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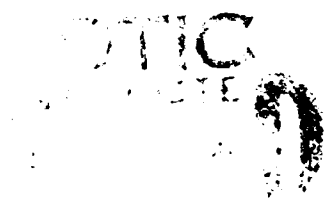
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Approximations to Some Generalized Functions; Application to Array Processing

Albert H. Nuttall
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PREFACE

This research was conducted under NUSC Project No. A75205, Sub-project No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 3302), Program Manager Mr. Robert M. Hillyer, Naval Material Command (MAT 05).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block numbers) Approximations to some generalized functions generated by the process of differentiation are developed and used to interpret some integrals involving generalized functions. Application to the realization of the ideal voltage response pattern of an array is made and illustrated for both an impulsive and a smooth approximation to the required generalized weighting function. ↙		

TABLE OF CONTENTS

	<u>Page</u>
LIST OF ILLUSTRATIONS	ii
LIST OF SYMBOLS	iii
INTRODUCTION	1
APPROXIMATION PROCEDURE	2
EXAMPLES	8
APPLICATION TO ARRAY WEIGHTING	14
USE OF NEUTRALIZERS	19
SUMMARY	29
APPENDIX A - VOLTAGE RESPONSE PATTERN FOR WEIGHTING (36)	31
APPENDIX B - PROGRAM FOR THE EVALUATION OF (59)	37
REFERENCES	41

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LIST OF ILLUSTRATIONS

<u>Figure</u>	<u>Page</u>
1 Approximation $f_\epsilon(x)$ to $f(x)$	3
2 Approximation $g_\epsilon(x)$ to $g(x)$	3
3 Approximation to Generalized Function $x^\nu U(x)$	8
4 Approximation to Generalized Function $1/x^2$	11
5 Pattern of Spherical Array for Weighting (36)	17
6 Pattern of Circular Array for Weighting (36)	18
7 Neutralizer $n_\epsilon(x)$ and its Derivative	19
8 Approximation to Generalized Function via Neutralizer	21
9 Neutralizer (55) and its Derivative	24
10 Approximate Weighting $w_\epsilon(x)$ for $\nu = -1.5$, $B = 4$	25
11 Pattern of Spherical Array for Weighting (53)	27
12 Pattern of Circular Array for Weighting (53)	28

LIST OF SYMBOLS

$f(x)$	ordinary function
$g(x)$	generalized function
$h(x)$	good function
ϵ	parameter controlling the transition region
$f_\epsilon(x)$	approximation to $f(x)$
$g_\epsilon(x)$	approximation to $g(x)$
$U(x)$	unit step function
$\delta(x)$	delta function
I_ϵ	approximation to integral I
v, B	weighting parameters; (32)
u	dimensionality parameter of array; (38)
$w(x)$	required array weighting; (32), (37)
$w_\epsilon(x)$	approximate array weighting; (36), (53)
u	array parameter containing geometry, wavelength, and look and steering angles; (38)
$v(u)$	array voltage response; (38)
$J_a(z)$	Bessel function ratio; (40)
$v_j(u)$	ideal voltage pattern; (41)
$v_\epsilon(u)$	approximate voltage pattern; (59)
$n_\epsilon(x)$	neutralizer; figure 7
y	linear transformation of x ; (54)

INTRODUCTION

Evaluation of integrals involving generalized functions is often accomplished via an integration by parts without regard to an interpretation of the behavior of the function at its singular points. See, for example, ref. 1, eqs. (I-17), (I-18), and (I-32); thus, there follows an integral such as

$$\int_{-\infty}^{\infty} dx \frac{\cos(wx)}{x^2} = -\pi |w| . \quad (1)$$

It is difficult to interpret and attach physical significance to this integral; in fact, the major contribution to the integral in (1) comes from the neighborhood of $x = 0$, where the integrand appears to be positive and not integrable, yet the right-hand side of (1) is negative and finite. We would like to approximate the generalized function $1/x^2$ in (1) and get a physical interpretation that is consistent with the result given by (1).

In a recent study of the ideal patterns for arrays in one, two, and three dimensions, it was found that the required weightings were impulsive or more singular than an impulse, depending on the dimensionality of the application; see ref. 2. In order to make these results of practical utility, it is necessary to approximate these singular weightings (generalized functions) by finite-valued functions and thereby realize approximations to the ideal patterns. This approximation procedure and its performance will be detailed here.

APPROXIMATION PROCEDURE

We take as given the possibility of approximating a delta function $\delta(x)$ by finite-valued functions; see, for example, ref. 1, page 279, where a one-sided rectangular pulse and a two-sided Gaussian pulse are used for illustration purposes. Also, on page 280, a one-sided approximation to the doublet $\delta'(x)$ is given. Alternative approximations are presented in ref. 3, pages 11-12.

Let us suppose that $f(x)$ is an ordinary function which is integrable at $x = a+$, but that generalized function (ref. 3, page 30)

$$g(x) \equiv f'(x) \quad (2)$$

is not integrable at $x = a+$. For example, with $a = 0$,

$$f(x) = \begin{cases} -2x^{-1/2} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}, \quad (3)$$

$$g(x) = \begin{cases} x^{-3/2} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}, \quad (4)$$

is such a pair; see ref. 3, definition 8. Yet integration by parts yields

$$\int_{-\infty}^{\infty} dx g(x) h(x) = - \int_{-\infty}^{\infty} dx f(x) h'(x) = 2 \int_0^{\infty} dx x^{-1/2} h'(x), \quad (5)$$

which is integrable for any good function $h(x)$.

To make sense of this situation, we approximate function $f(x)$ by function $f_{\epsilon}(x)$ as shown in figure 1; that is, $f_{\epsilon}(x)$ has constant value $f(a+\epsilon)$ in the neighborhood ϵ of a . The approximation to generalized function $g(x)$ that we adopt is

$$g_\epsilon(x) = f'_\epsilon(x) = f(a+\epsilon) \delta(x-a) + f'(x) U(x-a-\epsilon), \quad (6)$$

as depicted in figure 2, where $U(x)$ is the unit step function; that is, $g_\epsilon(x)$ has an impulse of area $f(a+\epsilon)$ at $x = a$, and is equal to $f'(x)$ for $x > a+\epsilon$. (We can approximate the impulse as discussed at the beginning of this section.)

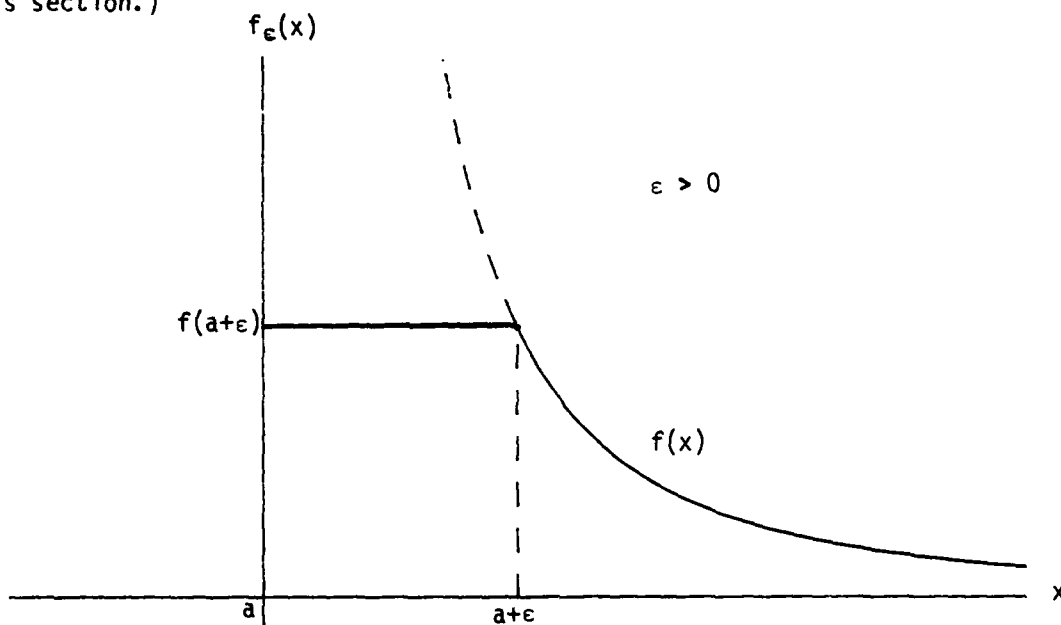


Figure 1. Approximation $f_\epsilon(x)$ to $f(x)$.

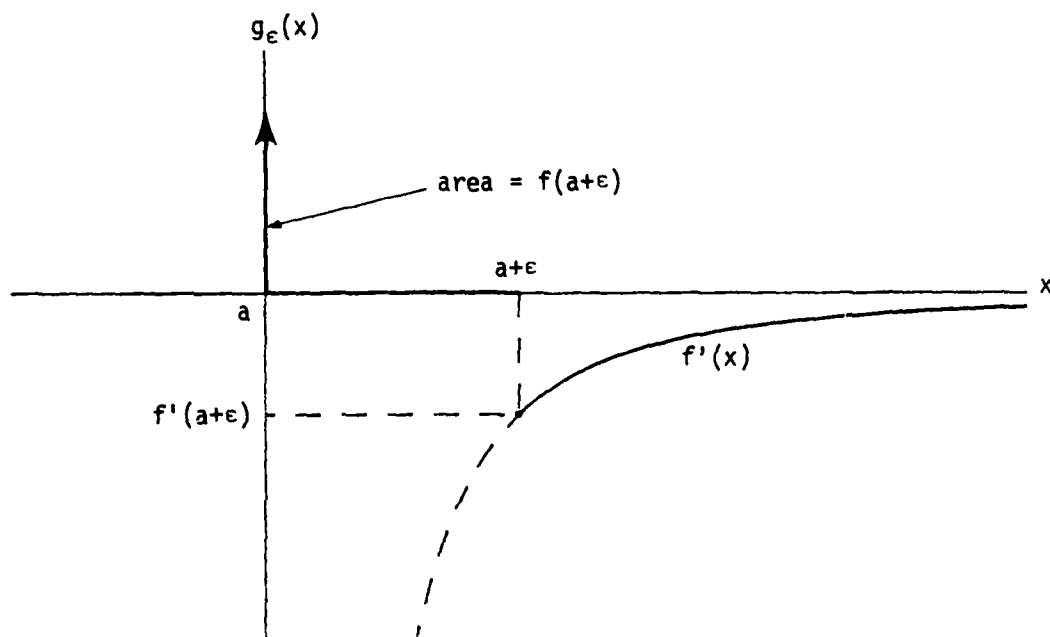


Figure 2. Approximation $g_\epsilon(x)$ to $g(x)$.

It is important to observe that if $f(x)$ is singular (ref. 3, definition 21) at $x = a$, as for example (3), the area $f(a+\epsilon)$ of the impulse at $x = a$ becomes progressively larger as $\epsilon \rightarrow 0$. However, the area under the remaining portion of $g_\epsilon(x)$ also becomes larger, but in such a fashion that the total area under $g_\epsilon(x)$ remains finite as $\epsilon \rightarrow 0$; this will be shown below in (11).

To see the effect of this approximation on an integral of a generalized function $g(x)$, consider integral

$$I \equiv \int_a^b dx g(x) h(x) . \quad (7)$$

The approximation to I is:

$$\begin{aligned} I_\epsilon &\equiv \int_a^b dx g_\epsilon(x) h(x) \\ &= f(a+\epsilon) h(a) + \int_{a+\epsilon}^b dx f'(x) h(x) , \end{aligned} \quad (8)$$

where we employed (6). Integration by parts on (8) yields two alternative forms:

$$I_\epsilon = f(a+\epsilon)[h(a) - h(a+\epsilon)] + f(b) h(b) - \int_{a+\epsilon}^b dx f(x) h'(x) \quad (9A)$$

$$= f(a+\epsilon)[h(a) - h(a+\epsilon)] + \int_a^{a+\epsilon} dx f(x) h'(x)$$

$$+ f(b) h(b) - \int_a^b dx f(x) h'(x) . \quad (9B)$$

Since $f(x)$ is integrable at $x = a+$, the leading term in (9A) and the two leading terms in (9B) both approach zero as $\epsilon \rightarrow 0$, yielding the limit

$$I = \lim_{\epsilon \rightarrow 0} I_{\epsilon} = f(b) h(b) - \int_a^b dx f(x) h'(x) . \quad (10)$$

This is the value of (7) expressed in terms of integrable functions.

The area under approximation $g_{\epsilon}(x)$, for $\epsilon > 0$, is available by substituting $h(x) = 1$ in (8) and (9):

$$\int_a^b dx g_{\epsilon}(x) = f(b) , \quad (11)$$

which is finite and independent of ϵ . Thus, the impulse in figure 2 is necessary in order to compensate for the increasing area that develops under $f'(x)$ near $x = a$, when ϵ approaches zero.

SINGULARITY AT $x = b-$

Suppose, instead, that $f(x)$ is integrable at $x = b-$, but that generalized function (2) is not. By employing a procedure similar to that above, we have approximation

$$g_{\epsilon}(x) = -f(b-\epsilon) \delta(x-b) + f'(x) U(b-\epsilon-x) \quad (12)$$

to $g(x)$. The integral I_{ϵ} can then be expressed as

$$\begin{aligned}
 I_\epsilon &= \int_a^b dx g_\epsilon(x) h(x) \\
 &= -f(b-\epsilon) h(b) + \int_a^{b-\epsilon} dx f'(x) h(x) \\
 &= -f(b-\epsilon) [h(b) - h(b-\epsilon)] - f(a) h(a) - \int_a^{b-\epsilon} dx f(x) h'(x). \quad (13)
 \end{aligned}$$

There follows

$$\int_a^b dx g(x) h(x) - I = \lim_{\epsilon \rightarrow 0} I_\epsilon = -f(a) h(a) - \int_a^b dx f(x) h'(x), \quad (14)$$

in terms of integrable functions.

The area under approximation $g_\epsilon(x)$ is finite and independent of ϵ : let $h(x) = 1$ in (13) and get directly

$$\int_a^b dx g_\epsilon(x) = -f(a). \quad (15)$$

RELATION TO "FINITE PART" OF INTEGRAL

If we start with integral (7) and integrate by part, there follows

$$I = \left[f(x) h(x) \right]_a^b - \int_a^b dx f(x) h'(x). \quad (16)$$

Now if $f(x)$ is singular at $x = a$, the finite part (ref. 3, page 32) of (16) is obtained by dropping the term involving $f(a)$, thereby obtaining identically (10). Conversely, if $f(x)$ is singular at $x = b$, the finite part of (16) is just (14). Thus, the limit of the approximation procedure developed here is exactly what is yielded by the finite part procedure.

EXAMPLES

EXAMPLE 1

$$f(x) = \frac{x^{\nu+1}}{\nu+1} \quad \text{for } x > 0 \quad \text{where } -2 < \nu < -1. \quad (17)$$

This function is singular but integrable at $x = 0$. In figure 1, we identify

$$a = 0 \quad (18)$$

and get

$$f'(x) = x^{\nu} \quad \text{for } x > 0. \quad (19)$$

Then, from (6), the approximation to the generalized function $x^{\nu} U(x)$ is

$$g_{\epsilon}(x) = \frac{\epsilon^{\nu+1}}{\nu+1} \delta(x) + x^{\nu} U(x-\epsilon) \quad (20)$$

and is depicted in figure 3.

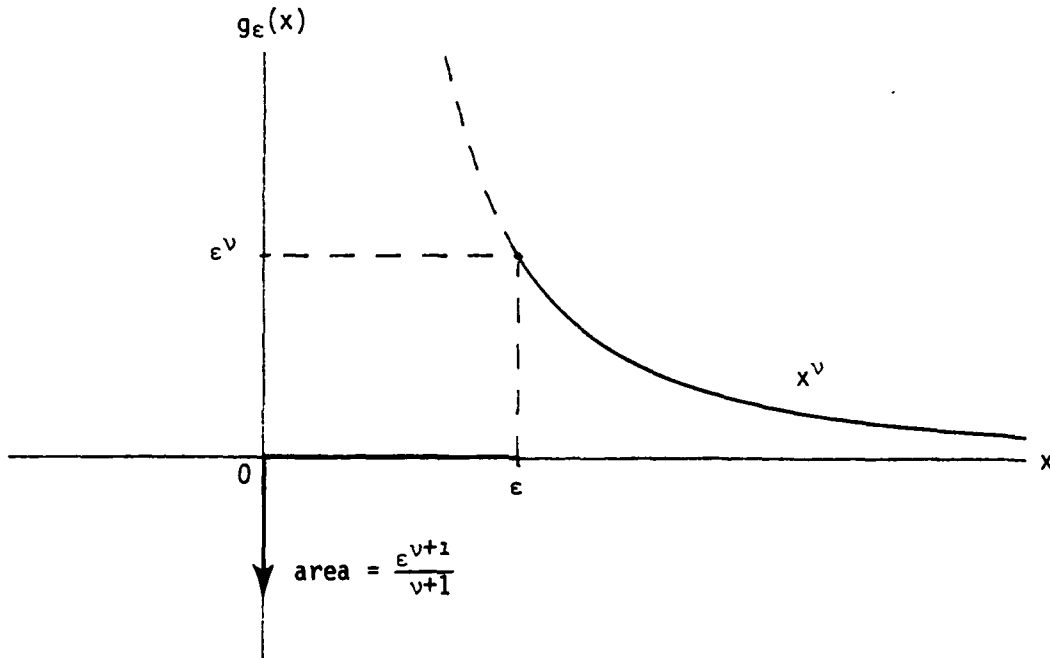


Figure 3. Approximation to Generalized Function $x^{\nu} U(x)$.

The result of applying (20) to a good function $h(x)$ is available from (8) and (17),

$$\int_0^b dx g_\epsilon(x) h(x) = \frac{\epsilon^{v+1}}{v+1} h(0) + \int_\epsilon^b dx x^v h(x), \quad (21)$$

or from either form in (9). The limit follows from (10) and (7):

$$I = \frac{b^{v+1}}{v+1} h(b) - \int_0^b dx \frac{x^{v+1}}{v+1} h'(x) = \int_0^b dx x^v h(x). \quad (22)$$

The following examples are derived in similar fashion; just the results are listed.

EXAMPLE 2

$$f(x) = \ln x \text{ for } x > 0$$

$$a = 0$$

$$f'(x) = \frac{1}{x} \text{ for } x > 0$$

$$g_\epsilon(x) = \ln \epsilon \delta(x) + \frac{1}{x} U(x-\epsilon)$$

$$I = \ln b h(b) - \int_0^b dx \ln x h'(x) = \int_0^b dx \frac{1}{x} h(x). \quad (23)$$

EXAMPLE 3

$$f(x) = \frac{1}{2}(\ln x)^2 \quad \text{for } x > 0$$

$$a = 0$$

$$f'(x) = \frac{\ln x}{x} \quad \text{for } x > 0$$

$$g_\epsilon(x) = \frac{1}{2}(\ln \epsilon)^2 \delta(x) + \frac{\ln x}{x} U(x-\epsilon)$$

$$I = \frac{1}{2}(\ln b)^2 h(b) - \int_0^b dx \frac{1}{2}(\ln x)^2 h'(x) = \int_0^b dx \frac{\ln x}{x} h(x) . \quad (24)$$

EXAMPLE 4

$$f(x) = \frac{x^{\nu+1}}{\nu+1} \left[\ln x - \frac{1}{\nu+1} \right] \quad \text{for } x > 0 ; \nu > -2$$

$$a = 0$$

$$f'(x) = x^\nu \ln x \quad \text{for } x > 0$$

$$g_\epsilon(x) = \frac{\epsilon^{\nu+1}}{\nu+1} \left[\ln \epsilon - \frac{1}{\nu+1} \right] \delta(x) + x^\nu \ln x U(x-\epsilon)$$

$$I = \frac{b^{\nu+1}}{\nu+1} \left[\ln b - \frac{1}{\nu+1} \right] h(b) - \int_0^b dx \frac{x^{\nu+1}}{\nu+1} \left[\ln x - \frac{1}{\nu+1} \right] h'(x)$$

$$= \int_0^b dx x^\nu \ln x h(x) . \quad (25)$$

EXAMPLE 5

Here we consider the result for the generalized function $1/x^2$ as presented in the integral (1). Consider the ordinary function

$$f_{\epsilon}(x) \equiv \begin{cases} -1/x & \text{for } |x| > \epsilon \\ -\frac{1}{\epsilon} \operatorname{sgn}(x) & \text{for } |x| < \epsilon \end{cases}, \quad (26)$$

and its derivative

$$g_{\epsilon}(x) = f'_{\epsilon}(x) = \begin{cases} 1/x^2 & \text{for } |x| > \epsilon \\ -\frac{2}{\epsilon} \delta(x) & \text{for } |x| < \epsilon \end{cases}; \quad (27)$$

see figure 4. The function $g_{\epsilon}(x)$ is an approximation to generalized function

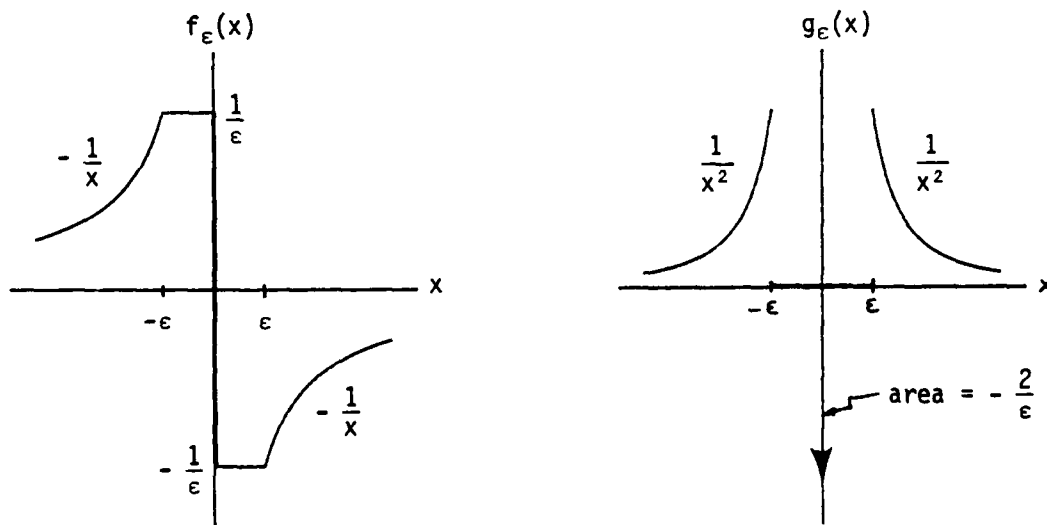


Figure 4. Approximation to Generalized Function $\frac{1}{x^2}$

$1/x^2$; it contains an impulse of area $-2/\epsilon$ (at $x = 0$) that tends to $-\infty$ as $\epsilon \rightarrow 0$.

The approximation to integral (7) is as defined in (8); namely, for $a = -\infty$, $b = +\infty$,

$$I_\epsilon = \int_{-\infty}^{+\infty} dx g_\epsilon(x) \cos(wx) = 2 \int_\epsilon^{+\infty} dx \frac{\cos(wx)}{x^2} - \frac{2}{\epsilon}$$

$$= -\pi |w| - 2 \frac{1 - \cos(w\epsilon)}{\epsilon} + 2w \text{Si}(w\epsilon), \quad (28)$$

where Si is the sine integral (ref. 4, eq. 5.2.1). The limit as $\epsilon \rightarrow 0$ gives the result for the generalized function $1/x^2$, namely,

$$\int_{-\infty}^{+\infty} dx \frac{1}{x^2} \cos(wx) = -\pi |w|. \quad (29)$$

This result is in agreement with ref. 1, eq. I-32, and with ref. 3, page 43, for x^{-m} with $m = 2$.

An alternative approximation to generalized function $1/x^2$ that uses finite functions is

$$g_\epsilon(x) = \begin{cases} 1/x^2 & \text{for } |x| > \epsilon \\ 0 & \text{for } |x| < \epsilon \end{cases} - \frac{2}{\epsilon} \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{x^2}{2\epsilon}\right). \quad (30)$$

There follows

$$I_\epsilon = \int_{-\infty}^{+\infty} dx g_\epsilon(x) \cos(wx)$$

$$= -\pi|w| + \frac{2}{\epsilon} \left[\cos(w\epsilon) - \exp\left(-\frac{1}{2}w^2\epsilon^2\right) \right] + 2w \operatorname{Si}(w\epsilon) \quad (31)$$

The limit as $\epsilon \rightarrow 0$ is again exactly (29). The essential feature of approximations (27) and (30) is the large, sharp negative pulse of area $-2/\epsilon$ that develops about $x = 0$ as ϵ gets small.

APPLICATION TO ARRAY WEIGHTING

This example relies heavily on material presented in ref. 2 on array weighting for good voltage response patterns in one, two, and three dimensions. In particular, from ref. 2, eq. E-7: for $-2 < \nu < -1$, we have, for the required array weighting $w(x)$ to realize pattern $J_{\mu+\nu+1}(\sqrt{u^2-B^2})$, in one, two, or three dimensions ($\mu = -\frac{1}{2}, 0, \text{ or } \frac{1}{2}$), the relation

$$-x w(x) = \frac{d}{dx} \left\{ \left(\frac{\sqrt{1-x^2}}{B} \right)^{\nu+1} I_{\nu+1} \left(B \sqrt{1-x^2} \right) U(1-x) \right\} \quad \text{for } 0 < x, \quad (32)$$

and zero otherwise. To match (2), we identify

$$g(x) = -x w(x), \quad (33)$$

and

$$f(x) = \left(\frac{\sqrt{1-x^2}}{B} \right)^{\nu+1} I_{\nu+1} \left(B \sqrt{1-x^2} \right) U(1-x) \quad \text{for } 0 < x. \quad (34)$$

The function $f(x)$ is singular but integrable at $x = 1-$, since $-1 < \nu+1 < 0$; thus the derivative in (32) will generate a generalized function (ref. 3, page 30) for the weighting $w(x)$.

We now identify $b = 1$ in (12) and obtain, via (34) and ref. 4, eq. 9.6.28, the approximation

$$g_\epsilon(x) = - \left(\frac{\sqrt{2\epsilon-\epsilon^2}}{B} \right)^{\nu+1} I_{\nu+1} \left(B \sqrt{2\epsilon-\epsilon^2} \right) \delta(x-1) \\ - x \left(\frac{\sqrt{1-x^2}}{B} \right)^\nu I_\nu \left(B \sqrt{1-x^2} \right) U(1-\epsilon-x) \quad \text{for } 0 < x. \quad (35)$$

Then there follows from (33), the approximate weighting

$$w_{\epsilon}(x) = \left(\frac{\sqrt{2\epsilon-\epsilon^2}}{B}\right)^{\nu+1} I_{\nu+1}\left(B\sqrt{2\epsilon-\epsilon^2}\right) \delta(x-1) \\ + \left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu} I_{\nu}\left(B\sqrt{1-x^2}\right) U(1-\epsilon-x) \quad \text{for } 0 < x. \quad (36)$$

This is the result presented in ref. 2, eq. F-14. It is an approximation to the generalized function

$$w(x) = \left(\frac{\sqrt{1-x^2}}{B}\right)^{\nu} I_{\nu}\left(B\sqrt{1-x^2}\right) U(1-x) U(x), \quad (37)$$

which is the required array weighting according to ref. 2, eq. E-7.

We will carry this example further than the previous ones, by determining the voltage response pattern that is actually realized by an array employing approximation weighting (36) rather than (37). The array voltage response pattern is given by ref. 2, eqs. 1-12, for any μ , as

$$v(u) = \int_0^1 dx x \left(\frac{x}{u}\right)^{\mu} J_{\mu}(ux) w(x), \quad (38)$$

where the parameter μ determines the array dimensionality, and u contains the array geometry, the plane-wave arrival wavelength, and the various look and steering angles. Substitution of generalized function (37) in (38) yields pattern (ref. 2, eq. E-1)

$$v(u) = J_{\mu+\nu+1}\left(\sqrt{u^2-B^2}\right) \quad \text{for all } u, \quad (39)$$

where we define entire function (ref. 2, eqs. 17-21)

$$J_{\alpha}(z) \equiv \frac{J_{\alpha}(z)}{z^{\alpha}}. \quad (40)$$

The particular case of $\mu + \nu = -1.5$ in (39) yields ideal voltage pattern

$$v_i(u) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\sqrt{u^2 - B^2}) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos(\sqrt{u^2 - B^2}) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cosh(\sqrt{B^2 - u^2}). \quad (41)$$

This relation is true for all u , whether larger or smaller than B . It indicates a mainlobe at $u = 0$ of amplitude $\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cosh(B)$, and sidelobes for $u > B$, all of equal amplitude $\pm \left(\frac{2}{\pi}\right)^{\frac{1}{2}}$. Plots of this ideal pattern are available in ref. 2, figures 13-16.

When approximation $w_\epsilon(x)$ in (36) is substituted, instead, in (38), we get several equivalent expressions for the corresponding pattern $v_\epsilon(u)$, as given in appendix A. Plots of typical results are presented in figures 5 and 6 for $\nu = -1.5$. In both figures, the curve labeled $\epsilon = 0$ is the desired pattern (39). Figure 5 corresponds to a volumetric-spherical array ($\mu = .5$), and the desired pattern is, from (39),

$$J_0(\sqrt{u^2 - B^2}) = J_0(\sqrt{u^2 - B^2}) = I_0(\sqrt{B^2 - u^2}), \quad (42)$$

which decays at a 3 dB/octave rate for large u . Figure 6 corresponds to a planar-circular array ($\mu = 0$) with desired pattern equal to ideal pattern (41). The approximations in both of these figures for $\epsilon = .1$ are quite good, but those for $\epsilon = .2$ have undergone significant degradation. The possibility of replacing the delta function in (36) by a narrow pulse is considered in the next section.

Figure 5 furnishes an approximation to the bottom asterisked case in ref. 2, figure 12; figure 6 does the same for the middle asterisked case in ref. 2, figure 13. The three asterisked cases for $\nu = -1$ in ref. 2, figures 11-13, merely require delta functions at $x = 1$ and are considered solved. The last remaining asterisked case is the bottom one in ref. 2, figure 13, for $\nu = -2$. But this has already been shown in ref. 2, eqs. E-37 - E-39, to involve a delta function and its derivative, both of which are easily approximated; see, for example, ref. 1, pages 279-280, or ref. 2, pages 11-12.

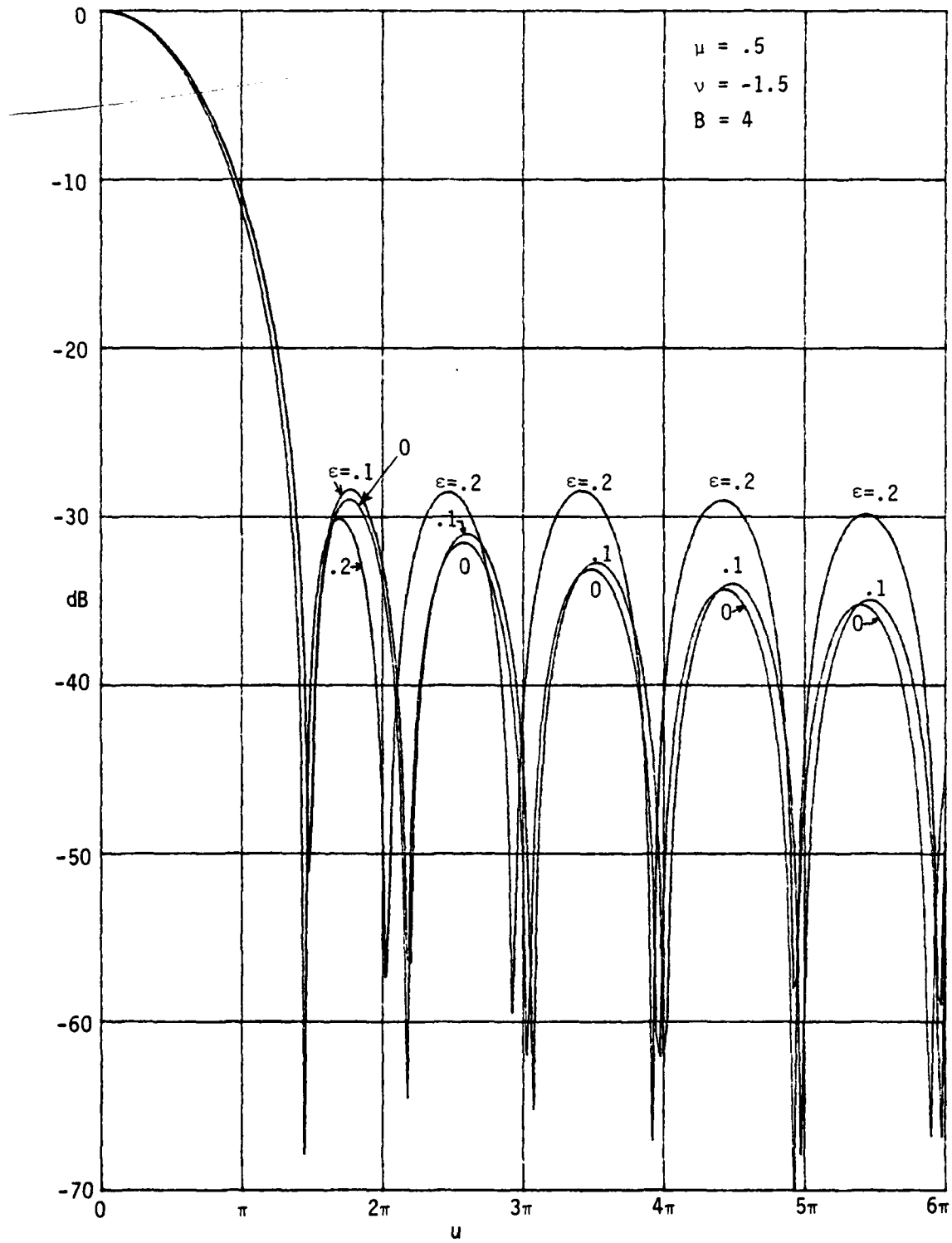


Figure 5. Pattern of Spherical Array for Weighting (36)

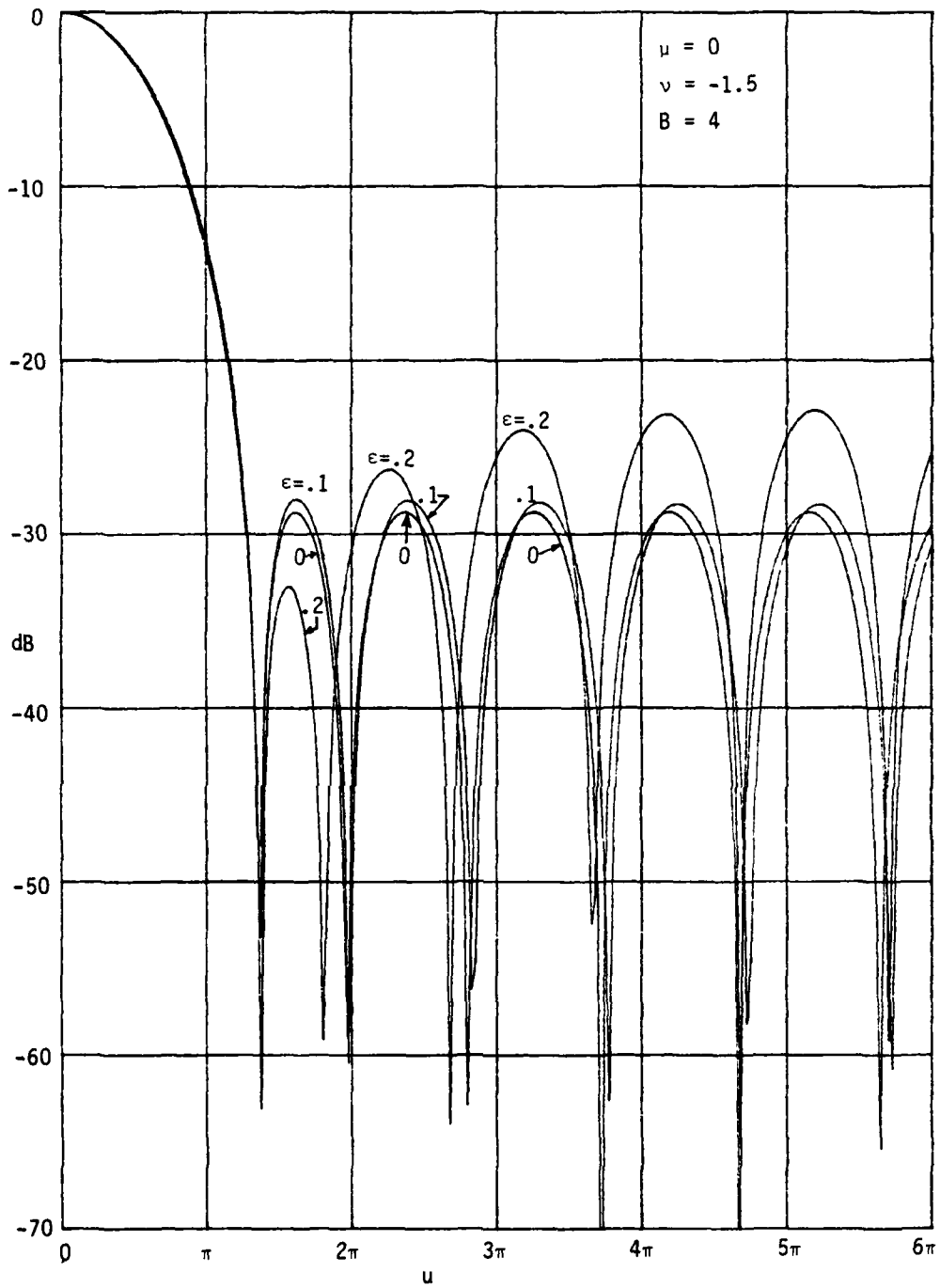
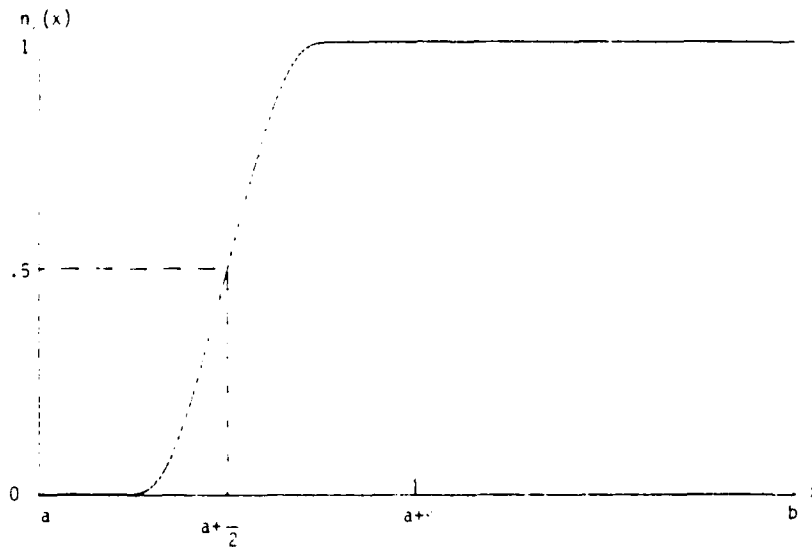
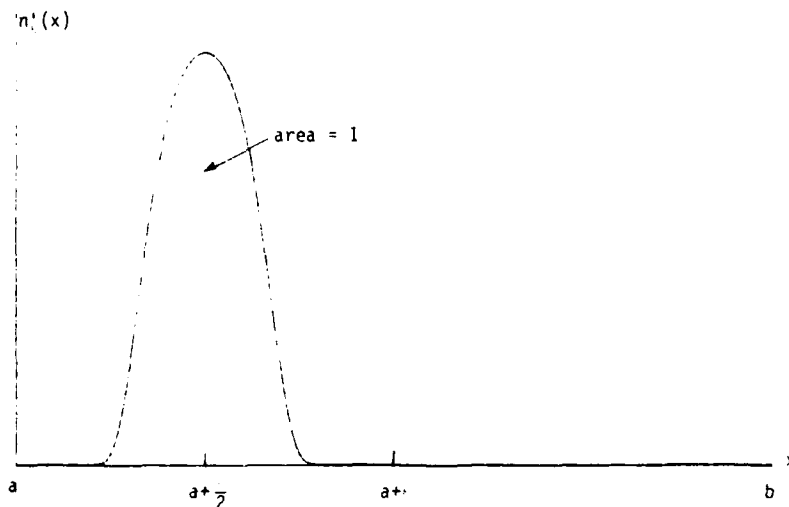


Figure 6. Pattern of Circular Array for Weighting (36)

USE OF NEUTRALIZERS

An alternative smoother approximation to $f(x)$ and $g(x)$ than afforded by figures 1 and 2 is by the use of neutralizers; see ref. 5, section 3.3. Consider the neutralizer $n_\epsilon(x)$ and its derivative shown in figure 7; the neutralizer is 0 at $x = a$, and 1 at $x = b$. Furthermore, it has derivatives

Figure 7A. Function $n_\epsilon(x)$ Figure 7B. Function $n'_\epsilon(x)$ Figure 7. Neutralizer $n_\epsilon(x)$ and its Derivative

of all orders, all of which are zero at end points a and b . The positive parameter ϵ characterizes the critical point, $x_c = a + \frac{\epsilon}{2}$, where the neutralizer is $1/2$; this point x_c will approach a as $\epsilon \rightarrow 0^+$. The neutralizer has completed its transition to 1 by the value $x = a + \epsilon$. $n_\epsilon(x)$ will approach 1 for all $x > a$, as $\epsilon \rightarrow 0$.

The approximation we take to $f(x)$ is the product

$$f_\epsilon(x) \equiv f(x) n_\epsilon(x) , \quad (43)$$

and the corresponding approximation to generalized function (2) is the smooth function

$$g_\epsilon(x) = f_\epsilon'(x) = f'(x) n_\epsilon(x) + f(x) n_\epsilon'(x) . \quad (44)$$

This approach is similar to the regular sequence of good functions used to define a generalized function in ref. 3, pages 16-17. A representative example is depicted in figure 8. As before, the area under approximation $g_\epsilon(x)$ is independent of ϵ :

$$\int_a^b dx g_\epsilon(x) = \left[f_\epsilon(x) \right]_a^b = f_\epsilon(b) = f(b) . \quad (45)$$

The result of applying $g_\epsilon(x)$ to function $h(x)$ is now

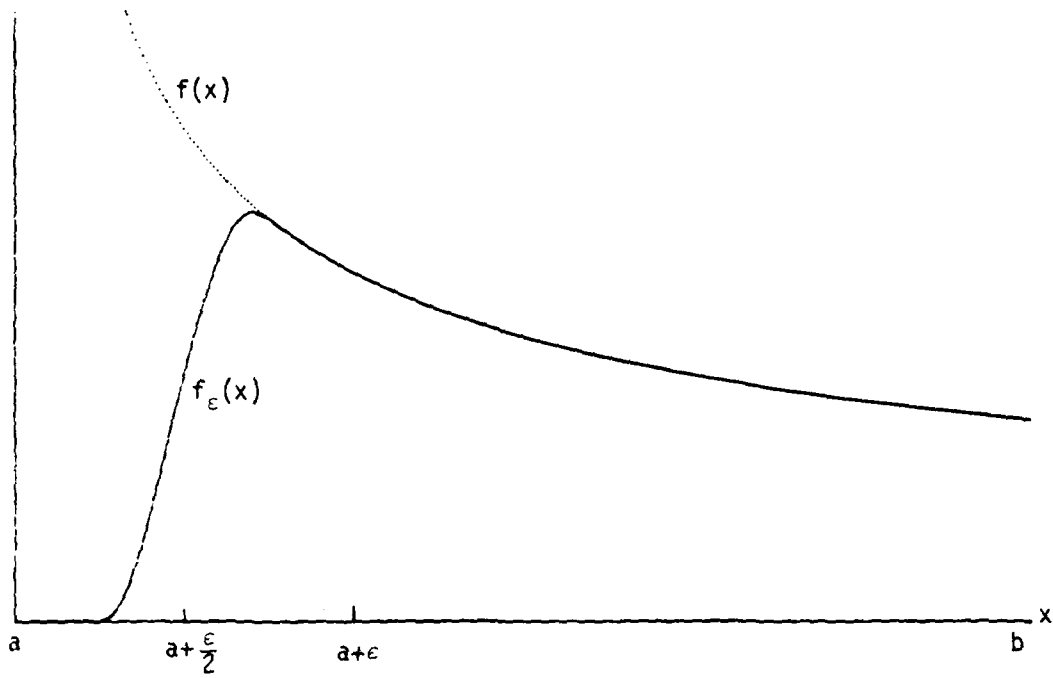


Figure 8A. Function $f_\epsilon(x)$

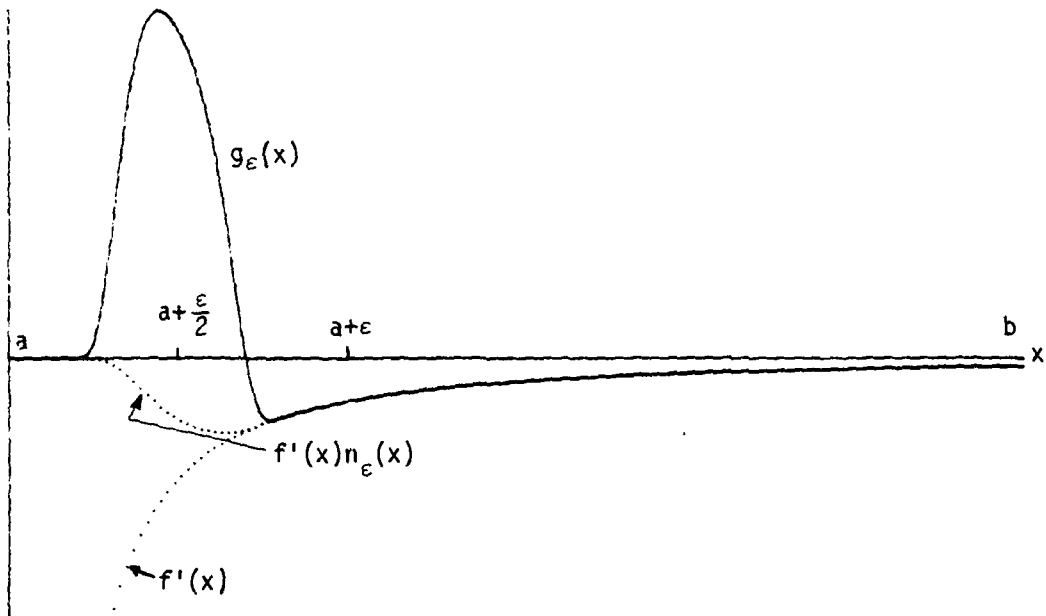


Figure 8B. Function $g_\epsilon(x)$

Figure 8. Approximation to Generalized Function via Neutralizer

$$\begin{aligned}
I_\epsilon &= \int_a^b dx g_\epsilon(x) h(x) = \int_a^b dx f_\epsilon'(x) h(x) \\
&= \left[f_\epsilon(x) h(x) \right]_a^b - \int_a^b dx f_\epsilon(x) h'(x) \\
&= f(b) h(b) - \int_a^b dx f(x) n_\epsilon(x) h'(x), \tag{46}
\end{aligned}$$

where we have utilized the properties of the neutralizer. Equation (46) now replaces (8). In the limit of $\epsilon \rightarrow 0$, (46) yields

$$I = \lim_{\epsilon \rightarrow 0} I_\epsilon = f(b) h(b) - \int_a^b dx f(x) h'(x), \tag{47}$$

in agreement with (10).

A limiting case of figure 7 is a step-function at $x = a + \frac{\epsilon}{2}$; i.e.,

$$n_\epsilon(x) = U(x - a - \frac{\epsilon}{2}). \tag{48}$$

Then

$$n_\epsilon'(x) = \delta(x - a - \frac{\epsilon}{2}) \tag{49}$$

and (44) yields

$$g_\epsilon(x) = f'(x) U(x - a - \frac{\epsilon}{2}) + f(a + \frac{\epsilon}{2}) \delta(x - a - \frac{\epsilon}{2}). \tag{50}$$

This is almost identical to (6), which has its impulse located at $x = a$ rather than at $x = a + \frac{\epsilon}{2}$.

APPLICATION TO ARRAY

The starting point is (32), again, for the required weighting. The function $f(x)$ is identified as in (34), and we get, from (33) and (44),

$$-xw_{\epsilon}(x) = g_{\epsilon}(x) = f'(x) n_{\epsilon}(x) + f(x) n'_{\epsilon}(x). \quad (51)$$

There follows, from (34) and ref. 4, eq. 9.6.28,

$$f'(x) = -x \left(\frac{\sqrt{1-x^2}}{B} \right)^{\nu} I_{\nu} \left(B \sqrt{1-x^2} \right) \quad \text{for } 0 < x < 1. \quad (52)$$

Substitution of (34) and (52) in (51) yields, for the approximate weighting,

$$\begin{aligned} w_{\epsilon}(x) &= \left(\frac{\sqrt{1-x^2}}{B} \right)^{\nu} I_{\nu} \left(B \sqrt{1-x^2} \right) n_{\epsilon}(x) \\ &\quad - \frac{1}{x} \left(\frac{\sqrt{1-x^2}}{B} \right)^{\nu+1} I_{\nu+1} \left(B \sqrt{1-x^2} \right) n'_{\epsilon}(x) \quad \text{for } 0 < x < 1. \quad (53) \end{aligned}$$

However, the neutralizer in this case must be chosen to be 1 at $x = 0$, and 0 at $x = 1$; that is, it is a reflected version of figure 7A. For our purposes, it is not necessary for the neutralizer to have derivatives of all orders. Rather, we select $n_{\epsilon}(x)$ so that $n''_{\epsilon}(x)$ is continuous for all x , and such that $n'_{\epsilon}(x)$ and $n''_{\epsilon}(x)$ are zero at the edges of the transition region $(1-\epsilon, 1)$; see figure 9. Here, letting

$$y = \frac{x-1 + \frac{\epsilon}{2}}{\epsilon}, \quad (54)$$

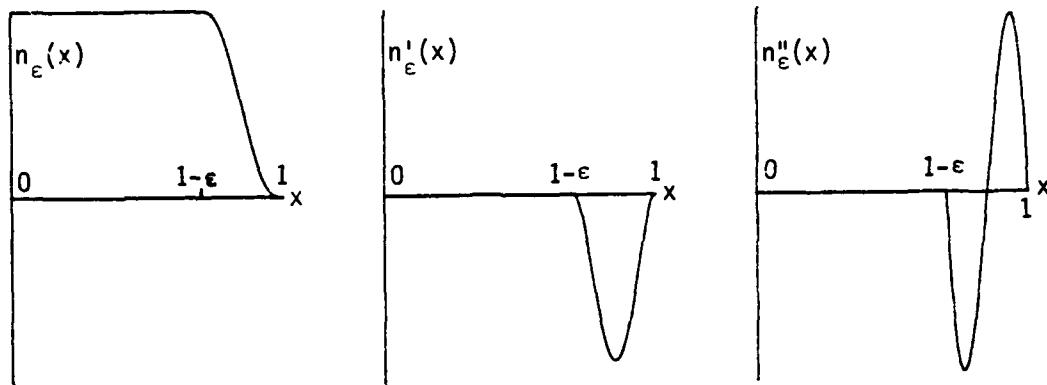


Figure 9. Neutralizer (55) and its Derivatives

we take

$$n_{\epsilon}(x) = \frac{1}{2} - \frac{1}{8}(15y - 40y^3 + 48y^5) \quad \text{for } 1-\epsilon \leq x \leq 1, \quad (55)$$

and 1 for $0 \leq x \leq 1-\epsilon$. Then

$$n'_{\epsilon}(x) = -\frac{15}{8\epsilon}(1-8y^2+16y^4) \quad \text{for } 1-\epsilon \leq x \leq 1, \quad (56)$$

and zero otherwise. Also,

$$n''_{\epsilon}(x) = \frac{30}{\epsilon^2}(y-4y^3) \quad \text{for } 1-\epsilon \leq x \leq 1, \quad (57)$$

and zero otherwise. From (55)-(57), there follows

$$\left. \begin{aligned} n_{\epsilon}(x) &\sim \frac{10}{\epsilon^3}(1-x)^3 \\ n'_{\epsilon}(x) &\sim -\frac{30}{\epsilon^3}(1-x)^2 \\ n''_{\epsilon}(x) &\sim \frac{60}{\epsilon^3}(1-x) \end{aligned} \right\} \text{ as } x \rightarrow 1- . \quad (58)$$

Combining this with the behavior of the Bessel function near zero argument (ref. 4, eq. 9.6.7), we find that both terms of the approximate weighting $w_{\epsilon}(x)$ in (53) are proportional to $(1-x)^{3+\nu}$ as x approaches 1. Thus if $\nu \geq -2$, the approximate weighting will approach zero at least linearly at $x = 1$. An example of $w_{\epsilon}(x)$ for $\nu = -1.5$ and $B = 4$ is given in figure 10

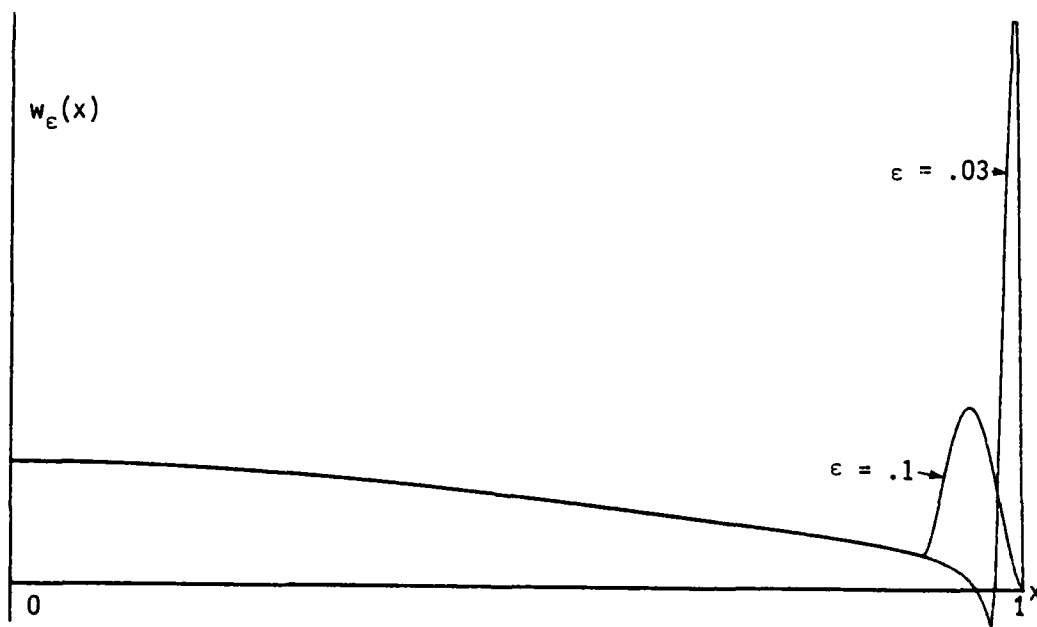


Figure 10. Approximate Weighting $w_{\epsilon}(x)$ for $\nu = -1.5$, $B = 4$

for two different selections of ϵ . The large positive spike near $x = 1$ is very pronounced for small ϵ .

The array voltage response corresponding to approximate weighting $w_\epsilon(x)$ in (53) is given by (38) as

$$\begin{aligned} v_\epsilon(u) &= \int_0^1 dx x \left(\frac{x}{u}\right)^\mu J_\mu(ux) w_\epsilon(x) \\ &= \int_0^1 dx x^{2\mu+1} J_\mu(ux) w_\epsilon(x), \end{aligned} \quad (59)$$

where we used (40). A program for the evaluation of (59) (along with (53)-(56)) is presented in appendix B. Sample responses are plotted in figure 11 for a spherical array and in figure 12 for a circular array. Comparison with the corresponding results in figures 5 and 6 reveals that the impulsive weighting of (36) yields a better approximation to the ideal pattern ($\epsilon = 0$) than the smoother weighting of (53). That is, ϵ must be chosen smaller in (53) than in (36), in order to realize approximately the same voltage response pattern in the first few sidelobes.

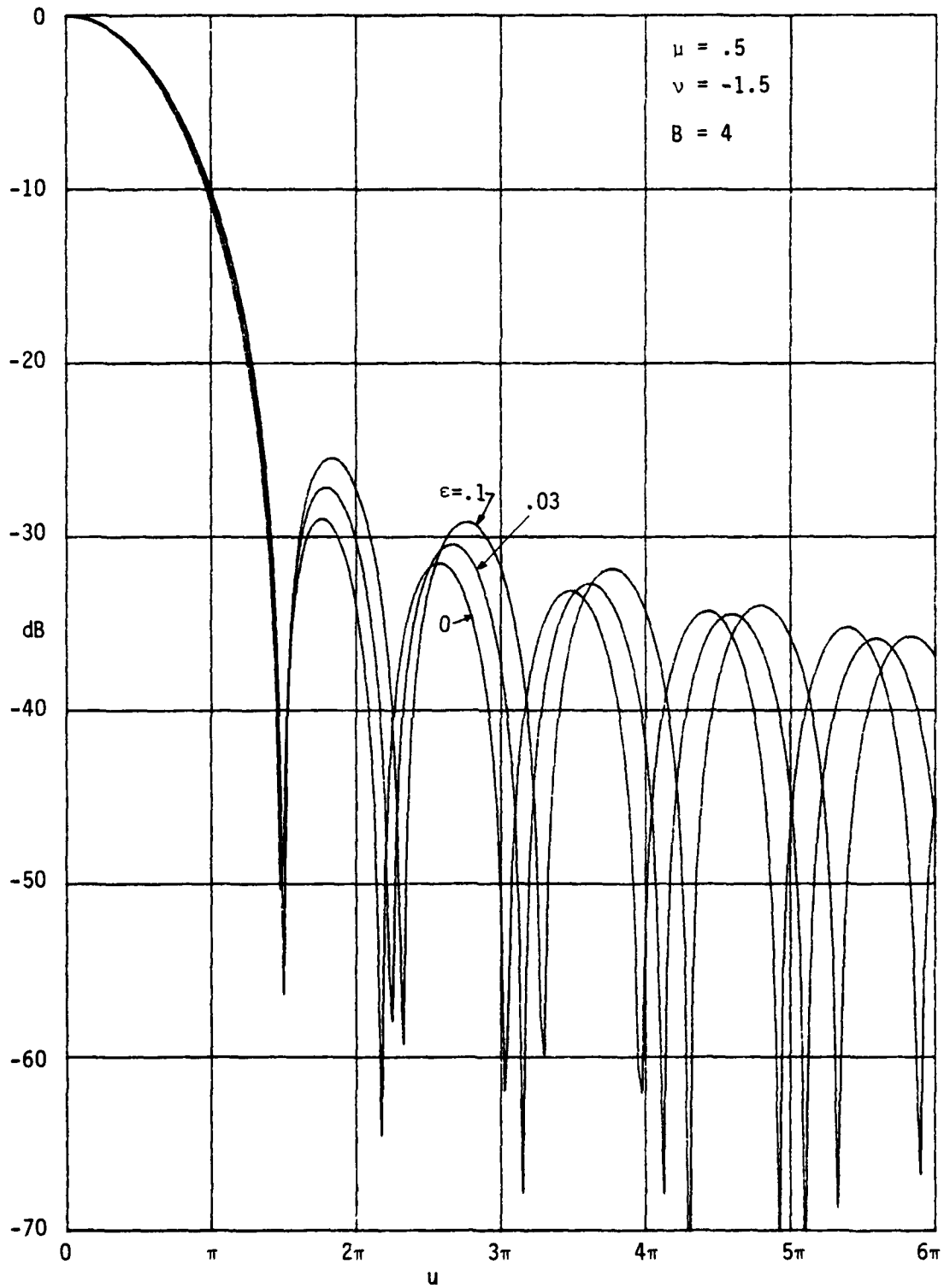


Figure 11. Pattern of Spherical Array for Weighting (53)

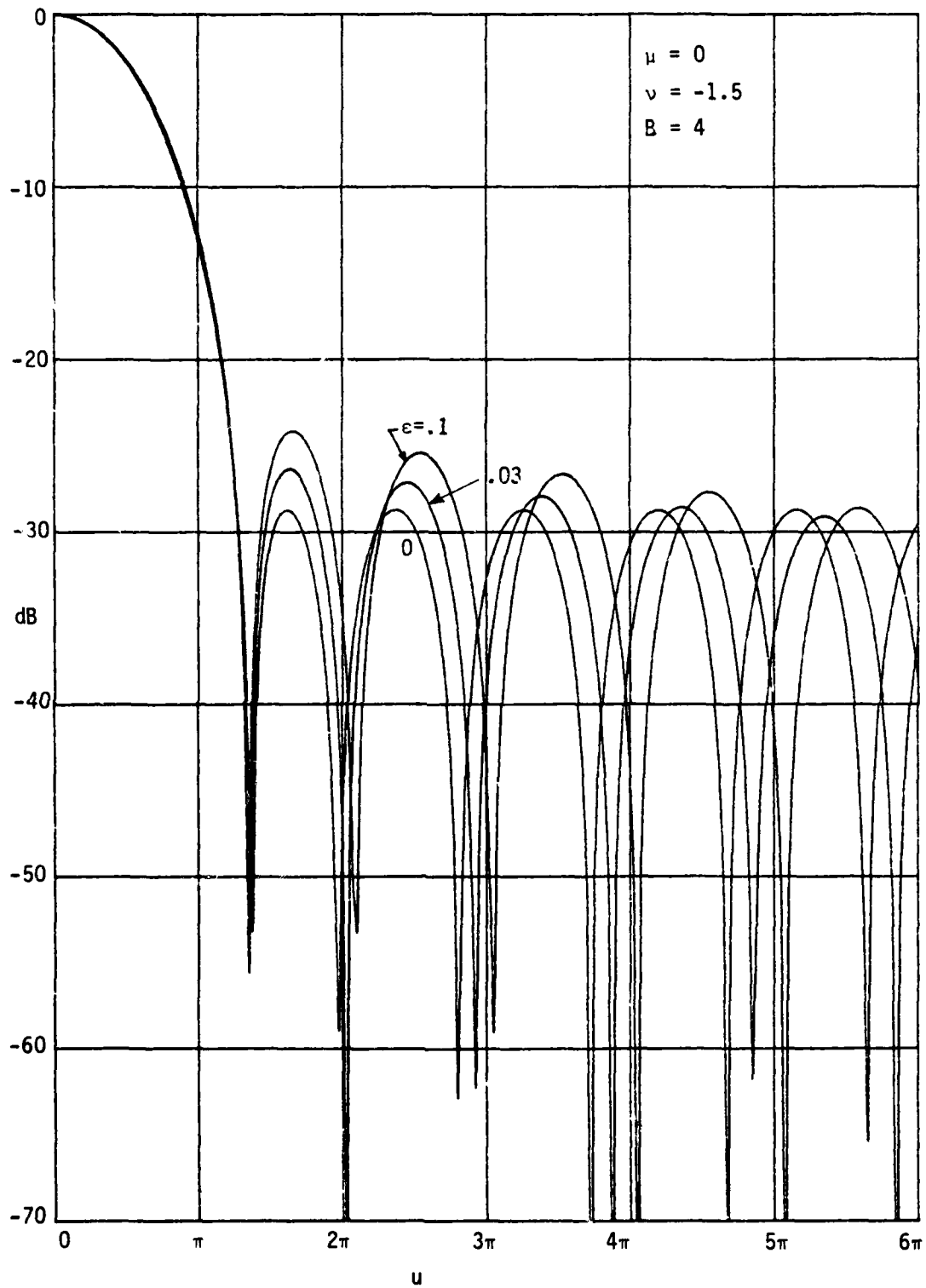


Figure 12. Pattern of Circular Array for Weighting (53)

SUMMARY

We have indicated how approximations to some generalized functions generated by the process of differentiation may be realized, and then have applied the procedure to the approximation of the weighting required to realize the ideal response patterns of arrays in several dimensions. Two examples, one impulsive and the other smooth, were used for circular and spherical arrays, and the resultant approximate patterns were plotted for different choices of the parameter ϵ governing the transition region near the singularity of the generalized function. How small ϵ must be chosen depends on the form of the approximate weighting and the desired closeness to the specified pattern.

Appendix A
VOLTAGE RESPONSE PATTERN FOR WEIGHTING (36)

With the shorthand notation

$$f_v(x) \equiv \left(\frac{\sqrt{1-x^2}}{B} \right)^v I_v \left(B \sqrt{1-x^2} \right) \quad \text{for } 0 < x < 1, \quad (\text{A-1})$$

approximate weighting (36) takes the form

$$w_\epsilon(x) = f_{v+1}(1-\epsilon) \delta(x-1) + f_v(x) U(1-\epsilon-x) \quad \text{for } 0 < x. \quad (\text{A-2})$$

Substitution of (A-2) in (38) yields, upon use of (40), the corresponding voltage response

$$v_\epsilon(u) = f_{v+1}(1-\epsilon) J_\mu(u) + \int_0^{1-\epsilon} dx x^{2\mu+1} J_\mu(ux) f_v(x) \quad (\text{A-3})$$

for any μ ; this is the relation programmed in ref. 2, appendix F.

For small ϵ (the case of most interest), the cusp of $f_v(x)$ in (A-3) at $x = 1$ causes numerical integration problems. We can alleviate this problem by integrating by parts on (A-3), using

$$U = x^{2\mu} J_\mu(ux), \quad V = -f_{v+1}(x), \quad (\text{A-4})$$

to get

$$v_\epsilon(u) = f_{v+1}(1-\epsilon) \left[J_\mu(u) - (1-\epsilon)^{2\mu} J_\mu(u(1-\epsilon)) \right] + \int_0^{1-\epsilon} dx x^{2\mu-1} J_{\mu-1}(ux) f_{v+1}(x) \quad \text{for } \mu > 0. \quad (\text{A-5})$$

The function $f_{\nu+1}(x)$ is integrable at $x = 1$ if $\nu > -2$.

The completed integral over $(0,1)$ in (A-5) can be evaluated in closed form (ref. 2), leading to

$$\begin{aligned} v_\epsilon(u) &= J_{\mu+\nu+1}(\sqrt{u^2-B^2}) \\ &+ f_{\nu+1}(1-\epsilon) [J_\mu(u) - (1-\epsilon)^{2\mu} J_\mu(u(1-\epsilon))] \\ &- \int_{1-\epsilon}^1 dx x^{2\mu-1} J_{\mu-1}(ux) f_{\nu+1}(x), \end{aligned} \quad (A-6)$$

which is advantageous because the interval $(1-\epsilon,1)$ is small for small ϵ . Finally, the cusp at $x = 1$ can be eliminated for $\nu \geq -3/2$, by changing the variable of integration according to $x = \sin t$, getting

$$\begin{aligned} v_\epsilon(u) &= J_{\mu+\nu+1}(\sqrt{u^2-B^2}) \\ &+ f_{\nu+1}(1-\epsilon) [J_\mu(u) - (1-\epsilon)^{2\mu} J_\mu(u(1-\epsilon))] \\ &- \int_A^{\pi/2} dt (\sin t)^{2\mu-1} (\cos t)^{2\nu+3} J_{\mu-1}(u \sin t) \mathcal{Q}_{\nu+1}(B \cos t), \end{aligned} \quad (A-7)$$

where $A = \arcsin(1-\epsilon)$, and

$$\mathcal{Q}_\alpha(z) \equiv \frac{I_\alpha(z)}{z^\alpha}. \quad (A-8)$$

Form (A-7) is good numerically for small ϵ and any value of μ . There is no cusp at $t = \pi/2$ if $\nu \geq -3/2$. A program for (A-7) follows, where ϵ , μ , ν , and B are arbitrary ($\epsilon \geq 0$, $\nu \geq -3/2$, $B \geq 0$).

```

10  Eps=.1      E      ! Pattern Ve(i) via (A-7)
20  Mu=0       M
30  Nu=-1.5    N
40  Bc=4       B      ! B in (26)
50  OUTPUT 0;"Eps =";Eps;" Mu = ";Mu;" Nu = ";Nu;" Bc =";Bc
60  DIM Ve(0:480)
70  COM U,Bc,M1,N1,M2,N2
80  M1=Mu-1
90  N1=Nu+1
100 M2=2*Mu-1
110 N2=2*Nu+3
120 Alpha=Mu+Nu+1
130 T=2*Eps-Eps^2
140 IF Eps=0 THEN 160
150 F1e=T/N1*FNInuxnu(N1,Bc*SQR(T))      ! for line 2 of (A-7)
160 E1=1-Eps
170 Te=E1^(2*Mu)
180 A=ASN(E1)
190 B=PI/2
200 FOR Iu=0 TO 480
210 U=Iu*PI/40
220 Sq=SQR(ABS(U*U-Bc*Bc))
230 IF U<=Bc THEN T1=FNInuxnu(Alpha,Sq)      ! line 1 of (A-7)
240 IF U>Bc THEN T1=FNJnuxnu(Alpha,Sq)
250 IF Eps>0 THEN 290
260 T3=T1
270 V=0
280 GOTO 480
290 T2=FNJnuxnu(Mu,U)-Te*FNJnuxnu(Mu,U*E1)      ! for line 2 of (A-7)
300 T3=T1+F1e*T2
310 S=(FNS(A)+FNS(B))*0.5
320 N=2
330 H=(B-A)*0.5
340 F=(B-A)/3
350 Vo=9E99
360 T=0
370 FOR K=1 TO N-1 STEP 2
380 T=T+FNS(A+H*K)
390 NEXT K
400 S=S+T
410 V=(S+T)*F      ! line 3 of (A-7)
420 IF ABS(V-Vo)<=ABS(V)*1E-5 THEN 480
430 Vo=V
440 N=N*2
450 H=H*0.5
460 F=F*0.5
470 GOTO 360
480 Ve(Iu)=T3-V      ! Voltage Response (A-7)
490 PRINT Iu, Ve(Iu)
500 NEXT Iu
510 PLOTTER IS "GRAPHICS"
520 GRAPHICS
530 SCALE 0,480,-70,0
540 GRID 40,10
550 PENUP
560 FOR Iu=0 TO 480
570 PLOT Iu,20*LGT(ABS(Ve(Iu)/Ve(0)))
580 NEXT Iu
590 PENUP
600 END
610 !

```

```

620 DEF FNS(T) ! Integrand of (A-7)
630 COM U, Bc, M1, N1, M2, N2
640 St=SIN(T)
650 Ct=COS(T)
660 T1=FNJnu(nu=M1, U*St)
670 T2=FNJnu(nu=N1, Bc+Ct)
680 RETURN St*M2*Ct+N2*T1*T2
690 FNEND
700 !
710 DEF FNGamma(X) ! Gamma(X) via HART, page 279, #5231
720 N=INT(X)
730 R=X-N
740 IF (N=0) OR (R=0) THEN 770
750 PRINT "FNGamma(X) IS NOT DEFINED FOR X = ", X
760 STOP
770 IF R=0 THEN 800
780 Gamma2=1
790 GOTO 840
800 P=3.36954359131+R*(1.09850630453+R*(0.142928637949+R*(3.93013464186E-2))
810 P=43.9410209139+R*(22.9680800836+R*(12.5021698112+R*(P)))
820 Q=43.9410209191+R*(4.39050474596-R*(7.15075063299-R))
830 Gamma2=P/Q ! Gamma(2+R) for 0 < R < 1
840 IF N>2 THEN 880
850 IF N=2 THEN 930
860 Gamma=Gamma2
870 GOTO 980
880 Gamma=Gamma2
890 FOR K=1 TO N-2
900 Gamma=Gamma*(X-K)
910 NEXT K
920 GOTO 980
930 R=1
940 FOR K=0 TO 1-N
950 R=R*(X+K)
960 NEXT K
970 Gamma=Gamma2/R
980 RETURN Gamma
990 FNEND
1000 !

```

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```

1010 DEF FNJoukhu(Nu,X)      ! Jk(X) via 9.4.1 & 9.4.3
1020 IF ABS(X)<1 THEN 1130
1030 A=.797884560803
1040 IF Nu=0 THEN RETURN FNJo(X)
1050 IF Nu=.5 THEN RETURN A+SIN(X)/X
1060 IF Nu=-.5 THEN RETURN A+COS(X)
1070 IF Nu=1 THEN RETURN FNJ1(X)*X
1080 IF Nu=-1 THEN RETURN -FNJ1(X)*X
1090 IF Nu=1.5 THEN RETURN A*(SIN(X)-X+COS(X))/X^2
1100 IF Nu=-1.5 THEN RETURN -A*(X+SIN(X)+COS(X))
1110 IF Nu=2 THEN RETURN (2+FNJ1(X)-X+FNJo(X))/X^3
1120 IF Nu=-2 THEN RETURN (2+FNJ1(X)-X+FNJo(X))/X
1130 R=Nu
1140 IF (INT(R)/R) OR (R)=0 THEN 1160
1150 K=R=-Nu
1160 S=T=1/(2*R+FNGamma(R+1))
1170 R=-.25+K*X
1180 Big=ABS(S)
1190 FOR N=1 TO 100
1200 T=T+R/(N+(N+R))
1210 S=S+T
1220 Big=MAX(Big,ABS(S))
1230 IF ABS(T)<=1E-11*ABS(S) THEN 1270
1240 NEXT N
1250 PRINT "100 TERMS IN FNJoukhu(Nu,X) AT X:1Nu:"
1260 PAUSE
1270 D=12-LGT(ABS(Big/S)) ! NO. OF SIGNIF. DIGITS
1280 IF D=0 THEN S=S*(4+R)/K
1290 RETURN S
1300 FNEND
1310 !
1320 DEF FNJo(X)             ! Jo(X) via 9.4.1 & 9.4.3
1330 Y=ABS(X)
1340 IF Y>3 THEN 1390
1350 T=Y*Y/9
1360 Jo=.0444479-T*(.0039444-T+.00021)
1370 Jo=1-T*(2.2499997-T*(1.2656208-T*(.3163066-T+Jo)))
1380 GOTO 1450
1390 T=3/Y
1400 Jo=.9512E-5-T*(.00137237-T*(.00072805-T+.00014475))
1410 Jo=.79788456-T*(7.7E-7+T*(.00552740+T*Jo))
1420 S=.00262573-T*(.00054125+T*(.00029333-T+.00013558))
1430 T=Y-.78539816-T*(.04166397+T*(3.954E-5-T*S))
1440 Jo=Jo*COS(T)/SQRY
1450 RETURN Jo
1460 FNEND
1470 !
1480 DEF FNJ1(X)            ! J1(X) via 9.4.4 & 9.4.6
1490 Y=ABS(X)
1500 IF Y>3 THEN 1550
1510 T=Y*Y/9
1520 J1=.00443319-T*(.00031761-T+.00001109)
1530 J1=X*(.5-T*(.56249985-T*(.21093573-T*(.03954239-T+J1)))
1540 GOTO 1610
1550 T=3/Y
1560 J1=.00017105-T*(.00249511-T*(.00113653-T+.00020033))
1570 J1=.79788456+T*(1.5E-6+T*(.01659667+T*J1))
1580 S=.00637879-T*(.00074348+T*(.00079824-T+.00029166))
1590 T=Y-2.35619449+T*(.12499612+T*(5.65E-5-T*S))
1600 J1=SGN(X)*J1*COS(T)/SQRY
1610 RETURN J1
1620 FNEND
1630 !

```

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```
1640 DEF FNInu(nu,Nu,K)      ! Inu(N) via 9.8.1 & 9.8.2
1650 IF ABS(N) > 1 THEN 1770
1660 A=.398942280401
1670 E=EXP(N)
1680 IF Nu=0 THEN RETURN FNInu(N)
1690 IF Nu=.5 THEN RETURN A*(E-1)*E*(K)
1700 IF Nu=-.5 THEN RETURN A*(E+1)*E*(K)
1710 IF Nu=1 THEN RETURN FNInu(N)*K
1720 IF Nu=-1 THEN RETURN FNInu(N)/K
1730 IF Nu=1.5 THEN RETURN A*(K-1)*(E+K+1)*E*(K+.5)
1740 IF Nu=-1.5 THEN RETURN A*(K-1)*(E-1)*(K+.5)
1750 IF Nu=2 THEN RETURN (K*FNInu(N)-2*FNInu(N)/K)*K+.5
1760 IF Nu=-2 THEN RETURN (K*FNInu(N)-2*FNInu(N)/K)*K
1770 A=Nu
1780 IF (INT(A)/A) OR (A)=0 THEN 1800
1790 K=A-Nu
1800 S=T=1/(2*A*FNGamma(A+1))
1810 R=.25*(K+K)
1820 Big=ABS(S)
1830 FOR N=1 TO 100
1840 T=T*R/(N*(N+R))
1850 S=S+T
1860 Big=MAX(Big,ABS(S))
1870 IF ABS(T)/S <= 1E-11*ABS(S) THEN 1910
1880 NEXT N
1890 PRINT "100 TERMS IN FNInu(nu,Nu,K) AT ";Nu;"
1900 PAUSE
1910 D=12-LGT(ABS(Big/S)) ! NO. OF SIGNIF. DIGITS
1920 IF K=0 THEN S=S*(4*R)^K
1930 RETURN S
1940 FNEND
1950 !
1960 DEF FNIno(X)      ! Ino(X) via 9.8.1 & 9.8.2
1970 Y=ABS(X)
1980 IF Y > 3.75 THEN 2030
1990 T=Y*Y/14.0625
2000 Io=.2659732+T*(.0360768+T*(.0045813))
2010 Io=1+T*(3.5156229+T*(3.0899424+T*(1.2067492+T*(Io))))
2020 GOTO 2070
2030 T=3.75/Y
2040 Io=.00916281-T*(.02057706-T*(.02635537-T*(.31647613-T*(.00292377))))
2050 Io=.39894228+T*(.01328592+T*(.00225319-T*(.00157585-T*(Io))))
2060 Io=Io*EXP(Y)/SQRT(Y)
2070 RETURN Io
2080 FNEND
2090 !
2100 DEF FNIn(X)      ! In(X) via 9.8.3 & 9.8.4
2110 Y=ABS(X)
2120 IF Y > 3.75 THEN 2170
2130 T=Y*Y/14.0625
2140 I1=.02658733+T*(.00301532+T*(.00032411+
2150 I1=K*(.5+T*(.87890594+T*(.51498869+T*(.15034934+T*(I1))))
2160 GOTO 2210
2170 T=3.75/Y
2180 I1=.01031555-T*(.02282967-T*(.02895312-T*(.31711654-T*(.00420059))))
2190 I1=.39894228-T*(.03988024+T*(.00362016-T*(.00163801-T*(I1))))
2200 I1=SGN(X)*I1*EXP(Y)/SQRT(Y)
2210 RETURN I1
2220 FNEND
```

Appendix B
PROGRAM FOR THE EVALUATION OF (59)

The integral of interest is, from (59),

$$v_{\epsilon}(u) = \int_0^1 dx \mathcal{J}_u(ux) x^{2u+1} w_{\epsilon}(x). \quad (\text{B-1})$$

We approximate this integral by sampling at increment $\Delta = \frac{1}{N_x}$ and using Simpson's rule. There follows

$$v_{\epsilon}(u) \cong \frac{\Delta}{3} \sum_{n=0}^{N_x} \bar{w}_n \mathcal{J}_u(nu\Delta) (n\Delta)^{2u+1} w_{\epsilon}(n\Delta), \quad (\text{B-2})$$

where

$$\{\bar{w}_n\} = 1, 4, 2, 4, \dots, 4, 2, 4, 1. \quad (\text{B-3})$$

The program below evaluates (B-2). Values of $x^{2u+1} w_{\epsilon}(x)$ are stored in array Xmuw.

```

10  Eps=.1      ! TRANSITION PARAMETER  $\epsilon$ 
20  Nu=0       ! DIMENSIONALITY PARAMETER  $\mu$ 
30  Nu=-1.5    ! WEIGHTING PARAMETER  $\nu$ 
40  Bc=4       ! WEIGHTING PARAMETER  $B$ 
50  Nn=248     ! NUMBER OF INCREMENTS IN  $X$ 
60  Niu=240    ! NUMBER OF INCREMENTS IN  $U$ 
70  DIM Xmuwe(0:1024),Ve(0:240)
80  REDIM Xmuwe(0:Nn),Ve(0:Niu)
90  Mu21=Mu+2+1
100 Nu1=Nu+1
110 Del=1/Nn
120 Xmuwe(Nn)=0
130 FOR I=0 TO Nn-1
140  K=I+Del
150  R2=1-X*X
160  Rs=SQR(R2)
170  Br=Bc+Rs
180  Iv=R2^Nu*FNInuxnu(Nu,Br)
190  IF X>1-Eps THEN 220
200  Xmuwe(I)=X^Mu21*Iv
210  GOTO 250
220  Iv1=R2^Nu1*FNInuxnu(Nu1,Br)
230  Ne=FNNeut(Eps,X)
240  Nep=FNNeutp(Eps,X)
250  Xmuwe(I)=X^Mu21*(Iv*Ne-Iv1*Nep/X)
260  NEXT I
270  FOR Iu=0 TO Niu
280  U=Iu/Niu*6*PI
290  Ud=U*Del
300  T=Xmuwe(0)+FNJnuxnu(Mu,U)*Xmuwe(Nn)
310  So=Se=0
320  FOR Ns=1 TO Nn-1 STEP 2
330  So=So+FNJnuxnu(Mu,Ns*Ud)*Xmuwe(Ns)
340  NEXT Ns
350  FOR Ns=2 TO Nn-2 STEP 2
360  Se=Se+FNJnuxnu(Mu,Ns*Ud)*Xmuwe(Ns)
370  NEXT Ns
380  Ve(Iu)=T+4*So+2*Se
390  PRINT Iu,Ve(Iu)
400  NEXT Iu
410  PLOTTER IS "GRAPHICS"
420  GRAPHICS
430  SCALE 0,Niu,-70,0
440  GRID Niu/6,10
450  PENUP
460  T=20*LGT(Ve(0))
470  FOR Iu=0 TO Niu
480  Y=20*LGT(ABS(W.Iu))-T
490  PLOT Iu,Y
500  NEXT Iu
510  PENUP
520  END
530  !

```

```
540 DEF FNNeut(E,X)
550 IF X<=1-E THEN RETURN 1
560 Y=(X-1+.5*E)/E
570 T=Y*Y
580 T=15-T*(40-T*48)
590 RETURN .5-.125*Y*T
600 FNEND
610 !
620 DEF FNNeutp(E,X)
630 IF X<=1-E THEN RETURN 0
640 Y=(X-1+.5*E)/E
650 T=Y*Y
660 T=1-T*(8-16*T)
670 RETURN -1.875*T/E
680 FNEND
690 !
```

ALL OTHER FUNCTIONS ARE LISTED IN APPENDIX A.

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-8