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On The Instability of the Slotted ALOHA
Multiaccess Algorithm

by

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Abstract: The slotted ALOHA multiaccess algorithm for the infinite user model is shown to be unstable via a martingale method of independent interest. Consequently the hypothesis of statistical equilibrium used to calculate the maximum throughput is not valid.

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I. Introduction

In his paper [7] Massey noted that there was neither mathematical nor experimental justification for the hypothesis of statistical equilibrium assumed by Abramson and others in their calculation of the maximum throughput of the slotted ALOHA multiaccess algorithm. Now the opposite of statistical equilibrium is instability which in the present context means the number of blocked terminals becomes infinite with probability one as time tends to infinity. Using simulations Kleinrock and Lam [5] noted that the uncontrolled slotted ALOHA scheme is unstable. A mathematical proof of this fact was offered by Fayolle et al. [2] using a Markov chain model to describe the number of blocked terminals. Translated into the language of Markov chains instability as defined above is equivalent to showing that the Markov chain is transient, see e.g. Karlin and Taylor [4] for the necessary definitions and background. In particular we remind the reader of the well known fact that an irreducible aperiodic Markov chain falls into one of three mutually exclusive (and exhaustive) classes: positive recurrent, null recurrent and transient, see e.g. Karlin and Taylor op. cit.

Now Fayolle et al [2] showed that the Markov chain occurring in their model of the slotted ALOHA scheme is not positive recurrent from which they erroneously include that the Markov chain is transient. Unfortunately they did not exclude the possibility that it might be null recurrent. Nevertheless their assertion that the Markov chain is transient is correct and we present in this paper a novel proof of this fact based upon a martingale method of independent interest. The elementary facts concerning martingales and Markov chains that we shall need are summarized in part II and the application to the Markov chain model of Fayolle et al op. cit. will be given in part III.

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II. Transient Markov Chains - A Martingale Approach

In this section we present a well known method (at least to Probabilists) for constructing a supermartingale associated to a Markov chain which leads to a simple sufficient condition for a Markov chain to be transient.

Let $\{X_k, k = 0, 1, 2, \dots\}$ denote a Markov chain with state space the non negative integers $I^+ = \{0, 1, 2, \dots\}$ and transition matrix $P_{ij} = P(X_{k+1} = j | X_k = i)$. In addition we assume that the Markov chain is irreducible and aperiodic. Now if $\{X_k\}$ is recurrent (whether it is positive or null recurrent is immaterial) then the event $\{X_k = j\}$ occurs infinitely often (abbreviated i.o.) with probability one for any state j . Suppose we can construct a non negative, non constant function $f: I^+ \rightarrow R^+ = [0, \infty)$ satisfying the following sequence of inequalities:

$$(1) \sum_{j=0}^{\infty} P_{ij} f(j) \leq f(i) \text{ for every } i \in I^+.$$

Now $E(f(X_{k+1}) | X_k = i) = \sum_{j=0}^{\infty} P_{ij} f(j) \leq f(i) = f(X_k)$ and by the Markov property

(2) $E(f(X_{k+1}) | X_0, \dots, X_k) = E(f(X_{k+1}) | X_k) \leq f(X_k)$. Consequently the sequence of random variable $f(X_k)$ is a non negative supermartingale with respect to $\{X_k\}$, see e.g. Karlin and Taylor [4], Definition 1.2, p. 239 or the more general definition to be found in Doob [1]. We shall call f a supermartingale generating function.

(3) Theorem (Doob): A non negative supermartingale converges with probability one.

This is a basic result in martingale theory and we refer the reader to Doob [1] for the proof.

(4) Theorem. Suppose the irreducible aperiodic Markov chain $\{X_k\}$ admits a non constant, non negative supermartingale generating function f . Then the Markov chain is transient i.e., $P(\lim_{k \rightarrow \infty} X_k = \infty) = 1$.

Proof: Assume to the contrary that $\{X_k\}$ is recurrent and let $j_0 \neq j_1$ be any two states for which $f(j_0) \neq f(j_1)$ (since f is non constant such a pair exists, by hypothesis). Now recurrence implies that $X_k = j_0$ and $X_k = j_1$ i.o. with probability one and hence $f(X_k) = f(j_0)$ and $f(X_k) = f(j_1)$ i.o. with probability one. But this implies that $\lim_{k \rightarrow \infty} f(X_k)$ does not exist because $f(j_0) \neq f(j_1)$ and this contradicts Doob's martingale convergence theorem (3).

III. Instability of the slotted ALOHA scheme via a martingale generating function.

We now apply the results of part 2 to the study of the slotted ALOHA scheme as presented in Fayolle et al [2]. First, some notation: let $X_k \triangleq$ the number of packets awaiting to be transmitted at time k and $A_k \triangleq$ the number of new packets that arrive during the k^{th} time interval. We make the following standard assumptions:

- (5) $A_0, A_1, \dots, A_k$, are i.i.d. random variables with probability distribution $P(A_k = j) = \lambda_j$, $j = 0, 1, 2, \dots$ and moment generating function

$$\psi(\xi) = \sum_{j=0}^{\infty} \lambda_j \xi^j.$$

- (6) The A_k 's are independent of the X_k 's and when a new packet arrives during the k^{th} interval it is transmitted in the next time interval; thus, two or more arrivals at time k lead to a collision at time $k + 1$.
- (7) If a terminal is blocked at time k it retransmits with probability p , $0 < p < 1$, at time $k + 1$ and each terminal acts independently of the others.

Under these assumptions it is easy to see that the process $\{X_k\}$ is a Markov chain with transition Matrix P_{ij} given by

$$P_{0j} = \lambda_j, \quad j \geq 0$$

$$P_{ii-1} = \lambda_0 i p (1-p)^{i-1}$$

$$P_{ij} = 0, \quad j \leq i-2$$

$$P_{ij} = \lambda_{j-i} (1-p)^i + \lambda_{j-i} (1-(1-p))^i - i p (1-p)^{i-1}$$

$$\begin{aligned}
 & + \lambda_{j-i+1} i p(1-p)^{i-1}, j \geq i \\
 & = \lambda_{j-i} (1-i p(1-p)^{i-1}) + \lambda_{j-i+1} i p(1-p)^{i-1}, j \geq i
 \end{aligned}$$

To insure the Markov chain is aperiodic and irreducible it suffices to assume

(9) $0 < \lambda_0 < 1_0$

Construction of the supermartingale generating function f :

We try a solution of the form

(10) $f(j) = \xi^j$ for some ξ , $0 < \xi < 1$. There are two cases to consider:

Case 1: $a = \sup_i i p(1-p)^{i-1} < \psi'(1)$

Case 2: Since $\lim_{i \rightarrow \infty} i p(1-p)^{i-1} = 0$ there exists i^* such that $i p(1-p)^{i-1} < \psi'(1)$ for $i \geq i^*$.

(11) Theorem: (a) If $\psi'(1) > a$ then there exists a ξ , $0 < \xi < 1$ such that $f(j) = \xi^j$ is a supermartingale generating function; (b) If $\psi'(1) \leq a$ then there exists an integer i^* such that $f(j) = 1$, $0 \leq j \leq i^*$ and $f(j) = \xi^{j-i^*}$, $j \geq i^*$ is a supermartingale generating function.

Proof: Case 1: Note that $\sum_{j=0}^{\infty} P_{0j} f(i) = \sum_{j=0}^{\infty} \lambda_j \xi^j = \psi(\xi) \leq 1 = f(0)$ so inequality is certainly satisfied when $i=0$. Turning now to the case $i > 0$ we have

$$\sum_{j=0}^{\infty} P_{ij} \xi^j = \sum_{j=i-1}^{\infty} P_{ij} \xi^j = \lambda_0 i p(1-p)^{i-1} \xi^{i-1} + \sum_{k=0}^{\infty} \lambda_k (1-i p(1-p)^{i-1}) \xi^{k+i}$$

$$+ \sum_{k=0}^{\infty} \lambda_{k+1} i p(1-p)^{i-1} \xi^{k+i} =$$

$$\xi^{i-1} \{ \lambda_0 i p(1-p)^{i-1} + \sum_{k=0}^{\infty} \lambda_k (1-i p(1-p)^{i-1}) \xi^{k+1} + \sum_{k=0}^{\infty} \lambda_{k+1} i p(1-p)^{i-1} \xi^{k+1} \}.$$

Note that $\sum_{k=0}^{\infty} \lambda_k (1-i p(1-p)^{i-1}) \xi^{k+1} = \xi (1-i p(1-p)^{i-1}) \sum_{k=0}^{\infty} \lambda_k \xi^k =$

$$= \xi (1-i p(1-p)^{i-1}) \psi(\xi) \text{ and that } \sum_{k=0}^{\infty} \lambda_{k+1} i p(1-p)^{i-1} \xi^{k+1} =$$

$$i p(1-p)^{i-1} \sum_{k=1}^{\infty} \lambda_k \xi^k = i p(1-p)^{i-1} (\psi(\xi) - \lambda_0). \text{ Upon suitably rearranging}$$

terms we obtain the formula

(12) $\sum_{j=0}^{\infty} P_{ij} \xi^j = \xi^{i-1} \psi(\xi) \{ \xi (1-i p(1-p)^{i-1}) + i p(1-p)^{i-1} \}$

consequently $f(j) = \xi^j$ will be a supermartingale generating function provided

$$(13) \quad \xi^{i-1} \psi(\xi) \{ \xi(1-ip(1-p)^{i-1}) + ip(1-p)^{i-1} \} \leq \xi^i, \text{ equivalently}$$

(14) $\psi(\xi) \leq \xi / \{ \xi(1-ip(1-p)^{i-1}) + ip(1-p)^{i-1} \}$, $i=1,2,\dots$. Denote the function occurring on the right hand side of (14) by $\phi_i(\xi)$ and set $ip(1-p)^{i-1} = a_i$ noting that $\lim_{i \rightarrow \infty} ip(1-p)^{i-1} = 0$. Observe that ϕ_i is concave on $[0,1]$, $\phi_i'(1) = \frac{1}{1-a_i} \leq \sup_i \frac{1}{1-a_i} = \frac{1}{1-a} < \psi'(1)$ and that $a \geq a_i$ implies $\phi_i(\xi) \leq \xi / \{ \xi(1-a) + a \}$ on $[0,1]$. Consequently $h(\xi) = \psi(\xi) - [\xi / \{ \xi(1-a) + a \}]$ is convex on $[0,1]$ with $h(0) = \lambda_0 > 0$, $h(1) = 0$ and $h'(1) = 0$ and $h'(1) = \psi'(1) - a > 0$. Therefore there exists a $\xi^*, 0 < \xi^* < 1$ such that $h(\xi) < 0$ on the open interval $(\xi^*, 1)$ and this implies that $\psi(\xi) - \phi_i(\xi) < h(\xi) < 0$ for all ξ in $(\xi^*, 1)$ and all i , so $\psi(\xi) < \phi_i(\xi)$ for at least one ξ in $(\xi^*, 1)$. This completes the construction of the supermartingale generating function f in case 1.

Case 2: Since $\lim_{i \rightarrow \infty} \phi_i'(1) = \lim_{i \rightarrow \infty} a_i = 0$ it follows there exists an i^* such that $a_i < \psi'(1)$ all $i \geq i^*$. If we now set $f(j) = 1$ for $0 \leq j \leq i^*$ and $f(j) = \xi^{j-i^*}$ for $j \geq i^*$ and $0 < \xi < 1$ then the inequalities $\sum_{j=0}^{\infty} P_{ij} \xi^j \leq 1 = f(i)$, $0 \leq i \leq i^*$ are automatically satisfied whilst for $i > i^*$ the same calculation as in case 1 leads to the condition

$$(14') \quad \psi(\xi) \leq \xi / [\xi(1-ip(1-p)^{i-p}) + ip(1-p)^{i-1}], \quad i > i^* \text{ for some } \xi \text{ in } (0,1).$$

By choosing i^* so that $\sup_{i > i^*} \phi_i'(1) = \sup_{i > i^*} a_i < \psi'(1)$ the same reasoning using in case 1 may be applied here to conclude that condition (14') is satisfied. The proof is finished.



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IV. Concluding Remarks

It is interesting to note that martingale methods may also be used to prove stability. In the context of Markov Chains this was already done by Lamperti [6] and for applications to the control of queueing processes see the forth coming paper by Hajek [3].

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