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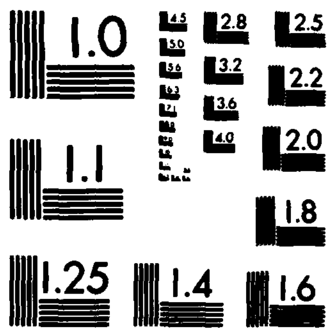
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$C^{1+\alpha}$ LOCAL REGULARITY OF WEAK SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT

It is demonstrated the local $C^{1+\alpha}$ nature of weak solutions of elliptic equations of the type (1.1) in the introduction under the degeneracy (or singularity) assumptions $[A_1]-[A_3]$.

AMS (MOS) Subject Classifications: 35J15, 35B65, 35B45, 35J70

Key Words: elliptic degenerate and singular equations

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

The equations treated in the paper represent a quasilinear generalization of the elliptic p.d.e. $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$, $p > 1$. Such an equation is degenerate for $|\nabla u|$ close to zero if $p > 2$ and is singular for $1 < p < 2$. It is demonstrated that the weak solutions are continuously differentiable and the derivatives are Hölder continuous.

These equations arise in the theory of non-Newtonian fluids. In view of this it is of interest to investigate the local smoothness of the solutions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

$C^{1+\alpha}$ LOCAL REGULARITY OF WEAK SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

E. DiBenedetto[1]

1. Introduction

The main result of this paper is the $C^{1+\alpha}$ nature of local weak solutions of elliptic equations of the type

$$-\operatorname{div} \vec{a}(x,u,\nabla u) + b(x,u,\nabla u) = 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^N$, $N \geq 2$, $\vec{a}$ is a map from $\mathbb{R}^{2N+1}$ into $\mathbb{R}^N$ and b maps $\mathbb{R}^{2N+1}$ into $\mathbb{R}$.

The point here is that we do not assume uniform ellipticity of the leading part of (1.1), which is allowed to be degenerate for certain values of $|\nabla u|$. In a precise way we assume (the summation notation is throughout used)

[A_1] $$a^k_{u_{x_j}} \xi_j \xi_k \geq \gamma_0(|u|)|\nabla u|^{p-2}|\xi|^2; \quad \xi \in \mathbb{R}^N, \quad p > 1$$

[A_2] $$|a^k_{u_{x_j}}| \leq \gamma_1(|u|)|\nabla u|^{p-2}, \qquad k,j = 1,2,\ldots,N$$

[A_3] $$|a^k_u, a^k_{x_j}| \leq \gamma_1(|u|)|\nabla u|^{p-1}, \quad k,j = 1,2,\ldots,N$$

[A_4] $$|b(x,u,\nabla u)| \leq \gamma_1(|u|)|\nabla u|^p .$$

The functions $\gamma_0(\cdot)$ and $\gamma_1(\cdot)$ are continuous in $\mathbb{R}^+$; $\gamma_0(\cdot)$ is decreasing and strictly positive and $\gamma_1(\cdot)$ is increasing.

Thus, the degeneracy of (1.1) is of the same nature as

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad p > 1 , \tag{1.2}$$

moreover (1.2) satisfies [A_1]-[A_3], $\forall p > 1$.

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The $C^{1+\alpha}$ local regularity of weak solutions of (1.2) for $p > 2$ has been investigated by Uhlenbeck [22], Ural'tzeva [23] and Evans [7]. While this paper was in preparation, Lewis [13] has informed me that he has obtained the same result for the case $1 < p < 2$. Lewis proof is quite different with respect to the one we give here.

The fact that (1.2) appears as the Euler equation of the variational integral

$$F(u) = \int_\Omega |\nabla u|^p dx, \quad p > 1$$

under certain side conditions, plays one way or another some role in the arguments of [13,22,23].

Due to the generality we consider, (1.1) need not be the Euler equation of a variational problem.

The proof reflects the following general idea. Consider a ball B around a point $x_0 \in \Omega$; if the set where (1.1) is degenerate is confined in a small portion of B, then somehow it can be controlled. If conversely $|\nabla u|$ is small in a large portion of B, then it can be compared with the radius of B.

This point of view, originated in 1957 with the work of De Giorgi [5], is now quite standard in dealing with degenerate (or singular) equations, and is the one that has been employed in [2,6,7,23]. The difference is the technical handling which might give richer or poorer informations on the solution.

Here we propose a substantially different technical version of this fact which permits a unitary treatment of the cases $1 < p < 2$ and $p > 2$, along with the full quasi-linear variational structure of (1.1). The proof of the boundedness of $|\nabla u|$ is based on controlling the growth of $|\nabla u|$ in $[A_2]-[A_4]$ with the oscillation of the solution u.

An advantage of our approach is that it does not require a different analysis for "degenerate" points and "regular" points. The behaviour of the solution around any point $x_0 \in \Omega$ is analyzed unitarily at once.

By a local weak solution of (1.1) we mean a function $u \in W^{1,p}_{loc}(\Omega)$ such that

$$\int_\Omega \{\vec{a}(x,u,\nabla u)\cdot\nabla\varphi + b(x,u,\nabla u)\varphi\}dx = 0 \tag{1.3}$$

for all $\varphi \in W^{1,p}(\Omega)$; supp $\varphi \subset \Omega$.

We will assume throughout that u is locally bounded in Ω. If $\gamma_1(s) \le \tilde{\gamma}_1 < \infty$, $\forall s \in R^+$, then the local boundedness is implied by Serrin's results [21] if the lower order terms $|b(x,u,\nabla u)| \le \gamma|\nabla u|^{p-1}$, and by the arguments of [12] if $|b(x,u,\nabla u)| \le \gamma|\nabla u|^{\sigma}$, $\sigma < p - N/(N + p)$.

Let us fix Ω' a subdomain of Ω such that $\overline{\Omega'} \subset \Omega$ and let $M = \operatorname{ess\,sup}_{\Omega'} |u|$. All our arguments are local in nature and will be carried over Ω', so that we might replace $\gamma_0(|u|)$ and $\gamma_1(|u|)$ in $[A_1]-[A_4]$ with $\gamma_0 \equiv \gamma_0(M)$, $\gamma_1 \equiv \gamma_1(M)$.

In order to justify the calculations to follow it is sufficient to have

$$|\nabla u|^{\frac{p-2}{2}} u_{x_i x_j} \in L_2^{loc}(\Omega) \ , \quad i,j = 1,2,\dots,N \ .$$

This fact would be implied by the assumptions

$$[B] \qquad |b_{u_{x_j}}| \le \gamma|\nabla u|^{p-1}, \ |b_u, b_{x_j}| \le \gamma|\nabla u|^p \ ,$$

modulo an argument involving difference-quotients, modeled on the results of [12] page 270-277. Here we will assume instead that u can be approximated by smooth solutions of regularized problems. This choice is motivated by simplicity of exposition and by the possible applications of our results to compactness arguments. Thus we will assume that

$[A_5]$ The local weak solution u under consideration, can be locally constructed as the weak $W^{1,p}(K)$ limit of a net $\{u_\varepsilon\}$ such that

$$\|u_\varepsilon\|_{\infty,K} \le M, \quad \forall \varepsilon > 0, \quad K \subset \Omega' \ ,$$

$u_\varepsilon \in C^2(\Omega')$, $\forall \varepsilon > 0$ and the u_ε are solutions (in the classical sense) of

$$(1.4) \qquad -\operatorname{div} \vec{a}_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) + b_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) = 0 \quad \text{in} \quad \Omega' \ .$$

Here $\vec{a}_\varepsilon, b_\varepsilon$ are regularizations of $\vec{a}$ and b, such that $\vec{a}_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$, $b_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \to \vec{a}(x,u,\nabla u)$, $b(x,u,\nabla u)$ weakly in $L_{p'}(\Omega')$, $\frac{1}{p} + \frac{1}{p'} = 1$, whenever $u_\varepsilon, \nabla u_\varepsilon \to u, \nabla u$ weakly in $L_p(\Omega')$ and uniformly in K. The regularizations $\vec{a}_\varepsilon, b_\varepsilon$ are so constructed as to satisfy

$[A_1]^\varepsilon$ $$a^k_{\varepsilon u_{x_j}}\xi_j\xi_k \geq \gamma_0[\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p-2}{2}}|\xi|^2; \quad \xi \in \mathbb{R}^N, \quad p > 1$$

$[A_2]^\varepsilon$ $$|a^k_{\varepsilon u_{x_j}}| \leq \gamma_1[\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p-2}{2}}, \qquad k,j = 1,2,\dots,N$$

$[A_3]^\varepsilon$ $$|a^k_{\varepsilon u}, a^k_{x_j}| \leq \gamma_1[\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p-1}{2}}, \qquad k,j = 1,2,\dots,N$$

$[A_4]^\varepsilon$ $$|b_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)| \leq \gamma_1[\varepsilon - |\nabla u_\varepsilon|^2]^{p/2} .$$

Such an approximation assumption is not restrictive in view of the available existence theory (see [10,12]).

To stress further this point, in Section 2 we will show that if [B] holds, locally bounded weak solution of (1.1) are locally unique and that in fact can be approximated as in $[A_5]$.

We can now state our main results.

<u>Theorem 1</u>: Let $u \in W^{1,p}_{loc}(\Omega) \cap L^{loc}_\infty(\Omega)$, $p > 1$ be a local weak solution of (1.1) under the assumptions $[A_1]$-$[A_5]$. Then $|\nabla u| \in L^{loc}_\infty(\Omega)$ and for every compact $K \subset \Omega'$, there exists a constant C_0 depending only upon γ_0,γ_1,p,N,M and $\operatorname{dist}(K,\partial\Omega')$ such that

$$\|\nabla u\|_{\infty,K} \leq C_0 .$$

<u>Theorem 2</u>: Let $u \in W^{1,p}_{loc}(\Omega) \cap L^{loc}_\infty(\Omega)$, $p > 1$ be a local weak solution of (1.1) under the assumptions $[A_1]$-$[A_5]$. Then $x \to \nabla u(x)$ is locally Hölder continuous in Ω', i.e. for every compact $K \subset \Omega'$, there exist constants C_1 and $\alpha \in (0,1)$, depending only upon γ_0,γ_1,p,N,M and $\operatorname{dist}(K,\partial\Omega')$, such that

$$|u_{x_i}(x) - u_{x_i}(y)| \leq C_1|x - y|^\alpha; \quad x,y \in K; \quad i = 1,2,\dots,N .$$

<u>Remark</u>: The theorems still hold if $b(x,u,\nabla u)$ is not homogeneous with respect to $|\nabla u|$. Let us suppose that $b(x,u,\nabla u) = b_0(x,u,\nabla u) + \varphi$ (or $|b(x,u,\nabla u)| \leq \gamma_1|\nabla u|^p + \psi$). Then if φ (or ψ respectively) belongs to $L^{loc}_q(\Omega)$, $q > p'N$, Theorems 1, 2 remain valid.

We will carry the proofs for the homogeneous case, and then it will be apparent how to modify the arguments to include the mentioned non-homogeneous situation.

Corollary: Let $u \in W^{1,p}_{loc}(\Omega)$, be a local weak solution of

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \varphi, \quad p > 1; \quad \varphi \in L_q^{loc}(\Omega)$$

$q > p'N$. Then $u \in C^{1+\alpha}_{loc}(\Omega)$.

Remarks: (i) Theorem 2 does not hold for $p = 1$ as shown by the following counterexample. The function

$$u(x_1,x_2) = \begin{cases} x_2 + 1, & x_2 < 0 \\ 2x_2 + 1, & x_2 > 0 \end{cases},$$

satisfies $\operatorname{div}(|\nabla u|^{-1}\nabla u) = 0$ in the weak sense in the disc $D \equiv \{x_1^2 + x_2^2 < 1\}$. However, the gradient is bounded but discontinuous. Geometrically the equation says that the mean curvature of each level set of u is zero. Existence for $\operatorname{div}(|\nabla u|^{-1}\nabla u) = 0$ is related to functions of least gradient (see [17,20]).

(ii) The $C^{1+\alpha}_{loc}$ regularity is the best possible (see [14]). The results of Uhlenbeck [22] and Ural'tzeva [23] hold also for systems. Because of the generality of (1.1), the $C^{1+\alpha}$ regularity for systems with the structure $[A_1]-[A_4]$ is not to be expected. Even for the nondegenerate case $(p = 2)$, the best results available are expressed in terms of "partial regularity" in the sense of Morrey [18,8,9].

An investigation of the partial regularity for systems is of interest; it falls however, beyond the scope of this work.

(iii) The equation (1.1) with the degeneracy $[A_1]-[A_4]$ has geometrical interest for $p > 2$ (see references in [22]) and arises in the theory of non-Newtonian fluids both for the case $p > 2$ (dilatant fluids) and $1 < p < 2$ (pseudo plastic fluids) [1,15,16].

(iv) The results of this paper have a parabolic counterpart. The subject will be treated in a forthcoming paper.

The theorems will be proved in terms of local equiboundedness and equi-Hölder continuity of the net $\{\nabla u_\varepsilon\}$.

The only estimate which we assume, uniform in ε is

$$\int_K |\nabla u_\varepsilon|^p dx \le C(K) .$$

The statement that a constant γ depends only upon the data, will mean that γ can be determined only in terms of $\gamma_0, \gamma_1, p, N, M$ and is independent of ε. With γ we will denote a generic positive constant depending only upon the data, which might be different in different contexts.

The paper is organized as follows. Section 2 contains some remarks on the approximation assumption $[A_5]$, along with some preliminary material to be used as we proceed. Section 3 contains the proof of local boundedness of the gradient, whereas in Section 4 we show the $C^{1+\alpha}_{loc}$ regularity for equations with restricted structure

$$-\mathrm{div}\ \vec{a}(\nabla u) = 0 ,$$

where $\vec{a}$ satisfies $[A_1]-[A_2]$. Finally, in Section 5 we recover the result for the general structure (1.1) by making use of the results of Campanato [3,4].

It is a pleasure to acknowledge conversations with W. Ziemer, during the preparation of the manuscript.

2. Remarks on local approximations

The remarks of this section reflect a modification of results and proofs of facts collected in [12] for the nondegenerate case.

Assumptions $[A_1]$-$[A_3]$ imply that in Ω'

$$(2.1)\qquad \begin{cases} \vec{a}(x,u,\nabla u)\cdot\nabla u \geq \dfrac{\gamma_0}{p-1}|\nabla u|^p, & p > 1 \\ |\vec{a}(x,u,\nabla u)| \leq \dfrac{\gamma_1}{p-1}|\nabla u|^{p-1} . \end{cases}$$

To prove (2.1) observe that

$$[a^k(x,u,\nabla u) - a^k(x,u,0)]u_{x_k} = [\int_0^1 \frac{\partial}{\partial t} a^k(x,u,t\nabla u)dt]u_{x_k} =$$

$$= \int_0^1 a^k_{u_{x_j}}(x,u,t\nabla u)u_{x_k}u_{x_j}dt \geq \frac{\gamma_0}{p-1}|\nabla u|^p .$$

For $x_0 \in \Omega'$ we let $B(R) \equiv \{|x - x_0| < R\}$, where R is so small that $B(R) \subset \Omega'$. Also with κ_N we denote the measure of the unit sphere in $\mathbb{R}^N$, so that $\text{meas}\, B(R) = \kappa_N R^N$.

Consider concentric balls $B(R)$, $B(R - \sigma R)$, $\sigma \in (0,1)$ and construct a smooth cutoff function $x \to \zeta(x)$ *such that* $\zeta \equiv 1$ *on* $B(R - \sigma R)$, $\text{supp}\,\zeta \subset B(R)$, $|\nabla\zeta| \leq (\sigma R)^{-1}$. In what follows $x \to \zeta(x)$, will always denote such a cutoff function. In the weak formulation (1.3) select test functions

$$\varphi = \pm (u - k)^{\pm}\exp[\lambda(u - k)^{\pm}]\zeta^2 .$$

Routine calculations (see [21,12]) and a suitable choice of λ yield the inequalities

$$(2.2)\qquad \|\nabla(u - k)^{\pm}\|^p_{p,B(R-\sigma R)} \leq \gamma(\sigma R)^{-p}\|(u - k)^{\pm}\|^p_{p,B(R)} ,$$

where γ is a constant depending only upon the data.

Inequalities (2.2) hold for every ball $B(R) \subset \Omega'$, every $\sigma \in (0,1)$ and every $-M \leq k \leq M$. By virtue of the results of [12] page 81-90, they imply the local Hölder continuity of u in Ω'. Therefore for every compact $K \subset \Omega'$ there exist constants C and $\beta \in (0,1)$ depending only upon the data and $\text{dist}(K,\partial\Omega')$ such that

$$(2.3)\qquad |u(x) - u(y)| \leq C|x - y|^{\beta}, \quad (x,y) \in K .$$

If in (2.2) we set $\sigma = \frac{1}{2}$ and $k = \inf_{B(R)} u$, in view of (2.3) we deduce that

$$(2.4) \qquad \int_{B(R)} |\nabla u|^p dx \le \gamma R^{N-p+\beta p} ,$$

for every ball $B(R) \subset K$, and for some constant γ depending only upon the data and $\mathrm{dist}(K,\partial\Omega')$.

Lemma 2.1: Let u be a local weak solution of (1.1) with $1 < p \le 2$. Then there exists a constant γ depending only upon the data such that for every function $\xi \in \overset{\circ}{W}{}^{1,p}(B(R))$

$$(2.5) \qquad \int_{B(R)} |\nabla u|^p \xi^2 dx \le \gamma R^{2\beta} \int_{B(R)} |\nabla u|^{p-2} |\nabla\xi|^2 dx .$$

Proof: The lemma follows from (2.4) via Lemma 1.3 and Lemma 1.4 of [12] page 59-61.

Lemma 2.2: Let u be a local weak solution of (1.1) with $p \ge 2$. Then there exists a constant γ depending only upon the data such that, $\forall \xi \in \overset{\circ}{W}{}^{1,p}(B(R))$

$$(2.6) \qquad \int_{B(R)} |\nabla u|^p \xi^2 dx \le \gamma R^{\beta} \int_{B(R)} |\nabla u|^{p-2} |\nabla\xi|^2 dx ,$$

provided that $R^{\beta} < \gamma_0/2\gamma_1 C$.

Proof: In the weak formulation (2.3) set $\varphi = [u - u(x_0)]\xi^2$ where x_0 is an arbitrary point of $B(R)$. Using (2.1) we obtain

$$\gamma_0 \int_{B(R)} |\nabla u|^p \xi^2 dx \le \gamma_1 \int_{B(R)} |\nabla u|^{p-1} [u(x) - u(x_0)] 2\xi |\nabla\xi| dx +$$

$$+ \gamma_1 \int_{B(R)} |\nabla u|^p [u(x) - u(x_0)] \xi^2 dx .$$

The lemma follows from an application of Cauchy inequality $ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$.

Lemma 2.3 (local uniqueness): Let u_1, u_2 be any two local weak solutions of (1.1), and assume that [B] holds. Then there exist a number R_0, depending only upon the data such that if $u_1 = u_2$ on $\partial B(R)$, $R \le R_0$, then $u_1 \equiv u_2$ in $B(R)$.

Proof: Writing the weak formulations for u_1 and u_2 and subtracting, we obtain

$$\int_{B(R)} \{[\vec{a}(x,u_1,\nabla u_1) - \vec{a}(x,u_2,\nabla u_2)] \cdot \nabla\varphi + [b(x,u_1,\nabla u_1) - b(x,u_2,\nabla u_2)]\varphi\} dx = 0$$

for all $\varphi \in \overset{\circ}{W}{}^{1,p}(B(R))$.

Select $\varphi = u_1 - u_2$, and observe that

$$[a_1^k(x,u_1,\nabla u_1) - a^k(x,u_2,\nabla u_2)](u_1 - u_2)_{x_k} =$$

$$= \int_0^1 a^k_{u_{x_j}}(x,tu_1 + (1-t)u_2, t\nabla u_1 + (1-t)\nabla u_2)(u_1 - u_2)_{x_k}(u_1 - u_2)_{x_j}dt +$$

$$+ \int_0^1 a^k_u(x,tu_1 + (1-t)u_2, t\nabla u_1 + (1-t)\nabla u_2)(u_1 - u_2)_{x_k}(u_1 - u_2)dt \geq$$

$$\geq \gamma_0\Big(\int_0^1 |t\nabla u_1 + (1-t)\nabla u_2|^{p-2}dt\Big)|\nabla\varphi|^2 -$$

$$- \gamma_1\Big(\int_0^1 |t\nabla u_1 + (1-t)\nabla u_2|^{p-1}dt\Big)|\varphi||\nabla\varphi| \; .$$

Treating similarly the lower order terms, we obtain after standard calculations

$$\int_{B(R)} [|\nabla u_1| + |\nabla u_2|]^{p-2}|\nabla\varphi|^2dx \leq \gamma(p)\int_{B(R)} [|\nabla u_1| + |\nabla u_2|]^p\varphi^2dx \; .$$

We majorize the integral on the right hand side by applying Lemma 2.1 if $p \in (1,2]$ and Lemma 2.2 if $p > 2$. In either case we obtain the existence of a constant γ depending only upon *the data, such that*

$$\int_{B(R)} [|\nabla u_1| + |\nabla u_2|]^{p-2}|\nabla\varphi|^2dx \leq \gamma R^\beta \int_{R(B)} [|\nabla u_1| + |\nabla u_2|]^{p-2}|\nabla\varphi|^2dx \; .$$

From this it follows that if $\gamma R^\beta < 1$, $\varphi = u_1 - u_2 \equiv 0$ in $B(R)$.

We are now in the position to construct local approximations to u if [B] holds. Let $\vec{a}_\varepsilon$ and b_ε be the regularizations of $\vec{a}$ and b satisfying $[A_1]^\varepsilon$-$[A_4]^\varepsilon$ and consider the boundary value problem

$$(2.7) \qquad \begin{cases} -\mathrm{div}\, \vec{a}_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) + b(x,u_\varepsilon,\nabla u_\varepsilon) = 0 & \text{in } B(R) \\ u_\varepsilon = u \quad \text{on } \partial B(R), & R \leq R_0 \; . \end{cases}$$

In view of $[A_1]^\varepsilon$-$[A_4]^\varepsilon$, the remarks above, and Theorem 8.7 of [12] page 311, (2.7) has a unique solution $u_\varepsilon \in W^{1,p}(B(R))$ such that

(a) $$u_\varepsilon \in C^2(B(R)) \cap C^\alpha(\overline{B(R)})$$

(b) $$\|u_\varepsilon\|_{\infty,B(R)} \leq M = \sup_{\Omega'}|u|, \quad \forall\varepsilon > 0 \; .$$

An uniform bound for $\int_{B(R)} |\nabla u_\varepsilon|^p dx$ in terms of M and $\int_{B(R)} |\nabla u_\varepsilon|^p dx$, is readily derived by standard theory (see [10,12]) and hence for a subnet (relabeled with ε) $u_\varepsilon \to w$ weakly in $W^{1,p}(B(R))$. After we show that $u_\varepsilon \in C^{1+\alpha}_{loc}$, we also have $u_\varepsilon \to w$ and $\nabla u_\varepsilon \to \nabla w$ uniformly on compacts of $B(R)$, and hence passing to the limit in the integral identities

$$\int_{B(R)} \{\vec{a}_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi + b_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi\}dx = 0$$

$\forall\varphi \in \overset{\circ}{W}{}^{1,p}(B(R))$, in view of the local uniqueness we obtain (1.3).

These remarks prove that it is not restrictive in order to prove our theorem, to work on the smooth approximations $\{u_\varepsilon\}$ introduced in $[A_5]$. Let therefore $\{u_\varepsilon\}$ be a net satisfying $[A_5]$; then considerations analogous to the ones above, imply that $\{u_\varepsilon\}$ are uniformly locally Hölder continuous in Ω', i.e. for every compact $K \subset \Omega'$, there exist constants C, $\beta \in (0,1)$ depending only upon the data and $\mathrm{dist}(K,\partial\Omega')$ such that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C|x - y|^\beta, \quad (x,y) \in K, \quad \forall\varepsilon > 0 .$$

<u>Lemma 2.4</u>: Let $B(R) \subset \Omega'$. There exists a constant γ depending only upon the data such that for every $\xi \in \overset{\circ}{W}{}^{1,p}(B(R))$

$$\int_{B(R)} [\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p+2}{2}} \xi^2 dx \le \gamma R^{2\beta} \int_{B(R)} \{[\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p-2}{2}} (\sum_{i=1}^N |\nabla u_{x_i}|^2)\xi^2 +$$

$$+ [\varepsilon + |\nabla u_\varepsilon|^2]^{p/2}|\nabla\xi|^2\}dx + \gamma \int_{B(R)} [\varepsilon + |\nabla u_\varepsilon|^2]^{p/2}\xi^2 dx .$$

<u>Proof</u>: For notational simplicity we will drop the subscript ε and set $w = (\varepsilon + |\nabla u_\varepsilon|^2)$. Consider the integral

$$I = \int_{B(R)} w^{p/2}|\nabla u|^2\xi^2 dx ,$$

and rewrite $|\nabla u|^2$ in the form

$$|\nabla u|^2 = \sum_{i=1}^N u_{x_i}(u - u(x_0))_{x_i} ,$$

where x_0 is any point in $B(R)$. Integrating by parts

$$I = -\int_{B(R)} [u(x) - u(x_0)]\{w^{p/2}\Delta u\xi^2 + pw^{\frac{p-2}{2}} u_{x_i}u_{x_\ell}u_{x_\ell x_i}\xi^2 + 2w^{p/2}u_{x_i}\xi\xi_{x_i}\}dx \le$$

$$\le \gamma R^\beta \int_{B(R)} \{w^{p/2}(\sum_i |\nabla u_{x_i}|^2)^{1/2}\xi^2 + 2w^{\frac{p+1}{2}}\xi|\nabla\xi|\}dx \le$$

$$\le \gamma R^\beta \int_{B(R)} \{\eta w^{\frac{p+2}{2}}\xi^2 + \eta^{-1}w^{\frac{p-2}{2}}(\sum_i |\nabla u_{x_i}|^2)\xi^2 +$$

$$+ \eta w^{\frac{p+2}{2}}\xi^2 + \eta^{-1}w^{p/2}|\nabla\xi|^2\}dx .$$

By taking $\eta = (2\gamma R^\beta)^{-1}$ we obtain

$$\int_{B(R)} w^{\frac{p+2}{2}}\xi^2 dx = \int_{B(R)} w^{p/2}|\nabla u|^2\xi^2 dx + \varepsilon\int_{B(R)} w^{p/2}\xi^2 dx \le$$

$$\le \frac{1}{2}\int_{B(R)} w^{\frac{p+2}{2}}\xi^2 dx + \gamma R^{2\beta}\int_{B(R)} \{w^{\frac{p-2}{2}}(\sum_i |\nabla u_{x_i}|^2)\xi^2 +$$

$$+ w^{p/2}|\nabla\xi|^2\}dx + \varepsilon\int_{B(R)} w^{p/2}\xi^2 dx .$$

The lemma follows.

For $s \in R$ set $A^+_{s,R} \equiv \{x \in B(R) | u(x) > s\}$; $A^-_{s,R} \equiv \{x \in B(R) | u(x) < s\}$.

The proof of the following lemma can be found in [5].

<u>Lemma 2.5</u>: Let $u \in W^{1,1}(B(R))$ and let k,ℓ be real numbers such that $\ell > k$. Then

$$(\ell - k)[\text{meas } A^+_{\ell,R}]^{1-\frac{1}{N}} \le \gamma \frac{R^N}{\text{meas}[B(R)\setminus A^+_{k,R}]}\int_{A^+_{k,R}\setminus A^+_{\ell,R}} |\nabla u| dx ,$$

where γ is a constant depending only on the dimension N.

3. Boundedness of the gradient

We prepare for the proof of Theorem 1 by establishing some integral inequalities needed in what follows. The subscript ε will be dropped and we set $w = (\varepsilon + |\nabla u_\varepsilon|^2)$.

Differentiating formally the approximating equation (1.4) with respect to x_i gives

$$-\operatorname{div}(\vec{a}_{u_{x_j}} u_{x_i x_j} + \vec{a}_u u_{x_i} + \vec{a}_{x_i}) + \frac{d}{dx_i} b(x,u,\nabla u) = 0 \tag{3.1}$$

in $\mathcal{D}'(\Omega)$. The equation holds in the sense of the integral identity

$$\int_\Omega \{[\vec{a}_{u_{x_j}} u_{x_i x_j} + \vec{a}_u u_{x_i} + \vec{a}_{x_i}]\cdot\nabla\varphi - b(x,u,\nabla u)\varphi_{x_i}\}dx = 0 \tag{3.2}$$

for all $\varphi \in \overset{\circ}{W}{}^{1,q}(\Omega)$, $q > 1$.

In (3.2) select the test functions

$$\varphi = u_{x_i}\eta(w)\zeta^2$$

where $\eta(\cdot)$ is a non-negative smooth function such that $\eta'(w) \geq 0$, and ζ is the standard cutoff function in $B(R)$ and $B(R - \sigma R)$. Adding over $i = 1,2,\ldots,N$, identities (3.2) with the indicated choice of φ and using $[A_1]^\varepsilon$-$[A_4]^\varepsilon$ we have the estimates:

$$\text{(i)} \quad \int_{B(R)} a^k_{u_{x_j}} u_{x_i x_j}[u_{x_i x_k}\eta(w)\zeta^2 + u_{x_i} w_{x_k}\eta'(w)\zeta^2 + 2u_{x_i}\eta(w)\zeta\zeta_{x_k}]dx$$

$$\geq \gamma_0 \int_{B(R)} w^{\frac{p-2}{2}} (\sum_i |\nabla u_{x_i}|^2)\eta(w)\zeta^2 dx + \frac{1}{2}\gamma_0 \int_{B(R)} w^{\frac{p-2}{2}} |\nabla w|^2\eta'(w)\zeta^2 dx -$$

$$- 2\gamma_1 \int_{B(R)} w^{\frac{p-2}{2}} |\sum_{i,j} u_{x_i} u_{x_i x_j}| \zeta\eta(w)|\nabla\zeta| dx .$$

From now on we will set

$$I(R) = \int_{B(R)} w^{\frac{p-2}{2}} (\sum_i |\nabla u_{x_i}|^2)\eta(w)\zeta^2 dx + \int_{B(R)} w^{\frac{p-2}{2}} |\nabla w|^2\eta'(w)\zeta^2 dx .$$

(ii) $$\int_{B(R)} [a_u^k u_{x_i} + a_{x_i}^k][u_{x_i x_k}\eta(w)\zeta^2 + u_{x_i} w_{x_k}\eta'(w)\zeta^2 + 2u_{x_i}\eta(w)\zeta\zeta_{x_k}]dx \le$$

$$\le \delta I(R) + \gamma_1^2\delta^{-1}\int_{B(R)} w^{\frac{p+2}{2}}[\eta(w) + (1+\sqrt{w})^2\eta'(w)]\zeta^2 dx +$$

$$+ \gamma_1 \int_{B(R)} w^{p/2}(1+\sqrt{w})\eta(w)\zeta|\nabla\zeta|dx .$$

(iii) $$\int_{B(R)} b(x,u,\nabla u)\frac{\partial}{\partial x_i}[u_{x_i}\eta(w)\zeta^2]dx \le \delta I(R) +$$

$$+ \gamma_1^2\delta^{-1}\int_{B(R)} w^{\frac{p+2}{2}}[\eta(w) + w\eta'(w)]\zeta^2 dx + 2\gamma_1\int_{B(R)} w^{\frac{p+1}{2}}\eta(w)\zeta|\nabla\zeta|dx .$$

Collecting these estimates as parts of (3.2) with the choice $\delta = \gamma_0/4$, we deduce the existence of a constant γ depending only upon the data, such that

$$I(R) \le \gamma\int_{B(R)} w^{\frac{p-2}{2}}|\sum_{i,j} u_{x_i}u_{x_i x_j}|\eta(w)\zeta|\nabla\zeta|dx + \tag{3.3}$$

$$+ \gamma\int_{B(R)} w^{\frac{p+2}{2}}[\eta(w) + (1+\sqrt{w})^2\eta'(w)]\zeta^2 dx +$$

$$+ \gamma\int_{B(R)} w^{\frac{p+1}{2}}\eta(w)\zeta|\nabla\zeta|dx + \gamma\int_{B(R)} w^{p/2}\eta(w)\zeta|\nabla\zeta|dx .$$

We will make particular choices of $\eta(\cdot)$. First select $\eta(w) = w^s$, $s > 0$, and estimate the terms in (3.3) as follows

(a) $$\gamma\int_{B(R)} w^{\frac{p-2}{2}}|\sum_{i,j} u_{x_i}u_{x_i x_j}|\eta(w)\zeta|\nabla\zeta|dx \le \delta\int_{B(R)} w^{\frac{p-2}{2}}(\sum_i |\nabla u_{x_i}|^2)\eta(w)\zeta^2 dx$$

$$+ \gamma^2\delta^{-1}\int_{B(R)} w^{p/2}\eta(w)|\nabla\zeta|^2 dx ,$$

(b) $$\int_{B(R)} w^{\frac{p+2}{2}}(1+\sqrt{w})^2\eta'(w)\zeta^2 dx = \int_{B(R)\cap[w>1]}(\text{——})\,dx + \int_{B(R)\cap[w<1]}(\text{——})\,dx \le$$

$$\le 4s\int_{B(R)} w^{\frac{p+2+2s}{2}}\zeta^2 dx + 4s\kappa_N R^N ,$$

$$\text{(c)} \qquad \int_{B(R)} w^{\frac{p+1}{2}} \eta(w)\zeta|\nabla\zeta|dx \le \frac{1}{2}\int_{B(R)} w^{\frac{p+2+2s}{2}} \zeta^2 dx + \frac{1}{2}\int_{B(R)} w^{\frac{p+2s}{2}} |\nabla\zeta|^2 dx .$$

These remarks in (3.3) yield

$$\text{(3.4)} \qquad I(R) \le \gamma(s+1)\int_{B(R)} w^{\frac{p+2+2s}{2}} \zeta^2 dx + \gamma\int_{B(R)} w^{\frac{p+2s}{2}} |\nabla\zeta|^2 dx + s\gamma\kappa_N R^N$$

for a new constant γ depending only upon the data and independent of s and R.

Next in (3.3) we will choose

$$\eta(w) = (w^{p/2} - k)^+ \equiv \max\{w^{p/2} - k; 0\}; \qquad k > 1 .$$

For simplicity we set $w^{p/2} = v$ so that $\eta(w) = (v-k)^+$, and estimate the parts of (3.3) as follows

$$\text{(d)} \qquad \int_{B(R)} w^{\frac{p-2}{2}} |\sum_{i,j} u_{x_i} u_{x_i x_j}|(v-k)^+\zeta|\nabla\zeta| \le \frac{1}{2}\int_{B(R)} w^{\frac{p-2}{2}} |\nabla w|(v-k)^+\zeta|\nabla\zeta| =$$

$$= \frac{1}{p}\int_{B(R)} |\nabla(v-k)^+|(v-k)^+\zeta|\nabla\zeta|dx \le \frac{2}{p}\delta\int_{B(R)} |\nabla(v-k)^+|^2 \zeta^2 dx +$$

$$+ \frac{4}{p}\delta^{-1}\int_{B(R)} (v-k)^{+^2}|\nabla\zeta|^2 dx ,$$

$$\text{(e)} \qquad \int_{B(R)} w^{\frac{p+2}{2}} [\eta(w) + (1+\sqrt{w})^2\eta'(w)]\zeta^2 dx \le \gamma(p)\int_{B(R)} w^{p+1}\chi(v > k)\zeta^2 dx ,$$

$$\text{(f)} \qquad \int_{B(R)} (w^{p/2} + w^{\frac{p+1}{2}})\eta(w)\zeta|\nabla\zeta|dx \le \gamma\int_{B(R)} w^{p+1}\chi(v > k)\zeta^2 dx +$$

$$+ \gamma\int_{B(R)} (v-k)^{+^2}|\nabla\zeta|^2 .$$

Consequently dropping the term involving $\eta(\cdot)$ in $I(R)$

$$(3.5)\qquad \int_{B(R)} w^{\frac{p-2}{2}} w^{\frac{p-2}{2}} |\nabla w|^2 \chi(v > k)\zeta^2 dx \le \frac{2}{p}\delta \int_{B(R)} |\nabla(v-k)^+|^2\zeta^2 dx +$$

$$+ \gamma(\delta)\int_{B(R)} (v-k)^{+^2}|\nabla\zeta|^2 dx + \gamma\int_{B(R)} w^{p+1}\chi(v > k)\zeta^2 dx .$$

Finally we observe that

$$\int_{B(R)} w^{\frac{p-2}{2}} w^{\frac{p-2}{2}} |\nabla w|^2 \chi(v > k)\zeta^2 dx = \frac{2}{p}\int_{B(R)} |\nabla(v-k)^+|^2\zeta^2 dx ,$$

and hence for $\delta = \frac{1}{2}$ we see that there exists a constant γ depending only on the data and independent of R and k such that

$$(3.6)\qquad \int_{B(R)} |\nabla(v-k)^+|^2\zeta^2 dx \le \gamma\int_{B(R)} (v-k)^{+^2}|\nabla\zeta|^2 dx +$$

$$+ \gamma\int_{B(R)} w^{p+1}\chi(v > k)\zeta^2 dx; \qquad k \ge 1 .$$

Inequalities (3.4) and (3.6) will be employed to prove the local boundedness of the gradient.

<u>Proposition 3.1</u>: $|\nabla u_\varepsilon| \in L_q^{loc}(\Omega)$, $\forall q \in [1,\infty)$ uniformly in ε.

<u>Proof</u>: By virtue of Lemma 2.4 applied with $\xi = w^{\frac{s}{2}}\zeta$ we have

$$\int_{B(R)} w^{\frac{p+2+2s}{2}} \zeta^2 dx \le \gamma R^\beta (1+s)^2 I(R) + \gamma(R^\beta + 1)\int_{B(R)} w^{\frac{p+2s}{2}} (\zeta^2 + |\nabla\zeta|^2) dx .$$

Combining this with (3.4), for a new constant γ we have

$$(3.7)\qquad \int_{B(R)} w^{\frac{p+2+2s}{2}} \zeta^2 dx \le \gamma R^\beta (1+s)^3 \int_{B(R)} w^{\frac{p+2+2s}{2}} \zeta^2 dx +$$

$$+ \gamma(1+s)^2\int_{B(R)} w^{\frac{p+2s}{2}} [\zeta^2 + |\nabla\zeta|^2] dx + \gamma(1+s)^2 \kappa_N R^N .$$

Let now $\{R_s\}$ be a decreasing sequence of numbers such that

$$R_0 = (2\gamma)^{-1/\beta}, \qquad R_s = R_0(1 + s)^{-3/\beta},$$

and consider the concentric balls $B(R_s)$ and $B(R_{s+1})$.

If $x \to \zeta_s(x)$ is a standard cutoff function in $B(R_s)$ which equals one in $B(R_{s+1})$ we have $|\nabla\zeta_s|^2 \le R_0^{-2}(s + 2)^{8/\beta}$ and therefore (3.7) implies

$$\int_{B(R_{s+1})} w^{\frac{p+2+2s}{2}} dx \le \gamma R_0^{-2}(s + 2)^{10/\beta} \int_{B(R_s)} w^{\frac{p+2s}{2}} dx + \gamma(1 + s)^2 \kappa_N R^N, \tag{3.8}$$

where γ is independent of R_0 and s.

Iterating over s, starting from $s = 0$, the proposition follows.

<u>Remark</u>: Inequalities (3.9) imply that a local bound for $\int_{K'} |w|^q dx$ over a compact $K' \subset \Omega'$ is obtained only in terms of the data and $\int_K w^{p/2} dx$ for $K' \subset K$.

<u>Proposition 3.2</u>: $|\nabla u_\varepsilon| \in L_\infty^{loc}(\Omega)$, and for a compact $K' \subset \Omega'$ the quantity $\|\nabla u_\varepsilon\|_{\infty,K'}$ is estimated only in terms of $\|\nabla u\|_{p,K}$ for a compact K containing K'.

<u>Proof</u>: Consider inequalities (3.6) and estimate the last integral as follows

$$\int_{B(R)} w^{p+1}\chi(v > k)dx \le \left[\int_{B(R)} w^{\frac{N(p+1)}{2-N\kappa}} dx\right]^{\frac{2-N\kappa}{N}} \left[\text{meas } A^+_{k,R}\right]^{1-\frac{2}{N}+\kappa}$$

where $\kappa \in \left(0, \frac{1}{N}\right]$ and $A^+_{k,R} \equiv \{x \in B(R) | v(x) > k\}$.

If Ω'' is a subdomain of Ω', by virtue of proposition 3.1, the quantity

$$\left[\int_{\Omega''} w^{N(p+1)/2-N\kappa} dx\right]^{\frac{2-N\kappa}{N}}$$

is bounded uniformly in ε by a constant γ depending only upon the data and $\text{dist}(\Omega'', \partial\Omega')$. Therefore if $B(R) \subset \Omega''$ from (3.6) we deduce

$$\int_{B(R-\sigma R)} |\nabla(v - k)^+|^2 dx \le \gamma(\sigma R)^{-2} \int_{B(R)} (v - k)^{+^2} dx + \gamma\left[\text{meas } A^+_{k,R}\right]^{1-\frac{2}{N}+\kappa}.$$

These inequalities are valid for all $B(R) \subset \Omega''$ and all $\sigma \in (0,1)$. The proposition now follows from Lemma (5.4) of [12] page 76.

We conclude this section by giving a simple proof of the *boundedness of the gradient* for equations with restricted structure

$$-\text{div}\, \vec{a}_\varepsilon(\nabla u_\varepsilon) = 0, \quad \forall \varepsilon > 0 \tag{3.9}$$

where

$$a^k_{\varepsilon u_{x_j}} \xi_k \xi_j \geq \gamma_0 [\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p-2}{2}} |\xi|^2$$

$$|a^k_{\varepsilon u_{x_j}}| \leq \gamma_1 [\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p-2}{2}}, \qquad p > 1 .$$

Equations with this structure include $\text{div}(|\nabla u|^{p-2}\nabla u) = 0$, $p > 1$.

Even though the gradient estimate is a particular case of Propositions 3.1 and 3.2, the simple constant-dependence typical of (3.9) will be needed in what follows. In particular in Proposition 3.1 use was made (via Lemma 2.4) of the Hölder continuity of u, whereas for (3.9) is our goal to find a bound for $\|\nabla u\|_{\infty,K}$ independent of the local properties of u.

<u>Proposition 3.3</u>: Let u_ε be a local weak solution of (3.9) in Ω. For every ball $B(R) \subset \Omega$ and every $0 < \delta < 1$ we have,

$$\|[\varepsilon + |\nabla u_\varepsilon|^2]\|^{p/2}_{\infty,B(R-\delta R)} \leq \gamma(\delta) R^{-N} \int_{B(R)} [\varepsilon + |\nabla u_\varepsilon|^2]^{p/2} dx, \quad \forall \varepsilon > 0 ,$$

where γ depends upon the data and δ only and is independent of R.

<u>Proof</u>: Taking the x_i derivative of (3.9) in the weak sense we have (the subscript ε is dropped)

$$\int_{B(R)} a^k_{u_{x_j}} u_{x_i x_j} \varphi_{x_k} dx = 0 \qquad \forall \varphi \in \overset{\circ}{W}{}^{1,p}(B(R)) . \tag{3.10}$$

Set $\varphi = u_{x_i} w^{\frac{\alpha}{2}} \zeta^2$, $\alpha > 0$, where $w = (\varepsilon + |\nabla u_\varepsilon|^2)$ and ζ is the standard cutoff function in $B(R)$ and $B(R - \sigma R)$. Standard calculation yield, for γ independent of α,

$$\int_{B(R)} \left|\nabla w^{\frac{p+\alpha}{4}}\right|^2 \zeta^2 dx \leq \gamma \int_{B(R)} w^{\frac{p+\alpha}{2}} |\nabla\zeta|^2 dx .$$

Setting

$$v = w^{p/4}, \qquad \theta = 1 + \frac{\alpha}{p}$$

the above can be rewritten as

$$\int_{B(R)} |\nabla v^{\theta}|^2 \zeta^2 dx \leq \gamma \int_{B(R)} (v^{\theta})^2 |\nabla \zeta|^2 dx .$$

Since α is an arbitrary positive number, θ is an arbitrary number larger than 1. The Moser iteration technique [19] now gives

$$\|v\|^2_{\infty,B(R-\delta R)} \leq \gamma(\delta) R^{-N} \int_{B(R)} v^2 dx$$

for any $0 < \delta < 1$. The Proposition is proved.

This proof applied to $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ seems more direct than the ones in [7,13,23].

4. Hölder continuity of the gradient

We start by proving that local weak solutions of

$$-\operatorname{div} \vec{a}_\varepsilon(\nabla u) = 0$$

are $C^{1+\alpha}_{loc}(\Omega)$. Here $\vec{a}_\varepsilon$ satisfy $[A_1]^\varepsilon - [A_2]^\varepsilon$ and the equation for the component u_{x_i} of ∇u is viewed in the weak sense (3.10). For $k \in \mathbb{R}$ in (3.10) set

$$\varphi = \pm (u_{x_i} - k)^{\pm}\zeta^2$$

to obtain

$$\int_{B(R)} w^{\frac{p-2}{2}} |\nabla(u_{x_i} - k)^{\pm}|^2 \zeta^2 \le \gamma \int_{B(R)} w^{\frac{p-2}{2}} (u_{x_i} - k)^{\pm^2} |\nabla\zeta|^2 dx \tag{4.1}$$

for γ depending only upon the data.

For the case $1 < p < 2$ another inequality will be needed.

In (3.10) set

$$\varphi_\eta = -[(|u_{x_i}| + \eta)^{p-2}u_{x_i} - k]^-\zeta^2, \qquad k \in \mathbb{R}^+ ,$$

where η is a small positive number which will be let $\to 0$.

Let us set

$$\psi_\eta = (|u_{x_i}| + \eta)^{p-2}u_{x_i}; \qquad \psi = |u_{x_i}|^{p-2}u_{x_i}$$

and notice that

$$(p-1)(|u_{x_i}| + \eta)^{p-2}\chi[\psi_\eta < k] \le \frac{\partial}{\partial u_{x_i}} \varphi_\eta \le (|u_{x_i}| + \eta)^{p-2}\chi[\psi_\eta < k] .$$

Setting $A^-_{k,R}(\eta) \equiv \{x \in B(R) | \psi_\eta < k\}$ and $A^-_{k,R} \equiv \{x \in B(R) | \psi < k\}$ from (3.10) with the indicated choice of test function we have

$$\gamma_0 \int_{A^-_{k,R}(\eta)} w^{\frac{p-2}{2}} |\nabla u_{x_i}|^2 (|u_{x_i}| + \eta)^{p-2}\zeta^2 dx \le \tag{4.2}$$

$$\le 2\gamma_1 \int_{B(R)} w^{\frac{p-2}{2}} |\nabla u_{x_i}|(\psi_\eta - k)^- \zeta |\nabla\zeta| dx .$$

Notice that

$$|\nabla u_{x_i}|^2(|u_{x_i}| + \eta)^{p-2} = \frac{4}{p^2}\,|\nabla(|u_{x_i}| + \eta)^{p/2}|^2$$

and that if $\eta < \varepsilon$

$$w^{\frac{p-2}{2}}\,|\nabla u_{x_i}| \le \frac{2}{p}\,w^{\frac{p-2}{4}}\,|\nabla(|u_{x_i}| + \eta)^{p/2}| .$$

Therefore from (4.2) it follows that

$$\int_{A^-_{k,R}(\eta)} w^{\frac{p-2}{2}}\,|\nabla(|u_{x_i}| + \eta)^{p/2}|^2\zeta^2 dx \le \gamma\int_{B(R)} (\psi_\eta - k)^{-2}|\nabla\zeta|^2 .$$

Letting $\eta \to 0$ and setting

$$v = |u_{x_i}|^{p/2}\,\text{sign}\,u_{x_i}$$

the above gives

$$(4.3)\qquad \int_{B(R)} w^{\frac{p-2}{2}}\left|\nabla\left(v - k^{\frac{p}{2(p-1)}}\right)^-\right|^2\zeta^2 dx \le \gamma\int_{B(R)} |(\psi - k)^-|^2|\nabla\zeta|^2 .$$

Inequalities (4.3) hold for every ball $B(R) \subset \Omega'$ and every $k \in \mathbb{R}^+$.

<u>Proposition 4.1</u>: Let $x_0 \in \Omega'$ and let R be so small that $B(2R) \subset \Omega'$. Set

$$\lambda = \frac{1}{2}\max_{1\le i\le N}\,\sup_{B(2R)}\,|u_{x_i}| .$$

There exists a number c_0 depending upon the data but independent of ε, R, λ, such that if for some $1 \le i \le N$

$$\text{meas}\{x \in B(2R)\,|\,u_{x_i}(x) < \lambda\} \le c_0 R^N ,$$

then

$$u_{x_i}(x) \ge \frac{\lambda}{4} , \qquad \forall x \in B(R) .$$

Analogously if

$$\text{meas}\{x \in B(2R)\,|\,u_{x_i}(x) > -\lambda\} \le c_0 R^N ,$$

then

$$u_{x_i}(x) \le -\frac{\lambda}{4} , \qquad \forall x \in B(R) .$$

Proof: Either $\varepsilon \geq \lambda^2$ or $\varepsilon < \lambda^2$. In the first case, recalling that $w = (\varepsilon + |\nabla u_\varepsilon|^2) \leq \varepsilon + N\lambda^2$ on $B(\rho)$, $\forall\rho \leq 2R$, from inequalities (4.1) written for the ball $B(\rho) \subset \Omega'$ we have

$$\int_{B(\rho)} |\nabla(u_{x_i} - k)^-|^2 \zeta^2 dx \leq \gamma \int_{B(\rho)} (u_{x_i} - k)^{-2} |\nabla\zeta|^2 dx , \tag{4.4}$$

for every $k \leq \lambda$ and for a new constant γ depending only upon the data.

In the second case we consider inequalities (4.1) for $p > 2$ written for $(u_{x_i} - k)^-$, $k \leq \lambda$ on the balls $B(\rho)$, $\rho \leq 2R$, and estimate the left hand side from below as follows

$$\int_{B(\rho)} w^{\frac{p-2}{2}} |\nabla u_{x_i}|^2 \chi(u_{x_i} < k)\zeta^2 dx \geq \int_{B(\rho)} |u_{x_i}|^{p-2} |\nabla u_{x_i}|^2 \chi(u_{x_i} < k)\zeta^2 dx$$

$$= \frac{4}{p^2} \int_{B(\rho)} |\nabla |u_{x_i}|^{p/2}|^2 \chi(u_{x_i} < k)\zeta^2 dx .$$

Setting

$$v = |u_{x_i}|^{p/2} \operatorname{sign} u_{x_i}, \quad \text{above gives}$$

$$\int_{B(\rho)} w^{\frac{p-2}{2}} |\nabla u_{x_i}|^2 \chi(u_{x_i} < k)\zeta^2 dx \geq \frac{4}{p^2} \int_{B(\rho)} |\nabla(v - h)^-|^2 \zeta^2 dx$$

for every $h \leq h_0 = \lambda^{p/2}$. Therefore, recalling the definition of λ, if $x \to \zeta(x)$ is the standard cutoff function in $B(\rho)$, $B(\rho - \sigma\rho)$ $\sigma \in (0,1)$, we have from (4.1) and $p > 2$

$$\int_{B(\rho-\sigma\rho)} |\nabla(v - h)^-|^2 dx \leq \gamma h_0^2 (\sigma\rho)^{-2} [\text{meas } A^-_{h,\rho}] , \tag{4.5}$$

for a new constant γ depending only upon the data and independent of $\varepsilon, \rho, \sigma, h$. We have also set

$$A^-_{h,\rho} \equiv \{x \in B(\rho) | v(x) < h\} .$$

For the case $1 < p < 2$, from (4.3) we deduce $\forall\rho \leq 2R$

$$[(N + 1)\lambda]^{p-2} \int_{B(\rho)} \left|\nabla\left(v - k^{\frac{p}{2(p-1)}}\right)^-\right|^2 \zeta^2 dx \leq \gamma \int_{B(\rho)} [|u_{x_i}|^{p-2} u_{x_i} - k]^{-2} |\nabla\zeta|^2 dx ,$$

so that if we choose $k < \lambda^{p-1}$ and denote with h any number smaller than $\lambda^{p/2}$ and $h_0 = \lambda^{p/2}$, we will have (4.5) for a new constant γ independent of $\varepsilon,\rho,\sigma,h$. Hence (4.5) holds both for the case $p > 2$ and $1 < p < 2$.

We will prove the proposition for the case $\varepsilon < \lambda^2$, i.e. in the case (4.5) (valid $\forall p > 1$) hold. Then it will be clear how to achieve the proof in the (simpler) case when (4.4) are verified.

Inequalities (4.5) are verified $\forall\sigma \in (0,1)$, $\forall\rho < 2R$ and every choice of $h < h_0 = \lambda^{p/2}$. For $n > 0$ integer, consider the balls $B(\rho_n)$, $B(\bar{\rho}_n)$ where

$$\rho_n = R + \frac{R}{2^n} ;$$

and construct smooth cutoff functions $x \to \zeta_n(x)$ such that $\zeta_n(x) \equiv 1$ on $B(\rho_{n+1})$; supp ζ_n $B(\rho_n)$; $|\nabla\zeta_n| < (\rho_n - \rho_{n+1})^{-1} < 2^{n+1}R^{-1}$.

We will use (4.5) over the pair of balls $B(\rho_{n+1})$ and $B(\rho_n)$ for the sequence of decreasing levels

$$h_n = h_0 - \frac{H}{4}\left(1 - \frac{1}{2^n}\right); \qquad H = \sup_{B(2R)} (v - h_0)^- .$$

Applying Lemma 2.5 to the function $u = -v$ for the levels $\ell = -h_{n+1}$, $k = -h_n$ over $B(\rho_{n+1})$ we have

$$\frac{H}{2^{n+3}} \left[\text{meas } A^-_{h_{n+1},\rho_{n+1}}\right]^{\frac{N-1}{N}} < \gamma \frac{(2R)^N}{\text{meas}\left[B(\rho_{n+1}) \setminus A^-_{h_n,\rho_{n+1}}\right]} \int_{A^-_{h_n,\rho_{n+1}} \setminus A^-_{h_{n+1},\rho_{n+1}}} |\nabla v|\,dx .$$

By virtue of the assumption of the proposition

$$\text{meas}\left[B(\rho_{n+1}) \setminus A^-_{h_n,\rho_{n+1}}\right] > \kappa_N \rho^N_{n+1} - \text{meas } A^-_{h_0,2R} > \kappa_N\left(R^N - \frac{c_0}{\kappa_N} 2^N R^N\right) > \frac{1}{2}\kappa_N R^N ,$$

if we choose $c_0 < \dfrac{\kappa_N}{2^{N+1}}$.

In view of this, setting

$$\mu_n = \text{meas } A^-_{h_n,\rho_n} ,$$

the above implies (for a new constant γ depending only upon the data)

$$\mu_{n+1}^{\frac{N-1}{N}} \le \gamma 2^n \frac{1}{H} \int_{A^-_{h_n,\rho_{n+1}} \setminus A^-_{h_{n+1},\rho_{n+1}}} |\nabla v|\,dx \le \frac{\gamma 2^n}{H} \left[\int_{B(\rho_{n+1})} |\nabla(v-h_n)^-|^2 dx\right]^{1/2} [\mu_n]^{1/2} .$$

From (4.5)

$$\int_{B(\rho_{n+1})} |\nabla(v - h_n)^-|^2 dx \le \frac{\gamma 2^{2n+2}}{R^2} h_0^2 \mu_n ,$$

so that

$$\mu_{n+1}^{\frac{N-1}{N}} \le \frac{\gamma 2^{2n}}{R^2} \left(\frac{h_0}{H}\right) \mu_n . \tag{4.6}$$

We observe that if $H < \frac{1}{2} h_0$, then

$$\sup_{B(2R)} (v - h_0)^- = \sup_{B(2R)} (|u_{x_i}|^{p/2} \operatorname{sign} u_{x_i} - \lambda^{p/2}) < \frac{1}{2} \lambda^{p/2} ,$$

i.e. $$u_{x_i}(x) \ge 2^{-2/p}\lambda \ge \frac{\lambda}{4} , \quad \forall x \in B(2R).$$

Therefore we may assume $H \ge \frac{1}{2} h_0$. Dividing both the sides of (4.6) by R^N, and setting

$$\nu_n = \frac{\mu_n}{R^N}$$

we have in dimensionless form

$$\nu_{n+1} \le \gamma b^n \nu_n^{1+\frac{1}{N-1}} , \qquad b = 2^{\frac{2N}{N-1}} ,$$

for a new γ dependent only upon the data.

From these recursion inequalities it follows from Lemma 4.7 of [12] page 66, that there exists a number c_0 depending only on γ and b such that if $\nu_0 \le c_0$, then $\nu_n \to 0$ as $n \to \infty$. Consequently if

$$\operatorname{meas}\{x \in B(2R) | u_{x_i}(x) < \lambda\} \le c_0 R^N ,$$

then

$$\operatorname{meas}\{x \in B(R) \mid u_{x_i}(x) < \frac{1}{2^{2/p}} \lambda\} = 0 \quad \text{i.e.}$$

$$u_{x_i}(x) \geq \frac{\lambda}{4} = \frac{1}{8} \max_{1 \leq j \leq N} \sup_{B(2R)} |u_{x_j}(x)|, \qquad \forall x \in B(R) .$$

Suppose now that the assumptions of Proposition 5.1 fail. Then for all $i = 1,2,\ldots,N$ we have

(4.7) $$\operatorname{meas}\{x \in B(2R) \mid u_{x_i}(x) > \lambda\} < (1 - c_0)\kappa_N R^N, \quad \underline{\text{and}}$$

(4.8) $$\operatorname{meas}\{x \in B(2R) \mid u_{x_i}(x) < -\lambda\} < (1 - c_0)\kappa_N R^N .$$

Set

$$2\lambda = \mu(2R) = \max_{1 \leq i \leq N} \sup_{B(2R)} |u_{x_i}(x)| ,$$

and observe that (4.7)-(4.8) are still verified if we replace λ with a larger number. We will use (4.7)-(4.8) with λ replaced by $(1 - \frac{1}{2^s})\mu(2R)$, for $s \in \mathbb{N}$.

<u>Proposition 4.2</u>: Let (4.7)-(4.8) hold $\forall\ 1 \leq i \leq N$. There exists a number $\delta_0 \in (0,1)$ depending only upon the data and independent of λ,ε,R such that

$$\mu(R/2) = \max_{1 \leq i \leq N} \sup_{B(R/2)} |u_{x_i}| \leq \delta_0 \mu(2R) \equiv \delta_0 \max_{1 \leq i \leq N} \sup_{B(2R)} |u_{x_i}| .$$

<u>Proof</u>: Suppose (4.7) holds and consider inequalities (4.4), written for $(u_{x_i} - (1 - \frac{1}{2^s})\mu(2R))^+$, $s \in \mathbb{N}$, over balls $B(\rho)$, $\rho \leq 2R$. Since the integrals are extended over the set where u_{x_i} is larger than $(1 - \frac{1}{2^s})\mu(2R)$, we have on this set

$$\frac{1}{2}\,\mu(2R)^{p-2} \leq w^{\frac{p-2}{2}} = [\varepsilon + |\nabla u_\varepsilon|^2]^{\frac{p-2}{2}} \leq (2N)^{\frac{p-2}{2}}\,\mu(2R)^{\frac{p-2}{2}} .$$

Consequently (4.4) can be rewritten (for all $p > 1$)

(4.9) $$\int_{B(\rho-\sigma\rho)} |\nabla(u_{x_i} - (1 - \frac{1}{2^s})\mu)^+|^2 dx \leq \gamma(\sigma\rho)^{-2} \int_{B(\rho)} (u_{x_i} - (1 - \frac{1}{2^s})\mu)^{+^2} dx ,$$

where we have set $\mu = \mu(2R)$ for notational simplicity.

<u>Lemma 4.1</u>: For every $\theta_0 \in (0,1)$, there exists $s_0 \in \mathbb{N}$ such that

$$\operatorname{meas}\{x \in B(R) \mid u_{x_i}(x) > (1 - \frac{1}{2^{s_0}})\mu(2R)\} \leq \theta_0 R^N .$$

Proof of Lemma 4.1: We majorize $(u_{x_i}(x) - (1 - \frac{1}{2^s})\mu)^+$ by $\mu/2^s$ and write (4.9) for $\rho = 2R$ and $\sigma = \frac{1}{2}$ to obtain

$$(4.10) \qquad \int_{B(R)} |\nabla(u_{x_i} - (1 - \frac{1}{2^s})\mu)^+|^2 dx \le \gamma R^{-2}(\frac{\mu}{2^s})^2 \kappa_N R^N$$

for a new constant γ. Apply Lemma 2.5 to the function $u_{x_i}(\cdot)$ for the levels $\ell = \mu - 2^{-(s+1)}\mu$, $k = \mu - 2^{-s}\mu$ and notice that by virtue of (4.7)

$$\text{meas}\{B(R)\setminus A^+_{k,R}\} \ge c_0 R^N .$$

Then we have

$$2^{-(s+1)}\mu[\text{meas } A^+_{\ell,R}] \le \frac{\gamma R}{c_0} \int_{A^+_{k,R}\setminus A^+_{\ell,R}} |\nabla u_{x_i}| dx \le$$

$$\le \frac{\gamma R}{c_0} [\int_{B(R)} |\nabla(u_{x_i} - (1 - 2^{-s})\mu)^+|^2 dx]^{1/2} [\text{meas } A^+_{k,R}\ A^+_{\ell,k}]^{1/2} .$$

Squaring both the sides, using (4.10) and dividing by $(2^{-(s+1)}\mu)^2$ gives

$$[\text{meas } A^+_{\mu - \frac{1}{2^{s+1}}\mu, R}]^2 \le \gamma R^N [\text{meas } A^+_{k,R}\setminus A^+_{\ell,R}] .$$

We add over $s = 1, 2, \ldots, s_0 - 1$ to obtain

$$(s_0 - 2)[\text{meas } A^+_{\mu - \frac{1}{2^{s_0}}\mu, R}] \le \frac{\gamma}{c_0} (\kappa_N R^N)^2 ,$$

where γ depends only upon the data. To prove the lemma we have only to choose s_0 so large that

$$\frac{\gamma}{c_0(s_0 - 2)} \le \theta_0^2 .$$

Lemma 4.2: Let inequalities (4.9) hold $\forall \rho \le 2R$. There exist a number θ_0 such that if for some $s_0 \in \mathbb{N}$

$$\text{meas}\{x \in B(R) | u_{x_i}(x) > \mu - 2^{-s_0}\mu\} \le \theta_0 R^N ,$$

then

$$\text{meas}\{x \in B(R/2) | u_{x_i}(x) > \mu - 2^{-s_0}\mu + \frac{1}{2} H\} = 0$$

where

$$H = \sup_{B(2R)} (u_{x_i}(x) - (\mu - 2^{-s_0}\mu))^+ .$$

Proof of Lemma 4.2: The lemma is proved essentially in the same way as Proposition 5.1. We leave to the reader the few modifications needed.

Proof of Proposition 4.2 (concluded): Fix θ_0 as in Lemma 4.2 and choose s_0 consequently by the technique indicated in Lemma 4.1. Lemma 4.2 gives then

$$\sup_{B(R/2)} u_{x_i}(x) < \mu - \frac{1}{2^{s_0}}\mu + \frac{1}{2}H < \mu - \frac{1}{2^{s_0}}\mu + \frac{1}{2^{s_0+1}}\mu = \delta_0\mu;\ \delta_0 = (1 - 2^{-(s_0+1)}) .$$

Starting now from (4.8) by the same arguments we arrive at

$$\inf_{B(R/2)} u_{x_i}(x) > -\delta_0\mu .$$

Since (4.7)-(4.8) both hold $\forall\ 1 \le i \le N$ the conclusion follows.

Proposition 4.3: The solution of $-\operatorname{div} \vec{a}_\varepsilon(\nabla u_\varepsilon) = 0$ are $C^{1+\eta}_{loc}(\Omega)$, uniformly in ε, and for every ball $B(R) \subset \Omega$ there exist constants γ and $\eta \in (0,1)$ depending only upon the data and $\operatorname{dist}(B(R),\partial\Omega)$ such that

$$\operatorname*{osc}_{B(\rho)} u_{\varepsilon x_i} \le \gamma\left(\frac{\rho}{R}\right)^\eta, \qquad i = 1,2,\ldots,N$$

for every ball $B(\rho)$ concentric with $B(R)$, $\rho < R$.

Proof: Suppose the assumptions of Proposition 4.1 are verified. Then for some i either

$$u_{x_i}(x) > \frac{1}{8}\mu(2R) \qquad \forall x \in B(R)$$

or

$$u_{x_i}(x) < -\frac{1}{8}\mu(2R) \qquad \forall x \in B(R) .$$

In either case

$$\left(\frac{1}{8}\right)^{p-2}\mu(2R)^{p-2} \le w^{\frac{p-2}{2}} \le (N+1)^{\frac{p-2}{2}}\mu(2R)^{\frac{p-2}{2}}, \qquad \forall x \in B(R) .$$

Therefore writing (4.4) over balls $B(\rho)$, $\rho \le R$ for all $i = 1,2,\ldots,N$ and all levels k we have $(\forall p > 1)$

$$(4.11)\qquad \int_{B(\rho-\sigma\rho)} |\nabla(u_{x_i} - k)^{\pm}|^2 dx \le \gamma(\sigma\rho)^{-2} \int_{B(R)} (u_{x_i} - k)^{\pm 2} dx ,$$

for a new constant γ depending upon the data and independent of ε,ρ,k. These inequalities imply, with the aid of the results of [12] page 81-90 that there exist a constant $\delta_1 \in (0,1)$ depending only upon the data, such that for all $i = 1,2,\ldots,N$

$$\operatorname*{osc}_{B(R/2)} u_{x_i}(x) \le \delta_1 \operatorname*{osc}_{B(2R)} u_{x_i} .$$

If we set $\forall \rho \le 2R$

$$\omega(\rho) = \max_{1\le i\le N} \operatorname*{osc}_{B(\rho)} u_{x_i} ,$$

the above implies

$$(4.12)\qquad \omega(R/2) \le \delta_1 \omega(R) .$$

If the assumptions of Proposition 4.1 fail then for all $i = 1,2,\ldots,N$ we have that both (4.7)-(4.8) are verified. Proposition 4.2 then gives the existence of δ_0 depending only on the data such that

$$(4.13)\qquad \mu(R/2) \le \delta_0 \mu(2R) .$$

Fix some R_0 such that $B(R_0) \subset \Omega'$ and consider the sequence of radii $R_n = R_0/2^{2n}$, $n = 0,1,\ldots$ and balls $B(R_n)$ all concentric and shrinking. The proof shows that if for *some* n_0 *the hypothesis of* Proposition 4.1 are verified, then for all $n > n_0$, we can estimate $w^{\frac{p-2}{2}}$ from above and below in terms of $\mu(R_{n_0})$ and hence in $B(R_n)$ $\forall n > n_0$ the equation behaves like a non-degenerate elliptic equation. From (4.11) then it follows that (4.12) holds $\forall n > n_0$. Consequently we can assert that there exist $n_0 \in \mathbb{N}$ such that (4.13) holds $\forall\, 0 \le n \le n_0$ and (4.12) holds for all $n > n_0$, $n \in \mathbb{N}$.

The proof of Proposition 4.3 now follows from a standard modification of Lemma 4.8 of [12] page 66-67.

5. Proof of Theorem 2

Let u_ε be solution of (1.4) in Ω for $\varepsilon > 0$ arbitrary but fixed. By the remarks of Section 2, u_ε is Hölder continuous in Ω' with constants γ and β depending upon the data, $\mathrm{dist}(\Omega',\partial\Omega)$ but independent of ε.

Let us fix Ω'' a subdomain of Ω compactly contained in Ω'. By virtue of the results of Section 3

$$\|\nabla u_\varepsilon\|_{\infty,\Omega''} \le \gamma$$

for γ depending on $\mathrm{dist}(\Omega'',\partial\Omega')$ and the data. Consequently the u_ε are equi-Lipschitz continuous in Ω''.

We want to prove that $u_\varepsilon \in C^{1+\alpha}$ around any point $x_0 \in \Omega''$, uniformly in ε. Let $x_0 \in \Omega''$ be fixed, choose R so small that $B(R) \equiv \{|x - x_0| < R\} \subset \Omega''$, and for $\omega \in L_1(B(R))$, set

$$\omega_R(x_0) = \frac{1}{\text{meas } B(R)} \int_{B(R)} \omega(x)dx .$$

If $\varepsilon > 0$ is fixed we will write u instead of u_ε. Consider the problem

$$(5.1) \qquad \begin{cases} -\mathrm{div}\, \vec{a}_\varepsilon(x_0,u_R(x_0),\nabla v) = 0 & \text{in } B(R) \\ v \equiv u & \text{on } \partial B(R) \end{cases}$$

where R is so small for the Lemma of local uniqueness to hold.

<u>Lemma 5.1</u>: Problem (5.1) has a unique solution $v \in C^2(B(R)) \cap C^\beta(\overline{B(R)})$. Moreover

(i) $\quad \inf\limits_{\partial B(R)} u \le v(x) \le \sup\limits_{\partial B(R)} u \qquad \forall x \in B(R)$

(ii) $\quad \operatorname*{osc}\limits_{B(R)} v \le \gamma R$

(iii) $\quad \forall\delta \in (0,1)$, there exist a constant $\gamma(\delta)$ such that

$$\|\nabla v\|^p_{\infty,B(R-\delta R)} \le \gamma(\delta) \frac{1}{R^N} \int_{B(R)} |\nabla v|^p dx ,$$

the constants $\gamma,\beta,\gamma(\delta)$ do not depend upon ε.

<u>Proof</u>: The statement of existence is classical [10,12].

To prove (i) consider (5.1) written in the weak form

$$(5.2) \qquad \int_{B(R)} \vec{a}_\varepsilon(x_0,u_R(x_0),\nabla v)\cdot\nabla\varphi dx = 0, \quad \forall\varphi \in \overset{\circ}{W}{}^{1,p}(B(R)) ,$$

and select $\varphi = (v - \sup_{\partial B(R)} u)^+ \in \overset{\circ}{W}{}^{1,p}(B(R))$. Using (2.1) we deduce

$$\gamma_0 \int_{B(R)} [\varepsilon + |\nabla v|^2]^{\frac{p-2}{2}} |\nabla(v - \sup_{\partial B(R)} u)^+|^2 dx \le 0 .$$

The statement about the "inf" is proved analogously. As a consequence

$$\operatorname*{osc}_{B(R)} v = \max_{B(R)} v - \min_{B(R)} v \le \sup_{\partial B(R)} u - \inf_{\partial B(R)} u \le \operatorname*{osc}_{B(R)} u .$$

Since u_ε are equi-Lipschitz in Ω'', (ii) follows. Statement (iii) is a consequence of Proposition 3.3.

Let us write (1.4) in the weak form for test functions $\varphi \in \overset{\circ}{W}{}^{1,p}(B(R))$ and subtract (5.2) from it. Dropping the subscript ε we obtain

$$\int_{B(R)} [\vec{a}(x,u,\nabla u) - \vec{a}(x_0,u_R(x_0),\nabla v)]\cdot\nabla\varphi dx + \tag{5.3}$$

$$+ \int_{B(R)} b(x,u,\nabla u)\varphi dx = 0; \qquad \forall\varphi \in \overset{\circ}{W}{}^{1,p}(B(R)).$$

Choosing $\varphi = u - v \in \overset{\circ}{W}{}^{1,p}(B(R))$ we have

$$[\vec{a}(x,u,\nabla u) - \vec{a}(x_0,u_R(x_0),\nabla v)]\cdot\nabla\varphi =$$

$$= \int_0^1 \frac{\partial}{\partial t} \vec{a}(tx + (1-t)x_0, tu + (1-t)u_R(x_0), t\nabla u + (1-t)\nabla v)dt\cdot\nabla\varphi =$$

$$= \int_0^1 a^k_{u_{x_j}} (tx + (1-t)x_0, tu + (1-t)u_R(x_0), t\nabla u + (1-t)\nabla v)dt(u-v)_{x_k}(u-v)_{x_j} +$$

$$+ \int_0^1 a^k_u(tx + (1-t)x_0, tu + (1-t)u_R(x_0), t\nabla u + (1-t)\nabla v)dt(u - u_R(x_0))(u-v)_{x_k} +$$

$$+ \int_0^1 a^k_{x_i}(tx + (1-t)x_0, tu + (1-t)u_R(x_0), t\nabla u + (1-t)\nabla v)dt(x - x_0)_i(u-v)_{x_k} \ge$$

$$\ge \gamma_0 \int_0^1 [\varepsilon + |t\nabla u + (1-t)\nabla v|^2]^{\frac{p-2}{2}} |\nabla(u-v)|^2 dt -$$

$$- \gamma_1 \int_0^1 [\varepsilon + |t\nabla u + (1-t)\nabla v|^2]^{\frac{p-1}{2}} [|u - u_R(x_0)| + |x - x_0|]|\nabla(u-v)|dt .$$

Therefore (5.3) implies for all $p > 1$

(5.4) $$\int_{B(R)} [\varepsilon + |\nabla u|^2 + |\nabla v|^2]^{\frac{p-2}{2}} |\nabla(u - v)|^2 \leq$$

$$\leq \gamma \int_{B(R)} [\varepsilon + |\nabla u|^2 + |\nabla v|^2]^{\frac{p-1}{2}} [|u - u_R(x_0)| + |x - x_0|] |\nabla(u - v)| dx +$$

$$+ \gamma \int_{B(R)} |b(x,u,\nabla u)| |u - v| dx .$$

Applying Lemma 5.1, and denoting with $\bar{x}$ an arbitrary point of $\partial B(R)$ we have $\forall x \in B(R)$

$$|u(x) - v(x)| \leq |u(x) - u(\bar{x})| + |v(x) - v(\bar{x})| \leq \operatorname*{osc}_{B(R)} u + \operatorname*{osc}_{B(R)} v \leq \gamma R .$$

Moreover, for the lower order terms recalling $[A_4]^{\varepsilon}$ and Theorem 1

$$b(x,u,\nabla u)|u(x) - v(x)| \leq \gamma R ,$$

and since u_ε are equi-Lipschitz in Ω''

$$|u(x) - u_R(x_0)| \leq \gamma R .$$

Finally using Cauchy inequality $ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ on the first integral on the right hand side of (5.4) we deduce that for a new constant γ depending only upon the data and independent of R and ε

(5.5) $$\int_{B(R)} [\varepsilon + |\nabla u|^2 + |\nabla v|^2]^{\frac{p-2}{2}} |\nabla(u - v)|^2 dx \leq \gamma R^{N+1} +$$

$$+ \gamma R \int_{B(R)} [\varepsilon + |\nabla u|^2 + |\nabla v|^2]^{p/2} dx .$$

<u>Lemma 5.2</u>: (i) $\int_{B(R)} |\nabla v|^p \leq \gamma R^N$.

(ii) $\forall \delta \in (0,1)$, $\exists \gamma(\delta)$ independent of ε, such that

$$\|\nabla v\|^p_{\infty, B(R-\delta R)} \leq \gamma(\delta) .$$

<u>Proof</u>: A straightforward calculation from (5.5) gives

$$\int_{B(R)} |\nabla v|^p dx \leq \gamma R^{N+1} + \gamma(R + 1) \int_{B(R)} |\nabla u|^p dx + \gamma R \int_{B(R)} |\nabla v|^p dx$$

for a new γ independent of R and ε. Since R can be chosen so that $\gamma R < \frac{1}{2}$, by virtue of Theorem 1

$$\int_{B(R)} |\nabla v|^p dx \le \gamma R^N .$$

Statement (ii) follows from (i) and (iii) of Lemma 5.1.

<u>Lemma 5.3</u>: For every $\delta \in (0,1)$, there exist a constant $\gamma(\delta)$ independent of R and ε such that

$$\text{(5.6)} \qquad \int_{B(\rho)} |\nabla(u - v)|^2 dx \le \gamma(\delta) R^{N+\sigma}, \qquad \forall \ 0 < \rho < R - \delta R .$$

where

$$\sigma = \min\{1; \frac{2}{p}\} .$$

<u>Proof</u>: By Lemma 5.2, (5.5) implies $\forall p > 1$

$$\text{(5.7)} \qquad \int_{B(R)} [\varepsilon + |\nabla u|^2 + |\nabla v|^2]^{\frac{p-2}{2}} |\nabla(u - v)|^2 dx \le \gamma R^{N+1}$$

for γ independent of R and ε.

If $p \ge 2$ we have

$$\int_{B(R)} |\nabla(u - v)|^2 dx \le \Big(\int_{B(R)} |\nabla(u - v)|^p dx\Big)^{\frac{2}{p}} (\kappa_N R^N)^{1-\frac{2}{p}} \le$$

$$\le 2\Big(\int_{B(R)} [\varepsilon + |\nabla u|^2 + |\nabla v|^2]^{\frac{p-2}{2}} |\nabla(u - v)|^2\Big)^{\frac{2}{p}} (\kappa_N R^N)^{\frac{p-2}{p}} \le \gamma R^{(N+1)\frac{2}{p}} .$$

If $1 < p < 2$, for $\delta \in (0,1)$ fixed we have from (5.5)

$$\int_{B(R-\delta R)} |\nabla(u - v)|^2 dx \le \gamma[1 + \|\nabla u\|^p_{\infty,B(R-\delta R)} + \|\nabla v\|^p_{\infty,B(R-\delta R)}] R^{N+1} .$$

Therefore by (ii) of Lemma 5.2, inequality (5.6) follows.

We fix $\delta = \frac{1}{2}$ so that $\forall \ 0 < \rho < \frac{R}{2}$

$$\text{(5.8)} \qquad \int_{B(\rho)} |\nabla(u - v)|^2 dx \le \gamma R^{N+\sigma} .$$

Proof of Theorem 2 (concluded): We have

$$(5.9)\qquad \int_{B(\rho)} |\nabla u - (\nabla u)_\rho(x_0)|^2 dx \le \int_{B(\rho)} |\nabla u - (\nabla v)_\rho(x_0)|^2 dx \le$$

$$\le \int_{B(\rho)} |\nabla u - \nabla v|^2 dx + \int_{B(\rho)} |\nabla v - (\nabla v)_\rho(x_0)|^2 dx .$$

By virtue of Proposition 4.3, $x \to \nabla v(x)$ is Hölder continuous in $B(R/4)$ with constants γ and $\eta \in (0,1)$ depending only upon the data and $\|\nabla v\|_{\infty,B(R/2)}$. The latter quantity is estimated by (ii) of Lemma 5.2, and therefore we conclude that there exist constants γ,η depending only upon the data such that

$$(5.10)\qquad \int_{B(\rho)} |\nabla v - (\nabla v)_\rho(x_0)|^2 dx \le \gamma\rho^N\left(\frac{\rho}{R}\right)^{2\eta}, \quad 0 < \rho \le \frac{R}{4} .$$

Using (5.8) and these remarks in (5.9) we have

$$\int_{B(\rho)} |\nabla u - (\nabla u)_\rho(x_0)|^2 dx \le \gamma\left[\rho^N\left(\frac{\rho}{R}\right)^{2\eta} + R^{N+\sigma}\right]$$

for all $0 < \rho < \frac{R}{4}$. Let $\theta,\alpha \in (0,1)$ be defined by

$$(5.11)\qquad \theta = \frac{\sigma}{N + 2\eta}, \quad \alpha = \frac{\sigma\eta}{N + 2\eta + \sigma},$$

and in (5.10) choose $\rho = \frac{1}{4} R^{(1+\theta)}$. This gives

$$\rho^{-(N+2\alpha)} \int_{B(\rho)} |\nabla u - (\nabla u)_\rho(x_0)|^2 dx \le \gamma$$

for a constant γ independent of ρ and ε.

From a result of Campanato [3] (see also [4]) it follows that $x \to \nabla u_\varepsilon$ is locally Hölder continuous in Ω'' with exponent α, uniformly in ε.

Remark: The relations in (5.11) link the estimated Hölder exponent of $x \to \nabla u$, with η, the Hölder exponent of ∇v in (5.1) "with constant coefficients", with p (via Lemma 5.3), and with the dimension N.

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It is demonstrated the local $C^{1+\alpha}$ nature of weak solutions of elliptic equations of the type (1.1) in the introduction under the degeneracy (or singularity) assumptions $[A_1]-[A_3]$.

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