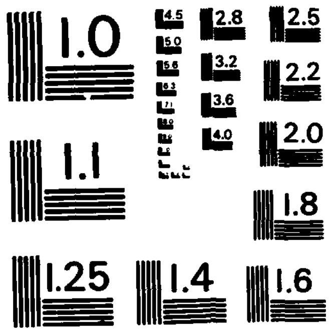

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AFOSR-TR- 82-0966

Reference 5

Some Generic Invariant Factor Assignment Results Using Dynamic Output Feedback

by

T. E. Djaferis* and S. K. Mitter**

Abstract

Working with input-output transfer functions in the frequency domain and exploiting a formulation involving Generalized Sylvester Resultants we are able to derive necessary and sufficient conditions for generic invariant factor assignment, in several cases, using proper dynamic output feedback compensators.

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I. Introduction

A very well known fact in system theory is that feedback can be used in order to improve system performance. In the theory of finite dimensional linear time invariant systems one frequently deals with the question of how to improve the performance of systems described by the following differential equation:

$$x(t) = Ax(t) + Bu(t), \tag{1.1}$$

where $x(t)$ is an n-vector, $u(t)$ an ℓ-vector A $(n \times n)$, $B(n \times \ell)$ matrices over the reals R. If constant state feedback is used,

$$u(t) = v(t) - Kx(t),$$

where K is an $\ell \times n$ real matrix, $v(t)$ a reference input, the closed-loop system is described by:

$$\dot{x}(t) = (A-BK)x(t) + Bv(t). \tag{1.2}$$

A central result in the area of pole assignment [21] is that: (A, B) is controllable iff for every symmetric set Λ of n complex numbers, there is a matrix K such that A-BK has Λ, for its set of eigenvalues. This implies that, under the assumption of controllability, arbitrary pole assignment can be accomplished by constant state feedback.

Rosenbrock in a subsequent publication, [19] showed that more than pole assignment can be accomplished for the system in (1.1). This result can be stated in the following manner:

Let (A, B) be a controllable pair with controllability indecies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. Let $\phi_i, 1 \leq i \leq \ell$, be given monic polynomials satisfying the divisibility conditions $\phi_i | \phi_{i-1}$, and with $\sum_{i=1}^{\ell} \theta(\phi_i) = n$, ($\theta(\cdot)$ denotes degree). Then there exists a constant matrix K such that the given polynomials are the non-unity invariant factors of $sI-A+BK$ if and only if

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \lambda_i \quad k = 1,2,\ldots,\ell, \text{ with equality at } k=\ell. \tag{1.3}$$

This result implies that we not only can arbitrarily assign the eigenvalues of A-BK, but the size and entries of the cyclic blocks appearing on the main diagonal of the rational canonical form of A-BK [17]. It is important to note that since (A, B) in controllable so is (A-BK, B) for any K (ie sI-A+BK, B are left coprime). From a frequency domain input-output point of view this means that if

$$Q(s) = (sI-A)^{-1}B$$

is the input-output transfer function of the system in (1.1), where the output is actually the state, and state feedback (1.2) is used, the closed loop transfer function becomes:

$$H(s) = (sI-A+BK)^{-1}B.$$

Making the invariant factors of sI-A+BK equal to a given set $\{\phi_i\}$ is equivalent to saying [12] that $M_H(s)$ the Smith–McMillan form of H(s) is given by

$$M_H(s) = \operatorname{diag}\left(\frac{\varepsilon_i(s)}{\phi_i(s)}\right) ,$$

(with appropriate modification if $n \neq \ell$).

In many practical applications, physical constraints frequently necessitate the use of output rather than state feedback,

$$u(t) = v(t) + Ky(t) \ , \ y(t) = Cx(t), \tag{1.4}$$

where y(t) is an m-vector and C an mxn constant real matrix. In many situations static output feedback is insufficient and dynamic output feedback is introduced:

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gy(t) \\ u(t) &= Hz(t) + Ky(t) \end{aligned} \tag{1.5}$$

where z(t) is a q-vector and F, G, H, K appropriate matrices with real entries.

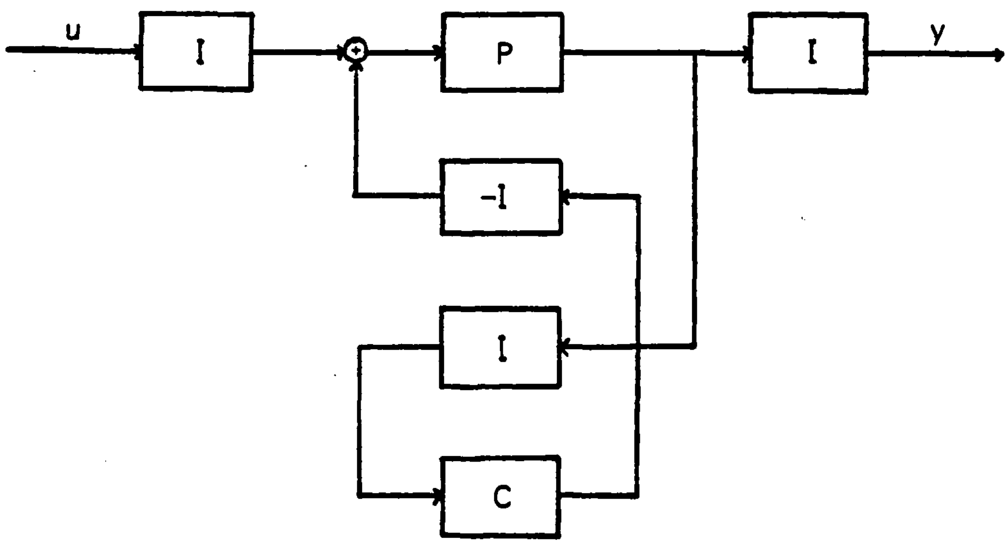
In light of the above Rosenbrock and Hayton [20] attempted to generalize Rosenbrock's earlier result to the output feedback case. They proceeded by using the frequency domain input-output point of view by considering the strictly proper system

$$P(s) = C(sI-A)^{-1}B = D_{LP}^{-1}N_{LP}$$

and the proper compensator $C(s) = A_{LC}^{-1}B_{LC}$ (no longer static) as given by the system matrices (Rosenbrock sense),

$$P_P(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & D_{LP} & N_{LP} \\ 0 & -I & 0 \end{bmatrix} \qquad P_C(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{LC} & B_{LC} \\ 0 & -I & 0 \end{bmatrix}.$$

The input-output transfer functions $P(s)$ $m \times \ell$ and $C(s)$ $(\ell \times m)$ have elements in $R(s)$, the field of rational functions in s over R. The matrices N_{LP}, D_{LP}, A_{LC}, B_{LC} have elements in $R[s]$, polynomials in s. If the two systems are connected as in the figure below (output feedback configuration),



a composite system for the resulting closed-loop system is obtained [20] and then brought by strict system equivalence to the form:

$$P_G(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{LC}D_{RP}+B_{LC}N_{RP} & A_{LC} \\ 0 & -N_{RP} & 0 \end{bmatrix}$$

where $D_{LP}^{-1}N_{LP} = N_{RP}D_{RP}^{-1}$ are left and right matrix fraction descriptions of the system and $A_{LC}^{-1}B_{LC}$ a left matrix fraction description of the compensator [5, 12]. The closed-loop transfer function is:

$$G(s) = N_{RP}(A_{LC}D_{RP}+B_{LC}N_{RP})^{-1}A_{LC} .$$

The basic result in [20] is the following:

Let $P(s) = N_{RP}D_{RP}^{-1}$ be an $m \times \ell$ strictly proper transfer function of order n, with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ column reduced with column degree $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ (λ_i are controllability indecies) and μ_1 the largest observability index. Let ϕ_i $1 \leq i \leq \ell$ be given polynomials satisfying the divisibility conditions $\phi_i | \phi_{i-1}$ and with $\sum_{i=1}^{\ell} \theta(\phi_i) = n+\ell(\mu_1-1)$. Then a sufficient condition for the existence of a proper $\ell \times m$ compensator $C(s) = A_{LC}^{-1}B_{LC}$ such that the invariant factors of $A_{LC}D_{RP}+B_{LC}N_{RP}$ are the ϕ_i is:

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \lambda_i + \mu_1 - 1 \qquad k = 1,2,\ldots,\ell \text{ with equality at } k=\ell. \tag{1.6}$$

One should immediately notice several basic differences of this result as compared with the earlier state feedback result.

a) In addition to the controllability indecies other indecies (namely observability) become important.

b) This is only a sufficient condition.

c) Dynamic feedback has been introduced.

d) As no coprimeness conditions have been imposed assigning the invariant factors of $A_{LC}D_{RP}+B_{LC}N_{RP}$ does not imply that the invariant factors of some sI-A* have been assigned where A* comes from a minimal realization, but rather that some realization (not minimal) can be found such that for the $\bar{A}$ matrix corresponding to it, $sI-\bar{A}$ has the given invariant factors.

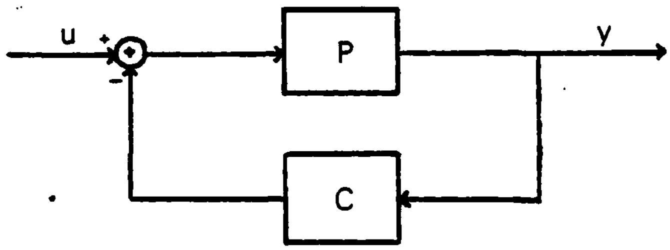
e) The order of the compensator used is $\ell(\mu_1-1)$. There exists the possibility that a better result can be stated employing a lower order compensator.

Several attempts have been made to "improve" the output feedback invariant factor result [6,7,9,10,15]. A partial list of some other recent publications on the general problem of pole assignment can be found in the references.

In this paper working in the frequency domain with input-output transfer functions and using a formulation employing matrix fraction descriptions and Generalized Sylvester Resultants, we are able to give short new proofs of Rosenbrock's state and output feedback results. Furthermore we demonstrate that such a structure easily lends itself to a "generic" formulation of the invariant factor problem. This allows us to derive necessary and sufficient conditions for generic invariant factor assignment in several cases, and prove some other interesting results as well.

2. Formulation

Throughout the paper we assume the following feedback configuration:



where $P(s)$ is then $m \times \ell$ input-output transfer function of the given strictly proper system and $C(s)$ the $\ell \times m$ transfer function of a proper compensator which is to be computed. Both $P(s)$ and $C(s)$ have elements in $R(s)$. Without loss of generality we assume that $m \geq \ell$. In the case that $\ell > m$ a "dual" formulation and results can be obtained. The closed loop transfer function is given by:

$$G(s) = P(s)(I + C(s)\,P(s))^{-1}$$

where we assume that $(I + C(s)\,P(s))^{-1}$ exists. Since $P(s)$ and $C(s)$ are rational matrices they can be "factored" into polynomial matrices, [5, 12]. We use the notation:

$P(s) = B_{RP}A_{RP}^{-1}$ a right matrix fraction description (MFD) of $P(s)$,

$= A_{LP}^{-1}B_{LP}$ a left MFD of $P(s)$,

$= N_{RP}D_{RP}^{-1}$ a right coprime (or irreducible) MFD of $P(s)$,

$= D_{LP}^{-1}N_{LP}$ a left coprime (or irreducible) MFD of $P(s)$,

where B_{RP}, A_{RP}, N_{RP}, D_{RP} etc are polynomial matrices, where the indeterminate s has been supressed for simplicity. The closed loop transfer function can then be expressed in the following ways:

$$
\begin{aligned}
G(s) &= P(I + CP)^{-1} \\
&= B_{RP}(A_{LC}A_{RP} + B_{LC}B_{RP})^{-1}A_{LC} \,. && (2.1) \\
&= N_{RP}(A_{LC}D_{RP} + B_{LC}N_{RP})^{-1}A_{LC} && (2.2) \\
&= N_{RP}(\underbrace{D_{LC}D_{RP} + N_{LC}N_{RP}}_{\Phi})^{-1}D_{LC} && (2.3) \\
&= N_{RP}\ \Phi^{-1}\ D_{LC} \\
&= \tilde{N}_{RP}\ \tilde{\Phi}^{-1}\ \tilde{D}_{LC} \quad \text{least order (or irreducible)} && (2.4)
\end{aligned}
$$

where $\tilde{N}_{RP}$, $\tilde{\Phi}$ are right coprime, $\tilde{\Phi}$, $\tilde{D}_{LC}$ left coprime. The description (2.4) is not unique since if $\overline{N} = \tilde{N}_{RP}E$, $\overline{D} = H\ \tilde{D}_{LC}$, $\overline{\Phi} = H\tilde{\Phi}E$, E, H unimodular then $\overline{N}\overline{\Phi}^{-1}\ \overline{D}$ is also a least order, (irreducible polynomial matrix description PMD [12]). Clearly since no coprimeness conditions have been imposed descriptions (2.1), (2.2), (2.3) are <u>not</u> least order.

If $M_G(s)$ is the Smith-McMillan form of G(s)

$$
M_G(s) = \begin{bmatrix} \frac{\varepsilon_1}{\phi_1} & & 0 \\ & \ddots & \\ & 0 & \frac{\varepsilon_\ell}{\phi_\ell} \\ & 0 & \end{bmatrix}
$$

where ϕ_i, $1 \le i \le \ell$ are monic and satisfy the divisibility conditions $\phi_i | \phi_{i-1}$ $2 \le i \le \ell$ using ideas of system equivalence one can show [7, 12] that

$$
\tilde{\Phi} = E \begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_\ell \end{bmatrix} H \,,
$$

where E, H are unimodular matrices. Therefore we call the $\{\phi_i\}$ the <u>invariant factors</u> (or polynomials) of the closed loop system. It is also clear that the $\{\phi_i\}$ are the non-unity invariant factors of sI-A where A comes from some minimal realization of G(s).

If on the other hand we use a description of G(s) which is <u>not</u> least order $G(s) = B\,\Psi^{-1}A$ with the Smith form of Ψ being $\begin{bmatrix} \psi_\ell & & 0 \\ & \ddots & \\ 0 & & \psi_1 \end{bmatrix}$,

where $\psi_i | \psi_{i-1}$ $2 \le i \le \ell$, we will have [7] $\phi_i | \psi_i$ $1 \le i \le \ell$. The $\{\psi_i\}$ will be the non-unity invariant factors of some $sI-\bar{A}$ where $\bar{A}$ comes from a non-minimal realization of G(s). The output feedback results of Rosenbrock and Hayton [20] deal with this problem.

The difference between the two approaches stems from the fact that one can work with either external or internal descriptions of systems [12]. From the results of section 6 it is evident that in the "generic" case the difference disappears. It should also be mentioned that if no coprimeness conditions need to be satisfied as in Rosenbrock and Hayton [20], the proofs of these results are much easier to construct.

It is clear from the above that a very natural way to proceed with the invariant factor assignment problem is the following:

Given a strictly proper system $P(s) = N_{RP}D_{RP}^{-1}$ find conditions for the existence of a polynomial solution X, Y to the polynomial equation:

$$X\,D_{RP} + Y\,N_{RP} = \Phi \qquad (2.5)$$

where:

1) Φ is equivalent to diag (ϕ_i), ϕ_i a given set of desired closed-loop invariant factors,
2) $X^{-1}Y$ exists and is proper,
3) N_{RP}, Φ are right coprime and X, Φ left coprime.

If one considers invariant factor assignment as Rosenbrock-Hayton [20], then the 3rd condition is dropped.

The way in which Rosenbrock-Hayton [20] proceed is to use the fact [18] that all polynomial solutions to (2.5) can be expressed as:

$$X = \Phi U - NN_{LP}$$

$$Y = \Phi V + ND_{LP}$$

where $UD_{RP} + VN_{RP} = I$ (which is guaranteed since D_{RP}, N_{RP} are right coprime [7, 12]) and N is an arbitrary polynomical matrix. By appropriately choosing N they show that $X^{-1}Y$ exists and is proper (if the given ϕ_i satisfy (1.6)).

We proceed in the following manner: Let N_{RP}, D_{RP}, X, Y be given as:

$$D_{RP} = D_t s^t + D_{t-1} s^{t-1} + \dots + D_o$$

$$N_{RP} = N_t s^t + N_{t-1} s^{t-1} + \dots + N_o$$

$$X = X_{k-1} s^{k-1} + X_{k-2} s^{k-2} + \dots + X_o$$

$$Y = Y_{k-1} s^{k-1} + Y_{k-2} s^{k-2} + \dots + Y_o$$

and let $XD_{RP} + YN_{RP} = \Phi$,

$$\Phi = \Phi_{k+t-1} s^{k+t-1} + \dots + \Phi_o .$$

Then equating coefficients:

$$[X_{k-1}, Y_{k-1}, \dots X_o, Y_o] S_k(D_{RP}, N_{RP}) = [\Phi_{k+t-1}, \dots \Phi_o]$$

where

$$S_k(D_{RP}, N_{RP}) = \begin{bmatrix} D_t & D_{t-1} & \cdots & D_o & 0 & \cdots & 0 \\ N_t & N_{t-1} & \cdots & N_o & 0 & \cdots & 0 \\ 0 & D_t & \cdots & D_1 & D_o & \cdots & 0 \\ 0 & N_t & \cdots & N_1 & N_o & \cdots & 0 \\ & & & \vdots & & & \\ 0 & 0 & \cdots & D_t & & \cdots & D_o \\ 0 & 0 & \cdots & N_t & & \cdots & N_o \end{bmatrix} \qquad (2.6)$$

The real matrix $S_k(D_{RP}, N_{RP})$ is the k^{th} order Sylvester Resultant of D_{RP}, N_{RP} [16] (a $k(m+\ell)\times\ell(t+k)$ matrix). Clearly if a Φ exists which is equivalent to diag (ϕ_i), in the range space of $S_k(D_{RP}, N_{RP})$ where $X^{-1}Y$ exists and is proper with N_{RP}, Φ right coprime, X, Φ left coprime we have a sufficient condition for invariant factor assignment. It would therefore be very helpful if we knew the rank of the resultant operator. The following result taken from [1, 16] is crucial for our investigation.

<u>Lemma 2.1</u> Let ND^{-1} be proper, $m\times\ell$ with observability indecies μ_i. Then

$$\text{rank } S_k(D, N) = (\ell+m)k - \sum_{i:\mu_i<k} (k-\mu_i).$$

It would also be very helpful to know under what conditions is diag $(\phi_i(s))$ equivalent to some element in the range space of $S_k(D_{RP}, N_{RP})$. The following lemma taken from [20] will be used for this purpose.

<u>Lemma 2.2</u> Let α_i, β_i, be given integers satisfying $\alpha_1\geq\alpha_2\geq\ldots\geq\alpha_\ell\geq 0$, $\beta_1\geq\beta_2\geq\ldots\geq\beta_\ell\geq 0$. Let $\{\phi_i\}$ $1\leq i\leq\ell$ satisfy the divisibility conditions $\phi_i|\phi_{i-1}$ $2\leq i\leq\ell$. Then a necessary and sufficient condition for the existence of an $\ell\times\ell$ polynomial matrix $\Phi(s)$ equivalent to diag (ϕ_i) and satisfying $\lim_{s\to\infty} [\text{diag}(s^{-\alpha_i})\Phi(s)\text{diag}(s^{-\beta_i})] = I$ is

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \alpha_i+\beta_i \qquad k=1,2,\ldots,\ell \text{ with equality at } k=\ell.$$

Throughout this paper we shall use the following definition of genericity:

<u>Definition</u>: A set $S\subseteq R^t$ is called <u>generic</u> if it contains a non-empty Zariski open set of R^t [24].

3. Rosenbrock's State Feedback Result

In this section we shall use the formulation introduced in the previous section to give a new short proof of Rosenbrock's state feedback result [19].

Theorem 3.1 Let $P(s) = (sI-A)^{-1}B$ (where (A, B) is a controllable pair) with controllability indecies $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0$. Let ϕ_i $1 \leq i \leq \ell$ be given monic polynomials satisfying the divisibility conditions $\phi_i | \phi_{i-1}$, $2 \leq i \leq \ell$ and with $\sum_{i=1}^{\ell} \theta(\phi_i) = n$.

Then there exists a constant C such that the given polynomials are the non-unity invariant factors of sI-A+BC if and only if

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \lambda_i \qquad k = 1,2,\ldots,\ell, \text{ with equality at } k=\ell. \tag{3.1}$$

Proof:

(Sufficiency). P(s) is $n \times \ell$, strictly proper with observability indecies all equal to $\mu=1$. Let $P(s) = N_{RP}D_{RP}^{-1}$ where

$$\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = \begin{bmatrix} D_{hc} \\ 0 \end{bmatrix} \operatorname{diag}(s^{\lambda_i}) + L(s),$$ D_{hc} invertible, L(s) contains lower order terms, [12]. This implies that

$$D_{RP} = D_{\lambda_1} s^{\lambda_1} + D_{\lambda_1 - 1} s^{\lambda_1 - 1} + \ldots + D_0$$

$$N_{RP} = \qquad N_{\lambda_1 - 1} s^{\lambda_1 - 1} + \ldots + N_0$$

$$S_1(D_{RP}, N_{RP}) = \begin{bmatrix} D_{\lambda_1} & D_{\lambda_1 - 1} & \ldots & D_0 \\ 0 & N_{\lambda_1 - 1} & & N_0 \end{bmatrix}$$

From Lemma 2.1 rank $S_1 = n+\ell$. Now the number of non-zero columns of S_1 is $(\lambda_1+1) + (\lambda_2+1) + \ldots + (\lambda_\ell+1) = n+\ell$. Let $C = X^{-1}Y$ where

$$X = D_{hc}^{-1}$$

$$Y = Y_0 \quad \text{(constant)}.$$

Then by appropriately choosing Y_o any polynomial $\Phi(s)$ can be reached which satisfies $\lim_{s\to\infty} (\Phi(s)\mathrm{diag}(s^{-\lambda_i})) = I$. But if the $\{\phi_i\}$ satisfy (3.1) then from Lemma 2.2 with $\alpha_i=\mu_i$, $\beta_i=0$ a polynomial $\Phi^*(s)$ equivalent to $\mathrm{diag}(\phi_i(s))$ can be constructed which satisfies $\lim_{s\to\infty} (\Phi^*(s)\mathrm{diag}(s^{-\lambda_i})) = I$. Let Y_o^* be the Y_o that corresponds to it. Then $C = D_{hc}Y_o^*$ is the desired compensator since $G(s) = (sI-A+BC)^{-1}B = N_{RP}\Phi^*(s)^{-1}$ both being irreducible representations ([12], Lemma 6.5-9, p. 446).

(Necessity). Suppose that C is a constant that makes the closed loop invariant factor be equal to $\{\phi_i\}$. Let $G(s) = N_{RP}(D_{RP}+CN_{RP})^{-1}$. Clearly N_{RP}, $\Psi = D_{RP}+CN_{RP}$ are right coprime.

This means that the invariant factors of Ψ are the $\{\phi_i\}$.

Now
$$\lim_{s\to\infty} (\Psi\ \mathrm{diag}(s^{-\lambda_i})) = \lim_{s\to\infty} \left([I\ C]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \mathrm{diag}(s^{-\lambda_i})\right)$$
$$= D_{hc}$$

where $\det D_{hc} \neq 0$. Clearly $\Psi_1 = D_{hc}^{-1}\ \Psi$ satisfies $\lim_{s\to\infty} (\Psi_1\ \mathrm{diag}(s^{-\lambda_i}))$ $= I$ and has invariant factors $\{\phi_i\}$. From Lemma 2.2 the degrees of the $\{\phi_i\}$ must satisfy (3.1).

4. Rosenbrock-Hayton's Output Feedback Result

Ideas developed in section 2 can be used to give a new short proof of Rosenbrock-Hayton's output feedback result [20].

Theorem 4.1 Let $P(s) = N_{RP}D_{RP}^{-1}$ be an $m \times \ell$ strictly proper transfer function of order n, with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ column reduced, with column indecies $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$ (λ_i controllability indecies), and μ_1 the largest observability index. Let ϕ_i $1 \leq i \leq \ell$ be given polynomials satisfying the divisibility conditions $\phi_i | \phi_{i-1}$ and with $\sum_{i=1}^{\ell} \theta(\phi_i) = n + \ell(\mu_1 - 1)$.

Then a sufficient condition for the existense of a paper $\ell \times m$ compensator $C(s) = A_{LC}^{-1}B_{LC}$ such that the invariant factors of $A_{LC}D_{RP} + B_{LC}N_{RP}$ are the $\{\phi_i\}$ is:

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} (\lambda_i + \mu_1 - 1) \qquad k = 1,2,\dots,\ell \text{ with equality at } h=\ell. \qquad (4.1)$$

Proof:
$$\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = \begin{bmatrix} D_{hc} \\ 0 \end{bmatrix} \operatorname{diag}(s^{\lambda_i}) + L(s),$$ D_{hc} invertible with $L(s)$ containing lower order terms. This implies that

$$D_{RP} = D_{\lambda_1} s^{\lambda_1} + D_{\lambda_1 - 1} s^{\lambda_1 - 1} + \dots + D_0$$
$$N_{RP} = N_{\lambda_1 - 1} s^{\lambda_1 - 1} + \dots + N_0$$

and $S_{\mu_1}(D_{RP}, N_{RP})$ the μ_1-order Sylvester resultant of D_{RP}, N_{RP} (2.6).

From Lemma 2.1 $S_{\mu_1}(D_{RP}, N_{RP})$ is rank $n+\ell\mu_1$. Now the number of non-zero columns of $S_{\mu_1}(D_{RP}, N_{RP})$ is $(\lambda_1+1)+(\lambda_2+1) + \dots + (\lambda_\ell+1) + (\mu_1-1)\ell = n+\ell\mu_1$.

Let $C = X^{-1}Y$ where

$$X = X_{\mu_1 - 1} s^{\mu_1 - 1} + \dots + X_0$$
$$Y = Y_{\mu_1 - 1} s^{\mu_1 - 1} + \dots + Y_0 \,.$$

This means:

$$[X_{\mu_1-1}, Y_{\mu_1-1}, \ldots, X_o, Y_o] S_{\mu_1}(D_{RP}, N_{RP}) = [\Phi_{\lambda_1+\mu_1-1}, \ldots, \Phi_o].$$

Since $S_{\mu_1}(D_{RP}, N_{RP})$ is rank $n+\ell\mu_1$ any polynomial $\Phi(s)$ which satisfies

$$\lim_{s\to\infty}(\mathrm{diag}(s^{-(\mu_1-1)})\Phi(s)\mathrm{diag}(s^{-\lambda_i})) = H$$

H constant can be reached. In particular any polynomial $\Phi(s)$ for which the corresponding H is the identity. Since

$$[X,\ Y]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = X_{\mu_1-1} D_{hc}\mathrm{diag}(s^{\lambda_i+\mu_1-1}) + \overline{L}(s)$$

($\overline{L}(s)$ contains lower order terms) the X corresponding to such a $\Phi(s)$ must have $X_{\mu_1-1} = D_{hc}^{-1}$ ie $C = X^{-1}Y$ must exist and be proper [12].

But if the $\{\phi_i\}$ satisfy 4.1 then from Lemma 2.2 with $\alpha_i=\lambda_i$ $\beta_i=\mu_1-1$ a polynomial $\Phi^*(s)$ equivalent to $\mathrm{diag}(\phi_i(s))$ can be constructed which satisfies

$$\lim_{s\to\infty}(\mathrm{diag}(s^{-(\mu_1-1)})\Phi^*(s)\mathrm{diag}(s^{-\lambda_i})) = I.$$

Then $C = X^{-1}Y$, for any X, Y which maps to $\Phi^*(s)$ is a compensator which satisfies the requirements of the theorem.

5. The Equation $XD + YN = \Phi$

It has been mentioned in section 2 that this polynomial matrix equation plays a key role in the analysis of the invariant factor problem. Using Sylvester Resultants we are able to prove the following two results.

Proposition 5.1 Let $P = N_{RP}D_{RP}^{-1}$ be an $m\times\ell$ strictly proper transfer function where $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ is column reduced, with column degrees $\lambda_i = \lambda$ and observability indecies $\mu_i = \mu$,

$$D_{RP} = Is^{\lambda} + D_{\lambda-1}s^{\lambda-1} + \ldots + D_o, \; N_{RP} = N_{\lambda-1}s^{\lambda-1} + \ldots + N_o \; .$$

Let $$Z = \{\Phi\varepsilon R^{(\lambda+q)\ell^2} | \Phi = Is^{\lambda+q} + \Phi_{\lambda+q-1}s^{\lambda-1} + \ldots + \Phi_o\}$$

$$Q = \{(X,Y) | X = Is^q + X_{q-1}s^{q-1} + \ldots X_o, \; Y = Y_q \; s^q \; + \ldots Y_o\}$$

A necessary and sufficient condition for the existence of a polynomial solution X, Y to $XD_{RP} + YN_{RP} = \Phi$ in the class Q, for every Φ in Z is $q \geq \mu-1$.

Equation $XD_{RP} + YN_{RP} = \Phi$ with the conditions imposed can be written as:

$$[I, Y_q, \ldots X_o, Y_o]S_{q+1}(D_{RP}, N_{RP}) = [I, \Phi_{\lambda+q-1}, \ldots, \Phi_o],$$

where $S_{q+1}(D_{RP}, N_{RP})$ is thought of as an operator from $R^{\ell(\ell+m)(q+1)}$ into $R^{\ell(\lambda+q+1)\ell}$. From Lemma 2.1

$$\begin{aligned} \text{rank } S_k &= (\ell+m)k & 1 \leq k \leq \mu \\ &= (\ell+m)k - m(k-\mu) & \mu < k \end{aligned} \; .$$

Therefore

a) $S_1, S_2, \ldots S_{\mu-1}$ are not onto,

b) S_μ is onto and one-one,

c) $S_{\mu+1}, S_{\mu+2} \ldots$ are onto.

Proof:

(Necessity). Assume that $q<\mu-1$ and that $XD_{RP} + YN_{RP} = \Phi$ has a solution in Q for every Φ in Z. Thinking of (X, Y) as an element in $R^{\ell((\ell+m)q+m)}$

and Φ as an element in $R^{\ell(\lambda+q)\ell}$ one can see that the set of Φ which can be reached from elements in Q, form a set of dimension less than $\ell(q+\lambda)\ell$. Therefore $q \geq \mu-1$.

(Sufficiency). $S_{\mu+k}$ in onto for $k \geq 0$. This means that for any Φ in Z there exist X, Y given by

$$X = X_{\mu+k-1}S^{\mu+k-1} + \ldots + X_0$$

$$Y = Y_{\mu+k-1}S^{\mu+k-1} + \ldots + Y_0 .$$

such that

$$[X_{\mu+k-1}, Y_{\mu+k-1}, \ldots X_0, Y_0]S_{\mu+k} = [I, \Phi_{\lambda+\mu+k-2}, \ldots \Phi_0]$$

For this to happen $X_{\mu+k-1} = I$ which implies that $(X, Y) \in Q$.

This result addresses the following question: Suppose we fix the order of the proper compensator (order is $q\ell$), as well as the observability indecies (all equal to q). What are the possible Φ (closed loop denominator matrix) that can be reached. Since $\lambda_i = \lambda$ and $\mu_i = \mu$ the result concerns the "generic" case. In the next section these concepts will be used for obtaining necessary and sufficient conditions for generic invariant factor assignment.

A polynomial solution X, Y to $XD + YN = \Phi$ is called an acceptable solution if $X^{-1}Y$ exists (ie $\det X \neq 0$) and is proper.

<u>Proposition 5.2</u> Let N ($m \times \ell$), D ($\ell \times \ell$), Φ ($\ell \times \ell$) be polynomial matrices, λ, q non-negative integers, $n = \lambda\ell$ $\mu = \frac{n}{m}$, and let W, Z, S be the sets:

$$W = \{(N,D) \in R^{(m+\ell)n} \mid D = Is^{\lambda}+D_{\lambda-1}s^{\lambda-1}+\ldots+D_0,\ N=N_{\lambda-1}s^{\lambda-1}+\ldots+N_0\}$$

$$Z = \{\Phi \in R^{(\lambda+q)\ell^2} \mid \Phi = Is^{\lambda+q}+\Phi_{\lambda+q-1}s^{\lambda+q-1}+\ldots+\Phi_0\}$$

$$S = \{(N,D) \in R^{(m+\ell)n} \mid \begin{array}{l}\text{For which there exists an acceptable solution}\\ \text{X, Y to } XD+YN=\Phi \text{ for every } \Phi \text{ in Z.}\end{array}\}$$

A necessary and sufficient condition for S to be a generic subset of $R^{(m+\ell)n}$ is $q \geq \mu-1$.

Proof:

(Necessity). Suppose that S is generic and that $q<\mu-1$. Show a contradiction.

The set $F \subseteq R^{(m+\ell)n}$ for which rank $S_i(D,N)=(m+\ell)i$ for $1\leq i\leq\mu$ and rank $S_{\mu+1}(D,N)=(m+\ell)\mu+\ell$ is generic. It certainly contains a Zariski open set and it is non-empty since any (N,D) which gives rise to an ND^{-1} with equal observability indecies must belong to it (Lemma 2.1). As a matter of fact any (N,D) in F must have equal observability indecies because otherwise one of the rank conditions would be violated (Lemma 2.1). It follows (Corollary 1 Bitmead et al.) that for any (N,D) in F, N,D are right coprime, and ND^{-1} has equal controllability indecies as well.

Let N,D be an element in $S\cap F$. For any such element (Corollary 2, [20], p.848) it must be that $\theta(Y)\leq q$, (X,Y) some acceptable solution. But this implies that any acceptable solution must be of the form:

$$X = Is^q+X_{q-1}s^{q-1}+\ldots+X_0 \qquad Y = Y_qs^q+\ldots+Y_0$$

which means $(X,Y)\in Q$, Q as in proposition 5.1 and $q\geq\mu-1$. This is a contradiction.

(Sufficiency). Let $q\geq\mu-1$. From Proposition 5.1 we have that $S_{q+1}(D,N)$ is onto for any (N,D) in F. This means that there exist X,Y such that

$$X = X_ss^q+\ldots+X_0 \qquad Y = Y_qs^q+\ldots Y_0$$

and $$[X_q,Y_q,\ldots X_0,Y_0]S_{q+1}(D,N)=[I,\Phi_{\lambda+q-1},\ldots,\Phi_0]$$

for every Φ in Z. It follows that $X_q=I$ and that (X,Y) is an acceptable solution.

This result is stronger than Proposition 5.1. The restriction of searching for a solution of $XD+YN=\Phi$ that belongs to a certain class Q can be removed. This is because an acceptable solution (X,Y) to $XD+YN=\Phi$ with N,D,Φ as above, must belong to the class Q, there are no acceptable solutions outside Q.

The previous two results dealt with the case when $P = N_{RP} D_{RP}^{-1}$ is a strictly proper transfer function with equal controllability indecies. Knowledge of the rank of the resultant operators $S_i(D_{RP}, N_{RP})$ played a key role in their proof. Proposition 5.3 addresses the question of the generic rank of $S_i(D, N)$ when $P = N D^{-1}$ is an $\ell x \ell$ strictly proper transfer function with an arbitrary set of controllability indecies $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell \geq 0$.

Proposition 5.3. Let $P = N D^{-1}$ be our $\ell x \ell$ strictly proper transfer function with controllability indecies $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell > 0$, $n = \lambda_1 + \ldots + \lambda_\ell$, $\mu = \frac{n}{\ell}$ and

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \text{diag } (s^{\lambda_i}) + L(s)$$

L(s) contains lower order terms. The pair (N, D) can be thought of as an element of $R^{2n\ell}$. The set $W \subset R^{2n\ell}$, of (N, D) for which $S_i(D, N)$ has rank $2\ell i$ for $1 \leq i \leq \mu$ is a generic subset of $R^{2n\ell}$.

Proof:

It is clear that the set of (N, D) for which $S_i(N, D)$ is rank $2i\ell$ for $1 \leq i \leq \mu$ is a Zariski open set. The difficulty lies in proving that it is indeed non-empty.

We first construct the submatrix $T_\mu(N)$ of $S_\mu(D, N)$ and show that it is generically full rank.

$$T_\mu(N) = \begin{bmatrix} N_{\lambda_1-\mu} & \cdots & & N_0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots \\ N_{\lambda_1-2} & & & & & N_0 & 0 \\ N_{\lambda_1-1} & N_{\lambda_1-2} & \cdots & & & N_1 & N_0 \end{bmatrix} = [B_{\lambda_1-1}, \ldots B_0]$$

where B_i is the ith $\mu\ell x\ell$ block column of $T_\mu(N)$. Form $T_\mu{}^*(N)$ from $T_\mu(N)$ deleting the columns indicated:

From the ℓth columns of the B_j's keep the ℓth columns of $B_0,\ldots,B_{\lambda_\ell-1}$ and remove the rest.

From the $(\ell-1)$st columns of the B_j's keep the $(\ell-1)$st columns of $B_0,\ldots,B_{\lambda_{\ell-1}-1}$ and remove the test. And in general from the $(\ell-i)$th columns of the B_j's keep the $(\ell-i)$th columns of $B_0,\ldots,B_{\lambda_{\ell-i}-1}$ and remove the rest.

What remains is the nxn matrix $T_\mu{}^*(N)$.

Clearly the set $V = \{N \subset R^{n\ell} \mid T_\mu{}^*(N) \text{ rank } n\}$ is a Zariski open set in $R^{n\ell}$. We claim that it is non-empty. This can be seen from the fact that an N^* exists that makes $T_\mu{}^*(N^*)$ after a proper rearrangement of its columns equal to a lower triangular matrix with 1's on the main diagonal.

It is now easy to see that $S_\mu(I \operatorname{diag}(s^{\lambda_i}), N^*)$ is rank $2\mu\ell$ (and that rank $S_i(I \operatorname{diag}(s^{\lambda_i}), N^*) = 2i\ell$ for $1 \le i \le \mu$).

Expressing it in a different way Proposition 5.3 guarantees that "almost all" $\ell x\ell$ strictly proper transfer functions of McMillan degree $n = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$, $\mu = \frac{n}{\ell}$ with arbitrary controllability indecies $\lambda_1 \ge \lambda_2 \ge ,,, \ge \lambda_\ell$, have equal observability indecies μ.

6. Invariant Factor Results

In section 2 we made a distinction between what we call the invariant factors of the closed loop system and what has been used by Rosenbrock-Hayton [20]. The difference arises because we insist on working with irreducible polynomial matrix descriptions (PMD's 12) whereas Rosenbrock-Hayton [20] choose not to. It can be said that since the difference is due to possible cancellations that in some "generic" sense the two definitions are the same. In this section we will exploit the ideas presented thus far in order to obtain necessary and sufficient conditions for invariant factor assignment in several cases.

Theorem 6.1 Let $P=ND^{-1}$ be an $\ell x \ell$ strictly proper transfer function where $\lambda>o$ and

$$D = Is^{\lambda}+D_{\lambda-1}s^{\lambda-1}+\ldots+D_o$$

$$N = \quad N_{\lambda-1}s^{\lambda-1}+\ldots+N_o$$

Let $q\geq 0$ and $\phi(s) = s^{\lambda+q}+\phi_{\lambda+q-1}s^{\lambda+q-1}+\ldots+\phi_o$.

Let $W = \{(N,D,\phi)\varepsilon R^{2\lambda\ell^2+\lambda+q} \mid$ For which there exists a proper compensator of order $\ell\cdot q$ making $\phi_1=\phi_2=\ldots\phi_\ell=\phi$ the invariant factors of the closed loop system. $\}$

Then $q\geq\lambda-1$ is a necessary and sufficient condition for W to be a generic subset of $R^{2\lambda\ell^2+\lambda+q}$.

Proof:

(Sufficiency). Let $q\geq\lambda-1$ and $t=2\lambda\ell^2+\lambda+q$. Show that W contains a non-empty Zariski open set. Let Φ be the diagonal matrix

$$\Phi = \text{diag}(\phi_i) \ , \ \phi_i = \phi. \tag{6.1}$$

Since $q\geq\lambda-1$ (we are dealing with the square case $\ell=m, \mu=\frac{\ell.\lambda}{m}=\lambda$). The set $F \subseteq R^t$, of (N,D,ϕ) for which $S_{q+1}(D,N)$ is full rank is generic. Let

$\bar{S}_{q+1}$ be the matrix $([q+1)m+\ell q]x(\lambda+q)\ell)$ obtained from S_{q+1} by removing the first ℓ rows and ℓ columns. Clearly $\bar{S}_{q+1}$ is generically full rank. This implies that for any Φ as in 6.1 there exists $X=Is^q+\ldots+X_o$, $Y=Y_q s^q+\ldots+Y_o$ such that

$$[Y_q,\ldots X_o,Y_o]\bar{S}_{q+1} = [\Phi_{\lambda+q-1},\ldots,\Phi_o]-[D_{\lambda-1},\ldots D_o,0\ldots 0]$$

One such $X,Y(C=X^{-1}Y$ proper of order $\ell q)$ is given by:

$$[Y_q,X_{q-1},\ldots,X_o,Y_o] = ([\Phi_{\lambda+q-1},\ldots,\Phi_o]-[D_{\lambda-1},\ldots 0]) \times(\bar{S}^T_{q+1}\cdot\bar{S}_{q+1})^{-1} S^T_{q+1} \tag{6.2}$$

$\bar{S}^T_{q+1}\cdot\bar{S}_{q+1}$ is invertible since $\bar{S}_{q+1}$ is full rank and with fewer columns than rows.

This means that X_i, Y_i are rational expressions in the parameters of N, D, ϕ. Now the set $E \subseteq R^t$ for which Φ, X are left coprime and N, Φ right coprime is a Zariski open set, since coprimeness is a condition satisfied when certain matrices are full rank (Corollary 1, [1]).

The key point to demonstrate is that E is non empty. We claim that $\alpha = (N_\alpha, D_\alpha, \phi_\alpha)$

$$D_\alpha = Is^\lambda \ , \ N_\alpha = I \ , \ \Phi_\alpha = Is^{\lambda+q}+I$$

is a point in E. Clearly N_α, Φ_α are right coprime since $\begin{bmatrix} \Phi_\alpha \\ N_\alpha \end{bmatrix}$ is full rank for all s [19]. Now for this specific Φ_α the solution given in (6.2) can easily be computed to be

$$X_\alpha = Is^q \ , \ Y_\alpha = I \ .$$

Clearly X_α , Φ_α are left coprime.

Since $G(s) = N(XD+YN)^{-1}X$ for any (N,D,ϕ) in E with X,Y given by (6.2) is a least order (irreducible) polynomial matrix fraction description of G(s) and

$$XD + YN = \Phi$$

the invariant factors of $G(s)$ are the ϕ_i. Clearly $E \subseteq W$ and thus W is a generic subset of R^t.

(Necessity). Let $W \subseteq R^t$ be the set of (N,D,ϕ) for which there exists a proper compensator $C = X^{-1}Y$ which makes $\phi_1=\phi_2=\ldots=\phi_\ell=\phi$ the closed loop invariant factors. Assume that it is generic.

For any $\alpha\epsilon W$ the following must be true. Let $C = X^{-1}Y$ be the proper compensator of order ℓq that accomplishes the task. We choose X to be row reduced and let $q_1\geq q_2\geq\ldots\geq q_\ell\geq o$ be the row degrees $q_1+q_2+\ldots+q_\ell=\ell q$, and $X_{hc} = I$, the highest row degree coefficient matrix of X.

The following three facts must be true for such an α.

1) The matrices N, $XD+YN$ must be right coprime and X, $XD+YN$ must be left coprime. Since

$$\theta(\det(XD+YN))= \lambda\ell+\ell q$$

if there are cancellations then the resulting denominator matrix will be $\overline{XD+YN}$ where

$$\theta(\det(\overline{XD+YN}))<\lambda\ell+\ell q.$$

But then the $\{\phi_i=\phi\}$ could not be the invariant factors of the closed-loop system since $\sum_{i=1}^{\ell}\theta(\phi_i) = \lambda\ell+\lambda q$.

2) In actuality the row degrees of a row reduced representation of $C = X^{-1}Y$ must all be equal to q.

Now $$\lim_{s\to\infty} \operatorname{diag}(s^{-q_i})[X\ Y]\begin{bmatrix} D \\ N \end{bmatrix}\operatorname{diag}(s^{-\lambda}) = I.$$

From 1) $XD+YN$ has invariant factors $\{\phi_i=\phi\}$. From Lemma 2.2 this implies that

$$\sum_{i=1}^{k}\theta(\phi_i)\geq\sum_{i=1}^{k}\lambda+q_i \quad k=1,2,\ldots \text{ with equality at } k=\ell.$$

This implies $\lambda+q \geq \lambda+q_1$ If $q_1<q$ then $q_1+\ldots+q_\ell\neq\ell q$.

Therefore $q_1=q$. But $2\lambda+2q \geq 2\lambda+q_1+q_2 \Rightarrow q_2=q$ and generally $q_i=q$.

3) Since X,Y must be of the form

$$X = Is^q+X_{q-1}s^{q-1}+\ldots+X_o\ ,\ Y = Y_qs^q+Y_{q-1}s^{q-1}+\ldots+Y_o$$

it must be that

$$XD+YN = \begin{bmatrix} \phi & & \\ & \phi & 0 \\ & & \ddots \\ 0 & & & \phi \end{bmatrix},\ \phi \text{ the given polynomial.}$$

In general

$$XD+YN = Is^{\lambda+q}+\phi_{\lambda+q-1}s^{\lambda+q-1}+\ldots+\phi_o = \overline{\Phi}$$

But since the invariant factors of $\overline{\Phi}$ are $\phi_1=\phi_2=\ldots=\phi_\ell=\phi$ this means that the gcd of 1x1 minors in particular must be ϕ. This is a polynomial of degree $\lambda+q$. All off-diagonal entries in $\overline{\Phi}$ are of lesser degree or zero. They cannot be of lesser degree therefore they are zero.

With all this in mind suppose now that $q<\lambda-1$. Then for any α in W, ie for generic α we must have

$$[I,Y_q,\ldots,X_o,Y_o]S_{q+1}(D,N)=[I,\underbrace{\phi_{\lambda+q-1},\ldots\phi_o}]$$
$$= \Phi \text{ diagonal}$$

Look at the first row of this matrix equation:

$$\underline{y}\cdot S_{q+1}(D,N)=[\underbrace{1,0\ldots0}_{\ell},\ \underbrace{\phi_{\lambda+q-1},0\ldots0}_{\ell},\ldots,\underbrace{\phi_o,0\ldots0}_{\ell}] = \underline{\phi}$$

where $\underline{y}$ is a $1\text{x}2(q+1)\ell$ vector, $S_{q+1}(D,N)$ a $2(q+1)\ell\text{x}(\lambda+q+1)\ell$ matrix and $\underline{\phi}$ a $(\lambda+q+1)\ell$ vector.

Let $u = 2(q+1)\ell\ ,\ v = (\lambda+q+1)\ell$.

If we partition $S_{q+1}(D,N) = [A',B]$

where A is uxu and B is uxv since $q<\lambda-1$ it is clear that $u<v$. The matrix A is invertible for a generic subset of R^t (A has a resultant structure).

$$y = \underline{\phi}_u A^{-1}$$

where $\underline{\phi}_u$ is the first u entries in the $\underline{\phi}$ vector. But since there are more equations than unknowns it must also be that

$$yB = \underline{\phi}_{v-u} \tag{6.3}$$

where $\underline{\phi}_{v-u}$ are the remaining v-u entries in $\underline{\phi}$. But relationship (6.3) is only satisfied on a Zariski closed set of R^t. This is a contradiction. This completes the necessity part of the theorem.

Theorem 6.1 in effect says that for almost all strictly proper transfer functions of McMillan degree $n=\lambda\ell$ and equal controllability indecies, a necessary and sufficient condition for the existence of a proper compensator of order ℓq which makes the closed-loop invariant factors equal to $\phi=\phi_1=\ldots=\phi_\ell$ for almost all ϕ of degree $\lambda+q$ is $q\geq\lambda-1$. It is necessary therefore that the order of the compensator be greater than or equal to $\ell(\lambda-1)$. It should be emphasized that here we are considering the square case where $\lambda=\mu$, μ the observability index of the transfer function P. Thus $\ell(\mu-1)$ is the more appropriate bound.

Theorem 6.1 addresses the case of strictly proper transfer function of McMillan degree $n = \lambda\ell$ and equal controllability indecies. Theorems 6.2, 6.3 and 6.4 deal with the more general case.

<u>Theorem 6.2</u>. Let $P = ND^{-1}$ be an $\ell x \ell$ strictly proper transfer function where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0$, $n = \lambda_1 + \lambda_2 + \ldots \lambda_\ell$, $\mu = \frac{n}{\ell}$, and

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \operatorname{diag}(s^{\lambda_i}) + L(s)$$

with L(s) containing lower order terms.

Let $\{\phi_i\}$ be ℓ monic polynomials $\phi_i | \phi_{i-1}$ $2 \leq i \leq \ell$ such that $\theta(\phi_i) = \lambda_i + q$.

Let $W = \{(N,D,\phi_1,\ldots,\phi_\ell\} \in R^{2n\ell+\lambda_1+q}$ $\Big|$ $\left\{\begin{array}{l}\text{For which there exists a proper compen-} \\ \text{sator of order lq making } \{\phi_i\} \text{ the invari-} \\ \text{ant factors of the closed loop system.}\end{array}\right\}$

Then $q \geq \mu - 1$ is a necessary condition for W to be a generic subset of $R^{2n\ell+\lambda_1+q}$.

<u>Proof</u>: Let $W \subset R^t$ $(t = 2n\ell+\lambda_1+q)$ be the set of $(N,D,\phi,\ldots,\phi_\ell)$ for which there exists a proper compensator $C = X^{-1}Y$, of order ℓq which makes $\{\phi_i\}$ the closed loop invariant factor. Assume that it is generic.

For any $\alpha \epsilon W$ the following must be true. Let $C = X^{-1}Y$ be the proper compensator that accomplishes the task. We choose X to be row reduced and let $q_1 \geq q_2 \geq \ldots q_\ell \geq 0$ be the row degrees $q_1 + q_2 + \ldots + q_\ell = \ell_q$ and $X_{hc} = I$ the highest row degree coefficient matrix of X.

The following three properties must hold for such an α.

1) The matrices N, $XD + YN$ must be right coprime and $X, X^{D} + YN$ mus. be lest coprime, (follows proof of Theorem 6.1).

2) The row degrees of a row reduced representation of $C = X^{-1}Y$ must all be equal to q.

   Now $\lim_{s\to\infty} \text{diag}(s^{-q_i})\,[X\ Y]\begin{bmatrix} D \\ N \end{bmatrix}\text{diag}(s^{-\lambda_i}) = I.$

   From 1) $XC + YN$ has invariant factors $\{\phi_i\}$. From Lemma 2.2 this implies that

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \lambda_i + q_i \qquad i = 1,2,\ldots k \text{ with equality at } k = \ell.$$

   This means that $\lambda_1 + q \geq \lambda_1 + q_i$. If $q_1 < q$ then $q_1 + \ldots + q_\ell \neq \ell q$. Therefore $q_1 = q$. Proceeding in the same fashion $\lambda_1 + \lambda_2 + 2q \geq \lambda_1 + \lambda_2 + q_1 + q_2$, means $q_2 = q$ and so on, resulting in $q_i = q$ $\quad 1 \leq i \leq \ell$.

3) Since X, Y must be of the form

$$X = Is^q + X_{q-1}s^{q-1} + \ldots + X_0, \qquad Y = Y_q\, s^q + \ldots + Y_0$$

it must be that

$$XD + YN = \begin{bmatrix} \phi_1 & & & & \\ X & \phi_2 & & & 0 \\ X & X & \phi_3 & & \\ & & & \ddots & \\ X & X & X & & \phi_\ell \end{bmatrix}$$

where $\{\phi_i\}$ are the given polynomials and the X's indicate possible non-zero locations. In general

$$XD + YN = I\ \mathrm{diag}(s^{\lambda_i+q}) + Q(s) = \Phi\ .$$

Since $\{\phi_i\}$ are the invariant factors of Φ, this means in particular that $\phi_\ell(\theta(\phi_\ell) = \lambda_\ell + q)$ is the gcd of 1x1 minors of Φ. Since off-diagonal entries in the ℓth column are of degree less than $\lambda_\ell + q$ they must be zero. Furthermore the gcd of 2x2 minors of Φ is $\phi_\ell\cdot\phi_{\ell-1}(\theta(\phi_\ell\cdot\phi_{\ell-1}) = \lambda_\ell + \lambda_{\ell-1} + 2q)$. Since off-diagonal entries in the ℓ-1 column are of degree less than $\lambda_{\ell-1} + q$, above the diagonal entries must be zero. Continuing in this fashion we see that claim 3) is true.

Suppose now that $q < \mu - 1$. Then for any α in W(i.e. for generic α) we must have:

$$[I,\ Y_q,\ldots,X_0,\ Y_0]\ \overline{S}_{q+1}(D,\ N) = [\phi_{\lambda_1+q},\ \ldots,\ \phi_0]$$

where $\overline{S}_{q+1}(D, N)$ is the $(q+1)2\ell \times n + \ell(q+1)$ matrix obtained from $S_{q+1}(D, N)$ by deleting its all zero columns. Looking at the first row of this matrix equation:

$$\underline{y}\ \overline{S}_{q+1}(D,\ N) = [1,\ldots,\phi_{\lambda_1+q-1},0,\ldots,0,\ldots,\phi_0,0,\ldots,0] = \underline{\phi}$$

A contradiction follows as in the proof of Theorem 6.1, where now use is made of Proposition 5.3. This completes the proof of Theorem 6.2.

We see that as in the case of equal controllability indecies $q \geq \mu - 1$ is a necessary condition for generic invariant factor assignment, where μ is the observability index of the transfer function P. We believe than $q = \mu - 1$ is a sufficient condition as well, as can be seen from the next two results, where it is shown to be the case when the controllability indecies are as in a) or b).

a) $2|\ell$, $k \geq 1$, $\lambda_1 = \ldots = \lambda_{\frac{\ell}{2}} = k + 2$, $\lambda_{\frac{\ell}{2}+1} = \ldots = \lambda_\ell = k$.

b) $3|\ell$, $k \geq 1$, $\lambda_1 = \ldots = \lambda_{\frac{\ell}{3}} = k + 2$, $\lambda_{\frac{\ell}{3}+1} = \ldots \lambda_{\frac{2\ell}{3}} = k + 1$,

$$\lambda_{\frac{2\ell}{3}+1} = \ldots = \lambda_\ell = k .$$

<u>Theorem 6.3</u>. Let $P = ND^{-1}$ be an $\ell x \ell$ strictly proper transfer function where ℓ is even, $k \geq 1$,

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{\frac{\ell}{2}} = k + 2 \;,\; \lambda_{\frac{\ell}{2}+1} = \ldots = \lambda_\ell = k \;,\; \mu = k + 1$$

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \operatorname{diag}(s^{\lambda_i}) + L(s) .$$

Let $\{\phi_i\}$ be ℓ monic polynomials $\phi_i | \phi_{i-1}$ $2 \leq i \leq \ell$ such that $\theta(\phi_i) = \lambda_i + q$. Let

$$W = \left\{ (N,D,\phi_1,\ldots,\phi_\ell) \; \varepsilon R^{2n\ell+\lambda_1+q} \;\middle|\; \begin{array}{l} \text{For which there exists a proper} \\ \text{compensator of order } \ell \cdot q \text{ making} \\ \{\phi_i\} \text{ the invariant factor of} \\ \text{the closed loop system} \end{array} \right\} .$$

Then $q = \mu - 1$ is a sufficient condition for W to be a generic subset of $R^{2n\;+\lambda_1+q}$.

Proof:

Let $q = \mu - 1$ and $t = 2n\ell + \lambda_1 + q$. Show that W contains a non-empty Zariski open set. Since $q = \mu - 1$ the set $F \subseteq R^t$ of $(N,D,\phi_1,\ldots,\phi_\ell)$ for which $S_\mu(D, N)$ is full rank is generic. This implies that for any Φ of the form

$$\Phi = \mathrm{diag}(\phi_i) ,$$

a unique solution to $X = Is^q + \ldots + X_0 \quad Y = Y_q s^q + \ldots + Y_0$ to $XD + YN = \Phi$ exists given by:

$$[I, Y_q, X_{q-1}, \ldots X_0, Y_0] = [\overline{\Phi}_{\lambda_1+q},\ldots,\overline{\Phi}_0]\overline{S}_\mu^{-1}$$

This means that X_i, Y_i are rational expressions in the parameters of $N,D,\phi_1,\ldots,\phi_\ell$. Now the set $E \subseteq R^t$ for which Φ, X are left coprime and N,Φ right coprime is a Zariski open set. They key point is to demonstrate that E is non empty. It can be verified after some algebraic manipulations that with

$$D = \begin{bmatrix} I_{\frac{\ell}{2}} s^{k+2} & 0 \\ 0 & I_{\frac{\ell}{2}} s^k \end{bmatrix} \qquad N = \begin{bmatrix} I_{\frac{\ell}{2}} & 0 \\ I_{\frac{\ell}{2}} s & I_{\frac{\ell}{2}} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} I_{\frac{\ell}{2}} \phi_1 & 0 \\ 0 & I_{\frac{\ell}{2}} \phi_2 \end{bmatrix} \text{ where } \phi_2 = (s+1)^{2k},\ \phi_1 = (s+1)^{2k}(s^2-1) ,$$

$\overline{S}_\mu(D,N)$ is invertible and X,Φ are left coprime and Φ,N right coprime.

Theorem 6.4. Let $P = ND^{-1}$ be an $\ell x \ell$ strictly proper transfer function where $3|\ell$, $k \geq 1$, $\mu = k+1$,

$$\lambda_1 = \ldots = \lambda_{\frac{\ell}{3}} = k + 2,\ \lambda_{\frac{\ell}{3}+1} = \ldots = \lambda_{\frac{2\ell}{3}} = k + 1,\ \lambda_{\frac{2\ell}{3}+1} = \ldots = \lambda_\ell = k$$

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \operatorname{diag}(s^{\lambda_i}) + L(s)\ .$$

Let $\{\phi_i\}$ be ℓ monic polynomials $\phi_i|\phi_{i-1}$ $2 \leq i \leq \ell$ such that $\theta(\phi_i) = \lambda_i + q$.

Let

$$W = \left\{(N,D,\phi_1,\ldots,\phi_\ell)\ \varepsilon R^{2n\ell+\lambda_1+q} \;\middle|\; \begin{array}{l}\text{For which there exists a}\\ \text{proper compensator of order}\\ \ell q \text{ making } \{\phi_i\} \text{ the invariant}\\ \text{factors of the closed loop}\\ \text{system}\end{array}\right\}$$

Then $q = \mu - 1$ is a sufficient condition for W to be a generic subset of $R^{2n\ell+\lambda_1+q}$.

Proof: The proof proceeds in a similar fashion as Thm. 6.3. The point which shows the non-emptiness of W is the following:

$$D = \begin{bmatrix} I_{\frac{\ell}{3}} s^{k+2} & 0 & 0 \\ 0 & I_{\frac{\ell}{3}} s^{k+1} & 0 \\ 0 & 0 & I_{\frac{\ell}{3}} s^k \end{bmatrix} \qquad N = \begin{bmatrix} I_{\frac{\ell}{3}} & 0 & 0 \\ 0 & I_{\frac{\ell}{3}} & 0 \\ I_{\frac{\ell}{3}} s & 0 & I_{\frac{\ell}{3}} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} I_{\frac{\ell}{3}}\phi_1 & 0 & 0 \\ 0 & I_{\frac{\ell}{3}}\phi_2 & 0 \\ 0 & 0 & I_{\frac{\ell}{3}}\phi_3 \end{bmatrix}$$

where $\phi_3 = (s+1)^{2k}$, $\phi_2 = (s+1)^{2k+1}$, $\phi_3 = (s+1)^{2k+1}\ (s-1)$.

Remark: Sufficiency proofs for many more controllability index configurations have been constructed by using a different test point in each case. This makes it difficult to construct a general test point that can be used in every case.

Now it is evident that there are two basic issues concerning invariant factor assignment. One is the allowable degrees of the closed loop invariant factors (i.e. the sizes of the attainable cyclic blocks of A, where A comes from a minimal realization of G(s)) and the other is the reachable invariant polynomials themselves. That is assuming an allowable set of degrees is it possible to reach all (or almost all) such polynomials. The necessary condition appearing in Rosenbrock-Hayton [20] is addressing the allowable degrees issue. Here we have assumed a particular degree configuration (which incidentally is compatible with their conditions) and are investigating the issue of the order of the compensator needed for almost arbitrary invariant factor assignment.

It is important to mention that different degree assignments require different order compensators. This is evident from Theorem 6.5.

Theorem 6.5 Let $P = ND^{-1}$ be an $m\times\ell$ strictly proper transfer function where $\lambda > o$, $n = \lambda\ell$ and

$$D = Is^{\lambda} + D_{\lambda-1}s^{\lambda-1} + \ldots + D_o$$

$$N = \quad N_{\lambda-1}s^{\lambda-1} + \ldots + N_o$$

Let $q \geq o$ and $\phi(s) = s^{n+q} + \phi_{n+q+1}s^{n+q+1} + \ldots + \phi_o$.

Let

$$W = \left\{ (N,D,\phi)\ R^{2\lambda\ell^2+n+q} \;\middle|\; \begin{array}{l} \text{For which there exists a proper} \\ \text{compensator of order } q \text{ making } \phi_1 = \phi, \\ \phi_2 = \phi_3 = \ldots = \phi_\ell = 1 \text{ the invariant factors} \\ \text{of the closed-loop system.} \end{array} \right\}$$

Then a sufficient condition for W to be a generic subset of $R^{2\lambda\ell^2+n+q}$ is $q \geq \mu-1$.

Proof:

Using the Sylvester Resultant formulation one can show [8] that if $q \geq \mu-1$ then the set $M \subsetneq R^{2\lambda\ell^2+n+q}$ of (N,D,ϕ) for which a proper compensator of order q exists and of the form:

$$X = \begin{bmatrix} 1, & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix} s^q + \begin{bmatrix} {}_{q-1}X_1, & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix} s^{q-1} \dots + \begin{bmatrix} {}_0X_1, & 0, & \dots & 0 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & \end{bmatrix}$$

$$Y = \begin{bmatrix} {}_qy_1, & \dots & {}_qy_m \\ & 0 & \\ & & \end{bmatrix} s^q + \dots + \begin{bmatrix} {}_0y_1, & \dots & {}_0y_\ell, & \dots & {}_0y_m \\ 0 & & & & \\ \vdots & & I_{\ell-1} & & 0 \\ 0 & & & & \end{bmatrix}$$

which makes ϕ the closed loop characteristic polynomial is generic.

$$XD+YN = \Phi \quad \text{where} \quad \det\Phi = \phi.$$

But because of the structure of Φ it can easily be shown that the set $J \subseteq R^{2\lambda\ell^2+n+q}$ for which the gcd's of the ixi minors of Φ for $1 \leq i \leq \ell-1$ are all equal to 1 is generic. Therefore the set $J \supseteq M$ for which $\phi_1=\phi$, $\phi_2=\phi_3=\dots=\phi_\ell=1$ are the invariant factors of Φ is a generic subset of $R^{2\lambda\ell^2+n+q}$.

Similar versions of this theorem have appeared in the past [2, 12, 20]. The approach taken here is different. As mentioned earlier in the case when $\ell > m$ a "dual" formulation and results can be obtained.

7. Conclusions

The problem of generalized pole assignment using output feedback has not been completely solved as yet. Great progress has been made as evidenced by many important contributions [see references]. In this paper using a formulation involving Generalized Sylvester Resultants we were able to give new short proofs of earlier results as well as suggest necessary and sufficient conditions for generic invariant factor assignment in several cases. We believe that the ideas presented here can be used to obtain many more results.

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References

1. R. R. Bitmead, S-Y Kung, B. D. O. Anderson, T. Kailath, "Greatest Common Divisors via Generalized Sylvester and Bezout Matrices", IEEE Trans. on AC, Vol. AC-23, No.6, December 1978.

2. F. M. Brasch, J. B. Pearson, "Pole Placement Using Dynamic Compensators", IEEE Trans. on AC, Vol. AC-15, No. 1, February 1970.

3. R. W. Brockett, C. I. Byrnes, "Multivariable Nyquist Criteria, Root Loci, and Pole Placement: A Geometric Viewpoint", IEEE Trans. cn AC, Vol. AC-26, No. 1, February 1981.

4. E. J. Davison, S. H. Wang, "On Pole Assignment in Linear Multivariable Systems Using Output Feedback", IEEE Trans. on AC, Vol. AC-20, August 1975.

5. C. A. Desoer, M. Vidyasagar, "Feedback Systems: Input-Output Properties", Academic Press, New York, 1975.

6. T. E. Djaferis, S. K. Mitter, "The Generalized Pole Assignment Problem", Proceedings of 1979 IEEE CDC, Fort Lauderdale, Florida, December 1979.

7. T. E. Djaferis, "General Pole Assignment by Output Feedback and Solution of Linear Matrix Equations from an Algebraic Viewpoint", Ph.D. Thesis, M.I.T., June 1979.

8. T. E. Djaferis, "Generic Pole Assignment Using Dynamic Output Feedback", Proc. 19th Allerton Conference on Communication, Control and Computing, Monticello, Illinois, September 1981 (to appear, Int. J. Control).

9. E. Emre, "The Polynomial Equation $QQ_c+RP_c=\Phi$ with Application to Dynamic Feedback", SIAM J Cont. Opt., Vol. 18, No. 6, November 1980.

10. E. Emre, "Pole assignment by dynamic output feedback", Int. J. Control, Vol. 33, No. 2, 1981.

11. R. Hermann, C. F. Martin, "Applications of algebraic geometry to linear system theory", IEEE Trans. on AC, Vol. AC-22, 1977.

12. T. Kailath, "Linear Systems", Prentice Hall, Englewood Cliffs, New Jersey, 1980.

13. H. Kimura, "Pole Assignment by Gain Output Feedback", IEEE Trans. on AC, Vol. 20, 1975.

14. H. Kimura, "Further Result in the Problem of Pole Assignment by Output Feedback", IEEE Trans. on AC, Vol. 22, 1977.

15. T. Koussiouris, "On the general problem of pole assignment", Int. J. Control, Vol. 30, No. 4, 1979.

16. S.-Y. Kung, T. Kailath, M. Morf, "A generalized resultant matrix for polynomial matrices", Proc. IEEE Conf. on Decision and Control, Florida, 1976.

17. P. Lancaster, "Theory of Matrices", Academic Press, New York, 1969.

18. L. Pernebo, "Algebraic Control Theory for Linear Multivariable Systems", Tekn. Dr. Thesis, Lund. I.T., May 1978.

19. H. H. Rosenbrock, "State Space and Multivariable Theory", J. Wiley & Sons, New York, 1970.

20. H. H. Rosenbrock, C. E. Hayton, "The general problem of pole assignment", Int. J. Control, Vol. 27, No. 6, 1978.

21. W. M. Wonham, "On Pole Assignment in Multi-Input Controllable Linear Systems", IEEE Trans. on AC, Vol. AC-12, No. 6, December 1967.

22. J. C. Willems, W. H. Hesselink, "Generic Properties of the Pole Placement Problem", Preprints 1978 IFAC Congress, Helsinki, Finland.

23. W. A. Wolovich, "Linear Multivariable Systems", Applied Mathematical Sciences, Vol. II, Springer-Verlag, New York, 1974.

24. O. Zariski, P. Samuel, "Commutative Algebra", Vol. 1, Van Nostrand, Princeton, 1958.

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