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Center for Multivariate Analysis

University of Pittsburgh

THIRD ORDER EFFICIENCY OF THE MLE
- A COUNTEREXAMPLE

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June 1982

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THIRD ORDER EFFICIENCY OF THE MLE
- A COUNTEREXAMPLE

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ABSTRACT

We give an example of a curved exponential where the maximum likelihood estimate is not third order efficient either in the sense of Fisher-Rao or Rao.

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1. INTRODUCTION

That the maximum likelihood estimate (m.l.e.) is second order efficient has been proved under various conditions by Ghosh and Subramanyam (1974), Efron (1975), Ghosh, Sinha and Subramanyam (1979), Ghosh. Sinha and Wieand (1980), Pfanzagl and Wefelmeyer (1976, 1979) and Takechi and Akahira (1978). In the last three references this property is referred to as third order efficiency; this use of the term should not be confused with ours, which will be formally introduced in the next section. The purpose of this note is to produce an example of what Efron (1975) calls curved exponentials where the efficiency of the mle fails in the class of Fisher consistent estimates when we take into consideration third order terms, (namely terms of $0(n^{-3})$), in the expression for the mean square of the error of estimates concerned or terms of order $0(n^{-1})$ in the expression for the loss of information), vide (3.1) and (4.1). On the whole we have conformed to the notations and conventions of Ghosh and Subramanyam (1974) and our definition of third order efficiency is in analogy with that of second order efficiency presented there, but this note can be read independently of it. All our computations are a straightforward application of the so called delta method and may be justified as in Ghosh and Subramanyam (1974) or Ghosh, Sinha and Subramanyam (1979); no further reference will be made to this aspect.

In Section 2 the counterexample is introduced and the mle expanded. Lack of third order efficiency is established in Sections 3 and 4.

2. THE COUNTEREXAMPLE

Consider a one-parameter trivariate density

$$f_\theta(x_1,x_2,x_3) = \exp\{-\tfrac{1}{2}(x_1-\theta)^2 - \tfrac{1}{2}(x_2-\theta^2)^2 - \frac{x_3^2}{2}\}$$

$$\{Ax_1^2 x_2^2 + Bx_3^2 + 2C\, x_1x_2x_3\} \tag{2.1}$$

$$\{A(1+\theta^2)\ (1+\theta^4)+ B\}^{-1}\{2\pi\}^{-3/2}$$

$$-\infty < x_i < \infty,\ i = 1,2,3$$

$$-\infty < \theta < \infty$$

$$A > 0,\ B > 0,\ AB > C^2$$

with A,B,C independent of θ.

Let $\{X_{1i},X_{2i},X_{3i}\}$ be n i.i.d copies of X_1,X_2,X_3. Then the likelihood $L(\theta)$ is the product of n terms exemplified by (2.1). Hence the derivatives of log likelihood are

$$Z_n \equiv \frac{1}{n}\,\frac{d \log L}{d\,\theta} = (\bar{x}_1-\theta) + 2\theta(\bar{x}_2-\theta^2) - \frac{A(2\theta+4\theta^3+6\theta^5)}{(A+B)+A\theta^2+A\theta^4+A\theta^6}$$

$$W_n^* \equiv \frac{1}{n}\,\frac{d^2 \log L}{d\theta^2} = -1 + 2(\bar{x}_2-\theta^2) - 4\theta^2 - \frac{A(2+12\theta^2+30\theta^4)}{(A+B)+A\theta^2+A\theta^4+A\theta^6}$$

$$+ \frac{A^2(2\theta+4\theta^3+6\theta^5)^2}{\{(A+B)+A(\theta^2+\theta^4+\theta^6)\}^2}$$

$$V_n^* \equiv \frac{1}{n}\,\frac{d^3 \log L}{d\theta^3} = -12\theta - \frac{A(24\theta+120\theta^3)}{(A+B)+A(\theta^2+\theta^4+\theta^6)} + \frac{A^2(2+12\theta^2+30\theta^4)(2\theta+4\theta^3+6\theta^5)}{\{A+B)+A(\theta^2+\theta^4+\theta^6)\}^2}$$

$$+ \frac{2A^2(2\theta+4\theta^3+6\theta^5)(2+12\theta^2+30\theta^4)}{\{(A+B)+A(\theta^2+\theta^4+\theta^6)\}^2} - \frac{2A^3(2\theta+4\theta^3+6\theta^5)^3}{\{(A+B)+A(\theta^2+\theta^4+\theta^6)\}^3}$$

$$U_n^* \equiv \frac{1}{n}\,\frac{d^4 \log L}{d\theta^4} = \eta(\theta), \text{ say.}$$

Clearly

$$I(\theta) \equiv E_\theta\left(-\frac{1}{n}\frac{d^2 \log L}{d\theta^2}\right) = -2\, E_\theta(\bar{X}_2-\theta^2) + 4\theta^2 + \frac{A(2+12\theta^2+30\theta^4)}{(A+B)+A(\theta^2+\theta^4+\theta^6)}$$

$$- \frac{A^2(2\theta+4\theta^3+6\theta^5)^2}{\{(A+B)+A(\theta^2+\theta^4+\theta^6)\}^2} .$$

To compute $I(\theta)$ we need

$$E_\theta(\bar{X}_2-\theta^2) = \frac{2A\theta^2(1+\theta^2)}{(A+B)+A(\theta^2+\theta^4+\theta^6)} .$$

Let $W_n = W_n^* + I(\theta)$ so that $E_\theta W_n = 0$.

We shall now get an expansion of the mle $\hat{\theta}$. Clearly

$$0 = Z_n + (\hat{\theta}-\theta)(W_n - I(\theta)) + \tfrac{1}{2}(\hat{\theta}-\theta)^2 V_n^* + \tfrac{1}{6}(\hat{\theta}-\theta)^3 U_n^* + \text{smaller terms.}$$

So writing,

$$\hat{\theta} - \theta = \frac{A_1}{\sqrt{n}} + \frac{A_2}{n} + \frac{A_3}{n^{3/2}} + \ldots. \tag{2.2}$$

and substituting in the previous equation we get

$$Z_n - \frac{A_1}{\sqrt{n}} I(\theta) = 0$$

$$\frac{A_1}{\sqrt{n}} W_n - \frac{A_2}{n} I(\theta) + \tfrac{1}{2}\frac{A_1^2}{n} V_n^* = 0$$

$$\frac{A_2}{n} W_n - \frac{A_2}{n^{3/2}} I(\theta) + \frac{A_1}{\sqrt{n}} \cdot \frac{A_2}{n} V_n^* + \frac{1}{6}\left(\frac{A_1}{\sqrt{n}}\right)^3 U_n^* = 0$$

From these we determine A_1, A_2, A_3 and substitute in (2.2) yielding

$$\hat{\theta} - \theta = \frac{Z_n}{I(\theta)} + \left\{\frac{Z_n W_n}{I^2(\theta)} + \frac{1}{2}\frac{Z_n^2 V_n^*}{I^3(\theta)}\right\} + \left[\left\{\frac{Z_n W_n}{I^2(\theta)} + \frac{Z_n^2 V_n^*}{I^3(\theta)}\right\}\left\{\frac{W_n}{I(\theta)} + \frac{Z_n V_n^*}{I^2(\theta)}\right\}\right.$$

$$\left. + \frac{1}{6}\frac{U_n^* Z_n^3}{I^4(\theta)}\right] + \ldots. \tag{2.3}$$

Let

$$Q_2(\theta) = \frac{Z_n W_n}{I^2(\theta)} + \frac{1}{2}\frac{Z_n^2 V_n^*}{I^3(\theta)} \tag{2.4}$$

$$Q_3(\theta) = \left\{\frac{Z_n W_n}{I^2(\theta)} + \frac{1}{2}\frac{Z_n^2 V_n^*}{I^3(\theta)}\right\}\left\{\frac{W_n}{I(\theta)} + \frac{Z_n V_n^*}{I^2(\theta)}\right\} + \frac{1}{6}\frac{U_n^* Z_n^3}{I^4(\theta)} \tag{2.5}$$

So

$$\hat{\theta} - \theta = \frac{Z_n}{I(\theta)} + Q_2(\theta) + Q_3(\theta) + o_p(n^{-3/2}). \tag{2.6}$$

3. LACK OF THIRD ORDER EFFICIENCY IN THE SENSE OF RAO

Define

$$T' = \hat{\theta} + \delta^*\{\bar{X}_3 - \pi_3(\hat{\theta})\}^3$$

where

$$\pi_3(\theta) = E_\theta(\bar{X}_3),$$

and δ^* is a constant to be suitably chosen. Then like $\hat{\theta}$, T' is Fisher consistent. Let

$$E_\theta\{\bar{X}_3 - \pi_3(\hat{\theta})\}^3 = \xi(\theta)/n^2 + o(n^{-2}).$$

Then T' adjusted to have same bias as the mle (up to $o(n^{-2})$) is

$$T = \hat{\theta} + \delta^*[\{\bar{X}_3 - \pi_3(\hat{\theta})\}^3 - \xi(T')/n^2] .$$

We shall show

$$E_{\theta=0}(\hat{\theta}-\theta)^2 > E_{\theta=0}(T-\theta)^2 \tag{3.1}$$

if we neglect terms of order $o(n^{-3})$. Of course a similar inequality $E_{\theta=0}(\hat{\theta}'-\theta)^2 > E_{\theta=0}(T'-\theta)^2$ holds if instead of adjusting T' we adjust $\hat{\theta}$ to $\hat{\theta}'$ to ensure same bias as T' up to $o(n^{-2})$. We shall refer to (3.1) as lack of third order efficiency in the sense of Rao.

For $\theta = 0$,

$$Z_n = \bar{X}_1,\ W_n = 2\bar{X}_2,\ V_n^* = 0,$$

$$U_n^* = -12(2A^2+4AB+B^2)/(A+B)^2 \tag{3.2}$$

$$I(0) = (3A+B)/(A+B),$$

$$\pi_3(0) = 0,\ \pi_3'(0) = 0,\ E_{\theta=0}(\bar{X}_3^3) = 0.$$

To calculate $\xi(\theta)$, we use

$$\{\bar{X}_3 - \pi_3(\hat{\theta})\}^3 = [\{\bar{X}_3-\pi_3(\theta)\}^3 - 3(\bar{X}-\pi_3(\theta))^2(\hat{\theta}-\theta)\ \pi_3'(\theta)$$
$$+ 3(\bar{X}_3-\pi_3(\theta))\ (\hat{\theta}-\theta)^2(\pi_3'(\theta))^2 - (\hat{\theta}-\theta)^3(\pi_3'(\theta))^3] + \text{smaller terms}$$

which, using (3.2), gives for $\theta = 0$

$$\{\bar{X}_3 - \pi_3(\hat{\theta})\}^3 = \bar{X}_3^3 + o_p(n^{-2})$$

and so $\xi(o) = 0$.

Observe

$$E_{\theta=0}(T-\theta)^2 = E_{\theta=0}(\hat{\theta}-\theta)^2 + \delta^{*2}E_{\theta=0}[\{\bar{X}_3-\pi_3(\hat{\theta})\}^3 - \xi(T')/n^2]^2$$
$$+ 2\delta^*\ E_{\theta=0}(\hat{\theta}-\theta)\{(\bar{X}_3-\pi_3(\hat{\theta}))^3 - \xi(T')/n^2\} \tag{3.3}$$

We first note that the second term on the RHS of (3.3) is (using (3.2))

$$
\begin{aligned}
&E_{\theta=0}(\bar{X}_3^6) + o(n^{-3}) \\
&= \frac{15}{n^3}\left\{\frac{A+3B}{A+B}\right\}^3 + o(n^{-3}) \;. \qquad (3.4)
\end{aligned}
$$

For the third term on the RHS of (3.3), we have,

$$
\begin{aligned}
&E_{\theta=0}(\hat{\theta}-\theta)\{(\bar{X}_3-\pi_3(\hat{\theta}))^3 - \xi(T')/n^2\} \\
&= E_{\theta=0}\left(\frac{Z_n}{I(\theta)} + Q_2 + Q_3\right)\left\{(\bar{X}_3-\pi_3(\hat{\theta}))^3 - \frac{\xi(T')}{n^2}\right\}+o(n^{-3}) \quad \text{by (2.6)} \\
&= E_{\theta=0}(Q_2+Q_3)\{(\bar{X}_3-\pi_3(\hat{\theta}))^3 - \xi(T')/n^2\} + o(n^{-3}) \\
&(\text{since } E_\theta Z_n\{\ldots.\} = \frac{1}{n}\frac{d}{d\theta}E_\theta\{\ldots.\} = o(n^{-3})), \\
&= E_{\theta=0}\left(\frac{Z_n W_n}{I^2(\theta)} + \frac{Z_n W_n^2}{I^3(\theta)} + \frac{U_n^*(0)Z_n^3}{6I^4(0)}\right)(\bar{X}_3^3) + o(n^{-3}) \\
&= E_{\theta=0}\left\{\frac{2\bar{X}_1\bar{X}_2}{I^2(0)} + \frac{4\bar{X}_1\bar{X}_2^2}{I^3(0)} + \frac{U_n^*(0)\bar{X}_1^3}{6I^4(0)}\right\} \bar{X}_3^3 + o(n^{-3}) \qquad (3.5)
\end{aligned}
$$

We observe, after some straightforward calculations,

$$
\begin{aligned}
&E_{\theta=0}\ \bar{X}_1\bar{X}_2^2\ \bar{X}_3^3 = o(n^{-3}) \\
&E_{\theta=0}\bar{X}_1^3\bar{X}_3^3 = o(n^{-3}) \qquad (3.6) \\
&E_{\theta=0}\bar{X}_1\bar{X}_2\bar{X}_3^3 = \frac{A+3B}{(A+B)^2}\cdot\frac{6C}{n^3} + o(n^3)
\end{aligned}
$$

Using (3.4), (3.5) and (3.6) we get from (3.3),

$$E_{\theta=0}(T-\theta)^2 = E_{\theta=0}(\hat{\theta}-\theta)^2 + \{\delta^{*2} 15(\frac{A+3B}{A+B})^3 + \frac{4\delta^*}{I^2(0)} \frac{A+3B}{(A+B)^2} \cdot 6C\} \frac{1}{n^3} + o(n^{-3}).$$

Recall that $I(0) = (3A+B)/(A+B)$ so that the coefficient of n^{-3} above is

$$15\delta^{*2}(\frac{A+3B}{A+B})^3 + 4\delta^* \frac{(A+3B)}{(3A+B)^2} 6C$$

which can be made < 0 if, say,

$$A = B = 1, \quad \delta^* < 0 \text{ and } 20\delta^* + C > 0.$$

This proves (3.1).

4. LACK OF THIRD ORDER EFFICIENCY IN THE SENSE OF FISHER-RAO

The third order loss of information for $\hat{\theta}$ may be defined as follows. Let

$$V_n^{\hat{\theta}} = \frac{d \log L}{d\theta} - \alpha\sqrt{n} - \beta n(\hat{\theta}-\theta) - \gamma n(\hat{\theta}-\theta)^2 - \delta n(\hat{\theta}-\theta)^3 .$$

Let $E_3(\hat{\theta},\theta)$ be the variance of above expanded up to $o(n^{-1})$ and then minimized with respect to the coefficients β,γ,δ. Of course α does not play any role in this. The values of β,γ,δ obtained in this way will depend on n and will be denoted as β_n etc. Similarly we define

$$V_n^T(\theta) = \frac{d \log L}{d\theta} - \alpha'\sqrt{n} - \beta' n(T-\theta) - \gamma' n(T-\theta)^2 - \delta' n(T-\theta)^3$$

and then $E_3(T,\theta)$ as above. We shall show

$$E_3(\hat{\theta},\theta) > E_3(T,\theta) \qquad \text{for } \theta = 0 \tag{4.1}$$

We sketch a proof.

Using (2.6) and (3.1) one can show that for the present purpose of evaluating E_3 up to $o(n^{-1})$, we may take

$$\begin{aligned} \beta_n &= I(\theta) + \frac{\beta_1}{\sqrt{n}} + \frac{\beta_2}{n} \\ \gamma_n &= \gamma_o + \frac{\gamma_1}{\sqrt{n}} + \frac{\gamma_2}{n} \\ \delta_n &= \delta_o + \frac{\delta_1}{\sqrt{n}} \end{aligned} \tag{4.2}$$

(where γ_o is in fact the coefficient λ arising from considerations of second order efficiency, vide Ghosh and Subramanyam (1974)).

The following facts will be needed :

$$\begin{aligned}
&\mathrm{Cov}_{\theta=0}[\bar{X}_1\bar{X}_2^2, \bar{X}_3^3] = o(n^{-3}) \\
&\mathrm{Cov}_{\theta=0}[\bar{X}_1^2\bar{X}_2, \bar{X}_3^3] = E_{\theta=0}[\bar{X}_1^2\bar{X}_2\bar{X}_3^3] = \mathrm{Cov}_{\theta=0}[\bar{X}_1\bar{X}_2, \bar{X}_1\bar{X}_3^3] = o(n^{-3}) \\
&\mathrm{Cov}_{\theta=0}[\bar{X}_1^3, \bar{X}_3^3] = E_{\theta=0}[\bar{X}_1^3\bar{X}_3^3] = \mathrm{Cov}_{\theta=0}[\bar{X}_1^2, \bar{X}_1\bar{X}_3^3] = o(n^{-3}) \\
&\mathrm{Cov}_{\theta=0}[\bar{X}_1^2, \bar{X}_3^3] = E_{\theta=0}[\bar{X}_1^2 X_3^3] = \mathrm{Cov}_{\theta=0}[\bar{X}_1, \bar{X}_1\bar{X}_3^3] = o(n^{-3}) \qquad (4.3) \\
&\mathrm{Cov}_{\theta=0}[\bar{X}_1\ \bar{X}_2, \bar{X}_3^3] = o(n^{-3}) \\
&\mathrm{Cov}_{\theta=0}[\bar{X}_2^2, \bar{X}_3^3] = o(n^{-3}) \\
&\mathrm{Cov}_{\theta=0}[\bar{X}_2^2, \bar{X}_3^3] = o(n^{-3}) \\
&\mathrm{Cov}_{\theta=0}(\bar{X}_1, \bar{X}_3^3) = 0
\end{aligned}$$

Now at $\theta = 0$ (with $A = B = 1$ and hence $I = 2$)

$$\hat{\theta} - \theta = \frac{\bar{X}_1}{I} + \left(\frac{2\bar{X}_1\bar{X}_2}{I^2}\right) + \left(\frac{4\bar{X}_1\bar{X}_2^2}{I^3} + \frac{U_n^*\bar{X}_1^3}{6I^4}\right).$$

Hence

$$\begin{aligned} V_n^{\hat{\theta}}(\theta=0) = {} & -\alpha\sqrt{n} - \{\sqrt{n}\,\beta_1 \frac{\bar{X}_1}{I} + 2n\frac{\bar{X}_1\bar{X}_2}{I} + \frac{n\gamma_o\bar{X}_1^2}{I^2}\} \\ & - \{\beta_2 \frac{\bar{X}_1}{I} + 2\sqrt{n}\,\beta_1 \frac{\bar{X}_1\bar{X}_2}{I^2} + 4n\frac{\bar{X}_1\bar{X}_2^2}{I^2} + n\,U_n^*\frac{\bar{X}_1^3}{6I^3} + \sqrt{n}\gamma_1\frac{\bar{X}_1^2}{I^2} + 4n\gamma_o\frac{\bar{X}_1^2\bar{X}_2}{I^3} + n\delta_o\frac{\bar{X}_1^3}{I^3}\} \\ & - \{2\delta_2\frac{\bar{X}_1\bar{X}_2}{I^2} + \sqrt{n}\beta_1\left(\frac{4\bar{X}_1\bar{X}_2^2}{I^3} + \frac{U_n^*\bar{X}_1^3}{6I^4}\right) + \frac{\gamma_2\bar{X}_1^2}{I^2} + 4\sqrt{n}\,\gamma_1\frac{\bar{X}_1^2\bar{X}_2}{I^3} + 4n\gamma_o\frac{\bar{X}_1^2\bar{X}_2^2}{I^4} \\ & + 2n\gamma_o\frac{\bar{X}_1}{I}\left(\frac{4\bar{X}_1\bar{X}_2^2}{I^3} + \frac{U_n^*\bar{X}_1^3}{6I^4}\right) + \sqrt{n}\,\delta_1\frac{\bar{X}_1^3}{I^3} + 6n\,\delta_o\frac{\bar{X}_1^3\bar{X}_2}{I^3}\} + \text{smaller terms} \end{aligned} \tag{4.4}$$

and

$$V_n^T(\theta=0) = V_n^{\hat{\theta}}(\theta=0) - \delta^*\{nI\ \bar{X}_3^3\} + \text{smaller terms} \tag{4.5}$$

Using (4.3), (4.4) and (4.5) we can prove

$$\mathrm{Var}_{\theta=0}(V_n^T(\theta=0) = E_3(\hat{\theta},\theta=0) + [\mathrm{Var}_{\theta=0}(nI\ \delta^*\ \bar{X}_3^3) + 4\delta^* n^2 \mathrm{Cov}_{\theta=0}(\bar{X}_1\bar{X}_2, \bar{X}_3^3)] + o(n^{-1}$$

$$= E_3(\hat{\theta},\theta=0) + I^2\ \frac{120}{n}(\delta^*)^2 + \frac{24\delta^* C}{n} + o(n^{-1}) \tag{4.6}$$

The sum of the last two terms on the RHS of (4.6) can be made < 0 by taking $\delta^* < 0$, $20\delta^* + C > 0$. Now (4.1) follows from (4.6).

5. CONCLUDING REMARKS

A close inspection of the proof will reveal that the reason for lack of third order efficiency is due to the non-zero covariance between (Q_2+Q_3) and $(T-\hat{\theta})$. This covariance is likely to be non-zero in most cases but except in specially constructed simple examples like the present one checking this would involve prohibitive calculations.

It would be interesting to show that not only the mle but no other estimate can possess third order efficiency.

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