat{K}(t)$ by $\hat{K}\,(t - \frac{\ell}{2r})_+$. The computation of the moments μ_n and μ_n^* follows the procedure described in [1].

Let $\{\eta_n\ ,\ n\ge1\}$ be the normalized moments of $V\{C\}$, i.e., $\eta_n = E\{V^n(C)/r^n(s''-s')^n\}$. The sequence $\{\eta_n\ ;\ n\ge1\}$ is decreasing and, as shown in [1], $\lim_{n\to\infty} \eta_n = P_1$, which is the probability of complete visibility of C. Furthermore, $M_V(\ln q/\ell) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\eta_n$, where $\theta = (\ln q)\, r(s''-s')/\ell$. One can approximate this MGF, to any degree of accuracy, in the following manner. Given an arbitrary ϵ, $\epsilon>0$, let m be a positive integer such that

$$\left(e^{|\theta|} - \sum_{i=0}^{m} \frac{|\theta|^i}{i!}\right)\left(\eta_m - P_1\right) \le \varepsilon \quad . \tag{4.1}$$

Let

$$\hat{M}_V\left(\frac{\ln q}{\ell}\right) = \sum_{i=0}^{m-1} \frac{\theta^i}{i!} \eta_i + \eta_m \left(e\tau - \sum_{i=0}^{m-1} \frac{\theta^i}{i!}\right) . \tag{4.2}$$

Then, $\delta = \left|\hat{M}_V\left(\frac{\ln q}{\ell}\right) - M_V\left(\frac{\ln q}{\ell}\right)\right| \le \varepsilon$. Indeed,

$$\delta = \left|\sum_{i=m+1}^{\infty} \frac{\theta^i}{i!}\left(\eta_m - \eta_i\right)\right| \le \sum_{i=m+1}^{\infty} \frac{|\theta|^i}{i!}\left[\eta_m - \eta_i\right]$$

$$\le \left(\eta_m - P_1\right)\left(e^{|\theta|} - \sum_{i=0}^{m} \frac{|\theta|^i}{i!}\right) \tag{4.3}$$

$M_{V*}(\frac{\ln q}{\ell})$ can be approximated in the same manner. In Table 1 we provide numerical values of the lower and upper bounds for Q , corresponding to the following parameters of an annular region: r=1.0 , w=.6 , u=.4 . The parameters of the distribution of Y are a=.1 and b=.5 . These bounds are given for two values of μ , two values of ℓ , two values of $\Delta = s'' - s'$, and q = .8 .

Table 1. Lower and Upper Bounds for Q , Circular Arc, C , And Annular Region of Disk Centers

	Δ = 120°		Δ = 60°	
	μ=1	μ=5	μ=1	μ=5
ℓ=.2	.126861 .117760	.258360 .210320	.353190 .341425	.502616 .454504
ℓ=.4	.356616 .337353	.521052 .439304	.596354 .580451	.721755 .661923

This table shows that in the present case the method developed here is very satisfactory.

REFERENCE

[1] Yadin, M. and Zacks, S. (1982). Visibility Probabilities and Moments of Measures of Visibility On Star Shaped Curves In The Plane For Poisson Shadowing Processes.
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A star shaped curve, C, in the plane is subject to a Poisson shadowing process. According to this process, disks of random size appear at random locations in a region between a source of light, which is at the origin, and the curve C. These disks cast shadows on C. Trials are conducted along the lighted portion of C. Each trial requires a fixed length, ℓ, of C. The different trials are independent and have a fixed probability, p, of success. The number of trials conducted along is a random variable, N, which depends

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20. ABSTRACT (continued)

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