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Leo Marcus

On the Equivalence of If-Then-Else and Max-Min

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CONTENTS

1. INTRODUCTION	1
2. THE CASE OF ONE VARIABLE	1
3. THE CASE OF TWO OR MORE VARIABLES	3

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1. INTRODUCTION

The construct if-then-else, in one syntactic guise or other, has become a bona fide logical connective in many computing environments.

For example, a definition by cases of the following form can be written (more or less directly) in many programming languages (let a be a bitstring of length l , with bit $l-1$ on the left and bit 0 on the right; let i, j be any integers; and let $a\langle i:j \rangle$ represent the substring of a consisting of bits i down to, and including, j):

$$\begin{aligned} \text{lh}(a\langle i:j \rangle) = & \text{if } i \geq l, \text{ then } \text{lh}(a\langle l-1:j \rangle) \\ & \text{else (if } j < 0, \text{ then } \text{lh}(a\langle i:0 \rangle) \\ & \text{else (if } i < j, \text{ then } 0 \\ & \text{else } i-j + 1)). \end{aligned}$$

This definition has an equivalent without explicit cases, but using max and min:

$$\text{lh}(a\langle i:j \rangle) = \min\{l, \max\{\min\{i, l-1\} - \max\{j, 0\} + 1, 0\}\}$$

A natural generalization of the above allows i and j to be any functions from integers to integers.

While it is obvious that any term of the form $\max(s,t)$ can be expressed as "if $s < t$, then t else s ," and that \min can be expressed similarly, it is natural to ask the converse question, namely, are max and min always sufficient substitutes for if-then-else, or is if-then-else strictly more expressive? Here we show that any term containing one variable built from linear functions and if-then-else is equivalent to a max-min expression, but this is not the case for terms containing two or more variables.

2. THE CASE OF ONE VARIABLE

To be precise, define the set of predicates and terms T_1, T_2 as follows:

1. If t_1, t_2 are terms, then $t_1 < t_2, t_1 = t_2$ are predicates.
2. $0, 1, x, y, \dots$ are (atomic) terms of T_1, T_2 .
3. All linear combinations (over the rationals) of terms in $T_1 (T_2)$ are in $T_1 (T_2)$.
4. If $t_1, t_2 \in T_1$, then $\max(t_1, t_2), \min(t_1, t_2) \in T_1$.
5. If $t_1, t_2 \in T_2$, and p is a predicate containing only terms of T_2 , then "if p , then t_1 else t_2 " $\in T_2$.

Theorem 1: If $t(x) \in T_2$ contains occurrences of only one variable, then there is $t'(x) \in T_1$ such that $t(x) = t'(x)$ over the rationals.

Proof: By induction on the structure of t .

Case 0 t is constant: trivial.

Case 1 $t = \text{"if } f(x) < 0, \text{ then } g(x) \text{ else } h(x)\text{"}$ where f, g, h are linear. We will find linear k_1 and k_2 such that

$$t = \min(\max(g(x), k_1(x)), \max(h(x), k_2(x))).$$

For this it is sufficient that

$$1. f(x) < 0 \Rightarrow k_2(x) \geq g(x), h(x) \text{ and } k_1(x) \leq g(x)$$

$$2. f(x) \geq 0 \Rightarrow k_2(x) \leq h(x) \text{ and } k_1(x) \geq g(x), h(x)$$

Since f is linear, $f(x) < 0$ is equivalent to $x < r$ (or $x > r$) for some r . Then k_2 can be taken to be any linear function such that $k_2(r-1) > g(r-1), h(r-1)$ and $k_2(r) < g(r), h(r)$; similarly for k_1 .

Case 2 $t = \text{"if } f(x) = 0, \text{ then } g(x) \text{ else } h(x)\text{"}$ where f, g, h are linear. As above, we may assume $f(x) = 0$ is equivalent to $x = r$ for some r . If $g(r) > h(r)$, then we can find k_1, k_2 such that

$$t = \max(h(x), \min(k_1(x), k_2(x))).$$

Simply take $k_2(r) = k_1(r) = g(r)$, $x < r \Rightarrow k_2(x) < h(x)$, and $x > r \Rightarrow k_1(x) > h(x)$.

If $g(r) < h(r)$, we can find k_1, k_2 such that

$$t = \min(h(x), \max(k_1(x), k_2(x))).$$

Induction case $t = \text{"if } f(x) < 0, \text{ then } g(x) \text{ else } h(x)\text{"}$ where $f, g, h \in T_2$, and their equivalents $f', g', h' \in T_1$ have already been defined.

Since $f(x)$ consists of a finite number of linear pieces, $f(x)$ changes sign a finite number of times. Thus there are $r_1 < r_2 < \dots < r_n$ such that $f(x)$ alternates from negative (say) to positive as x increases through the r_i . Thus t is equivalent to

"if $x < r_1$, then $g(x)$ else (if $x < r_2$, then $h(x)$ else (... else

(if $x < r_n$, then $g(x)$ (or $h(x)$) else $h(x)$ (or $g(x)$)...)"

First we find an equivalent to $t_n = \text{"if } x < r_n, \text{ then } g(x) \text{ else } h(x)\text{"}$ by finding $k_1(x), k_2(x)$ such that t_n is equivalent to

$$t'_n = \min(\max(g'(x), k_1(x)), \max(h'(x), k_2(x)))$$

by satisfying conditions similar to those used in case 1. Then we find equivalents for $t_{n-1} = \text{"if } x < r_{n-1}, \text{ then } h(x) \text{ else } t'_n\text{"}$ and so on until t' is found.

The case of "if $f(x) = 0$, then $g(x)$ else $h(x)$ " for general f, g, h is treated similarly.

□

3. THE CASE OF TWO OR MORE VARIABLES

Theorem 2: There is $t \in T_2$ containing two variables which has no equivalent T_1 term.

Proof: The conclusion follows from the lemma below by taking $t = \text{"if } x < 0, \text{ then } y \text{ else } 0,\text{"}$ a term of T_2 not satisfying the condition of the lemma (look at the points $(-1, M)$ and $(0, M)$).

Lemma: For all $t(x, y) \in T_1$ there is $M > 0$ such that for all x, y $|t(x, y) - t(x + 1, y)| < M$.

Proof: By induction on the structure of terms in T_1 .

If t is atomic, clear.

Let $t = t_1 + t_2$, where the claim is true for t_1, t_2 (with constants M_1, M_2). Then

$$\begin{aligned} |t(x, y) - t(x + 1, y)| &= |t_1(x, y) + t_2(x, y) - t_1(x + 1, y) - t_2(x + 1, y)| \leq \\ &\leq |t_1(x, y) - t_1(x + 1, y)| + |t_2(x, y) - t_2(x + 1, y)| < M_1 + M_2. \end{aligned}$$

So we can take $M = M_1 + M_2$.

Let $t = \max(t_1, t_2)$. Then

$$\begin{aligned} |t(x, y) - t(x + 1, y)| &= |\max(t_1(x, y), t_2(x, y)) - \max(t_1(x + 1, y), t_2(x + 1, y))| \leq \max(|t_1(x, y) - t_1(x + 1, y)|, \\ &|t_2(x, y) - t_2(x + 1, y)|) < \max(M_1, M_2). \end{aligned}$$

So we can take $M = \max(M_1, M_2)$.

□

Notice that if we allow multiplication, the proof breaks down since there is no way to bound

$|t_1(x,y) - t_2(x,y) - t_1(x+1,y) + t_2(x+1,y)|$ based on the bounds for

$|t_1(x,y) - t_1(x+1,y)|$ and $|t_2(x,y) - t_2(x+1,y)|$.

And indeed any expression of the form "if $t_1 < t_2$, then t_3 else t_4 ," where the t_i are any if-then-else terms, can be written in an equivalent max-min form using multiplication:

First write "if $t_1 < t_2$, then 0 else 1" as $m(t_1, t_2) = \max(\min(1, 1 + 2t'_1 - 2t'_2), 0)$, and then take

$m(t_1, t_2) \cdot t'_4 + (1 - m(t_1, t_2)) \cdot t'_3$.