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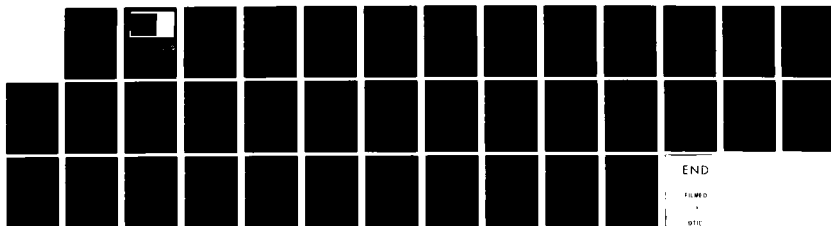
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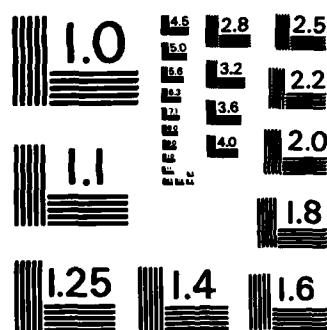
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BIVARIATE BOX SPLINES AND
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THREE DIRECTION MESH

C. de Boor and K. Höllig

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

Let S denote the space of bivariate piecewise polynomial functions of degree $\leq k$ and smoothness ρ on the regular mesh generated by the three directions $(1,0)$, $(0,1)$, $(1,1)$. We construct a basis for S in terms of box splines and truncated powers. This allows us to determine the polynomials which are locally contained in S and to give upper and lower bounds for the degree of approximation. For $\rho = \lfloor (2k-2)/3 \rfloor$, $k \neq 2(3)$, the case of minimal degree k for given smoothness ρ , we identify the elements of minimal support in S and give a basis for $S_{\text{loc}} = \{f \in S: \text{supp } f \subseteq \Omega\}$, with Ω a convex subset of $\mathbb{R}^2$.

AMS (MOS) Subject Classification: 41A15, 41A63, 41A25

Key Words: bivariate, B-splines, three direction mesh, degree of approximation, minimal support.

Work Unit Number 3 - Numerical Analysis and Computer Science

SIGNIFICANCE AND EXPLANATION

Local support bases for piecewise polynomial spaces are important for applications such as finite element methods, data fitting etc. In [BH₁] a general construction principle for such "B-splines" was used to obtain the so called box-splines. They have a particularly regular discontinuity pattern and coincide in special cases with standard finite elements.

→ This report investigates the use of translates of certain bivariate box-splines in the construction of a unified theory for piecewise polynomial functions on regular meshes.

A simple mesh is considered, derived from a square mesh by drawing in the same diagonal into every square. The space S of piecewise polynomial functions of a given degree and smoothness, and with discontinuities (in some derivative) only across lines of that mesh is considered. We show that the box splines and their translates provide a basis for the "local" part of S and use the techniques of [BH₁] to analyse the approximation properties of S .

The report stresses the importance of local support bases which are desirable for applications such as finite element methods, smoothing of data and approximation in general. Our results should be useful for the further investigation of smooth piecewise polynomials, in particular on regular meshes (c.f. [CW], [Si], [Sl] for related work).

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

Bivariate box splines and smooth pp functions on a three-direction mesh

C. de Boor and K. Höllig

0. Introduction. This paper records further results of our continuing investigation of certain multivariate B-splines. It follows [BH₁] in which we discussed general properties of box splines and the spaces spanned by translates of a box spline.

In the present paper, we explore the question to what an extent box splines may be useful in the study of spaces of smooth pp ($:=$ piecewise polynomial) functions in which they lie. We restrict attention to the simplest interesting situation, that of the space

$$S := \pi_{k,\Delta}^D$$

of bivariate pp functions in C^D , of degree $\leq k$, on the mesh Δ obtained from a uniform square mesh by drawing in the same diagonal in each square. Even in this simple setting, we find much challenge; in fact, we must leave some obvious questions unanswered for the present.

The specific questions we tried to answer are: (i) Are these B-splines "basic", i.e., to what an extent do box splines provide a basis for S ? The answer is that they provide a spanning set for the "local part", but have to be augmented by certain truncated powers to give a spanning set for all of S . In certain special circumstances, they even provide a basis for all finitely supported elements of S . But this happens rarely, because the answer is "Usually not" to the question: (ii) Are these B-splines "minimal", i.e., does S contain no element with support strictly inside that of a box spline? The box splines do provide material help in answering the question: (iii) What is the approximation order from S ?

In outline, the paper is as follows: In Section 1, we introduce the relevant notation in the process of specializing the general results of [BH₁] concerning box splines to the specific context of the bivariate 3-direction mesh Δ . We study the space spanned by certain translates of one such box spline, prove these translates to be linearly

independent even locally, and characterize all polynomials in their span. In Section 2, we show that S is spanned by certain box splines and their translates together with certain truncated powers. These latter functions are zero on a halfspace and agree with a suitable polynomial on the complementary halfspace. This permits us to show, in Section 3, that, in effect, the approximation order from S is entirely determined by how well one can approximate from $S_{loc} := \text{span of box splines contained in } S$. This, in turn, can be related to the question of which polynomials are contained in S_{loc} . We answer this question in full and thereby obtain upper and lower bounds on the approximation order from S which coincide in some cases and are, in any event, very close when ρ is as large as possible, i.e.,

$$\rho = \rho(k) := \lfloor (2k-2)/3 \rfloor .$$

We also give a conjecture concerning the approximation order for $\rho < \rho(k)$.

In Section 4, we look for elements of minimal support in S . These are provided by the box splines in case $k \equiv 1(3)$ and $\rho = \rho(k)$. For $k \equiv 0(3)$ and $\rho = \rho(k)$, there are in S elements of smaller support than that of the box splines. These were first discussed by Frederickson [Fr]. In either case, we show that these minimal support elements provide a basis for all finitely supported functions in S . We also discuss the case $k \equiv 2(3)$ and $\rho = \rho(k)$ in which the degree is not minimal for the given ρ to illustrate that the search for minimal support elements can be quite frustrating when k is not minimal. Only for sufficiently large k (with respect to ρ) does the minimal support question become simple again.

1. Box splines on a three-direction mesh. In $[BH_1]$, the box spline M_{Ξ} is defined as the distribution on $\mathbb{R}^m$ given by the rule

$$M_{\Xi} : \phi \mapsto \int_{[0,1]^n} \phi \left(\sum_{i=1}^n \lambda(i) \xi_i \right) d\lambda \quad (1)$$

for some sequence $\Xi := (\xi_i)_1^n$ in $\mathbb{R}^m$. In this section, we specialize the general results of $[BH_1]$ concerning M_{Ξ} and the span

$$S_{\Xi} := \text{span}(M_{\Xi}(\cdot - v))_{v \in \mathbb{Z}^m}$$

of its integer translates to the simple situation

$$m = 2, \quad \Xi = (d_1: r, d_2: s, d_3: t) \quad (2)$$

with the three directions given by

$$d_1 := e_1, d_2 := e_2, d_3 := e_1 + e_2.$$

By this we mean that $\text{ran } \Xi \subseteq \{d_1, d_2, d_3\}$ and that r, s, t are the relevant direction multiplicities which characterize Ξ , i.e.,

$$r := |\{i: \xi_i = d_1\}|, \quad s := |\{i: \xi_i = d_2\}|, \quad \text{and} \quad t := |\{i: \xi_i = d_3\}|.$$

This special choice of Ξ allows us to delve more deeply into the details in a setting of possibly practical importance.

In later sections, we will write

$$M_{r,s,t} \text{ instead of } M_{\Xi}.$$

For the remainder of this section, though, we write

$$M \text{ instead of } M_{\Xi}$$

and write

$$M_v := M(\cdot - v)$$

for any particular

$$v \in V := \mathbb{Z}^2.$$

We now study

$$S := S_{\Xi} = \text{span}(M_v)_v.$$

S is a subspace of

$$\pi_{k,\Delta} := \text{pp functions of degree } \leq k \text{ on the partition } \Delta,$$

with

$$k := n-2$$

and Δ the partition of $\mathbb{R}^2$ into triangles obtained from the three families of meshlines

$$v + zd_1, \quad v \in V, \quad z \in \mathbb{R}.$$

We have foregone the opportunity to make the symmetries in Δ more apparent by having the three families of meshlines intersect each other at an angle of 120° (as is done, e.g., in [Fr]). This would needlessly complicate the notation. It is sufficient to note that any permutation of the meshline families can be accomplished by some linear map on $\mathbb{R}^2$, and the corresponding change of variables leaves $\pi_{k,\Delta} \cap C^p$ invariant.

The smoothness of M depends on the direction multiplicities. We have

$$M \in L_{\infty}^{(d)} \subseteq C^{(d-1)},$$

with

$$d = (n - \max\{r, s, t\}) - 1$$

the number defined in [BH₁; (2.6)] as evaluated for our special case. Since $n = r + s + t$, it follows that, for fixed degree $k = n - 2$, we get maximal smoothness by choosing

$$\max\{r, s, t\} = \lceil (k+2)/3 \rceil.$$

Then, for $k = 3\mu + 1$, the corresponding maximal d is

$$d(k) := 2\mu + 1 = \lfloor (2k+1)/3 \rfloor, \quad (3)$$

$i = -1, 0, 1$. For $k = 3\mu + 1$, there is just one choice,

$$r = s = t = \mu + 1,$$

while, for $k = 3\mu$ or $k = 3\mu - 1$, there are three choices for (r, s, t) .

Recall from [BH₁; Cor.2 of Thm.5] that d also governs which polynomial spaces are contained in S . Precisely,

$$\pi_m \subseteq S \text{ iff } m \leq d. \quad (4)$$

Of course, as we will see shortly, some polynomials of degree higher than m may also be in S .

It follows from (1) that

$$\text{supp } M = \{ \sum_1^n \lambda(i) \xi_i : \lambda \in [0, 1]^n \} = \{ \sum_1^3 \lambda(i) d_i : \lambda \in [0, r] \times [0, s] \times [0, t] \}.$$

Thus, $\text{supp } M$ is a hexagon composed of

$$N := rs + rt + st$$

triangles of Δ which are translates of the triangle spanned by d_1 and d_3 and a like

number of triangles which are translates of the triangle spanned by d_2 and d_3 . This implies that exactly N M_V 's have any particular triangle of Δ in their support.

Since

$$\det(d_i, d_j) = 1 \quad \text{for } i \neq j,$$

we conclude from [BH₁; Prop.4] that $(M_V)_V$ need not be linearly dependent. We now prove much more.

Proposition 1. $(M_V)_{V \in V}$ is locally linearly independent, i.e.,

$$\{M_V : \text{supp } M_V \cap \Delta \neq \emptyset\} \quad (5)$$

is linearly independent over any open set Δ contained in some triangle of Δ .

Proof. Since (5) contains exactly N elements, it is necessary and sufficient to prove that S contains N functions which are linearly independent over Δ . This latter condition is shown to hold once we show that $\dim(S \cap \pi) > N$. It then follows, incidentally, that $\dim(S \cap \pi) = N$. Here,

$$S \cap \pi =: \pi_{\Xi}$$

is the linear space of all polynomials contained in S .

The proof consists in identifying various elements of π_{Ξ} . For the specific Ξ , we have from [BH₁; Theorem 5] that

$$\pi_{\Xi} = \ker D_1^r D_2^s \cap \ker D_1^r (D_1 + D_2)^t \cap \ker D_2^s (D_1 + D_2)^t. \quad (6)$$

Correspondingly, we would like to specify linearly independent elements of π_{Ξ} in the form

$$I_1^{\rho} I_2^{\sigma} I_3^{\tau} (1) \quad \text{for } \begin{cases} \rho < r, \sigma < s, \tau = t \\ \rho < r, \sigma = s, \tau < t \\ \rho = r, \sigma < s, \tau < t \end{cases},$$

with I_1, I_2, I_3 right inverses of $D_1, D_2, D_1 + D_2$ respectively, and 1 denoting the function $x \mapsto 1$. But, since each of these integral operators fails to commute with at least one of these differential operators, it is tricky to make the construction precise in this form.

Instead, we single out the two classes

$$\Lambda_1 := \{\phi_\alpha : \alpha(1) < r\} \quad \text{and} \quad \Lambda_2 := \{\phi_\alpha : \alpha(2) < s\}$$

of monomials

$$\phi_\alpha := ()^{\alpha}/\alpha!$$

Then $\Lambda_1 \subseteq \ker D_1^r$, $\Lambda_2 \subseteq \ker D_2^s$, therefore $\Lambda_1 \cap \Lambda_2$ provides a linearly independent set of rs elements in π_{Ξ} . In addition, we pick a set $B_1 \subseteq \text{span}(\Lambda_1 \setminus \Lambda_2)$ of rt elements and a set $B_2 \subseteq \text{span}(\Lambda_2 \setminus \Lambda_1)$ of st elements in π_{Ξ} and are then certain of the linear independence of the total collection

$$(\Lambda_1 \cap \Lambda_2) \cup B_1 \cup B_2$$

as soon as we prove that both B_1 and B_2 are themselves linearly independent.

To construct B_1 , we consider the right inverse J of $D_1 + D_2$ for which

$$(Jf)(\cdot, 0) = 0.$$

To find $J\phi_\alpha$, we write $J\phi_\alpha = \sum c_\beta \phi_\beta$ and consider the resulting linear system

$$\sum c_\beta (\phi_{\beta-e_1} + \phi_{\beta-e_2}) = \phi_\alpha$$

$$\sum c_\beta \phi_\beta(\cdot, 0) = 0$$

This gives $c_\beta = 0$ for $\beta(2) = 0$ and therefore $c_\beta = 0$ for $|\beta| \neq |\alpha| + 1$ and for $|\beta| = |\alpha| + 1$ with $\beta(1) > \alpha(1)$, hence also $c_{\alpha+e_2} = 1$. In conclusion,

$$J\phi_\alpha \in \phi_{\alpha+e_2},$$

with

$$\phi_\beta := \phi_\beta + \text{span} \{\phi_\gamma : |\gamma| = |\beta|, \gamma(1) < \beta(1)\}.$$

Therefore, more generally,

$$J[\phi_\beta] \subseteq \phi_{\beta+e_2}. \quad (7)$$

Now set

$$B_1 := \{J^i \phi_{(j,s-1)} : i=1, \dots, t; j=0, \dots, r-1\}.$$

By (7),

$$J^i \phi_{(j,s-1)} \in \phi_{(j,s-1+i)} \subseteq \text{span}(\Lambda_1 \setminus \Lambda_2) \quad \text{for } i > 0, j < r,$$

hence $B_1 \subseteq \text{span}(\Lambda_1 \setminus \Lambda_2)$, as desired. This also implies that $B_1 \subseteq \ker D_1^r$, hence

$$B_1 \subseteq \ker D_1^r D_2^s \cap \ker D_1^r (D_1 + D_2)^t.$$

But

$$(D_1 + D_2)^i (J^i \phi_{(j,s-1)}) = \phi_{(j,s-1)} \subseteq \ker D_2^s,$$

therefore $(D_1 + D_2)^t[B_1] \subseteq \ker D_2^s$, hence also $B_1 \subseteq \ker D_2^s(D_1 + D_2)^t$. We conclude that $B_1 \subseteq \pi_{\Xi}$.

Finally, we need to show the linear independence of B_1 . For this, consider the matrix C of polynomial coefficients for the elements of B_1 , i.e.,

$$J^i \phi_{j,s-1} = \sum C(i,j;\beta) \phi_{\beta}, \quad i=1, \dots, t; \quad j=0, \dots, r-1.$$

Choose the (reverse lexicographic) ordering

$$(i,j) < (h,k) := \begin{cases} i+j < h+k \\ \text{or} \\ i+j = h+k \text{ and } i < h \end{cases}.$$

Then C is unit lower triangular in the sense that

$$J^i \phi_{j,s-1} \in \phi_{j,s-1+i} + \text{span}\{\phi_{\beta} : \beta < (j,s-1+i)\}, \quad \text{all } j,i,$$

hence of full rank. Thus B_1 is linearly independent.

The construction of B_2 proceeds in exactly the same way, with the roles of the two independent variables interchanged. |||

For example, take $(r,s,t) = (2,2,1)$. Then $n = r + s + t = 5$, hence $k = 3$. Also, $d = d(3) = 2$, hence M is a piecewise cubic C^1 function. Now

$$\Lambda_1 \wedge \Lambda_2 = \{\phi_{\alpha} : \alpha(1), \alpha(2) = 0, 1\}$$

forms a basis for $\pi_{1,1}$:= bilinear polynomials. Further,

$$B_1 = \{J^i \phi_{j,1} : i=1; j=0,1\},$$

with

$$J\phi_{0,1} = \phi_{0,2} \quad \text{and} \quad J\phi_{1,1} = \phi_{1,2} + c\phi_{0,3}.$$

We determine c from the condition that $(D_1 + D_2)J\phi_{1,1} = \phi_{1,1}$. This gives

$$\phi_{0,2} + \phi_{1,1} + c\phi_{0,2} = \phi_{1,1},$$

hence $c = -1$. Thus $B_1 = \{\phi_{0,2}, \phi_{1,2} - \phi_{0,3}\}$. By symmetry,

$$B_2 = \{\phi_{2,0}, \phi_{2,1} - \phi_{3,0}\}.$$

$$\pi_{\Xi} = \pi_2 + \text{span}\{\phi_{1,2} - \phi_{0,3}, \phi_{2,1} - \phi_{3,0}\}$$

in case $\Xi = (e_1, e_1, e_2, e_2, e_1 + e_2)$.

Corollary 1. For any triangle τ of Δ , a basis for π_{τ} is provided by the N nontrivial polynomials which agree with M_{τ} on τ .

Corollary 2. If $p \in \pi$ agrees with M on some triangle of Δ , then $p \in \pi_{\tau}$.

Remark. While Corollary 1 is quite special, Corollary 2 is valid for an arbitrary box spline M in any number of dimensions. This is a consequence of [BH₁; Theorem 5] and is due to the fact that any polynomial p which agrees with M on some open set is necessarily mapped to 0 by any differential operator D_z for which $D_z M$ is supported only on certain hyperplanes.

2. **Spanning sets and local bases.** In this section, we give a truncated power basis for $\pi_{k,\Delta}^p(Q)$, with Q any rectangle bounded by Δ -mesh lines. This would allow us to verify the dimension formulas of $[CW_1]$ for this space. We also give a spanning set for $\pi_{k,\Delta}^p$ itself which, though finitely linearly independent, permits nontrivial infinite linear combinations which add to zero. Its main feature is that it consists of finitely supported functions, viz. box splines, on the one hand and of functions supported on half spaces and agreeing with some polynomial there on the other. These latter functions do not contribute to the approximation order obtainable from the scale $(\pi_{k,h\Delta}^p)$, as we will show in the next section. This means that the approximation order is no better than that obtainable from the span of the relevant box splines, and this fact allows us to give upper and lower bounds on the approximation order which differ by at most two in case p is as large as possible.

Recall from $[BH_1]$ that Dahmen's truncated powers $[D]$ can be thought of as shadows of the standard cone R_+^n : With $\Xi := (\xi_1)_1^n$ in R^n , the corresponding truncated power or cone spline C_Ξ is, by definition, the distribution on R^m given by the rule

$$C_\Xi : \phi \mapsto \int_{R_+^n} \phi(\sum_1^n \lambda(i) \xi_i) d\lambda.$$

Since $M_\Xi : \phi \mapsto \int_{[0,1]^n} \phi(\sum_1^n \lambda(i) \xi_i) d\lambda$, it follows that

$$C_\Xi = \sum_{v \in \mathbb{Z}_+^2} M_\Xi(\cdot - v) \quad (1)$$

Recall from $[BH_1]$ that, for Z in Ξ ,

$$D_Z C_\Xi = C_{\Xi \setminus Z}. \quad (2)$$

Now specialize to the setup of Section 1, i.e., to the specific sequence

$$\Xi = (d_1:r, d_2:s, d_3:t)$$

consisting only of the vectors $d_1 = e_1$, $d_2 = e_2$, and $d_3 = e_1 + e_2$ in R^2 , and therefore characterized by the corresponding direction multiplicities (r,s,t) . It follows from (2) that

$$C_\Xi \text{ has all derivatives of order } < \begin{Bmatrix} s+t-2 \\ r+t-2 \\ r+s-2 \end{Bmatrix} \text{ continuous across } \text{span}\left(\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}\right). \quad (3)$$

Correspondingly, the univariate function $N_{\frac{z}{2}}$ given by the rule

$$N_{\frac{z}{2}}(z) := C_{\frac{z}{2}}(z, 1-z), \text{ all } z \in \mathbb{R},$$

is a univariate B-spline, i.e.,

$$C_{\frac{z}{2}}(z, 1-z) = c_{\frac{z}{2}}M(z|0:r, \frac{1}{2}:t, 1:s) \quad (4)$$

for some positive $c_{\frac{z}{2}}$.

Here is an outline of what is to follow. We show that, near a lower left corner of its support, any $f \in \pi_{k,\Delta}^0$ can be written as a linear combination of certain truncated powers. For this, we split f into its homogeneous components. Being homogeneous, each such component is determined by its restriction to a line which "cuts across" the corner. Such a restriction is a univariate spline, hence uniquely representable as a linear combination of certain univariate B-splines, i.e., of restrictions of certain truncated powers.

Next, on subtracting from f this linear combination of truncated powers, we obtain a new element of $\pi_{k,\Delta}^0$ whose support is inside that of f and offers lower left corners, to the right and/or above, for further "peeling off".

We begin with a study of the simple pp space which models the behavior of $f \in \pi_{k,\Delta}^0$ near a lower left corner of its support. We denote this space by

$$S(\leq k, v) := \pi_{k,\Delta}^v$$

and mean by this the space of all pp functions of degree $\leq k$ with support in $\mathbb{R}_+^2$ and possible singularities only across the three rays

$$\mathbb{R}_+ d_i, \quad i=1,2,3.$$

In addition, we think of v here as a 3-vector, with $v(i)$ indicating that all derivatives of order $\leq v(i)$ are required to be continuous across $\mathbb{R}_+ d_i$, $i=1,2,3$.

Let

$$H_l := \{ f : f(zx) = z^l f(x), \text{ all } x \in \mathbb{R}^2, z \in \mathbb{R}_+ \}$$

denote the collection of all functions on $\mathbb{R}^2$ (positively) homogeneous of degree l . As is well known,

$$\pi_k = \bigoplus_{l \leq k} (H_l \cap \pi_k),$$

so it makes sense to talk about the homogeneous component of degree l of a polynomial. We

now make the same claim for $S(\leq k, v)$. With

$$S(l, v) := H_l \cap S(\leq k, v),$$

we claim

$$\text{Lemma 2. } S(\leq k, v) = \bigoplus_{l \leq k} S(l, v).$$

Proof. Only the inclusion " $\subseteq$ " requires proof. To prove this inclusion, it is sufficient to show that the pp function made up of the l th degree homogeneous components of an element of $S(\leq k, v)$ is again in $S(\leq k, v)$. This follows from the following

Claim. If a polynomial p vanishes to order ρ along the ray $R_+ d$, i.e.,

$$D^\alpha p = 0 \text{ on } R_+ d \text{ for } |\alpha| < \rho,$$

then each of its homogeneous components also vanishes to order ρ on $R_+ d$.

Proof. Assume without loss that $d = e_2$. Since $D_1^l p = 0$ on $R_+ e_2$ for $l < \rho$, we must have

$$p = ()^{(\rho+1, 0)} q$$

for some polynomial q . Writing

$$p = \sum_{l \leq k} p_l \text{ with } p_l \in H_l, \text{ all } l,$$

we conclude that each p_l has the factor $()^{(\rho+1, 0)}$, therefore vanishes on Re_2 together with every derivative of order $< \rho+1$. (In particular, $p_l = 0$ for $l < \rho+1$). |||

We took the trouble to express $S(\leq k, v)$ in terms of its homogeneous components $S(l, v)$ since, on $S(l, v)$, the linear map R given by the rule

$$(Rf)(z) := f(z, 1-z), \text{ all } z \in R,$$

is 1-1. This follows from the fact that, for any $f \in H_l$,

$$f(\lambda z, \lambda(1-z)) = \lambda^l (Rf)(z), \text{ all } \lambda \in R_+, z \in R,$$

hence such f is determined on the entire halfspace $x(1) + x(2) > 0$ once Rf is known.

We claim that R carries $S(l, v)$ onto the univariate spline space

$$S_u(\leq l, v)$$

which consists of all pp functions g of degree $\leq l$ on $\mathbb{R}$ with breakpoints $0, 1/2, 1$, with support in $[0,1]$, and with

$$g \in C^{v(i)} \text{ near } \xi_i := \begin{cases} 0 \\ 1/2 \\ 1 \end{cases} \text{ if } i = \begin{cases} 1 \\ 3 \\ 2 \end{cases}.$$

Indeed, R carries all of $S(\leq l, v)$ into $S_u(\leq l, v)$. In addition, we recall from (4) that, with

$$\Xi = (d_1:r, d_2:s, d_3:t),$$

R carries the cone spline C_Ξ to a positive multiple of the univariate B-spline

$$M(\cdot|0:r, 1/2:t, 1:s).$$

This implies that, with

$$(\zeta_i) := (d_1:l-v(1), d_3:l-v(3), d_2:l-v(2)),$$

the cone splines $C_{\zeta_1}, \dots, \zeta_{i+l+1}$ are in $S(l, v)$, and R carries these to a basis for $S_u(\leq l, v)$. Consequently, these cone splines must form a basis for $S(l, v)$. In particular,

$$\dim S(l, v) = \left[\sum_1^3 (l-v(i))_+ - l-1 \right]_+.$$

Therefore

$$\dim S(\leq k, v) = \sum_{l \leq k} \left[\sum_1^3 (l-v(i))_+ - l-1 \right]_+. \quad (5)$$

Remark. This formula shows that $\pi_{k, \Delta}^\rho$ contains no finitely supported functions unless ρ is suitably small: If $f \in \pi_{k, \Delta}^\rho \setminus 0$ has finite support, then its support must contain a "lower left corner", i.e., a mesh square $Q_v := [v(1), v(1)+1] \times [v(2), v(2)+1]$ along whose left and lower edge f vanishes to order ρ . This implies that f agrees on Q_v with $g(\cdot-v)$ for some $g \in S(\leq k, \rho, \rho, \rho)$. This in turn implies that $\dim S(\leq k, \rho, \rho, \rho) > 0$, and, by (5), this is equivalent to having $\rho \leq (2k-2)/3$. This conclusion was reached in [BD], using the same simple argument of cutting across such a lower left corner of the support, as is used here. We realized only recently that this conclusion can already be found in [Fa].

We are now ready to give a cone spline basis for

$$\pi_{k, \Delta}^\rho(Q) := \pi_{k, \Delta}|_Q \cap C^\rho(Q).$$

We use the translation map τ_v given by the rule

$$(\tau_v f)(x) := f(x - v).$$

Proposition 2. Let $Q = [0, M+1] \times [0, N+1]$. Then $\pi_{k, \Delta}^p(Q)$ is the direct sum of the spaces $\tau_v S(\langle k, v \rangle)_Q$ with

$$v_v := \begin{cases} (-1, -1, \rho) & , \quad v = (0, 0) \\ (-1, \rho, \rho) & , \quad v = (x, 0) \\ (\rho, -1, \rho) & , \quad v = (0, y) \\ (\rho, \rho, \rho) & , \quad v = (x, y) \end{cases}, \quad \text{all } v \in v_Q := v \cap ([0, M] \times [0, N]), \quad \text{and } x, y > 0.$$

Consequently,

$$\dim \pi_{k, \Delta}^p(Q) = \sum_{l \leq k} l+1 + (l-\rho)_+ + 2(M+N)(l-\rho)_+ + MN(3(l-\rho)_+ - l-1)_+, \quad (6)$$

and the restriction to Q of the cone splines

$$c_{\zeta_1, \dots, \zeta_{i+l+1}}^{(-v)}, \quad \text{all } i, \quad \text{with } (\zeta_j) := (d_1: l-v_v(1), d_3: l-v_v(3), d_2: l-v_v(2)) \quad (7) \\ \text{for } l \leq k, \quad v \in v_Q,$$

forms a basis for $\pi_{k, \Delta}^p(Q)$.

Proof. For any choice of v_v , the spaces $\tau_v S(\langle k, v \rangle)_Q$ are in direct sum. For the specific choices of v_v given, they are all in $\pi_{k, \Delta}^p(Q)$. Thus it only remains to show that $\pi_{k, \Delta}^p(Q)$ is contained in their sum.

We proceed by induction. For this, we use again the (reverse lexicographic) ordering

$$v < w := \begin{cases} |v| < |w| \\ \text{or} \\ |v| = |w| \quad \text{and} \quad v(1) < w(1) \end{cases}$$

which provides a total ordering for v_Q . We again use Q_v to denote the unit mesh square whose lower left corner is v . The induction hypothesis to be advanced is the following:

For all $v < w$, there exists $f_v \in \tau_v S(\langle k, v \rangle)_Q$ so that

$$\delta_w := f - \sum_{v < w} f_v$$

vanishes on $\bigcup_{v < w} Q_v$. In order to advance this hypothesis, we now show that, in its consequence, δ_w agrees on Q_w with some $f_w \in \tau_w S(\langle k, v \rangle)$. There are four cases:

- (i) $w = 0$. Then $\delta_w = f$, hence it agrees with some $f_w \in S(\langle k, -1, -1, \rho \rangle)$.
- (ii) $w = (i, 0)$ for some $i > 0$. Then δ_w vanishes to order ρ on the segment

$i \times [0,1]$, therefore agrees on Q_w with some $f_w \in \tau_S(\langle k, -1, \rho, \rho \rangle)$.

(iii) $w = (0, j)$ for some $j > 0$. Then δ_w vanishes to order ρ on the segment $[0,1] \times j$, therefore agrees on Q_w with some $f_w \in \tau_S(\langle k, \rho, -1, \rho \rangle)$.

(iv) $w = (1, j)$ for $1, j > 0$. Then δ_w vanishes to order ρ on the left as well as on the lower boundary segment of Q_w , therefore agrees on Q_w with some $f_w \in \tau_S(\langle k, \rho, \rho, \rho \rangle)$.

Since $\text{supp } f_w \subseteq \tau_{w+}^2 \subseteq \bigcup_{v < w} Q_v$, this advances the induction hypothesis, since it implies that $\delta_w - f_w$ vanishes also on Q_w as well as on $\bigcup_{v < w} Q_v$.

The dimension formula (6) now follows from (5). |||

Since M and N are arbitrary positive integers, we obtain the following

Corollary. For $Q = \mathbb{R}_+^2$, $\pi_{k,\Delta}^\rho(Q)$ is spanned by the restriction to Q of the cone splines listed in (7) (with $v_Q = \mathbb{R}_+^2$).

Next, we investigate the relationship of $\pi_{k,\Delta}^\rho(Q)$ to $\pi_{k,\Delta}^\rho|_Q$. These two spaces are, in fact, the same, but this is not clear a priori. It is obvious that

$$\pi_{k,\Delta}^\rho|_Q \subseteq \pi_{k,\Delta|Q}^\rho \cap C^\rho(Q) =: \pi_{k,\Delta}^\rho(Q).$$

For the converse containment, it is necessary and sufficient to show that every $f \in$

$\pi_{k,\Delta}^\rho(Q)$ can be extended to an element of $\pi_{k,\Delta}^\rho$. By Proposition 2, this is established once we show that, for each $v \in V_Q$, $\tau_S(\langle k, v \rangle)|_Q$ can be extended to a subset of $\pi_{k,\Delta}^\rho$, and this is obvious as long as $v_v = (\rho, \rho, \rho)$. This leaves three cases:

(i) $v_v = (-1, \rho, \rho)$. Then, for $i=1, \dots, \rho+1$,

$$C_i := C_{\zeta_1, \dots, \zeta_{i+l+1}}$$

involves the direction d_1 $l+2-i$ times, i.e., more than $l-\rho$ times, hence fails to be in C^ρ across $\mathbb{R}_+ d_1$. Recall from (4) that the restriction RC_i of C_i given by

$$(RC_i)(t) := C_i(t, 1-t), \text{ all } t \in \mathbb{R},$$

is a scaled univariate B-spline involving just the three knots 0, 1/2, and 1, and the

latter two no more than $l-p$ times. We can therefore write RC_1 on $[0,1]$ as a linear combination of the truncated powers

$$(1/2 - \cdot)_+^x, (1 - \cdot)_+^x, r=p+1, \dots, l.$$

Since C_1 is homogeneous of degree l , this implies that, on $\mathbb{R}_+^2$, C_1 itself is a linear combination of the truncated powers

$$T_{i,l,r}: x \mapsto (d_1 x)^{l-r} (d_1^* x)_+^r, \quad (8)$$

$i = 2, 3$, and $r = p+1, \dots, l$, with

$$d_2^* := e_1, \quad d_3^* := e_1 - e_2,$$

and these truncated powers are all in $\pi_{k,\Delta}^p$. We conclude that

$$\text{on } \mathbb{R}_+^2, \quad S(\langle k, v_v \rangle) \subseteq S(\langle k, \rho, \rho, \rho \rangle) + \text{span}\{T_{i,l,r}: i=2,3; \rho < r \leq l \leq k\}. \quad (9)$$

(ii) $v_v = (\rho, -1, \rho)$. In this case, we conclude that we can write the offending cone splines as linear combinations of the truncated powers

$$x \mapsto (d_1 x)^{l-r} (d_1^* x)_+^r \quad \text{and} \quad x \mapsto (d_3 x)^{l-r} (-d_3^* x)_+^r,$$

$\rho < r \leq l \leq k$, with

$$d_1^* := e_2.$$

This implies that,

$$\text{on } \mathbb{R}_+^2, \quad S(\langle k, v_v \rangle) \subseteq S(\langle k, \rho, \rho, \rho \rangle) + \text{span}\{T_{i,l,r}: i=1,3; \rho < r \leq l \leq k\} + \pi_k, \quad (10)$$

the last summand because the function $x \mapsto (d_3 x)^{l-r} (-d_3^* x)_+^r$ is in $\text{span}\{T_{3,l,r}\} + \pi_k$.

(iii) $v_v = (-1, -1, \rho)$. For this case,

$$S(l, v_v) = H_l \cap \pi \quad \text{on } \mathbb{R}_+^2$$

for $l < \rho$. For $l > \rho$, C_1 has either d_1 or d_2 but never both appearing more than $l-p$ times. This implies that

$$S(\langle k, v_v \rangle) \subseteq \pi_k + S(\langle k, \rho, \rho, \rho \rangle) + S(\langle k, -1, \rho, \rho \rangle) + S(\langle k, \rho, -1, \rho \rangle).$$

Thus, using the other cases, we find that,

$$\text{on } \mathbb{R}_+^2, \quad S(\langle k, v_v \rangle) \subseteq S(\langle k, \rho, \rho, \rho \rangle) + \text{span}\{T_{i,l,r}: i=1,2,3; \rho < r \leq l \leq k\} + \pi_k. \quad (11)$$

This establishes most of

Theorem 2. Let $Q = [0, M+1] \times [0, M+1]$. Then

$$\pi_{k,\Delta}^p(Q) = \pi_{k,\Delta}^p|_Q. \quad (12)$$

Further, on Q ,

$$\pi_{k,\Delta}^p = \pi_k + T + S \quad (13)$$

with

$$T := \text{span}\{ T_{i,r,\ell}(\cdot - Z d_i) : i=1,2,3; \rho < r \leq \ell \leq k \} \quad (14)$$

and

$$S := \text{span}\{ M_Z(\cdot - Z_+^2) : Z = (\zeta_1^\ell, \dots, \zeta_{1+\ell+1}^\ell), i=1, \dots, 3(\ell-\rho)-\ell-1; \ell \leq k \}, \quad (15)$$

and

$$(\zeta_i^\ell) := (d_1; \ell-\rho, d_3; \ell-\rho, d_2; \ell-\rho).$$

Proof. We only need to prove (13). But this follows from (12), from Proposition 2, and from (9)-(11), using (1) to convert the cone splines into linear combinations of corresponding box splines. |||

It is easy (but perhaps not all that useful) to obtain from Theorem 2 a spanning set of the same form for all of $\pi_{k,\Delta}^p$. Let $\Omega_1, \Omega_2, \Omega_3$ be the three domains into which $\mathbb{R}^2$ is subdivided by the three rays $\mathbb{R}_+ d_1, \mathbb{R}_+ d_2$, and $\mathbb{R}_- d_3$, with $\Omega_1 = \mathbb{R}_+^2$. For given $f \in \pi_{k,\Delta}^p$, we may choose by Theorem 2 an $f_1 \in \pi_k + T + S$ which agrees with f on Ω_1 . The function $f - f_1$ is in $\pi_{k,\Delta}^p$ and vanishes on Ω_1 , hence vanishes to order ρ on $\partial\Omega_1$. In effect, the component $f_{1,T}$ of f_1 from $\pi_k + T$ insures that $f - f_{1,T}$ vanishes to order ρ on $\partial\Omega_1$. This makes it possible to write $f - f_{1,T}$ on Ω_1 as an element $f_{1,S}$ of the span of the box splines listed in (15). An analogous argument therefore establishes the existence of an element $f_{2,T}$ in

$$\begin{aligned} & \text{span}\{ x \mapsto (d_i(x - Z_- d_i))^{k-r} (-d_i^* x)_+^r : i=1,2; \rho < r \leq \ell \leq k \} \\ & \subseteq \pi_k + \text{span}\{ T_{i,\ell,r}(\cdot - Z_- d_i) : i=1,2; \rho < r \leq \ell \leq k \} \subseteq \pi_k + T \end{aligned}$$

so that $g := f - f_{1,T} - f_{2,T}$ vanishes to order ρ on $\mathbb{R}_- d_3$. Since $f_{2,T}$ vanishes on Ω_1 , it follows that g vanishes to order ρ on $\partial\Omega_2$ and $\partial\Omega_3$, while $g = f - f_{1,T}$ on

Ω_1 , hence $g = f_{1,S}$ there. This makes it possible to write g on Ω_1 as a linear combination $f_{1,S}$ of box splines whose support is entirely in Ω_1 and which are obtained from the box splines listed in (15) by the linear change of variables which carries Ω_1 to Ω_1 . Because we chose the direction d_3 to appear exactly $l-p$ times in the box splines of degree l in (15), the box splines on Ω_1 so obtained are, in general, not translates of the ones in (15), but could be written as infinite linear combinations of such translates.

It follows that $f = f_{1,T} + f_{2,T} + f_{1,S} + f_{2,S} + f_{3,S}$. In this fashion, we can represent $w_{k,\Delta}^0$ as the span of w_k , the truncated powers listed in (14), and certain box splines.

3. Approximation order. In this section, we give upper and lower bounds for the approximation order of the smooth pp space

$$S := \pi_{k,\Delta}^p,$$

with Δ the three-direction mesh introduced in Section 1. The approximation order of S is, by definition, the integer m for which the following holds: For all sufficiently smooth functions f ,

$$\text{dist}(f, S_h) = O(h^m)$$

while, for some C^∞ -function f ,

$$\text{dist}(f, S_h) \neq o(h^m).$$

Here, the scale (S_h) of approximating spaces is generated from S by simple scaling,

$$S_h := \sigma_h(S),$$

with

$$(\sigma_h f)(x) := f(x/h), \text{ all } f, x, h.$$

Further,

$$\text{dist}(f, U) := \inf_{u \in U} \|f - u\|,$$

and $\|\cdot\|$ is the sup norm on some closed domain $\Omega \subseteq \mathbb{R}^2$,

$$\|f\| := \|f\|_{\infty, \Omega} := \sup_{x \in \Omega} |f(x)|.$$

In this definition, the approximation order depends on Ω , and rightly so. If, e.g., all the elements of S had their support in $\mathbb{R}_+^2$, then S_h would be entirely unable to approximate to functions having some support in $\mathbb{R}_+^2$, hence might well have different approximation order depending on whether or not Ω lies entirely in $\mathbb{R}_+^2$. For the specific spaces $\pi_{k,\Delta}^p$ or $S_\pm$ of interest here, though, the approximation order is the same for any closed and bounded Ω with some interior, since, for sufficiently small h , S_h is invariant under a suitable linear change of variables carrying one such Ω into another.

Here is a simple necessary condition for the approximation order to be m .

Lemma 3. Let U be a locally compact linear space of functions on $\mathbb{R}^n$, let Ω be a closed subset of $\mathbb{R}^n$ having 0 in its interior, and assume that

$$\text{dist}(f, U_h) = O(h^m)$$

for all sufficiently smooth f . Then

$$\pi_{m-1} \subseteq U.$$

Proof. Assume without loss that p is a polynomial homogeneous of degree $l < m$. By assumption, there exist const and $u_h \in U_h$ so that

$$\|p - u_h\|_{\omega, \Omega} < \text{const } h^m \text{ for all } h.$$

Therefore

$$\|p - w_h\|_{\omega, \Omega/h} < \text{const } h^{m-l} < \text{const } h,$$

with

$$w_h := u_h(h \cdot) / h^l \in U,$$

using the fact that

$$p(h \cdot) / h^l = p.$$

This shows that w_h converges to p uniformly on compact sets, hence $p \in U$. |||

This simple necessary condition is far from sufficient, obviously. For example, taking $U = \pi_{m-1}$, we obtain $U_h = U$, all h , and this scale has approximation order 0. It is not clear at present what other conditions one should add to get necessary and sufficient conditions for the approximation order to be at least m . Yet, Lemma 3 in conjunction with Theorem 2 leads to a close-to-exact estimate of the approximation order of $S = \pi_{k, \Delta}^0$ in case ρ is maximal for the given k .

A lower bound for the approximation order of S can already be found in [BD; Theorem 4] where it is shown that, for all sufficiently smooth f ,

$$\text{dist}(f, S_h) = O(h^{\rho+2}) \text{ in case } \rho = \rho(k) := \lfloor (2k-2)/3 \rfloor. \quad (1)$$

We saw already in Section 1 that this is as large a ρ ($= d-1$) as we can choose and still have box splines in S . Correspondingly, it is shown in [BD] that, for $\rho > \rho(k)$, S has approximation order 0.

Obviously, (1) provides also the lower bound $\rho(k)+2$ for the approximation order of S in case $\rho < \rho(k)$. But one would expect the approximation order of S to increase as

ρ decreases. This increase cannot be seen merely by studying the approximation order of $S_{\vec{s}} \subseteq S$ (the way (1) is obtained). For, [BH₁; Sec.6] shows that the approximation order of $S_{\vec{s}} := \text{span}(M_{\vec{s}}(\cdot - \vec{x}^2))$ is $d+1$, with

$$d = k + 1 - \max(r, s, t)$$

in case

$$\vec{s} = (d_1:r, d_2:s, d_3:t) \text{ and } r+s+t = k+2,$$

while, as discussed in Section 1, $S_{\vec{s}} \subseteq C^{(d-1)}$ and no better. This means that a decrease in ρ increases the number of different box splines $M_{\vec{s}}$ in S , but the approximation order of the additional subspaces $S_{\vec{s}}$ is less than the approximation order of those already in S when ρ is maximal, i.e., when $\rho = \rho(k)$.

It is the main goal of this section to provide an upper bound for the approximation order of S which, in the case $\rho = \rho(k)$, is close to the lower bound (1). The proof idea is simple: We show that the polynomials and truncated powers in S do not contribute to its approximation order. This means that the approximation order is already determined by the span

$$S_{\text{loc}}$$

of the box splines in S , and the discussion in Section 1 of the spaces $S_{\vec{s}}$ spanned by the translates of one such box spline $M_{\vec{s}}$ allows us to determine the maximal m for which π_{m-1} is contained in S_{loc} .

In view of (2.15), we set

$$S_{\text{loc}} := \sum_{l \leq k} \sum_i S_{\vec{s}_{i,l}} \quad (2a)$$

with

$$\vec{s}_{i,l} := (\zeta_1^l, \dots, \zeta_{i+l+1}^l) \text{ and } (\zeta_i) := (d_1:l-\rho, d_3:l-\rho, d_2:l-\rho). \quad (2b)$$

Proposition 3.1. If $\text{dist}(f, S_h) = O(h^m)$ for all sufficiently smooth functions f , then $\pi_{m-1} \subseteq S_{\text{loc}}$.

Proof. Let p be a sufficiently smooth function. By assumption, there exist const and $s_h \in S_h$, all h , so that

$$\|p - s_h\| < \text{const } h^m.$$

By Theorem 2, we can write

$$s_h = s'_h + s''_h$$

with

$$s'_h \in \sigma_h(S_{\text{loc}}) \quad \text{and} \quad s''_h \in \sigma_h(\pi_k + T)$$

and T the span of certain truncated powers,

$$T = \text{span}\{T_{i,l,r}(\cdot - \sum d_i) : i=1,2,3; \rho < r \leq k\},$$

where

$$T_{i,l,r}(x) := (d_i x)^{l-r} (d_i^* x)_+^r.$$

Therefore, for any positive η , the linear map

$$\delta_\eta := (\Delta_{nd_1} \Delta_{nd_2} \Delta_{nd_3})^{k+1}$$

with

$$\Delta_y f := f(\cdot + y) - f$$

carries all of $\sigma_h T$ to zero (since $k-\rho < k+1$), as well as all of π_k . We conclude that

$$\|\delta_\eta p - \delta_\eta s'\|_{\sigma, \Omega'} < 2^{3(k+1)} \text{const } h^m,$$

with $\Omega' := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 3(k-\rho)\eta\}$.

For $h \in \eta/\mathbb{N}$, we have $\delta_\eta s' \in \sigma_h(S_{\text{loc}})$. In view of Lemma 3, it therefore suffices to show that

$$\pi_{m-1} \subseteq \text{ran } \delta_\eta.$$

But this is obvious since, for any $y \neq 0$ and any r , Δ_y maps π_r onto π_{r-1} . |||

Theorem 3. For $\rho < \rho(k) := \lfloor (2k-2)/3 \rfloor$, the approximation order m of $S = \pi_{k,\Delta}^0$ satisfies $m \in [\rho(k)+2, m(k)]$, with

$$m(k) := \min\{2(k-\rho), k+1\}.$$

Proof. The lower bound for m is provided by (1). By Proposition 3.1, the upper bound is established once we show that

$$\pi_{m(k)} \subseteq S_{\text{loc}}.$$

Obviously, $\pi_{k+1} \not\subseteq S_{loc}$. Further, by (2), each box spline in S_{loc} involves each of the two directions d_1, d_2 at most $k-p$ times. This shows that $(D_1 D_2)^{k-p}$ carries $f \in S_{loc}$ to a distribution supported only on the meshlines of Δ . Consequently, S_{loc} cannot contain the particular polynomial $(\cdot)^{(k-p, k-p)}$ which is carried by $(D_1 D_2)^{k-p}$ to the nontrivial constant function $x \mapsto (k-p)!^2$. |||

The next proposition shows that the upper bound in Theorem 3 is sharp in the sense of Proposition 3.1.

Proposition 3.2. For S_{loc} as given by (2),

$$\max\{m : \pi_{m-1} \subseteq S_{loc}\} = m(k) := \min\{2(k-p), k+1\}. \quad (3)$$

Proof. The assertion is obvious for $k = 0, 1, 2$. Since we already know that $\pi_{m(k)} \not\subseteq S_{loc}$, we only have to show that $\pi_{m(k)-1} \subseteq S_{loc}$.

Consider the box splines listed in (2). Specifically, choose i so that $\Xi_i := \Xi_{i,k}$ involves direction d_1 as little as possible yet at least once. Let (r, s, t) be its direction multiplicities. Then $r > 1$. Further, $s+t = 2(k-p)$ as long as $k+2 > 2(k-p)$. In the contrary case, $r = 1$ and $s+t = k+1$. Hence, in either case,

$$r = k-p+1-i > 1 \text{ and } s+t = m(k). \quad (4)$$

We conclude with (1.6) that

$$(\cdot)^\alpha \in S_{\Xi_i} \text{ for } |\alpha| < m(k) \text{ and } \alpha(1) < r.$$

Indeed, each such polynomial is in $\ker D_1^r$, and, as a polynomial of degree $< m(k) = s+t$, trivially in $\ker D_2^{s+(D_1+D_2)^t}$.

Assume that we already know that

$$(\cdot)^\alpha \in S_{loc} \text{ for } |\alpha| < m(k) \text{ and } \alpha(1) < r+j \quad (5)$$

(as we do now for $j = 0$). If $k-p > r+j$, then, with (4), $k-p > k-p+1-i+j$, or, $i-j > 1$, hence we may consider Ξ_{i-j-1} . Let $(r+j+1, s', t')$ be its direction multiplicities. Then

$$s'+t' = m(k) - j - 1.$$

By the proof of Proposition 1, S_{Σ}^{i-j-1} contains an element of

$$\Phi_{\beta} := (\cdot)^{\beta/\beta!} + \text{span}\{(\cdot)^{\gamma} : |\gamma| = |\beta|, \gamma(1) < \beta(1)\} \quad (6)$$

provided $\beta(1) < r+j+1$ and $\beta(2) < s'+t' = m(k) - j - 1$. This implies that S_{Σ}^{i-j-1} contains an element of (6) for $\beta(1) = r+j$ and $|\beta| < m(k)$, and, on combining this with (5), we conclude that (5) holds also with j replaced by $j+1$.

This allows us to conclude that (5) holds for $r+j = k-\rho$. But then, by symmetry, also

$$(\cdot)^{\alpha} \in S_{\text{loc}} \text{ for } |\alpha| < m(k) \text{ and } \alpha(1) > k-\rho,$$

and this finishes the proof. |||

For $\rho = \rho(k)$, the bounds in Theorem 3 are particularly tight. If $i = -1, 0, 1$ and $k = 3\mu + i$, then $\rho(k) = 2\mu - 1 + i$. Therefore, for $k = 3\mu + 1$, the approximation order of $\pi_{k,\Delta}^{\rho(k)}$ equals $\rho(k)+2 = 2\mu+2$. For $k = 3\mu$, it lies between $\rho(k)+2$ and $\rho(k)+3$. The particular case $k = 3$ is discussed in detail in [BH₂] where it is shown that the approximation order of $\pi_{3,\Delta}^1$ is only 3 ($= \rho(3)+2$) rather than 4 ($= m(3)$). This is surprising and disappointing, since it shows that the simple mechanism on which Proposition 3.1 is based is not sufficient to predict the approximation order. One might be tempted to conclude from this example and from the case $k = 3\mu+1$ that the highest approximation order obtainable from an S_{Σ} in S determines the approximation order of S itself, at least when $\rho = \rho(k)$. The simple example $k = 2$ contradicts this. In this case, $\rho(k) = 0$, i.e., $\rho(k)+2 = 2$, yet local polynomial interpolation is well known to provide approximation order 3 from continuous piecewise parabolic functions on any triangulation Δ .

For $\rho < \rho(k)$, the lower bound stays constant while the upper bound increases until it reaches $k+1$, exactly at the point where ρ is small enough so that, already for a two-direction mesh Σ , $\pi_{k,\Sigma}^{\rho}$ contains finitely supported functions (see [BD]). We conjecture that the approximation order of $\pi_{k,\Delta}^{\rho}$ never differs from its upper bound $m(k)$ by more than 1. Proof of this conjecture would require construction of a local approximation scheme which makes use of much of S_{loc} rather than just one S_{Σ} .

4. Minimality of support. In this section, we show that box splines in

$$S := \pi_{k,\Delta}^0$$

may or may not have minimal support, even in the very restricted setting of maximal smoothness, i.e., when

$$\rho = \rho(k) ,$$

as we assume throughout this section. Precisely, we show that, for $k \equiv 1(3)$, the sole box spline in S has minimal support, while, for $k \equiv 0(3)$, the box splines in S do not. In the latter case, we show that a certain element first constructed in [Fr] has minimal support as does its 'flip'. In either case, the minimality allows us to conclude that translates of the minimal support elements span the subspace of all finitely supported elements of S . For the special cases $k = 3$ or 4 , this has also been proved in [CW₂]. The final case, $k \equiv 2(3)$, gives a hint of the complications awaiting those wishing to study the minimal support question for arbitrary ρ . We merely discuss the specific choices $k = 2, 5, 8$, and state some conjectures concerning the general case.

We make use of the notation

$$M_{r,s,t}$$

for the box spline M_{Ξ} with $\Xi = (d_1:r, d_2:s, d_3:t)$.

We say that f has minimal support in S if $f \in S$ and the only $g \in \mathbb{Z}$ having support strictly inside $\text{supp } f$ is $g = 0$. We say that f has unique minimal support in S if $f \in S$ and any $g \in S$ having support in $\text{supp } f$ is a multiple of f . Clearly, any f having unique minimal support in S has minimal support in S .

Theorem 4.1. Let $k = 3\mu - 2$ and $\rho = \rho(k) (= 2\mu - 2)$, and set $S_1 := \pi_{k,\Delta}^0$. Then the box spline $M := M_{\mu,\mu,\mu}$ has unique minimal support in S_1 .

The proof depends on the following lemma, for which we recall the abbreviation

$$Q_v := [v(1), v(1)+1] \times [v(2), v(2)+1]$$

used in Section 2 for the particular mesh square whose lower left corner is the vertex v .

Lemma 4.1. Let $\Omega := \text{conv}\{0, jd_1, jd_1+d_3, d_2\}$, i.e., $\Omega = \bigcup_{v=0}^j \Omega_{(v,0)} \cap \theta$,
with θ the triangle $\text{conv}\{jd_1, jd_1+d_3, (j+1)d_1\}$. Let $X :=$
 $\{f|_{\Omega} : f \in S_1, \text{supp } f \subseteq \{x(2) > 1\} \cup \Omega\}$. Then $\dim X = (j+1-\mu)_+$.

Proof. Since $k = 3\mu - 2$ and $k - \rho = \mu$, we conclude from (2.5) that $S(K, \rho, \rho, \rho)$ is spanned by the single cone spline $C := C_{\bar{\alpha}}$ with $\bar{\alpha} := (d_1: \mu, d_2: \mu, d_3: \mu)$. Thus, from the argument for Proposition 2,

$$X = \{f \in \text{span}(C(\cdot - (v, 0)))_{v=0}^j : f|_{\theta} = 0\}.$$

The cone spline C is homogeneous of degree $k = 3\mu - 2$ and vanishes to exact order $\rho = 2\mu - 2$ along $R_+ d_1$. Therefore

$$C(x - (v, 0)) = c(x - (v, 0))^{(\mu-1, 2\mu-1)} + o(x(2)^{2\mu-1}) \text{ for } x \in \theta$$

for some nonzero c . The condition $f|_{\theta} = 0$ therefore implies the condition

$$\sum_{v=0}^j a_v (x(1) - v)^{\mu-1} = 0 \text{ for } x \in \theta \quad (1)$$

in case $f = \sum_{v=0}^j a_v C(\cdot - (v, 0))$. Since the (univariate) polynomials $(\cdot - v)^{\mu-1}$, $v=0, \dots, \mu-1$, are linearly independent over any open set, (1) constitutes $\min\{j+1, \mu\}$ independent conditions on the coefficient vector (a_v) , and therefore

$$\dim X \leq (j+1 - \mu)_+.$$

The reverse inequality follows from the fact that, by Proposition 1, the box splines $M(\cdot - (v, 0))$, $v=0, \dots, j-\mu$, are independent over Ω and their restriction to Ω lies in X . |||

Corollary. If $f \in S_1$ has its support in R_+^2 , and its support in $[0, M]^2$ lies
between the rays $(\mu-1)d_1 + R_+ d_3$ and $\mu d_2 + R_+ d_3$, then f vanishes on all of $[0, M]^2$.

Proof. The given domain lies in the union of the sequence

$$\Omega_0, \Omega^0, \Omega_1, \Omega^1, \dots, \Omega_{M-1} \quad (2)$$

of sets

$$\begin{aligned} \Omega_v &:= (v, v) + \text{conv}\{0, (\mu-1)d_1, (\mu-1)d_1+d_3, d_2\} \\ \Omega^v &:= (v, v+1) + \text{conv}\{0, (\mu-1)d_2, (\mu-1)d_2+d_3, d_1\} \end{aligned}$$

(as illustrated in Figure 4.1 for $\mu = 2$ and $M = 4$) to which we can apply Lemma 4.1 with $j = \mu - 1$ in sequence, in order to conclude, step by step, that f must also vanish on each set in the sequence. |||

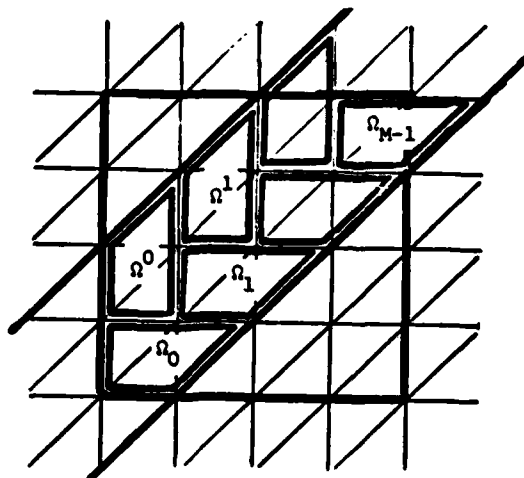


Figure 4.1

Proof of Theorem 4.1. Assume that $\text{supp } g \subseteq \text{supp } M$ for some $g \in S_1$. Lemma 4.1, with $j = \mu$, implies that $g = cM$ on $\Omega := \text{conv}\{(\mu-1, 0), (\mu, 0), (2\mu, \mu), (2\mu, \mu+1)\}$ for some scalar c . Thus $f := g - cM$ has support in the hexagon $\text{supp } M \setminus \Omega$. This hexagon lies in a domain of the type described in the corollary to Lemma 4.1, thus allowing the conclusion that $g = cM$. |||

The unique minimality of the support of M and its translates implies that they form a basis for the locally supported functions in S_1 .

Proposition 4.1. For given convex Ω , $\{M(\cdot - v) : v \in \mathbb{Z}^2, \text{supp } M(\cdot - v) \subseteq \Omega\}$ is a basis for

$$S_1[\Omega] := \{f \in S_1 : \text{supp } f \subseteq \Omega\}.$$

The proof of this corollary is analogous to the slightly more complicated proof of Proposition 4.2 below and is therefore omitted.

We now consider the slightly more complicated case $k = 3\mu - 3$ with $\rho = \rho(k) = 2\mu - 3$.

Set

$$S_0 := \pi_{k,\Delta}^0 = \pi_{3\mu-3,\Delta}^{2\mu-3}.$$

The box splines in S_0 do not have minimal support. But an element N_μ of unique minimal support in S_0 is given by the rule

$$N_1(x) := \begin{cases} 1 & \text{if } x \in \text{conv}\{0, d_1, d_3\} \\ 0 & \text{otherwise} \end{cases}, \quad (3)$$

$$N_{\nu+1} := N_\nu * M_{1,1,1} \quad \text{for } \nu \in \mathbb{N}.$$

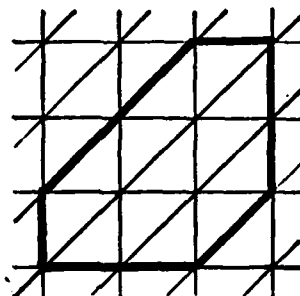


Figure 4.2

Here, $*$ indicates convolution. The function N_μ seems to have been considered first by Frederickson [Fr] and later, independently, by Sabin [Si], and thence in [Sl] and [Fa]. While the support of N_μ has some symmetry, it is asymmetric with respect to Δ . Figure 4.2 shows the support of N_2 . In general, the support of N_μ is circumambulated by walking alternatively μ and $\mu-1$ steps in the directions $d_1, d_3, d_2, -d_1, -d_3, -d_2$, starting at the origin. Because of the asymmetry, the element N'_μ given by the rule

$$N'_\mu(x) := N_\mu(x(2), x(1)) \quad (4)$$

is essentially different from N_μ . Together, they provide a local basis since they are closely related to the box splines in S_0 : Convolving the obvious identities

$$N_1 + N'_1 = M_{1,1,0}, \quad N_1 + N'_1(-d_1) = M_{1,0,1}, \quad N_1(-d_2) + N'_1 = M_{0,1,1}$$

with $M_{1,1,1}$, we obtain the identities

$$\begin{aligned} N_\mu + N'_\mu &= M_{\mu,\mu,\mu-1} \\ N_\mu + N'_\mu(-d_1) &= M_{\mu,\mu-1,\mu} \\ N_\mu(-d_2) + N'_\mu &= M_{\mu-1,\mu,\mu} \end{aligned} \quad (5)$$

Theorem 4.2. Let $k = 3\mu-3$ and $\rho = \rho(k) = 2\mu-3$, and let $S_0 := \pi_{k,\Delta}^\rho$. Then N_μ given by (3) has unique minimal support in S_0 .

The proof of Theorem 4.2 is based on the following lemma and its corollaries.

Lemma 4.2. Let Ω and X be as in Lemma 4.1, but with S_1 replaced by S_0 . Then

$$\dim X = (j+1 - \mu)_+ + (j+2 - \mu)_+.$$

Proof. Since $k = 3\mu-3$ and $k-\rho = \mu$, we conclude from (2.5) that $S(k, \rho, \rho, \rho)$ is spanned by the two cone splines C_0, C_1 corresponding to the direction multiplicities $(\mu, \mu-1, \mu)$ and $(\mu-1, \mu, \mu)$, respectively. Thus, from the argument for Proposition 2,

$$X = \{f \in \text{span}\{C_1(\cdot - (v, 0))\}_{i=0, v=0}^j : f|_\theta = 0\},$$

with $\theta := \text{conv}\{jd_1, jd_1+d_3, (j+1)d_1\}$. Because C_1 is homogeneous of degree $3\mu-3$ and vanishes to exact order $\rho+1 = 2\mu-3 + 1$ along $\mathbb{R}_+d_1$, we have

$$\begin{aligned} C_0(x-(v, 0)) &= c_0(x-(v, 0))^{(\mu-1, 2\mu-2)} + d_0(x-(v, 0))^{(\mu-2, 2\mu-1)} + o(x(2)^{2\mu-1}) \\ C_1(x-(v, 0)) &= c_1(x-(v, 0))^{(\mu-2, 2\mu-1)} + o(x(2)^{2\mu-1}) \end{aligned}$$

for $x \in \theta$ and some $c_0, c_1 \neq 0$. The condition

$$f := \sum a_{vi} C_i(\cdot - (v, 0)) = 0 \text{ on } \theta$$

therefore implies that

$$\begin{aligned} \sum_{v=0}^j a_{v0} (x(1) - v)^{\mu-1} &= 0 \\ \sum_{v=0}^j (a_{v0} d_0 + a_{v1} c_1) (x(1) - v)^{\mu-2} &= 0 \end{aligned} \quad \text{for } x \in \theta.$$

These are $\min\{j+1, \mu\} + \min\{j+1, \mu-1\}$ linearly independent conditions on the $2(j+1) -$ vector (a_{vi}) of coefficients and therefore

$$\dim X \leq (j+1 - \mu)_+ + (j+2 - \mu)_+.$$

The reverse inequality follows since

$$N_\mu(\cdot - (v, 0)), v=0, \dots, j-\mu, \text{ and } N'_\mu(\cdot - (0, v)), v=0, \dots, j+1-\mu$$

are independent over Ω and their restriction to Ω lies in X . Their linear independence follows from the fact that, by (5), N_μ and N'_μ agree near the origin with C_0 and C_1 , respectively. |||

Corollary 1. Let Ω and θ be as in the lemma, but with $j = \mu-1$, and let Ω' and θ' be their image under the 'flip' $x \mapsto (x(2), x(1))$. If $f \in S_0$ has support in $\{x(1), x(2) > 1\} \cup \Omega \cup \Omega'$, then f vanishes already on $\Omega \cup \Omega'$.

Proof. Much as in the proof of Lemma 4.2, we conclude that, on $\Omega \cup \Omega'$,

$$f = a_{00}C_0 + a_{01}C_1 + \sum_{\substack{0 \leq v < \mu \\ i=0,1}} (a_{vi}C_i(\cdot - (v, 0)) + a'_{vi}C_i(\cdot - (0, v)))$$

subject to the conditions that

$$\sum_{v < \mu} a_{v0}(x(1) - v)^{\mu-1} = 0 \quad \text{for } x \in \theta \quad (6a)$$

$$\sum_{v < \mu} (a_{v0}d_0 + a_{v1}c_1)(x(1) - v)^{\mu-2} = 0 \quad \text{for } x \in \theta \quad (6b)$$

for some $c_i \neq 0$, while, with $a'_{0i} := a_{0i}$, $i=0,1$, also

$$\sum_{v < \mu} a'_{v1}(x(2) - v)^{\mu-1} = 0 \quad \text{for } x \in \theta' \quad (6'a)$$

$$\sum_{v < \mu} (a'_{v1}d_0 + a'_{v0}c_1)(x(2) - v)^{\mu-2} = 0 \quad \text{for } x \in \theta'. \quad (6'b)$$

Note the reversal in the role of the second subscript, due to the fact that

$$C_i(x) = C_{i-1}(x(2), x(1)).$$

We conclude from (6a) that $a_{v0} = 0$, all v , and from (6'a) that $a'_{v1} = 0$, all v . In particular, $a'_{01} = a_{01} = 0$. Therefore (6b) implies that also $a_{v1} = 0$, all v , and, likewise, (6'b) implies that $a'_{v0} = 0$, all v . |||

Corollary 2. If $f \in S_0$ has its support in $\mathbb{R}_+^2$ and its support in $[0, M]^2$ lies between the rays $(\mu-1)d_1 + \mathbb{R}_+ d_3$ and $(\mu-1)d_2 + \mathbb{R}_+ d_3$, then $f = 0$ on $[0, M]^2$.

Proof. The given domain is contained in the union of the sequence

$$\Omega_0, \Omega_1, \dots, \Omega_{M-1}$$

of sets

$$\Omega_v := (v, v) + \Omega \cup \Omega',$$

with $\Omega \cup \Omega'$ as in Corollary 1, to which we can apply Corollary 1 in sequence, in order to conclude that f must vanish on each Ω_v . |||

Proof of Theorem 4.2. Assume that $\text{supp } g \subseteq \text{supp } N_\mu$ for some $g \in S_0$. By Lemma 4.2, $g = cN_\mu$ on $\Omega := \text{conv}\{(\mu-1, 0), (\mu, 0), (2\mu-1, \mu-1), (2\mu-1, \mu)\}$. This implies that $f := g - cN_\mu \in S_0$ has support in the domain described in Corollary 2 to Lemma 4.2, hence must be zero. |||

We now show that the elements with unique minimal support form a basis for all locally supported elements of S_0 .

Proposition 4.2. For given convex Ω , a basis for $S_0[\Omega] := \{f \in S_0 : \text{supp } f \subseteq \Omega\}$ is provided by the collection of all $N_\mu(-v), N'_\mu(-v)$, $v \in \mathbb{Z}^2$, in $S_0[\Omega]$.

Proof. Assume without loss that $\Omega \subseteq [0, M]^2$ and let $f \in S_0[\Omega]$ be given. Since N_μ, N'_μ agree near the origin with the respective cone splines C_0, C_1 , the argument for Proposition 2 leads to the conclusion that there is a unique linear combination

$$g := \sum_{v \in [0, M-2\mu+1]^2} (a_v N_\mu(-v) + a'_v N'_\mu(-v)) \quad (7)$$

which agrees with f on $[0, M-2\mu+2]^2$. This implies that $f-g$ has support only on $[0, M]^2 \setminus [0, M-2\mu+2]^2$. Application of Corollary 2 to Lemma 4.2 therefore proves that $f-g$

has no support in $\{x(1) < M-2\mu+2\}$ and, with this, a second application of that corollary shows that $f-g$ has no support in $\{M-2\mu+2 < x(1) < M\}$ either. Therefore $f = g$.

Let Ω' be the convex hull of the union of the supports of all the $N_\mu(-v)$ and $N'_\mu(-v)$ which appear in (7) with nonzero coefficients. Then Ω' is a polygon. We claim that $\Omega' \subseteq \Omega$. It is obvious from the construction of g that any lower left corner of Ω' must lie in $\text{supp } f$, hence in Ω . But since g is uniquely determined, this implies, using the mesh symmetries, that all six kinds of corners of Ω' lie in Ω , hence so does Ω' . |||

Finally, we consider the irregular and rich case $k = 3\mu-1$, for which $\rho(k) = 2\mu-2$. Set

$$S_2 := \pi_{k,\Delta}^{\rho(k)} = \pi_{3\mu-1,\Delta}^{2\mu-2}.$$

There are three independent cone splines of degree k at a lower left corner, but, because $\rho(k)$ is so low, there is also an additional cone spline of degree $k-1$ in S_2 . This means that a search for a basis for $S_2[\Omega]$ would have to come up with four unique minimal support elements per vertex. This, as it turns out, is not possible if we stick with the definition of "minimal support" given earlier.

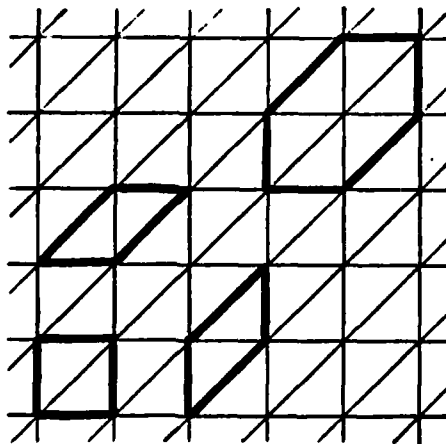


Figure 4.3

Already the case $k = 2$ of continuous parabolic splines provides the necessary illustrations: A suitable basis for $S_2[\Omega]$ is provided by the translates of four functions whose supports are drawn in Figure 4.3. These functions are obtainable as the Lagrange functions of standard local parabolic interpolation (at the vertices and the edge midpoints of each triangle). The first three have unique minimal support. But the fourth function's support is made up of six triangles and could accommodate each of the other three's much smaller support, hence it fails to be minimal.

In this case and others mentioned later, it is possible to recapture unique minimality of support by referring to the support of the associated B(ernstein or -ezier)-net of the pp functions instead. In any case, the support of the various box splines of degree 2 in S_2 is far from minimal since it contains ten triangles.

The next case, $k = 5$, hence $\rho(k) = 2$, provides the additional unhappy surprise that, in this case, S_2 contains an element supported on just one hexagon (i.e., on six triangles). This element occurs already in $[S_1]$. Because its support coincides with that of $M_{1,1,1}$, this element cannot be obtained from a parabolic one by convolution. The same is true of the next two "minimal support" elements whose supports coincide with that of N_μ and N'_μ respectively (see Figure 4.2). The fourth "minimal support" element is derived from, and has the same support as, $M_{2,2,2}$.

The next case, $k = 8$, is easy since its four "minimal support" elements can be obtained from those for $k = 5$ by convolution with $M_{1,1,1}$.

This pattern repeats: For odd μ , the four "minimal support" elements can be obtained from the preceding case by convolution with $M_{1,1,1}$. For even μ , enough local degrees of freedom have been accumulated to make possible elements in S_2 of yet smaller support than is had by the elements obtained from the preceding case by convolution with $M_{1,1,1}$.

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