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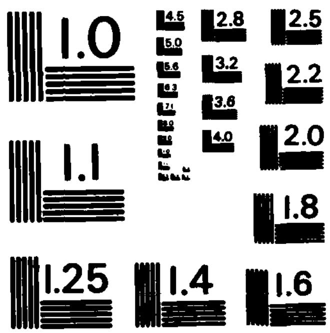


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MRC Technical Summary Report #2398

# ON THE CROSSING RULE

S. Friedland, J.W. Robbin, and
J.H. Sylvester

**Mathematics Research Center**
**University of Wisconsin—Madison**
**610 Walnut Street**
**Madison, Wisconsin 53706**



June 1982

(Received May 3, 1982)





Sponsored by

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

# ON THE CROSSING RULE

S. Friedland*,[1], J.W. Robbin**, and
J.H. Sylvester**,[1,2]

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## ABSTRACT

The paper gives conditions on a family of matrices which guarantee that some matrix in the family will have a multiple eigenvalue. In particular, the main theorem states exactly which dimensions admit $k$ dimensional subspaces of matrices for which all nonzero elements have distinct eigenvalues.

This question arises naturally in the theory of first order hyperbolic systems of partial differential equations; the main theorem, in this context, tells exactly for which integers $n$ an $n \times n$ system in $k$ space variables may be strictly hyperbolic.

AMS (MOS) Subject Classification: A18, L40.

Key Words: Eigenvalues, Hyperbolic Systems.

Work Unit Number 1 - Applied Analysis

* Hebrew University - Jerusalem, University of Wisconsin-Milwaukee.
** Department of Mathematics, University of Wisconsin-Madison.

[1] Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
[2] This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

## SIGNIFICANCE AND EXPLANATION

Hyperbolic systems of partial differential equations are those for which the Cauchy problem is well posed.

Strictly hyperbolic systems are a subclass for which: 1) solvability of the Cauchy problem is stable under perturbation 2) singularities propagate along curves 3) the condition of hyperbolicity can be verified, (i.e. to check if a system is strictly hyperbolic one need only verify an algebraic condition).

In this paper, it is shown that for first order $n \times n$ systems in more than two space variables, strict hyperbolicity can be obtained only for special dimensions $n$. For example, with 3 space variables there exist strictly hyperbolic $n \times n$ systems if and only if $n = 0, \pm 1$ modulo f.

The paper also includes many related results which are more algebraic in nature.



The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE CROSSING RULE

S. Friedland[*,1], J.W. Robbin[**], and J.H. Sylvester[**,1,2]

## 1. Introduction.

The classical theorem of Wigner-Von Neumann [1927] shows that the real symmetric matrices (resp. hermitian matrices) with a multiple eigenvalue form a real algebraic variety of codimension 2 (resp. 3) in the space of all real symmetric matrices (resp. all hermitian matrices). This implies the famous "non-crossing" rule which asserts that a "generic" one-parameter family of real symmetric matrices (or two-parameter family of Hermitian matrices) contains no matrix with a multiple eigenvalue. Our aim here is to give conditions on a family of matrices which force a "crossing" of the eigenvalues; i.e. assure that the family contains a member with a multiple eigenvalue. Our main result is:

There is a $(k+1)$-dimensional vector space of $(n \times n)$ real matrices such that each non-zero matrix has no multiple eigenvalue if and only if $k < \sigma(n)$ where $\sigma(n)$ is the function defined below.

For $n = 2 \pmod 4$ we have that $\sigma(n) = 2$; hence any 3-dimensional vector space of symmetric $(n \times n)$ matrices contains a non-zero matrix with a multiple eigenvalue. This result is due to Lax [1981].

The problem arises naturally in the theory of symmetric first-order hyperbolic systems with constant coefficients (see John [1977]); our theorem says exactly which dimensions admit examples of such systems having only simple characteristics.

* Hebrew University - Jerusalem, University of Wisconsin-Milwaukee.
**Department of Mathematics, University of Wisconsin-Madison.

1 Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
2 This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

Our theorem uses the work of Adams [1962] and is thus imtimately connected with the problem of vector fields on spheres. Thus in §2 we formulate a kind of non-linear Radon-Hurwitz-Adams theory and show how it follows easily from the work of Adams. In §3 we state and prove various forms of our main theorem. In §4 we consider the complex case and also the real case where the eigenvalues are allowed to be complex. Finally in §5 we remark that in the Hermitian case, the "middle eigenvalue" must cross if the dimensions are right .

The reader should note the general topological nature of the results. Namely all five theorems A, B, C, D, E below prove that there is a (k+1)-dimensional vector space of matrices having some property if and only if there is an odd continuous map from the k-sphere into the matrices with this property.

## §2. Families of invertible matrices.

de Denote by $M_m(\mathbf{R})$ the vector space of all real $m$ by $m$ matrices and by $GL(m,\mathbf{R})$ the open subset of invertible matrices. For $A \in M_m(\mathbf{R})$ let $A^* \in M_m(\mathbf{R})$ denote the transpose of $A$ and let $I \in M_m(\mathbf{R})$ denote the identity matrix. For $\alpha \in \mathbf{R}^{k+1}$, $|\alpha|$ denotes the usual Euclidean norm, $S^k$ denotes the $k$-sphere $|\alpha| = 1$, and $\mathbf{R}P^k$ denotes the real projective $k$-space obtained from $S^k$ by identifying antipodal points.

Let $\rho(m)$ denote the Radon-Hurwitz number:

$$\rho(m) = 2^c + 8d$$

for:

$$m = (2a+1)2^{c+4d}$$

where $a, c, d$ are integers and $c = 0, 1, 2, 3$.

THEOREM A: For integers $m > 0$ and $k \geq 0$ the following are equivalent:

(A1) $$k < \rho(m);$$

(A2) there exist $A_1, \cdots, A_k \in M_m(\mathbf{R})$ with:

$$A_i + A_i^* = 0$$

$$A_i A_i = -I$$

$$A_i A_j + A_j A_i = 0$$

for $i, j = 1, \cdots, k$ with $i \neq j$;

(A3) there is a linear map:

$$\Phi : \mathbf{R}^{k+1} \longrightarrow M_m(\mathbf{R})$$

such that:

$$|\Phi(\alpha)x| = |\alpha|\,|x|$$

for $\alpha \in \mathbf{R}^{k+1}$ and $x \in \mathbf{R}^m$;

(A4) there is a linear map:

$$\Phi : \mathbf{R}^{k+1} \longrightarrow M_m(\mathbf{R})$$

such that:

$$\Phi(\mathbf{R}^{k+1} \setminus 0) \subset GL(m, \mathbf{R});$$

(A 5) there is an odd continuous map:

$$\varphi : S^k \rightarrow GL(m, \mathbf{R});$$

(A6) the Whitney sum of $m$ copies of the canonical line bundle on $\mathbf{R}P^k$ is stably trivial: i.e. there exist continuous maps:

$$x_i : S^k \longrightarrow \mathbf{R}^n \qquad i = 1, \cdots, n$$

such that $x_1(\alpha), \cdots, x_n(\alpha)$ form a basis for $\mathbf{R}^n$ for each $\alpha \in S^k$ and:

$$x_i(-\alpha) = -x_i(\alpha) \qquad i = 1, \cdots, m$$
$$x_i(-\alpha) = x_i(\alpha) \qquad i = m+1, \cdots, n;$$

(A7) this Whitney sum is trivial; i.e. (A6) holds with $m = n$;

(A8) there are $k$ pointwise linearly independent vectorfields on the $(m-1)$-sphere $S^{m-1}$.

Proof: All of this is either well-known or easily obtained from well-known results. To orient the reader we discuss the proof but all we need for theorem B are the implications (A6) $\Rightarrow$ (A1) $\Rightarrow$ (A3).

The proof of (A1) $\Leftrightarrow$ (A2) $\Leftrightarrow$ (A3) is due to Radon [1922] and Hurwitz [1923]. We note (as they did) that (A2) $\Leftrightarrow$ (A3) is quite easy namely:

$$\Phi(\alpha) = \alpha_0 I + \Sigma \alpha_i A_i$$

satisfies (A3) if the $A_i$ satisfy (A2) while:

$$A_i = \Phi(e_0)^* \Phi(e_i)$$

(with $e_0, \cdots, e_k$ an orthonormal basis for $\mathbf{R}^{k+1}$) satisfy (A2) if $\Phi$ satisfies (A3). The point here is that (A2) says that the $A_i$ afford a representation of the Clifford Algebra on $k$ generators (though apparently Radon and Hurwitz did not know about Clifford [1876]). These algebras are semisimple and hence (by Wedderburn theory) the simple ones are matrix algebras over the reals, complexes, or quaternions and each representation is a multiple of the standard representation. In fact, the Clifford algebras are easily classified and the classification yields the numbers $\rho(m)$. For a nice exposition see Porteous [1969].

The implication (A2) $\Rightarrow$ (A8) is trivial; the vectorfields:

$$S^{m-1} \rightarrow TS^{m-1} : x \rightarrow A_i x$$

are pointwise orthonormal. The converse implication is of course the title theorem of Adams [1962].

Of the remaining implications, the following are obvious:

(A5) $\Leftrightarrow$ (A7) (Let $x_1(\alpha), \cdots, x_m(\alpha)$ be the columns of $\varphi(\alpha)$);

(A7) $\Rightarrow$ (A6) (Let $m = n$);

(A3) $\Rightarrow$ (A4) $\Rightarrow$ (A5) (Let $\varphi = \Phi \mid S^k$).

We shall prove (A6) $\Rightarrow$ (A7) and (using the main theorem of Adams [1962]) (A5) $\Rightarrow$ (A1).

Proof of (A6) $\Rightarrow$ (A7). Assume (A6). We first show that $k < m$. The easiest method is via Steifel-Whitney classes (see e.g. Husemoller [1966] for an exposition.) Let $J \to \mathbf{R}P^k$ denote the canonical real line bundle and $\mathbf{R} \to \mathbf{R}P^k$ the trivial line bundle. The hypothesis (A6) says that the n-plane bundle

$$E = \underbrace{J \oplus \cdots \oplus J}_{m} \oplus \underbrace{\mathbf{R} \oplus \cdots \oplus \mathbf{R}}_{n-m}$$

is trivial. Hence:

$$1 = w(E) = w(J)^m .$$

But

$$w(J) = 1 + \omega$$

where $\omega \in H^1(\mathbf{R}P^k, \mathbf{Z}_2)$ is such that $\omega^k \neq 0$. It follows immediately that $k < m$. Now (A6) $\Rightarrow$ (A7) follows immediately from the following well known:

PROPOSITION. Let $F \to P$ be an m-plane bundle over a k-dimensional manifold. Assume that $F$ is stably trivial and $k < m$. Then $F$ is trivial.

For proof see e.g. Husemoller [1962] page 100; it can also be proved using Sard's theorem as follows. We assume that for $\alpha \in P$ we have:

$$F_\alpha \oplus \operatorname{span}(x_{m+1}(\alpha), \cdots, x_n(\alpha)) = \mathbf{R}^n$$

where $F_\alpha$ is an m-dimensional subspace of $\mathbf{R}^n$ and $x_i : P \to \mathbf{R}^n$ are pointwise independent vector valued functions. We must find a continuous map:

$$Q : P \to GL(n)$$

such that:

$$Q(\alpha)F_\alpha = \mathbf{R}^m \times 0 \subset \mathbf{R}^n.$$

By induction we assume $n = m+1$, and by Weierstrass approximation we assume that $F_\alpha$ is smooth i.e. that:

$$F_\alpha = \xi(\alpha)^\perp$$

where $\xi : P \to S^m$ is smooth. Since $\dim(P) = k < m$ Sard's theorem yields a constant vector:

$$e \in S^m \setminus (\xi(P) \cup -\xi(P)).$$

We define an orthogonal transformation $Q(\alpha) : \mathbf{R}^n \to \mathbf{R}^n$ by:

$$Q(\alpha) \mid \mathrm{span}(e, \xi(\alpha))^{\perp} = \text{identity};$$

$$Q(\alpha)\xi(\alpha) = e;$$

$$Q(\alpha)\,\mathrm{span}(e, \xi(\alpha)) = \mathrm{span}(e, \xi(\alpha)).$$

This characterizes $Q(\alpha)$ uniquely if we demand that $Q(\alpha)$ should be a rotation through angle less than $\pi$ on the plane $\mathrm{span}(e, \xi(\alpha))$. Clearly $Q$ is continuous and:

$$Q(\alpha)F_{\alpha} = e^{\perp}$$

as required.

Proof of (A5) $\Longrightarrow$ (A1). We shall reduce this to the following theorem (1.2) of Adams [1962]:

THEOREM. If there is a continuous map:

$$f : \mathbf{R}P^{m+k}/\mathbf{R}P^{m-1} \to S^m$$

such that the composite:

$$S^m \xrightarrow{i} \mathbf{R}P^{m+k}/\mathbf{R}P^{m-1} \xrightarrow{f} S^m$$

has degree one, then $k < \rho(m)$. Here $X/Y$ denotes the space obtained from $X$ by smashing the subset $Y$ to a point so that there is a natural identification:

$$S^m = \mathbf{R}P^m/\mathbf{R}P^{m-1}$$

and $i$ denotes the inclusion:

$$i : \mathbf{R}P^m/\mathbf{R}P^{m-1} \to \mathbf{R}P^{m+k}/\mathbf{R}P^{m-1}.$$

Given $\varphi : S^k \to GL(m)$ as in (A3) we must define $f$ as in the theorem of Adams. For $x \in \mathbf{R}^m$ and $\alpha \in \mathbf{R}^{k+1}$ with $\alpha \neq 0$ let:

$$\tilde{f}(\alpha,x) = (|\alpha|,(\varphi(\alpha/|\alpha|)x).$$

As:

$$\tilde{f}(\lambda\alpha, \lambda x) = |\lambda|\,\tilde{f}(\alpha,x)$$

for $\lambda \in \mathbb{R}$, $\lambda \neq 0$ the map $\tilde{f}$ induces a map:

$$f: \mathbb{R}P^{m+k}\backslash \mathbb{R}P^{m-1} \to \mathbb{R}P^{m}\backslash \mathbb{R}P^{m-1}.$$

Extend $f$ to a map:

$$f: \mathbb{R}^{m+k}/\mathbb{R}P^{m-1} \to \mathbb{R}P^{m}/\mathbb{R}P^{m-1}$$

by mapping the point $H = \mathbb{R}P^{m-1}$ of $\mathbb{R}P^{m+k}/\mathbb{R}P^{m-1}$ into the point $\mathbb{R}P^{m-1}$ of $\mathbb{R}P^{m}/\mathbb{R}P^{m-1}$. It is left to show that $f$ is continuous at $H$. Indeed for $[\alpha,x] \in \mathbb{R}P^{m+k}$ we may assume that $|\alpha|^2+|x|^2 = 1$ while from the continuity of $\varphi$ and the compactness of $S^k$ we have $\varepsilon > 0$ with

$$\varepsilon|x| \leq |\varphi(\alpha)x|$$

for all $\alpha \in S^k$ and $x \in \mathbb{R}^m$. This gives:

$$\varepsilon^2(1-|\alpha|^2) \leq |\varphi(\alpha/|\alpha|)x|^2;$$

so that $\tilde{f}(\alpha,x)$ approaches $H \subset \mathbb{R}P^m$ as $(\alpha,x)$ approaches $0 \times S^{m-1}$. Thus $f$ is continuous at $H$ as required. For fixed $\alpha \in S^k$ the map $x \to f(\alpha,x)$ is linear invertible and hence of degree $\pm 1$. (If the negative sign occurs, compose with a reflection.) This completes the proof.

## §3. Families of matrices with simple eigenvalues.

We now state and prove our main theorem.

THEOREM B: For positive integers $n$ and $k$ the following are equivalent:

(B1) $k < \sigma(n)$ where $\sigma(n)$ is given by:

$$\sigma(n) = 2 \quad \text{for} \quad n \neq 0, \pm 1 \mod 8,$$
$$\sigma(n) = \rho(4b) \quad \text{for } n = 8b, \ 8b \pm 1;$$

(B2) either $k = 1$ or else there is an integer $m$ with $k < \rho(m)$ and $n$ is one of $2m-1$, $2m$, $2m+1$;

(B3) there is a linear map:

$$\Psi : \mathbf{R}^{k+1} \to M_n(\mathbf{R})$$

such that each matrix $\Psi(\alpha)$ $(\alpha \in \mathbf{R}^{k+1} \setminus 0)$ has $n$ distinct real eigenvalues;

(B4) there exists an odd continuous map:

$$\psi : S^k \to M_n(\mathbf{R})$$

such that each $\psi(\alpha)$ $(\alpha \in S^k)$ has $n$ distinct real eigenvalues.

Proof: The pattern of proof is (B1) $\Longleftrightarrow$ (B2) and (B2) $\Longrightarrow$ (B3) $\Longrightarrow$ (B4) $\Longrightarrow$ (B2).

(B1) $\Longrightarrow$ (B2). Assume $k < \sigma(n)$. If $k \neq 1$ we must have $k < \rho(4b)$ where $n$ is one of $8b-1$, $8b$, $8b+1$. Take $m = 4b$.

(B2) $\Longrightarrow$ (B1). Let $n = 8b + c$ where $-3 \le c \le 4$ and suppose also that $n = 2m-1$, $2m$, or $2m+1$. As $\rho(4b) > 1$ there is nothing to prove if $k = 1$. Hence suppose $1 < k < \rho(m)$. Then $m = 0 \mod 4$ so $n = -1, 0, 1 \mod 8$ whence $m = 4b$ so $\sigma(n) = \rho(m) > k$ as required.

(B2) $\Longrightarrow$ (B3). First assume that $k = 1$. Let $A_n \in M_n(\mathbb{R})$ be a diagonal matrix with distinct eigenvalues say:

$$A_n = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & n \end{bmatrix}$$

and $B_n \in M_n(\mathbb{R})$ be the tridiagonal matrix which is zero on the diagonal and one on super and subdiagonal:

$$B_n = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & & \\ & & & & 1 \\ & & & 1 & 0 \end{bmatrix} .$$

Take

$$\Psi_n(\alpha) = \alpha_0 A_n + \alpha_1 B_n .$$

Now $\Psi_n(\alpha)$ is symmetric and by expanding by minors in the last row:

$$P_n(\lambda) = (\lambda - n\alpha_0) P_{n-1}(\lambda) - \alpha_1^2 P_{n-2}(\lambda)$$

where:

$$P_n(\lambda) = \det(\lambda - \Psi_n(\alpha)) .$$

Thus if $\alpha_1 \neq 0$ the polynomials $P_n(\lambda)$ are pairwise orthogonal in a suitable measure (see e.g. Freund [1960] page 60) and hence have simple roots (Freund [1960] page 17); if $\alpha_1 = 0$ but $\alpha_0 \neq 0$ this is obvious. Hence in either case the eigenvalues of $\Psi_n(\alpha)$ are distinct so that $\Psi_n : \mathbb{R}^2 \to M_n(\mathbb{R})$ satisfies (B3).

Now assume $k < \rho(m)$ and $n = 2m-1,\ 2m,$ or $2m+1$. By theorem A choose $\Phi : \mathbb{R}^{k+1} \to M_m(\mathbb{R})$ so that

$$\Phi(\alpha)\,\Phi(\alpha)^* = |\alpha|^2 I$$

for $\alpha \in \mathbb{R}^{k+1}$ and define $\Gamma : \mathbb{R}^{k+1} \to M_{2m}(\mathbb{R})$ by:

$$\Gamma(\alpha) = \begin{bmatrix} I_m & -I_m \\ \Phi(\alpha) & \Phi(\alpha) \end{bmatrix}$$

where $I_m \in M_m(\mathbb{R})$ is the identity. One easily verifies that for $|\alpha| = 1$ we have:

$$\Gamma(\alpha)\,\Gamma(\alpha)^* = 2\, I_{2m}$$

so that if $D \in M_{2m}(\mathbb{R})$ is a constant matrix with distinct eigenvalues $\lambda_1, \cdots, \lambda_{2m}$ then:

$$\Psi(\alpha) = \Gamma(\alpha)\, D\Gamma(\alpha)^*$$

has distinct eigenvalues $2\lambda_1, \cdots, 2\lambda_{2m}$. Now $\Psi(\alpha)$ is a polynomial of degree $\leq 2$ in $\alpha$; the trick is to choose $D$ so that the zeroth order and second order terms drop leaving $\Psi(\alpha)$ linear in $\alpha$. This is accomplished by choosing:

$$D = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$$

so that:

$$\Psi(\alpha) = \begin{bmatrix} 0 & 2A\Phi(\alpha)^* \\ 2\Phi(\alpha)A & 0 \end{bmatrix} .$$

Note that the eigenvalues of $\Psi(\alpha)$ are $\pm 2|\alpha|^2 \nu_i$ $i = 1, \cdots, m$ where $\nu_1 > \nu_2 > \cdots > \nu_m > 0$ are the eigenvalues of $A$.

The map $\Psi$ clearly satisfies (B3) when $n = 2m$. In case $n = 2m + 1$ the map

$$\Psi_+(\alpha) = \begin{bmatrix} \Psi(\alpha) & 0 \\ 0 & 0 \end{bmatrix}$$

obtained from $\Psi$ by adding a zero row and column also satisfies (B3). To handle the case $n = 2m - 1$ let $\Psi(\alpha)$ be defined as above but let $A$ have eigenvalues $\nu_1 > \nu_2 > \cdots > \nu_m = 0$. Then $\Psi(\alpha)$ has $\nu_m = 0$ as a double eigenvalue with eigenspace spanned by the m-th and 2m-th columns of $\Gamma(\alpha)$ (assuming $A = \mathrm{diag}(\nu_1, \cdots, \nu_m)$). The difference of these two columns is a constant vector (i.e. independent of $\alpha$) which spans a one-dimensional subspace invariant by each $\Phi(\alpha)$. As $\Psi(\alpha)$ is symmetric the orthogonal complement $E^{2m-1}$ of this vector is also invariant by each $\Psi(\alpha)$ and so:

$$\Psi_-(\alpha) = \Psi(\alpha) | E^{2m-1}$$

gives the required map $\Psi_- : \mathbb{R}^{k+1} \to M_{2m-1}(\mathbb{R})$ satisfying (B3).

(B3) $\Longrightarrow$ (B4). Take $\psi = \Psi \mid S^k$.

(B4) $\Longrightarrow$ (B2). If $k = 1$ there is nothing to prove hence assume $k > 1$. Let $\lambda_1(\alpha) > \cdots > \lambda_n(\alpha)$ be the eigenvalues and choose corresponding unit eigenvectors $v_1(\alpha), \cdots, v_n(\alpha)$:

$$\psi(\alpha) v_i(\alpha) = \lambda_i(\alpha) v_i(\alpha) \qquad (i = 1, \cdots, n).$$

The vectors $v_i(\alpha)$ are defined up to a sign and hence (since $S^k$ is simply connected) may be taken to be continuous. Since $\psi(-\alpha) = -\psi(\alpha)$ we have that:

$$\lambda_i(-\alpha) = -\lambda_{n-i+1}(\alpha)$$

for $i = 1, \cdots, n$ and hence that:

$$(*) \qquad v_i(-\alpha) = \pm v_{n-i+1}(\alpha)$$

where the choice of $\pm$ is independent of $\alpha$ (by continuity). Changing the signs of some of the $v_i$ if necessary we may assume:

$$v_i(-\alpha) = v_{n-i+1}(\alpha)$$

for $i = 1, 2, \cdots, [n/2]$ but in case $n$ is odd either sign may occur in (*) for the middle eigenvalue $(i = n-i+1)$. Take $m = n/2$ if $n$ is even, $m = i - 1$ if $n$ is odd and the sign in (*) is $+$ when $i = n-i+1$ and $m = i$ if $n$ is odd and the sign is $-$. Let:

$$x_i(\alpha) = v_i(\alpha) - v_{n-i+1}(\alpha) \qquad i = 1, \cdots, [n/2];$$
$$x_i(\alpha) = v_i(\alpha) + v_{n-i+1}(\alpha) \qquad i = [n/2]+1, \cdots, n;$$
$$x_i(\alpha) = v_i(\alpha) \qquad i = n-i+1.$$

Then:

$$x_i(-\alpha) = -x_i(\alpha) \qquad i = 1, \cdots, m$$

and :

$$x_i(-\alpha) = x_i(\alpha) \qquad i = m+1, \cdots, n$$

so the Whitney sum of $m$ copies of the canonical bundle over $\mathbb{R}P^k$ is stably trivial. Hence by (A6) $\Longleftrightarrow$ (A1) we have $k < \rho(m)$ as required.

The following example illustrates the difference between the cases $k = 1$ and $k > 1$. Let $\psi : S^1 \to M_2(\mathbb{R})$ be given by :

$$\psi(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

so

$$\psi(\alpha)\, v(\beta) = v(\alpha - \beta)$$

for

$$v(\beta) = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} .$$

Thus the eigenvectors are given by $v(\beta)$ with $\beta = \alpha/2,\ (\alpha + \pi)/2$ and are not well-defined on $S^1$. This cannot happen for $S^k$ with $k > 1$.

## §4. The complex case.

Now let $\mathbf{F}$ be one of the fields $\mathbf{R}$ (reals), $\mathbb{C}$ (complexes), or $\mathbb{H}$ (quaternions); $M_m(\mathbf{F})$ be the real vector space of $m$ by $m$ matrices with entries from $\mathbf{F}$; $A^*$ denote the conjugate transpose of $A \in M_m(\mathbf{F})$; and $GL(m, \mathbf{F})$ = the invertible matrices in $M_m(\mathbf{F})$. Let $H_n(\mathbf{F}) = \{B \in M_n(\mathbf{F}) : B = B^*\}$ denote the Hermitian matrices.

Define the Radon-Hurwitz-Adams-Lax-Phillips numbers $\rho(m, \mathbf{F})$ by table 1 for $m = 1, 2, 4, 8$ and for general $m$ by the conditions:

$$\rho((2a+1)m, \mathbf{F}) = \rho(m, \mathbf{F});$$

$$\rho(16m, \mathbf{F}) = \rho(m, \mathbf{F}) + 8.$$

We have the following generalization of theorem A:

THEOREM C. *For integers* $m > 0$ *and* $k \geq 0$ *the following are equivalent:*

(C1) $k < \rho(m, \mathbf{F})$;

(C2) *there exist* $A_1, \ldots, A_k \in M_m(\mathbf{F})$ *with:*

$$A_i + A_i^* = 0$$

$$A_i A_i = -I$$

$$A_i A_j + A_j A_i = 0$$

for $i \neq j = 1, \ldots, m$.

(C3) *there is a linear map:*

$$\Phi : \mathbf{R}^{k+1} \to M_m(\mathbf{F})$$

satisfying:

$$\Phi(\alpha)\Phi(\alpha)^* = |\alpha|^2 I;$$

(C4) there is a linear map:

$$\Phi : \mathbf{R}^{k+1} \to M_m(\mathbf{F})$$

satisfying:

$$\Phi(\mathbf{R}^{k+1} \setminus 0) \subset GL(m, \mathbf{F});$$

(C5) there is an odd continuous map:

$$\varphi : S^k \to GL(m, \mathbf{F});$$

(C6) there are continuous maps:

$$x_i : S^k \to \mathbf{F}^n \qquad (i = 1, \ldots, n)$$

such that $x_1(\alpha), \ldots, x_n(\alpha)$ form a basis for $\mathbf{F}^n$ as a vector space over $\mathbf{F}$ and:

$$x_i(-\alpha) = -x_i(\alpha) \qquad i = 1, \ldots, m$$

$$x_i(-\alpha) = x_i(\alpha) \qquad i = m+1, \ldots, n;$$

(C7) (C6) holds with $m = n$.

TABLE 1

| $m \backslash \mathbb{F}$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ |
|---|---|---|---|
| 1 | 1 | 2 | 4 |
| 2 | 2 | 4 | 5 |
| 4 | 4 | 6 | 6 |
| 8 | 8 | 8 | 8 |

TABLE 2

| k | $M_m(\mathbb{F})$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ |
|---|---|---|---|---|
| 0 | $M_1(\mathbb{R})$ | 1 | 1 | 1 |
| 1 | $M_1(\mathbb{C})$ | 2 | 1 | 1 |
| 2 | $M_1(\mathbb{H})$ | 4 | 2 | 1 |
| 3 | $M_1(\mathbb{H})$ | 4 | 2 | 1 |
| 4 | $M_2(\mathbb{H})$ | 8 | 4 | 2 |
| 5 | $M_4(\mathbb{C})$ | 8 | 4 | 4 |
| 6 | $M_8(\mathbb{R})$ | 8 | 8 | 8 |
| 7 | $M_8(\mathbb{R})$ | 8 | 8 | 8 |

Proof: (C2)$\Leftrightarrow$(C3) $\Rightarrow$ (C4) $\Rightarrow$ (C5) and (C5)$\Leftrightarrow$(C6)$\Leftrightarrow$(C7) are exactly as before. (In (C6) $\Rightarrow$ (C7) note that the Steifel-Whitney class argument gives $k < m \dim_{\mathbb{R}}(\mathbb{F})$.) The main theorem of Adams-Lax-Phillips (1965) is (C1)$\Leftrightarrow$(C4); the same argument (using (A1$\Leftrightarrow$ (A5) instead of (A1)$\Leftrightarrow$(A4)) shows (C1)$\Leftrightarrow$ (C5).

We discuss (C1) $\Rightarrow$(C2). The first two columns of table 2 list some Clifford algebras (i.e. spaces $M_m(\mathbb{F})$ satisfying (C2)) for $k = 0, 1, \ldots, 7$. Using the inclusions:

$$M_m(\mathbb{R}) \subset M_m(\mathbb{C}) \subset M_m(\mathbb{H}),$$
$$M_m(\mathbb{C}) \subset M_{2m}(\mathbb{R}),$$
$$M_m(\mathbb{H}) \subset M_{2m}(\mathbb{C}),$$

we easily deduce the last three columns of table 2: an $m$ in the column labeled $\mathbb{F}$ and the row labeled $k$ means that there is a Clifford algebra on $k$ generators in $M_m(\mathbb{F})$. Now $\rho(m, \mathbb{F})$ as defined in table 1 is one greater than the largest value of $k$ for which $m$ appears under $\mathbb{F}$ in table 2; this proves (C1) $\Rightarrow$ (C2) in case $m = 1, 2, 4, 8$. The general case follows from the following fact: if $M_m(\mathbb{F})$ is a Clifford algebra on $k$ generators and $b$ is any positive integer then:

$$M_{bm}(\mathbb{F}) = M_m(\mathbb{F}) \otimes M_b(\mathbb{R})$$

is a Clifford algebra on $k$ generators and:

$$M_{16m}(\mathbb{F}) = M_m(\mathbb{F}) \otimes M_{16}(\mathbb{R})$$

is a Clifford algebra on $k + 8$ generators. (This is the "periodicity theorem"; see Porteous [1969] page 249. Table 2 appears there on page 250.) This proves (C1) $\Rightarrow$ (C2).

We next give a complex version of theorem B and, at the same time, a real version of theorem B which allows for complex eigenvalues.

THEOREM D. *Let* $n$ *and* $k$ *be positive integers and* $\mathbf{F} = \mathbf{R}, \mathbf{C}$. *In case* $\mathbf{F} = \mathbf{R}$ *assume* $k > 2$. (*The case* $k = 2$ *and* $\mathbf{F} = \mathbf{R}$ *is considered in Theorem E*.) *Let* $d = \dim_{\mathbf{R}}(\mathbf{F})$ $(= 1$ or $2)$. *Then the following are equivalent*:

(D1) $k < \sigma(n, \mathbf{F})$ *where*:

$$\sigma(n, \mathbf{F}) = d + 1 \quad \text{for} \quad n \neq 0, \pm 1 \mod 8$$

$$= \rho(4b, \mathbf{F}) \quad \text{for} \quad n = 8b,\ 8b \pm 1;$$

(D2) *either* $k \leq d$ *or there exists* $m$ *with* $k < \rho(m, \mathbf{F})$ *and* $n = 2m - 1,\ 2m,$ *or* $2m + 1$;

(D3) *there is a linear map*:

$$\Psi : \mathbf{R}^{k+1} \to H_n(\mathbf{F})$$

*such that each matrix* $\Psi(\alpha)$ $(\alpha \neq 0)$ *has* $n$ *distinct (necessarily real) eigenvalues*;

(D4) *there is an odd continuous map*:

$$\psi : S^k \to M_n(\mathbf{F})$$

*such that each* $\psi(\alpha)$ *has* $n$ *distinct (possibly complex) eigenvalues*.

Proof: The implications (D1) $\Longleftrightarrow$ (D2) $\Longrightarrow$ (D3) $\Longrightarrow$ (D4) are essentially as before; in case $k = 2$ and $\mathbb{F} = \mathbb{C}$ we can take:

$$\Psi_n(\alpha) = \alpha_0 A_n + \alpha_1 B_n + a_2 C_n$$

where:

$$C_n = \begin{bmatrix} 0 & i & & & & \\ -i & 0 & i & & & \\ & -i & 0 & & & \\ & & & \cdots & 0 & i \\ & & & & -i & 0 \end{bmatrix} .$$

(We remark that this works for the case $\mathbb{F} = \mathbb{H}$ as well. Take $d = 4$ and let $\alpha_2$ be a pure quaternion in the definition of $\Psi_n$.)

We now prove (D4) $\Longrightarrow$ (D1) in case $\mathbb{F} = \mathbb{C}$. If $k \leq 2$ there is nothing to prove, hence assume $k > 2$. Let $\lambda_1(\alpha), \ldots, \lambda_n(\alpha) \in \mathbb{C}$ be the eigenvalues of $\Psi(\alpha)$. Since $\Psi(-\alpha) = -\Psi(\alpha)$ we have:

$$\{\lambda_1(-\alpha), \ldots, \lambda_n(-\alpha)\} = \{-\lambda_1(\alpha), \ldots, -\lambda_n(\alpha)\}$$

as sets; we claim that after a suitable re-indexing we have:

$$\lambda_i(-\alpha) = -\lambda_{n-i+1}(\alpha)$$

as in Theorem B. Indeed, if this re-indexing is not possible we must have:

$$\lambda_i(-\alpha) = -\lambda_i(\alpha)$$

for more than one value of $i$. But then the map:

$$\eta : S^k \to \mathbb{C} = \mathbb{R}^2$$

given by:

$$\eta(\alpha) = \lambda_i(\alpha) - \lambda_j(\alpha)$$

is an odd map which is (by (D4)) nowhere vanishing; this contradicts the Borsuk-Ulam theorem (Spanier [1966], pages 104 and 266).

Now the eigenspaces:

$$E^i_\alpha = \{v \in \mathbb{C}^n : \Psi(\alpha)v = \lambda_i(\alpha)v\}$$

are complex line bundles of $S^k$ and hence (as $k > 2$) trivial. This follows from:

$$\pi_{k-1}(GL(1,\mathbb{C})) = \pi_{k-1}(S^1) = 0$$

and the homotopy characterization of bundles over a sphere (see Husemoller [1962], page 86 corollary 8.4). Hence the bundles $E^i_\alpha$ have nowhere zero sections $v_i(\alpha)$ and as:

$$E^i_{-\alpha} = E^{n-i+1}_\alpha$$

we may take:

$$(\#) \quad v_{n-i+1}(-\alpha) = v_i(\alpha)$$

for $i = 1, 2, \ldots, [n/2]$ (and hence for $i \neq n - i + 1$). In case $n$ is odd the middle eigenvector $v = v_i$ $(i = n - i + 1)$ satisfies:

$$v_i(-\alpha) = g(\alpha)v_i(\alpha)$$

for some continuous map $g : S^k \to S^1$. Clearly $g$ satisfies the condition that $g(\alpha)\, g(-\alpha) = 1$. As before we take:

$$\begin{aligned} x_i(\alpha) &= v_i(\alpha) - v_{n-i+1}(\alpha), && i = 1, \ldots, [n/2]; \\ x_i(\alpha) &= v_i(\alpha) + v_{n-i+1}(\alpha), && i = [n/2] + 1, \ldots, n; \\ x_i(\alpha) &= h(\alpha)\, v_i(\alpha), && i = n - i + 1 \end{aligned}$$

where $h : S^k \to S^1$ is so chosen that:

$$x_i(-\alpha) = \pm x_i(\alpha)$$

for $i = n - i + 1$. We can then choose $m$ (as in Theorem B) so that

$$x_i(-\alpha) = -x_i(\alpha)\,, \qquad i = 1,\ldots,m\,,$$
$$x_i(-\alpha) = x_i(\alpha) \qquad i = m+1,\ldots,n\,.$$

The existence of $h$ follows from the following:

LEMMA. A continuous map $g : S^k \to S^1$ has the form:

$$g(\alpha) = \pm h(\alpha)h(-\alpha)^{-1}$$

for some continuous map $h : S^k \to S^1$ if and only if it satisfies:

$$g(\alpha)g(-\alpha) = 1\,.$$

Proof of Lemma: "Only if" is immediate and by induction on $k$ "if" is easy for $k > 2$: define $h$ on the equator by the induction hypothesis, extend to the northern hemisphere by $\pi_{k-1}(S^1) = 0$ and extend to the southern hemisphere by $h(-\alpha) = \pm h(\alpha)\, g(\alpha)^{-1}$. This argument works also for $k = 2$ provided that the restriction of $h$ to $S^1$ (given by the induction hypothesis) extends to the disk. Hence consider the case $k = 1$. Take $h$ to be a square root of $g$:

$$h(\alpha)^2 = g(\alpha)$$

so that:

$$h(-\alpha)^{-2} = g(-\alpha)^{-1} = g(\alpha)$$

whence:

$$\{h(\alpha)h(-\alpha)^{-1}\}^2 = g(\alpha)^2$$

as required. To see that $h$ is single valued note that:

$$h(e^{i(\theta+\pi)})^2 = h(-e^{i\theta})^2 = g(-e^{i\theta}) = g(e^{i\theta})^{-1} = h(e^{i\theta})^{-2}$$

so that:

$$h(e^{i(\theta+\pi)}) = \pm h(e^{i\theta})^{-1}$$

whence:

$$h(e^{i(\theta+2\pi)}) = h(e^{i\theta})$$

as required. To see that $h$ extends to the disk note that:

$$2\deg(h) = \deg(g|S^1) = 0 .$$

This proves the lemma.

We have now verified (C6). As (C6) $\Longrightarrow$ (C1) we have proved Theorem D in case $\mathbb{F} = \mathbb{C}$.

We now consider the case $\mathbb{F} = \mathbb{R}$ and $k > 2$. Since $M_n(\mathbb{R}) \subset M_n(\mathbb{C})$ the conclusions of the complex case are available to us; moreover, if $\lambda_i(\alpha)$ is an eigenvalue so is $\overline{\lambda_i(\alpha)}$. If $\lambda_i(\alpha)$ is real for some $\alpha$ i.e. is real for all $\alpha$; otherwise at the transition we would have a real eigenvalue of multiplicity two. If $n$ is odd, the middle eigenvalue $(i = n-i+1,\ \lambda_i(-\alpha) = -\lambda_i(\alpha))$ must be real, else there would be two. Thus after a suitable reindexing we have three cases for each index $i = 1, \ldots, n$:

(1) $\bar{\lambda}_i = \lambda_i$

(2) $\bar{\lambda}_i = \lambda_{n-i+1}$

(3) $\bar{\lambda}_i = \lambda_j \quad j \neq n-i+1$

where in case (3) we may assume that $j \leq n/2$ if $i \leq n/2$.

In cases (1) and (3) it is obvious that we can choose the eigenvectors so as to satisfy (#) above and:

(##) $\quad \bar{v}_i = v_j \quad$ if $\quad \bar{\lambda}_i = \lambda_j$;

we show below that this is possible in case (2) as well. Assuming this we may define for $i, j = 1, \ldots, [n/2]$:

$$\left.\begin{aligned} x_i &= v_i - v_{n-i+1} \\ x_{n-i+1} &= v_i + v_{n-i+1} \end{aligned}\right\} \quad \text{if } \bar{\lambda}_i = \lambda_i$$

$$\left.\begin{aligned} x_i &= \operatorname{Im}(v_i) \\ x_{n-i+1} &= \operatorname{Re}(v_i) \end{aligned}\right\} \quad \text{if } \lambda_i = \lambda_{n-i+1}$$

$$\left.\begin{aligned} x_i &= \operatorname{Re}(v_i - v_{n-i+1}) \\ x_j &= \operatorname{Im}(v_i - v_{n-i+1}) \\ x_{n-i+1} &= \operatorname{Re}(v_j + v_{n-j+1}) \\ x_{n-j+1} &= \operatorname{Im}(v_j + v_{n-j+1}) \end{aligned}\right\} \quad \text{if } \bar{\lambda}_i = \lambda_j, \quad i < j.$$

and in case $n$ is odd and $i = n - i + 1$:

$$x_i = v_i .$$

Thus $x_1, \ldots, x_n$ are real and span $R^n$ and satisfy:

$$x_i(-\alpha) = -x_i(\alpha), \qquad i = 1, \ldots, m,$$

$$x_i(-\alpha) = x_i(\alpha), \qquad i = m+1, \ldots, n$$

where $m$ is either $[n/2]$ or $[n/2] + 1$. Then (as in Theorem B) we have proved (A6) and are done.

We prove that in case (2) we may satisfy (#) and (##). Let $w_i$ be any (normalized) section of $E^i$ and define $w_{n-i+1}$ by:

$$w_{n-i+1}(-\alpha) = w_i(\alpha).$$

Now $w_{n-i+1}$ and $\bar{w}_i$ are both sections of $E^{n-i+1}$ so:

$$\bar{w}_i = g w_{n-i+1}$$

where $g : S^k \to S^1$ is continuous. It follows that $g$ is even. We seek $h : S^k \to S^1$ continuous so that if:

$$v_i = h w_i$$

and $v_{n-i+1}$ is defined by (#), then (##) obtains. This follows immediately from the following:

LEMMA: A continuous map $g : S^k \to S^1$ has the form:

$$g(\alpha) = h(\alpha)h(-\alpha)$$

for some continuous $h : S^k \to S^1$ if and only if $g$ is even.

Proof: As in the last lemma take $h$ to be a square root of $g$; we get:

$$g(\alpha) = \pm h(\alpha)h(-\alpha).$$

If the minus sign occurs replace $h$ by $ih$. As before $h$ is well-defined.

We remark that it seems quite difficult to prove an analog of (D4) $\Rightarrow$ (D1) in case $\mathbb{F} = \mathbb{H}$, even (especially) in case the eigenvalues are real (unless one adopts the position that the real eigenvalues of a quaternionic matrix are always multiple). The reason is that $\pi_{k-1}(S^3) \neq 0$ for infinitely many $k$ so that the eigenspace quaternions line bundles over $S^k$ need not be trivial.

We now turn to the loose end left by Theorem D: the case $k = 2$ and $\mathbb{F} = \mathbb{R}$.

<u>THEOREM E</u>. <u>For each integer</u> $n \geq 2$ <u>the following are equivalent</u>:

(E1) $n \not\equiv 2 \mod 4$;

(E2) <u>there is a linear map</u>:

$$\Psi : \mathbb{R}^3 \to M_n(\mathbb{R})$$

<u>such that each $\Psi(\alpha)$ $(\alpha \neq 0)$ has $n$ distinct (possibly complex) eigenvalues</u>;

(E3) <u>there is an odd continuous map:</u>

$$\Psi : S^2 \to M_n(\mathbb{R})$$

<u>such that each</u> $\Psi(\alpha)$ $(\alpha \in S^2)$ <u>has $n$ distinct (possibly complex) eigenvalues.</u>

Proof: (E1) $\Longrightarrow$ (E2). For $\alpha \in \mathbb{R}^3$ the complex matrix:

$$U(\alpha) = \begin{pmatrix} \alpha_0 & \alpha_1 + i\alpha_2 \\ \alpha_1 - i\alpha_2 & -\alpha_0 \end{pmatrix}$$

is Hermitian and has trace zero. Hence its eigenvalues are $\pm\lambda(\alpha)$ where $\lambda(\alpha) > 0$ for $\alpha \neq 0$. The matrix $(1+i)U(\alpha)$ has eigenvalues $\pm(1+i)\lambda(\alpha)$. View this matrix as a 4 by 4 real matrix using the inclusion $M_2(\mathbb{C}) \subset M_4(\mathbb{R})$. It has four distinct eigenvalues $\pm(1+i)\lambda(\alpha)$ and $\pm(1-i)\lambda(\alpha)$. Hence:

$$\Psi_{4b}(\alpha) = \mathrm{diag}((1+i)U(\alpha),\ 2(1+i)U(\alpha), \ldots, b(1+i)U(\alpha))$$

has $4b$ distinct eigenvalues $\pm j(1 \pm i)\lambda(\alpha)$ $(j = 1, \ldots, b)$ and satisfies (E2) for $n = 4b$. For $n = 4b+1$ add a zero row and column:

$$\Psi_{4b+1} = \Psi_{4b} \oplus \Psi_1$$

where $\Psi_1(\alpha) = 0 \in M_1(\mathbb{R})$. For $n = 4b+3$ take the direct sum with the cross product:

$$\Psi_{4b+3} = \Psi_{4b} \oplus \Psi_3$$

where:

$$\Psi_3(\alpha)v = \alpha \times v$$

For $\alpha, v \in \mathbf{R}^3$. Note that the non-zero eigenvalues of $\Psi_3(\alpha)$ are pure imaginary and thus distinct from the eigenvalues of $\Psi_{4b}(\alpha)$.

(E2) $\Longrightarrow$ (E3). Take $\psi = \Psi \mid S^2$.

(E3) $\Longrightarrow$ (E1). Assume $n \equiv 2 \bmod 4$ but that $\psi$ satisfies (E3). By Theorem A some eigenvalue $\lambda_i(\alpha)$ must vanish for some value of $\alpha$; in particular $\lambda_i(\alpha)$ must be real for some $\alpha$. But as in Theorem D, $\lambda_i(\alpha)$ is real for some $\alpha$ if and only if it is real for all $\alpha$. Hence (as the complex eigenvalues occur in conjugate pairs) there must be $2m > 0$ real eigenvalues:

$$\lambda_1(\alpha) > \lambda_2(\alpha) > \cdots > \lambda_{2m}(\alpha).$$

But now:

$$\varphi(\alpha) = \psi(\alpha) - \frac{1}{2}(\lambda_m(\alpha) + \lambda_{m+1}(\alpha))I$$

is an odd map into $GL(n, \mathbf{F})$ contradicting Theorem A.

## §5. Which Eigenvalues Cross?

Theorem C has a Hermitian version as well. Let $\rho_H(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) be given by:

$$\rho_H(n, \mathbb{R}) = \rho(\tfrac{n}{2}) + 1;$$
$$\rho_H(n, \mathbb{C}) = \rho(n, \mathbb{C}) - 1;$$
$$\rho_H(n, \mathbb{H}) = \rho(\tfrac{n}{4}) + 5;$$

where we take $\rho(r) = 0$ if $r$ is not an integer.

<u>THEOREM F. Let $n > 0$ and $k > 0$ be integers and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then the following are equivalent</u>:

(F1) $$k < \rho_H(n, \mathbb{F});$$

(F2) <u>there is a linear map</u>:

$$\Phi : \mathbb{R}^{k+1} \to H_n(\mathbb{F})$$

with:

$$\Phi(\mathbb{R}^{k+1} \setminus 0) \subset GL(n, \mathbb{F});$$

(F3) <u>there is an odd continuous map</u>:

$$S^k \to GL(n, \mathbb{F}) \cap H_n(\mathbb{F}).$$

(The equivalence (F1) $\Leftrightarrow$ (F2) is proved in Adams-Lax-Phillips (1965); the same reasoning shows (F1) $\Leftrightarrow$ (F3).) This has an easy implication for the crossing problem:

COROLLARY. Let $n = 2m$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Then there is an odd continuous map:

$$\psi : S^k \to H_n(\mathbb{F})$$

with:

$$\lambda_m(\alpha) > \lambda_{m+1}(\alpha)$$

for all $\alpha \in S^k$ (where $\lambda_1(\alpha) > \cdots > \lambda_n(\alpha)$ are the eigenvalues of $\Psi(\alpha)$) if and only if $k < \rho_H(n, \mathbb{F})$.

Proof: "only if" is exactly as in Theorem E for "if" let $\psi = \varphi$ be any odd map into $GL(n, \mathbb{F}) \cap H_n(\mathbb{F})$. Then the integer $p$ defined by:

$$\lambda_1(\alpha) > \cdots > \lambda_p(\alpha) > 0 > \lambda_{p+1}(\alpha) > \cdots > \lambda_n(\alpha)$$

is independent of $\alpha$ since no $\lambda_i(\alpha)$ vanishes. But $\lambda_i(-\alpha) = -\lambda_{n-i+1}(\alpha)$ so we must have $p = m$.

We conclude with the following remark. Theorem B implies (for $k \geq \rho(n)$) the existence of a multiple eigenvalue but does not give any information about which eigenvalue is multiple; the above corollary implies that the "middle" eigenvalue is multiple. This is not accidental. Indeed in Friedland-Loewy (1976) it was shown that any $n$ (resp. $2n-1$) dimensional subspace of $H_n(\mathbb{R})$ (resp. $H_n(\mathbb{C})$) contains a non-zero matrix with a non-simple first eigenvalue and this result is sharp (i.e. can fail for subspaces of lower dimension). This suggests that the "higher the eigenvalue" the "rarer the crossing".

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| 4. TITLE (and Subtitle)<br>On The Crossing Rule | | 5. TYPE OF REPORT & PERIOD COVERED<br>Summary Report - no specific reporting period |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>S. Friedland, J.W. Robbin, and J.H. Sylvester | | 8. CONTRACT OR GRANT NUMBER(s)<br>DAAG29-80-C-0041, MCS-7927062, Mod. 1 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Mathematics Research Center, University of Wisconsin<br>610 Walnut Street<br>Madison, Wisconsin 53706 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>Work Unit No. 1 - Applied Analysis |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>* see below | | 12. REPORT DATE<br>June 1982 |
| | | 13. NUMBER OF PAGES<br>32 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

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U.S. Army Research Office
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Research Triangle Park
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National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Eigenvalues, Hyperbolic Systems.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper gives conditions on a family of matrices which guarantee that some matrix in the family will have a multiple eigenvalue. In particular, the main theorem states exactly which dimensions admit k dimensional subspaces of matrices for which all nonzero elements have distinct eigenvalues.

This question arises naturally in the theory of first order hyperbolic systems of partial differential equations; the main theorem, in this context, tells exactly for which integers n an n x n system in k space variables may be strictly hyperbolic.

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