

AD-A128 545

THE OUTPUT OF M(T)/G(T)/INFINITY QUEUES(U) ARMY
INVENTORY RESEARCH OFFICE PHILADELPHIA PA H KOTKIN
JUL 82 USAIRO-TR-82/2

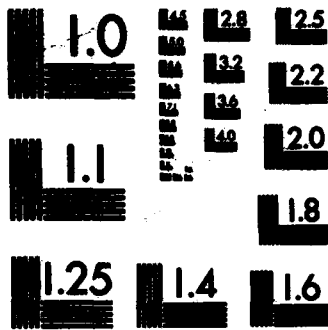
1/1

UNCLASSIFIED

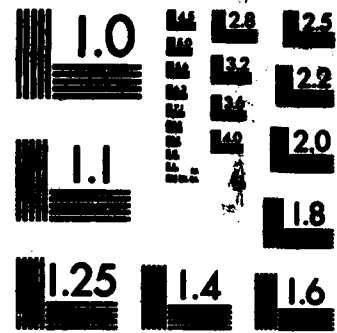
F/G 12/1

NL

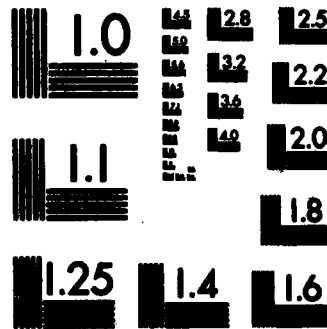
										END			



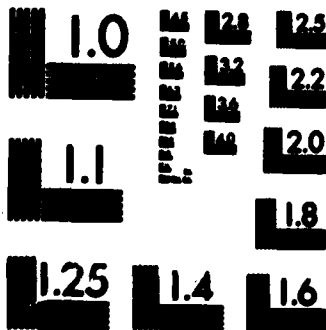
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



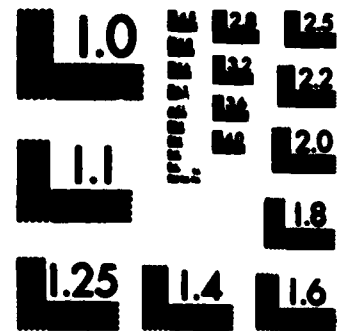
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



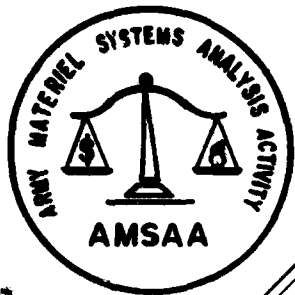
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



AD-
TECHNICAL REPORT
USAIRO-TR-82/2

2

AD A120545

AMSAA

THE OUTPUT OF
 $M(t) / G(t) / \infty$ QUEUES

INVENTORY
RESEARCH
OFFICE

July 1982

DTIC
OCT 20 1982
H

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

DTIC FILE COPY

US Army Inventory Research Office
US Army Materiel Systems Analysis Activity
800 Custom House, 2d & Chestnut Sts.
Philadelphia, PA 19106

82 10 20 038

Information and data contained in this document are based on the input available at the time of preparation. The results may be subject to change and should not be construed as representing the DARCOM position unless so specified.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A120545	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE OUTPUT OF M(t)/G(t) = Queues		5. TYPE OF REPORT & PERIOD COVERED Technical Report
7. AUTHOR(s) MEYER KOTKIN		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Inventory Research Office Room 800 - US Custom House - 2nd & Chestnut Sts. Philadelphia, PA 19106		8. CONTRACT OR GRANT NUMBER(s)
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Materiel Development & Readiness Command 5001 Eisenhower Avenue Alexandria, VA 22333		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Army Materiel Systems Analysis Activity Aberdeen Proving Ground, MD 21005		12. REPORT DATE July 1982
		13. NUMBER OF PAGES 18
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release; Distribution Unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES Information and data contained in this document are based on input available at the time of preparation. Because the results may be subject to change, this document should not be construed to represent the official position of the US Army Materiel Development & Readiness Command unless so stated.		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Non-Homogeneous Poisson Process Infinite Server Queues Departure/Output Process Laplace Transforms		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We study an infinite server queue in which the arrival process is a Non-Homogeneous Poisson Process (NHPP) and in which the service times of customers may depend on the time service was initiated. We establish a splitting theorem for NHPP similar to the well known splitting theorem for Stationary Poisson Process. With this theorem, we show that the departure process of the M(t)/G(t) queue is a NHPP, and that the number of departures during any <i>infinitely</i>		

**SDTC
SELECTED
OCT 20 1982
H**

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

ABSTRACT (CONT)

interval is independent of the number of remaining customers at the end of the interval. The splitting theorem can also be used to show that the number of busy servers at any time is Poisson distributed. ←

COPIES
STOLEN
1950

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

TABLE OF CONTENTS

	<u>Page</u>
TABLE OF CONTENTS.....	1
1. Introduction.....	2
2. Non-Homogeneous Poisson Processes.....	2
3. A Splitting Theorem.....	3
4. The Output of an M(t)/G(t)/∞ Queue.....	8
5. Special Cases.....	10
6. Summary.....	15
REFERENCES.....	16
DISTRIBUTION.....	17



Accession For	
NTIS CRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A	

1. Introduction

We consider an infinite server queue to which customers arrive according to a non-homogeneous Poisson Process (NHPP) with known intensity $\lambda(t)$, $t \geq 0$. A customer arriving at time s immediately enters service and with probability $F(s,w)$ will complete service by time w where, of course, $F(s,w) = 0$ for $w < s$. Service times for customers are assumed to be statistically independent of each other. Such a queue will be referred to as an $M(t)/G(t)/\infty$ queue to indicate that the arrival process is a NHPP, that a customer's service time may depend on the time he entered service and that there are an infinite number of servers. If $F(s,w) = G(w-s)$ is a function only of the difference $w-s$ and $\lambda(t) = \lambda$ for all t , we have, as a special case, the familiar $M/G/\infty$ queue.

An important stochastic process of interest in queuing theory is the counting process $\{D(t), t \geq 0\}$ where $D(t)$ is the number of departures from the queue by time t . In this note we establish and use a splitting theorem for NHPP to show that for the $M(t)/G(t)/\infty$ queue, $\{D(t), t \geq 0\}$ is a NHPP. Furthermore, we show that the number of departures from the queue during any finite time interval is independent of the number of customers still in service at the end of that interval. As special cases of this result we show that the transient output of an $M/G/\infty$ queue is a NHPP and that in steady state the output process of an $M/G/\infty$ queue is a stationary Poisson Process with rate equal to the rate of the Poisson input process. These results for the $M/G/\infty$ queue were first proved more laboriously by Mirasol [4].

Hillestad and Carillo [1] have shown that the number of busy servers at time w in the $M(t)/G(t)/\infty$ queue is Poisson distributed with mean depending on w and the form of the service time distribution. This result can be simply obtained by judiciously using the splitting theorem established in Section 3.

To the author's knowledge, no study has been made of the output process of the $M(t)/G(t)/\infty$ or $M(t)/G(t)/s$ queues. In a sequel to this report, we shall study the output process of $M(t)/M/s$ queues.

2. Non-Homogeneous Poisson Processes

Definition 2.1. Let $\{N(t), t \geq 0\}$ be a stochastic counting process and let

- a. $N(0) = 0$
- b. $\{N(t), t \geq 0\}$ have independent increments.

- c. $\Pr(2 \text{ or more events in } (t, t+h)) = O(h).$
- d. $\Pr(\text{exactly one event in } (t, t+h)) = \lambda(t)h + O(h).$

Then $\{N(t), t \geq 0\}$ is said to be a non-homogeneous Poisson Process with intensity function $\lambda(t)$.

Let $M(t) = \int \lambda(s) ds$. Then it is straightforward to show that Definition 2.1 is equivalent to Definition 2.2.

Definition 2.2. $\{N(t), t \geq 0\}$ is said to be a NHPP with mean value function (MVF) $M(t)$ if

- a'. $N(0) = 0$
- b'. $\{N(t), t \geq 0\}$ has independent increments.
- c'. for all $s < t$ and $n \geq 0$

$$\Pr(N(t) - N(s) = n) = e^{-(M(t) - M(s))} \frac{(M(t) - M(s))^n}{n!}$$

In this report, we shall assume the MVF of all NHPP to be completely differentiable. The proofs presented here can be extended to the case where the MVF is not completely differentiable but this would add unnecessary technical complexity to the discussion herein.

3. A Splitting Theorem

It is well known (see [5] or [6]) that if $\{N(t), t \geq 0\}$ is a stationary Poisson Process with rate λ and an event that occurs at time t is classified as type i with probability P_i , $i=1, \dots, n$, independently of the classification of other events ($\sum P_i = 1$), then the stochastic processes $\{N_i(t), t \geq 0\}$, $i=1, \dots, n$ are independent stationary Poisson Processes with rates λP_i , $i=1, \dots, n$. In this section we prove an analogous result for NHPP.

Theorem 1: Let $\{N(t), t \geq 0\}$ be a NHPP with intensity $\lambda(t)$ and MVF $M(t)$. An event that occurs at time $w \geq 0$ is classified as a type 1 event with probability $P_1(w)$ and as a type 2 event with probability $P_2(w) = 1 - P_1(w)$. Classification of each event is independent of the classification of other events. Let $N_1(t)$ be the number of type 1 events that have occurred by time t . Then $\{N_1(t), t \geq 0\}$ is a NHPP with intensity $\lambda(t)P_1(t)$ and MVF $M_1(t) = \int \lambda(s)P_1(s) ds$, $i = 1, 2$. Furthermore, $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are statistically independent.

Before proceeding with the proof of Theorem 1, we need to establish the following Lemma which is given as an exercise in both Ross [6, p29] and Parzen [5, p143].

Lemma 1: Let $\{N(t), t \geq 0\}$ be a NEPP with intensity $\lambda(t)$ and MGF $M(t)$. Given that $N(t) = n$, the joint distribution of the n epochs at which the events occurred, $\tau_1, \tau_2, \dots, \tau_n \leq t$ is the same as if they were the order statistics corresponding to n independent random variables y_1, y_2, \dots, y_n with common distribution function

$$F(s) = \frac{M(s)}{M(t)} \quad 0 \leq s \leq t \quad (1)$$

Proof: Let $y_{[1]} \leq y_{[2]} \leq \dots \leq y_{[n]}$ be the order statistics corresponding to y_1, y_2, \dots, y_n . Then clearly, the joint density of these order statistics is given by

$$g(w_1, w_2, \dots, w_n) = n! f(w_1) f(w_2) \dots f(w_n), w_1 \leq w_2 \leq \dots \leq w_n$$

where

$$f(w) = \frac{d}{dw} F(w) = \frac{\lambda(w)}{M(t)}$$

Thus,

$$\Pr(y_{[i]} = w_i, i = 1, 2, \dots, n) = \frac{n!}{M(t)^n} \prod_{i=1}^n \lambda(w_i) \quad (2)$$

Now we show that the joint distribution of the τ_1, \dots, τ_n given $N(t) = n$, is identical to (2). Let $w_1 \leq w_2 \leq \dots \leq w_n \leq t$ and let h_i be small enough so that $w_i + h_i < w_{i+1}$. Then

$$\begin{aligned} & \Pr(w_i \leq \tau_i < w_i + h_i, i=1, 2, \dots, n | N(t) = n) \\ &= \Pr(w_i \leq \tau_i < w_i + h_i, i = 1, 2, \dots, n) / \Pr(N(t) = n) \\ &= \Pr(N(w_i + h_i) - N(w_i) = 1, i=1, \dots, n; \\ & \quad N(w_i) - N(w_{i-1} + h_{i-1}) = 0, i=2, \dots, n; \\ & \quad N(w_1) - N(0) = 0; \\ & \quad N(t) - N(w_n + h_n) = 0) / \Pr(N(t) = n) \quad (3) \end{aligned}$$

From c' of Definition 2.2,

$$\begin{aligned} & \Pr(N(w_i) - N(w_{i-1} + h_{i-1}) = 0) = e^{-(M(w_i) - M(w_{i-1} + h_{i-1}))} \\ & \Pr(N(w_1) - N(0) = 0) = e^{-M(w_1)} \\ & \Pr(N(w_i + h_i) - N(w_i) = 1) = (M(w_i + h_i) - M(w_i)) e^{-(M(w_i + h_i) - M(w_i))} \\ & \Pr(N(t) - N(w_n + h_n) = 0) = e^{-(M(t) - M(w_n + h_n))} \quad (4) \end{aligned}$$

Since the intervals $[w_i, w_i+h_i]$ do not overlap, we have from (b) of Definition 2.2 and (4) that (3) becomes after some reduction

$$\frac{e^{-M(t)} \prod_{i=1}^n (M(w_i+h_i) - M(w_i))}{e^{-M(t)} \frac{M(t)^n}{n!}}$$

$$= \frac{n!}{M(t)^n} \prod_{i=1}^n (M(w_i+h_i) - M(w_i))$$

Therefore,

$$\frac{\Pr(w_{i-1} < \tau_i < w_i+h_i, i=1, \dots, n | N(t)=n)}{\prod_{i=1}^n h_i} = \frac{n!}{M(t)^n} \prod_{i=1}^n \frac{M(w_i+h_i) - M(w_i)}{h_i} \quad (5)$$

Now take the limit of both sides as the $h_i \rightarrow 0, i=1, \dots, n$ uniformly. Since

$$\lim_{h_i \rightarrow 0} \frac{M(w_i+h_i) - M(w_i)}{h_i} = \frac{d}{dw_i} M(w_i) = \lambda(w_i)$$

and since the MGFs are differentiable, the limit of the right hand side of (5) exists which therefore implies the limit on the left side of equation (5) also exists and it is, in fact, an ordinary probability density function. Thus,

$$f(w_1, \dots, w_n | N(t) = n) = \lim_{\substack{h_i \rightarrow 0 \\ i=1, \dots, n}} \frac{\Pr(w_{i-1} < \tau_i < w_i+h_i, i=1, \dots, n | N(t)=n)}{\prod_{i=1}^n h_i}$$

$$= \frac{n!}{M(t)^n} \prod_{i=1}^n \lambda(w_i)$$

which is precisely the probability density function (2).

Lemma 1 states that given $N(t) = n$, the n event times, $\tau_1, \tau_2, \dots, \tau_n$ when considered as unordered random variables have the same joint distribution as n independent random variables with common distribution $F(s)$ given by (1). Therefore, as a consequence of Lemma 1, given that we know an event occurred in $(0, t)$, the actual time that the event occurred has distribution function $F(s)$ and is independent of the time of the other events occurring in $(0, t)$.

Proof of Theorem 1. We prove Theorem 1 by first showing that $\{N_i(t), t \geq 0\}$ $i = 1, 2$ satisfy Definition 2.2 for NHPP. We shall show that $N_i(t)$ is Poisson distributed with mean $\int \lambda(s) p_i(s) ds$ by calculating the generating function of $N_i(t)$, $i = 1, 2$. The independence of the two processes is established by calculating the joint generating function of $N_1(t)$ and $N_2(t)$ and showing that this joint generating function is the product of the individual generating functions.

Clearly, $N_1(0) = N_2(0) = 0$ so that (a') of Definition 2.2 holds. For $i = 1, 2$, $t \geq 0$ and $|z| < 1$, let

$$g_i(z, t) = E[z^{N_i(t)}]$$

be the generating function for $N_i(t)$. By conditioning on $N(t)$ and applying the Law of Total Probability we have that

$$g_i(z, t) = E_{N(t)} [E[z^{N_i(t)} | N(t)]] \quad (6)$$

From Lemma 1, given that an event occurred in $(0, t]$, the probability it occurred at time x , $0 \leq x \leq t$ is $\frac{\lambda(x)}{M(t)}$. Given an event occurred at time x , the probability it was classified as a type i event is $p_i(x)$. Therefore, given that an event occurred in $(0, t]$ the probability it was a type i event is

$$\gamma_i = \int_0^t p_i(x) \cdot \frac{\lambda(x)}{M(t)} dx \quad (7)$$

independently of the other events that occurred in $(0, t]$. Therefore, given $N(t)$, $N_i(t)$ is a binomial random variable with parameters $N(t)$ and γ_i so that the conditional generating function of $N_i(t)$ is

$$E[z^{N_i(t)} | N(t)] = (1 - \gamma_i + \gamma_i z)^{N(t)}$$

and using this in (6) we have

$$g_i(z, t) = E_{N(t)} [(1 - \gamma_i + \gamma_i z)^{N(t)}]$$

Since $N(t)$ is Poisson distributed with mean $M(t)$ and $|1 - \gamma_i + \gamma_i z| < 1$,

$$g_i(z, t) = e^{-M(t)} \sum_{n=0}^{\infty} (\gamma_i - \gamma_i z)^n = e^{-M(t)} \gamma_i (1 - z) \quad (8)$$

But (8) is just the generating function of a Poisson random variable with mean

$$\gamma_i M(t) = \int_0^t \lambda(x) p_i(x) dx \quad (9)$$

Thus, $N_i(t)$ $i = 1, 2$ is Poisson distributed with mean given by (9) for all $t \geq 0$. By a similar argument one can show that for any interval $(t_1, t_2]$, $N_i(t_2) - N_i(t_1)$ is Poisson distributed with mean $\int_{t_1}^{t_2} \lambda(x) p_i(x) dx$ since given that an event occurred in $(t_1, t_2]$, it is type i with probability

$$p_i(t_1, t_2) = \int_{t_1}^{t_2} p_i(x) \frac{\lambda(x)}{(M(t_2) - M(t_1))} dx \quad i = 1, 2$$

independently of the other events occurring in $(t_1, t_2]$.

That each process has independent increments follows straightforwardly from the fact that $\{N(t), t \geq 0\}$ has independent increments and that each event is typed independently of other events. If we consider any finite number of non-overlapping time intervals, the number of type i events that occur in each interval depends only on the number of events of the process $\{N(t), t \geq 0\}$ that occurred in that interval and the typing mechanism since only events of the process $\{N(t), t \geq 0\}$ occurring in a particular interval can be classified and counted as a type i event in that interval. Since $\{N(t), t \geq 0\}$ has independent increments, the fact that the processes $\{N_i(t), t \geq 0\}$ $i = 1, 2$ have independent increments follows immediately by conditioning on the number of events of $\{N(t), t \geq 0\}$ that occurred in each interval.

All the conditions of Definition 2.2 have been satisfied and therefore $\{N_i(t), t \geq 0\}$ $i=1, 2$ are NHPP.

All that remains now in proving Theorem 1 is to show that $N_1(t)$ and $N_2(t)$ are independent for all t . Consider the joint generating function

$$\begin{aligned} g(z_1, z_2, t) &= E(z_1^{N_1(t)} z_2^{N_2(t)}) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \{z_1^n z_2^m \Pr(N_1(t) = n, N_2(t) = m | N(t) = n + m) \\ &\quad \cdot \Pr(N(t) = n + m)\} \end{aligned}$$

From our previous discussions we realize that

$$\Pr(N_1(t) = n, N_2(t) = m | N(t) = n + m) = \binom{n+m}{n} \gamma_1^n \gamma_2^m$$

since given $N(t) = m + n$, there are $\binom{m+n}{n}$ ways of choosing n events to be type 1 (and thereby choosing the m to be type 2). Thus,

$$g(z_1, z_2, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z_1^n z_2^m \binom{m+n}{n} \gamma_1^n \gamma_2^m e^{-M(t)} M(t)^{n+m} / (n+m)!$$

$$= \sum_{n=0}^{\infty} e^{-M(t)\gamma_1} \frac{z_1^n \gamma_1^n}{n!} M(t)^n \sum_{m=0}^{\infty} e^{-M(t)\gamma_2} \frac{z_2^m \gamma_2^m}{m!} M(t)^m$$

where we have used the fact that

$$e^{-M(t)} = e^{-M(t)\gamma_1} e^{-M(t)\gamma_2} \text{ since } \gamma_1 + \gamma_2 = 1$$

Thus,

$$g(z_1, z_2, t) = g(z_1, t) g(z_2, t)$$

Since this holds for all z_1, z_2 and t , $N_1(t)$ and $N_2(t)$ are independent and this completes the proof.

Note that it is trivial to extend Theorem 1 to the case when an event at time t can be classified, independently of other events, into one of n categories with the probability that the event is classified as type i being $p_i(t)$ with $\sum p_i(t) = 1$ for all t . This splitting theorem for NHPP is useful in proving many interesting properties of NHPP. In particular, we shall use it to show that the output process of the $M(t)/G(t)/\infty$ queue is indeed a NHPP.

4. The Output of an $M(t)/G(t)/\infty$ Queue

In this section we will determine the distribution of the number of departures in an arbitrary interval $(t, t+T)$ of length T in the $M(t)/G(t)/\infty$ queue. Recall that in this queue, service times are independent of each other and that if a customer arrives at time s , the probability he will complete service by time w is $F(s, w)$, with $\int_s^\infty d(F(s, w)) = 1$.

Let $\bar{F}(s, w) = 1 - F(s, w)$. We shall classify any arrival into one of three types:

- Type 1: if the arrival will depart the system in $(t, t+T]$
- Type 2: if the arrival will depart the system in $(t+T, \infty)$
- Type 3: if the arrival will depart the system in $(0, t]$.

Let $p_i(x)$ be the probability that a customer arriving at time x is classified as a Type i customer, $i = 1, 2, 3$. Then, for all $x \geq 0$

$$p_1(x) = F(x, t+T) - F(x, t)$$

$$p_2(x) = \bar{F}(x, t+T) = 1 - F(x, t+T)$$

$$p_3(x) = F(x, t)$$

where, we have, of course, adopted the convention that $F(x, y) = 0$ if $x > y$. Note that $\sum p_i(x) = 1$ for all x and since, by assumption, each arrival is classified independently of other arrivals, the conditions of Theorem 1 have been satisfied. Therefore, if $N_i(w)$ = the number of customers who have arrived by time w and are categorized as Type i , we have from Theorem 1 that $\{N_i(w), w \geq 0\}$, $i = 1, 2, 3$ are NHPP with MGF $\int \lambda(s) p_i(s) ds$, $i = 1, 2, 3$, and are mutually independent. We recognize that $\{N_1(w), w \geq 0\}$ is simply the counting process describing departures from the queue in $(t, t+T]$. In particular, $N_1(t+T)$ is the number of arrivals by $t+T$ who have departed in $(t, t+T]$ and $N_1(t+T)$ is independent of $N_2(t+T)$ which is the number of arrivals by $t+T$ that are still in the system at time $t+T$. Thus, we have the important and interesting result that for an $M(t)/G(t)/\infty$ the number of departures in any interval is independent of the number remaining in the system at the end of the interval. This generalizes Mirasol's result for $M/G/\infty$ queues. Furthermore, $N_1(t+T)$ is independent of $N_3(t+T) = N_3(t)$ which is the number of departures in $(0, t]$. In fact, if we take any countable number of non-overlapping intervals, using the same analysis as above, it is straightforward to show that the departures during any of these intervals is Poisson distributed and is independent of departures during any other interval. (If there are N non over-lapping intervals let $p_n(x)$ be the probability that an arrival at time x will complete service in the $n^{\text{th}} \leq N$ interval. Then argue as above). Clearly then, the conditions of Definition 2.2 are satisfied, and by letting $t=0$ we see $N_1(T) = D(T)$ so that we have established the fact that $\{D(w), w \geq 0\}$ is a NHPP and for all $T \geq 0$, $D(T)$ is independent of the number still in service at time T . Also, we have as a by-product that $N_2(t+T)$ is Poisson distributed so that the number of busy servers at time t in the $M(t)/G(t)/\infty$ queue is Poisson distributed with mean $\int \lambda(x) \bar{F}(x, t) dx$.

For all $t \geq 0$ then, $D(t)$ is Poisson distributed with MGF

$$M_D(t) = \int_0^t \lambda(x) F(x, t) dx \quad (10)$$

and intensity function

$$\lambda_D(t) = \frac{d}{dt} N_D(t) = \int_0^t \lambda(x) f(x,t) dx + \lambda(t) F(t,t) \quad (11)$$

where $f(x,t) = \frac{\lambda F(x,t)}{\partial t}$ and $F(t,t) = 0$ only if the service distribution at time t has an atom at 0 .

Returning to the process $\{N_1(w), w \geq 0\}$ we note that for all $t, T \geq 0$

$$\begin{aligned} E[N_1(t,T)] &= E \left[\begin{array}{l} \text{number of arrivals by } t+T \text{ who} \\ \text{departed the system in } (t, t+T) \end{array} \right] \\ &= E \left[\text{number of departures in } (t, t+T) \right] \\ &= \int_0^{t+T} \lambda(x) p_1(x) dx = \int_0^{t+T} \lambda(x) [F(x, t+T) - F(x, t)] dx \\ &= \int_0^t \lambda(x) [F(x, t+T) - F(x, t)] dx \\ &\quad + \int_t^{t+T} \lambda(x) F(x, t+T) dx \quad (12) \end{aligned}$$

The first term in (12) can easily be seen to be the number of arrivals in $(0, t]$ who depart in $(t, t+T]$ and the second term is the number of arrivals in $(t, t+T]$ who depart in $(t, t+T]$. (12) can also be obtained by noting that $E[N_1(t+T)] = \int_t^{t+T} \lambda_D(x) dx$.

5. Special Cases

a. M(t)/G/∞ Queue

For $w \geq s$ let $F(s, w) = G(w-s)$ be a function only of the difference $w-s$. This is just the classical definition of service times in queues. From (12) (or equivalently, from (11)) we have that $N_1(t+T)$ is Poisson distributed with mean

$$\begin{aligned} &\int_0^t \lambda(x) [G(t+T-x) - G(t-x)] dx + \int_t^{t+T} \lambda(x) G(t+T-x) dx \\ &= \int_0^{t+T} \lambda(x) G(t+T-x) dx - \int_0^t \lambda(x) G(t-x) dx \quad (13) \end{aligned}$$

b. M/G/∞ Queue

Let the service time distribution be as in Case a. and let $\lambda(t) = \lambda$

for all $t \geq 0$. From (13) we see that

$$E[N_1(t+T)] = \int_t^{t+T} \lambda G(y) dy \quad (14)$$

From (14) we note that

$$\lim_{t \rightarrow \infty} E[N_1(t+T)] = \lim_{t \rightarrow \infty} \int_t^{t+T} \lambda G(y) dy = \lambda T$$

so that in steady state, the expected number of departures over an interval of length T is simply λT . Thus, we would expect that for the $M/G/\infty$ queue, the steady state output rate would equal the input rate λ . This is verified directly from (11) since the intensity function of the departure process is

$$\begin{aligned} \lambda_D(t) &= \int_0^t \lambda g(t-x) dx + \lambda G(0) \\ &= \lambda G(t) \end{aligned}$$

and
$$\lim_{t \rightarrow \infty} \lambda_D(t) = \lambda$$

This interesting result for $M/G/\infty$ queues was first obtained by Mirasol [5].

c. $M(t)/D/\infty$ Queues

$$\text{Let } F(x,t) = \begin{cases} 0 & \text{if } t < x + T \\ 1 & \text{o.w.} \end{cases}$$

for all x represent the distribution for a deterministic service time of length T .

From (10),
$$M_D(t) = \int_{t-T}^t \lambda(x) dx = \text{the expected number of arrivals in } (t-T, t)$$

and from (11)

$$\lambda_D(t) = \lambda(t-T)$$

That is, the departure intensity at t is simply the arrival intensity T time units previous. This is as it should be since for h small,

$$\begin{aligned} \text{Pr}(\text{departure in } (t, t+h)) &= \text{Pr}(\text{arrival in } (t-T, t-T+h)) \\ &= \lambda(t-T)h + O(h) \end{aligned}$$

by our assumptions on the arrival process.

d. Queues With Dying Input

A NHPP with intensity function $\lambda(t)$ is said to be "dying" if

$$(1) \quad 0 \leq \lambda(t) < \infty \quad \text{for all } t \geq 0$$

$$(2) \quad \lim_{t \rightarrow \infty} \lambda(t) = 0$$

Thus, a NHPP is dying if its intensity function is dying out with time and if during any finite interval the expected number of events of that process that occur is finite. Intuitively, we would expect that if the input process to a queue is dying that the output process should be dying as well. Theorem 2, a Tauberian type theorem, verifies our intuition.

Theorem 2: For the $M(t)/G(t)/\infty$ queue, the departure process is dying if and only if the arrival process is dying.

Proof: We shall prove Theorem 2 for queues whose arrival intensity function and its derivatives are Laplace Transformable for all $\text{Re}(s) \geq 0$. (See [7]). Let $\lambda_A(t)$, $\lambda_D(t)$ be the intensity functions of the arrival and departure processes, respectively. From (11)

$$\begin{aligned} \lim_{t \rightarrow \infty} \lambda_D(t) &= \lim_{t \rightarrow \infty} \int_0^t \lambda_A(x) f(x,t) dx + \lim_{t \rightarrow \infty} \lambda_A(t) F(t,t) \\ &= \lim_{t \rightarrow \infty} \int_0^t \lambda_A(x) f(x,t) dx \end{aligned} \quad (15)$$

since $0 \leq \lim_{t \rightarrow \infty} \lambda_A(t) F(t,t) \leq \lim_{t \rightarrow \infty} \lambda_A(t) = 0$. Therefore, to study the limiting behaviour of $\lambda_D(t)$ it suffices to study the limiting behaviour of the function

$$b(t) = \int_0^t \lambda_A(x) f(x,t) dx \quad (16)$$

Let

$$\begin{aligned} \hat{\lambda}_A(s) &= \int_0^{\infty} e^{-st} \lambda_A(t) dt \\ \hat{\lambda}_D(s) &= \int_0^{\infty} e^{-st} \lambda_D(t) dt \\ \hat{f}(x,s) &= \int_x^{\infty} e^{-s(t-x)} f(x,t) dt \\ \hat{b}(s) &= \int_0^{\infty} e^{-st} b(t) dt \end{aligned}$$

Taking the Laplace Transform (LT) of both sides of (16) we see

$$\begin{aligned}\hat{b}(s) &= \int_0^{\infty} e^{-st} \left[\int_0^t \lambda_A(x) f(x,t) dx \right] dt \\ &= \int_0^{\infty} e^{-sx} \lambda_A(x) \left[\int_x^{\infty} e^{-s(t-x)} f(x,t) dt \right] dx \\ &= \int_0^{\infty} e^{-sx} \lambda_A(x) \hat{f}(x,s) dx\end{aligned}\quad (17)$$

Let

$$\bar{\gamma}(s) = \sup_x \hat{f}(x,s)$$

$$\underline{\gamma}(s) = \inf_x \hat{f}(x,s)$$

Recall that for fixed x , $\int_0^{\infty} f(x,y) dy = 1$ since it represents the (proper) service time distribution of a customer entering service at time x . Therefore, for all $0 < s < \infty$

$$0 < \underline{\gamma}(s) \leq \bar{\gamma}(s) \leq 1$$

From (17) then we have that

$$\underline{\gamma}(s) \hat{\lambda}_A(s) \leq \hat{b}(s) \leq \hat{\lambda}_A(s) \bar{\gamma}(s)$$

and therefore,

$$\lim_{s \rightarrow 0^+} s \underline{\gamma}(s) \hat{\lambda}_A(s) \leq \lim_{s \rightarrow 0^+} s \hat{b}(s) \leq \lim_{s \rightarrow 0^+} s \bar{\gamma}(s) \hat{\lambda}_A(s) \quad (18)$$

From (18) we can easily prove Theorem 2 since by the Final Value Theorem for LT we know

$$\lim_{t \rightarrow \infty} \lambda_A(t) = \lim_{s \rightarrow 0^+} s \hat{\lambda}_A(s)$$

and

$$\lim_{t \rightarrow \infty} b(t) = \lim_{s \rightarrow 0^+} s \hat{b}(s)$$

Say $\lim_{t \rightarrow \infty} \lambda_A(t) = 0$. Now, from (18) we have that

$$\lim_{s \rightarrow 0^+} \underline{\gamma}(s) \lim_{s \rightarrow 0^+} s \hat{\lambda}_A(s) \leq \lim_{s \rightarrow 0^+} s \hat{b}(s) \leq \lim_{s \rightarrow 0^+} s \hat{\lambda}_A(s) \lim_{s \rightarrow 0^+} \bar{\gamma}(s) \quad (19)$$

and since $\lim_{s \rightarrow 0^+} \underline{\gamma}(s) = \lim_{s \rightarrow 0^+} \overline{\gamma}(s) = 1$ we have, by applying the Final Value Theorem,

$$0 \leq \lim_{t \rightarrow \infty} b(t) \leq 0$$

and thus $\lim_{t \rightarrow \infty} b(t) = \lim_{t \rightarrow \infty} \lambda_D(t) = 0$

From the left inequality in (19) we also see that if $\lim_{t \rightarrow \infty} \lambda_D(t) = 0$ it must be that $\lim_{s \rightarrow 0^+} s \hat{\lambda}_A(s) = 0$ and the theorem is established.

The techniques developed in the proof of Theorem 2 lead to an interesting result for general Laplace transformable functions that can be represented as the convolution of two functions.

Theorem 3: Let $b(t)$ be Laplace Transformable and let

$$b(t) = \int_0^t g(x)h(t-x)dx \quad (20)$$

be the convolution of two non-negative, non-identically zero, proper functions g and h such that $\int_0^\infty g(x)dx = \infty = \int_0^\infty h(x)dx$. Then $\lim_{t \rightarrow \infty} b(t) = 0$ if and only if

$$\lim_{z \rightarrow \infty} g(z) = \lim_{z \rightarrow \infty} h(z) = 0$$

Proof: Since $b(t)$ is a convolution,

$$\hat{b}(s) = \hat{g}(s)\hat{h}(s)$$

where $\hat{b}(s)$, $\hat{g}(s)$, $\hat{h}(s)$ are the LT of b , g , and h , respectively. Then by the Final Value Theorem we have

$$\lim_{s \rightarrow 0^+} s\hat{b}(s) = \left(\lim_{s \rightarrow 0^+} s\hat{g}(s)\right)\left(\lim_{s \rightarrow 0^+} \hat{h}(s)\right) = \left(\lim_{s \rightarrow 0^+} \hat{g}(s)\right)\left(\lim_{s \rightarrow 0^+} s\hat{h}(s)\right)$$

Since $\lim_{s \rightarrow 0^+} \hat{g}(s) = \int_0^\infty g(x)dx = \infty = \lim_{s \rightarrow 0^+} \hat{h}(s) = \int_0^\infty h(x)dx$

Theorem 3 follows immediately.

Inspection of the convolution integral (20) intuitively shows why Theorem 3 is true. If both functions $g(x)$ and $h(x)$ did not go to zero some significantly positive mass would be accumulated when integrating from 0 to t and hence $b(t)$ would have a positive limiting value.

We note that in Theorem 2, since $F(x,t)$ is assumed to be a proper probability distribution function, it is trivially true that $\lim_{t \rightarrow \infty} f(x,t) =$

$$\lim_{t \rightarrow \infty} \frac{\partial}{\partial t} F(x,t) = 0.$$

6. Summary

In this report we have established a splitting theorem for NHPP similar to the splitting theorem for stationary Poisson Processes. Using this powerful result we showed that the output process of an $M(t)/G(t)/\infty$ queue is a NHPP and that the number of departures from the queue during a finite interval are independent of the number of customers remaining in service at the end of the interval. Furthermore, using the splitting theorem we showed that the number of busy servers in the $M(t)/G(t)/\infty$ queue is Poisson distributed with mean $\int \lambda(x) \bar{F}(x,t) dx$. These results provide a tool for analyzing queues for which the arrival process and service times are non-stationary.

REFERENCES

- 1 Hillestad, R. J. and M. J. Carillo, "Models and Techniques for Recoverable Item Stockage When Demand and the Repair Process are Non-Stationary -- Part I: Performance Measures," RAND Note N-148Z-AF, May 1980.
- 2 Kleinrock, L., "Queueing Systems, Vol 1: Theory," John Wiley & Sons, 1975.
- 3 Kitchen, J. W., "Calculus of One Variable," Addison-Wessely Publishing Co., 1968.
- 4 Mirasol, N. M., "The Output of an M/G/ ∞ Queueing System is Poisson," Letter to the Editor, Operations Research (11:2) pp282-284.
- 5 Parzen, E., "Stochastic Processes," Holden-Day, Inc., 1962.
- 6 Ross, S. M., "Applied Probability Models with Optimization Applications," Holden-Day, Inc., 1970.
- 7 Widder, D. V., "The Laplace Transform," Princeton University Press, 1946.

DISTRIBUTION

COPIES

1 Deputy Under Sec'y of the Army, ATTN: Office of Op Resch
Headquarters, US Army Materiel Development & Readiness Command

2 DRCDM-S

1 Commandant, US Army Logistics Management Center, Ft. Lee, VA 23801

1 Defense Logistics Studies Info Exchange, DEXSMC-D, ALMC, Ft. Lee, VA 23801

2 Defense Technical Info Center, Cameron Station, Alexandria, VA 22314

1 Commander, USA Armament Materiel Readiness Cmd, Rock Island, IL 61299

1 DRSAR-SA

1 Commander, USA Communications-Electronics Cmd, Ft. Monmouth, NJ 07703

1 DRSEL-PL-SA

1 Commander, USA Missile Command, Redstone Arsenal, AL 35898

1 DRSMI-S

1 DRSMI-D

1 Commander, USA Troop Support & Aviation Materiel Readiness Command,
St. Louis, MO 63120

1 DRSTS-BA

1 Commander, US Army Tank-Automotive Command, Warren, MI 48090

1 ATTN: DRSTA-S

1 Commander, USA Armament R&D Cmd, ATTN: DRDAR-SE, Dover, NJ 07801

1 Commander, USA Aviation R&D Cmd, 4300 Goodfellow Blvd, St. Louis, MO 63120

1 Commander, USA Aviation R&D Cmd, ATTN: DRDAV-BD (Marl Glenn), 4300
Goodfellow Blvd, St. Louis, MO 63120

1 Commander, USA Electronics R&D Cmd, ATTN: DRDEL-ST-SA, Rm 2D315,
Bldg. 2700, Ft. Monmouth, NJ 07703

1 Commander, USA Mobility Equipment R&D Cmd, ATTN: DRDME-O, Ft. Belvoir,
VA 22060

1 Commander, USA Natick R&D Cmd, ATTN: DRXDM-O, Natick, MA 01760

1 Director, Army Materiel Systems Analysis Activity, APG, MD 21005

1 DEXSY-G

1 DEXSY-A

1 DEXSY-C

1 DEXSY-R

1 DEXSY-EM

1 DEXSY-F

1 DEXSY-FX

1 DEXSY-FM

1 DEXSY-FR

1 DEXSY-FO

1 DEXSY-DA

1 Commander, US Army Logistics Center, Ft. Lee, VA 23801

1 Commander, US Air Force Logistics Cmd, WPAFB, ATTN: AFLC/XRS, Dayton,
Ohio 45433

1 US Navy Fleet Material Support Office, Naval Support Depot,
Mechanicsburg, PA 17055

1 George Washington University, Inst. of Management Science & Engr.,
707 22nd St., N.W., Wash., DC 20006

1 Naval Postgraduate School, ATTN: Dept of Opns Anal, Monterey, CA 93940

1 Air Force Institute of Technology, ATTN: SLGQ Head Quantitative Studies
Dept., Dayton, OH 43433

1 US Army Military Academy, West Point, NY 10996

1 Scientific Advisor, ATCL-SCA, Army Logistics Center, Ft. Lee, VA 23801

1 Librarian, Logistics Mgt Inst., 4701 Sangamore Rd., Wash., DC 20016

1 University of Florida, ATTN: Dept of Industrial Systems Engr.,
Gainesville, FL 32601

1 RAND Corp., ATTN: S. M. Drezner, 1700 Main St., Santa Monica, CA 90406

1 Director, Army Management Engineering Training Agency, Rock Island
Arsenal, Rock Island, IL 61299

1 Defense Logistics Agcy, ATTN: DLA-LO, Cameron Sta, Alexandria, VA 22314

1 Dep Chf of Staff (I&L), HQ USMC-LMP-2, ATTN: LTC Sonneborn, Jr.,
Wash., DC 20380

1 Operations Research Center, 3115 Etcheverry Hall, University of
California, Berkeley, CA 94720

1 Dr. Jack Muckstadt, Dept of Industrial Engineering & Operations Research,
Upson Hall, Cornell University, Ithaca, NY 14890

1 Dept of Industrial Engineering, University of Michigan, Ann Arbor,
MI 48104

1 Prof Robert M. Stark, Dept of Stat & Computer Sciences,
University of Delaware, Newark, DE 19711

1 Prof E. Gerald Hurst, Jr., Dept of Decision Science, The Wharton
School, University of Penna., Phila., PA 19104

1 Dept of Industrial Engr. & Engr. Management, Stanford University,
Stanford, CA 94305

1 Commander, US Army Communications Command, ATTN: Dr. Forry,
CC-LOG-LEO, Ft. Huachuca, AZ 85613

1 Commander, US Army Test & Evaluation Cmd, ATTN: DRSTE-SY,
Aberdeen Proving Ground, MD 21005

1 Prof Harvey M. Wagner, Dean, School of Business Adm, University
of North Carolina, Chapel Hill, NC 27514

1 DARCOM Intern Training Center, Red River Army Depot, Texarkana, TX 75501

1 Prof Leroy B. Schwarz, Dept of Management, Purdue University,
Krannert Bldg, West Lafayette, Indiana 47907

1 US Army Training & Doctrine Command, Ft. Monroe, VA 23651

1 Operations & Inventory Analysis Office, NAVSUP (Code 04A) Dept
of Navy, Wash., DC 20376

1 US Army Research Office, ATTN: Robert Launer, Math. Div.,
P.O. Box 12211, Research Triangle Park, NC 27709

1 Prof William P. Pierskalla, 3641 Locust Walk CE, Phila., PA 19104

1 Air Force Logistics Mgt Center, Gunter Air Force Station, AL 36144
AFLMC/LGY

1 AFLMC/XRP, Bldg. 205

1 Engineer Studies Center, 6500 Brooks Lane, Wash., DC 20315

1 Dr. Carl Weisman, C.A.C.I., 1815 N. Fort Myer Drive, Arlington, VA 22209

1 Commander, US Army Missile Cmd, ATTN: Ray Dotson, DRSMI-DS, Redstone
Arsenal, AL 35898

1 Prof Peter Kubat, Graduate School of Mgmt, University of Rochester,
Rochester, NY 14627

1 Prof Barney Bissinger, The Pennsylvania State University, Middletown,
PA 17057

1 Prof Garry D. Scudder, Mgmt Sciences Dept, College of Business Adm,
University of Minnesota, 271 19th Ave., South, Minneapolis, MN 55455