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AN UPPER BOUND ON ERRORS IN THE  
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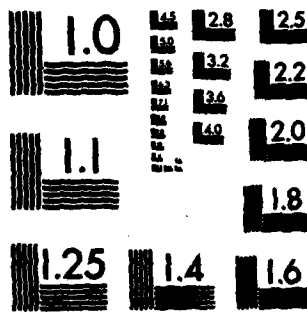
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## FOREWORD

This report has been prepared to present the results of a study that was performed to determine the effect of nutation angle interpolation errors upon artificial earth satellite positional accuracy. The work was performed under the auspices of task number HM0050-2-315 from the Defense Mapping Agency. This report was reviewed and approved by R. J. Anderle and R. W. Hill.

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## I. INTRODUCTION

Vector transformations between the mean-of-date and instantaneous earth-fixed reference systems are performed through application of the  $\mathbb{B}$  and  $\mathbb{C}$  transformation matrices, or their transposes, to the vector of interest (these matrices are described in detail in the next section). These transformation matrices are functionally dependent upon the nutation in longitude,  $\Delta\psi$ ; the nutation in obliquity,  $\Delta\epsilon$ ; and the equation of the equinoxes,  $\Delta H$ . The  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  angles are time dependent and can be computed in a straightforward manner at each time for which they are required. However, since each individual computation is quite lengthy, such an approach is impractical if  $\mathbb{B}$  and  $\mathbb{C}$  matrices are required for a great many times.

A more practical approach can be used to generate the required  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  angles for situations in which many  $\mathbb{B}$  and  $\mathbb{C}$  matrices are needed. Required values for these angles are obtainable by interpolating between values which have been precomputed at discreet times over the timespan of interest. However, the interpolation introduces errors into  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$ , which are manifested as errors in the associated transformed vector.

The following sections of this work discuss in detail these manifested errors in the transformed vector, as well as the magnitude of the interpolation errors in  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$ . In the next section a general first-order error equation is developed that analytically describes the upper error bound on the vector moduli obtained from mean-of-date/instantaneous earth-fixed vector transformations due to small errors in the  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  angles. This analytical result is applied to a six-point Lagrange interpolator in section III where numerical data are used to verify the accuracy of the analytical result, as well as assess the computational quality of the interpolator for this application. The last section summarizes the results and conclusions drawn from this analysis.

## II. ERROR EQUATION

A first-order analytic expression for the upper bound of the error introduced into a vector modulus by the coordinate transformation from the mean-of-date reference system to the instantaneous earth-fixed system is developed in this section. In this discussion it is assumed that this error is an interpolation error induced by using transformation matrix elements computed from angular arguments obtained by an interpolation process.

The transformation of a vector in the mean-of-date reference system,  $\vec{X}_M$ , to a vector in the true of date system,  $\vec{X}_T$ , is accomplished by the following application of the nutation matrix,  $\mathbb{C}$ :

$$\vec{X}_T = \mathbb{C} \vec{X}_M \quad (1)$$

Similarly, the transformation of  $\vec{X}_T$  to a vector in the instantaneous earth-fixed system,  $\vec{X}_E$ , is performed through the rotation,

$$\vec{X}_E = \mathbb{B} \vec{X}_T \quad , \quad (2)$$

where  $\mathbb{B}$  is the earth-fixed transformation matrix. Using Equation (1) in Equation (2) provides the complete mean of date to instantaneous earth-fixed transformation given by

$$\vec{X}_E = \mathbb{B} \mathbb{C} \vec{X}_M \quad . \quad (3)$$

It should be mentioned that both  $\mathbb{B}$  and  $\mathbb{C}$  are orthogonal transformations, i.e.,

$$\mathbb{B}^{-1} = \mathbb{B}^t \quad (4)$$

and

$$\mathbb{C}^{-1} = \mathbb{C}^t \quad (5)$$

so that

$$\vec{X}_M = \mathbb{C}^t \vec{X}_T \quad , \quad (6)$$

$$\vec{X}_T = \mathbb{B}^t \vec{X}_E \quad , \quad (7)$$

and

$$\vec{X}_M = \mathbb{C}^t \mathbb{B}^t \vec{X}_E \quad . \quad (8)$$

In the above expressions the superscript "t" means matrix transpose.

The  $\mathbb{B}$  and  $\mathbb{C}$  matrices have the following forms:

$$\mathbb{B} = \begin{bmatrix} \cos \Lambda & \sin \Lambda & 0 \\ -\sin \Lambda & \cos \Lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (9)$$

and

$$\mathbb{C} = \begin{bmatrix} \cos \Delta\psi & -\sin \Delta\psi \cos \bar{e} & -\sin \Delta\psi \sin \bar{e} \\ \cos e \sin \Delta\psi & \cos e \cos \Delta\psi \cos \bar{e} + \sin e \sin \bar{e} & \cos e \cos \Delta\psi \sin \bar{e} - \sin e \cos \bar{e} \\ \sin e \sin \Delta\psi & \sin e \cos \Delta\psi \cos \bar{e} - \cos e \sin \bar{e} & \sin e \cos \Delta\psi \sin \bar{e} + \cos e \cos \bar{e} \end{bmatrix} \quad , \quad (10)$$

where  $\Lambda$  is the longitude of the Greenwich meridian from the true vernal equinox at time  $t$  given by

$$\Lambda = H_0 + \Delta H(t) + \tilde{\omega} (t - \Delta t_2) \quad ; \quad (11)$$

$\Delta\psi$  is the nutation in longitude;  $\Delta\epsilon$  is the nutation in obliquity;  $\bar{\epsilon}$  is the mean obliquity of date; and  $\epsilon$  is true obliquity of date given by

$$\epsilon = \bar{\epsilon} + \Delta\epsilon \quad (12)$$

The terms in Equation (11) have the following definitions:

$H_0$  = mean hour angle of Greenwich at zero hours universal time of the day of interest

$\Delta H(t)$  = equation of the equinoxes =  $\Delta\psi \cos \epsilon$

$\tilde{\omega}$  = mean sidereal rotation rate of the earth

$\Delta t_2$  = correction for irregular earth rotation

Suppose that the angles  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  are obtained from interpolations between  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  values, which have been precomputed at discreet times over the timespan of interest, and that these interpolations introduce the errors  $\delta\psi$ ,  $\delta\epsilon$ , and  $\delta H$  into their values, i.e.,

$$\Delta\psi = \Delta\psi_0 + \delta\psi \quad (13)$$

$$\Delta\epsilon = \Delta\epsilon_0 + \delta\epsilon \quad (14)$$

$$\epsilon = \bar{\epsilon} + \Delta\epsilon_0 + \delta\epsilon = \epsilon_0 + \delta\epsilon \quad (15)$$

and

$$\Lambda = \Lambda_0 + \delta H \quad (16)$$

where  $\Delta\psi_0$ ,  $\Delta\epsilon_0$ ,  $\epsilon_0$ , and  $\Lambda_0$  are the "errorless" values of  $\Delta\psi$ ,  $\Delta\epsilon$ ,  $\epsilon$ , and  $\Lambda$ , respectively. Substituting the last four equations into the expressions for  $\mathbb{R}$  and  $\mathbb{C}$ ; assuming that the errors are very small; and using the small angle approximations

$$\cos \delta\theta \approx 1, \quad \sin \delta\theta \approx \delta\theta \quad \text{for } \delta\theta \ll 1 \quad (17)$$

provide the following first-order results:

$$\mathbb{R} = \mathbb{R}_0 + \delta H \mathbb{R}_1 \quad (18)$$

and

$$\mathbb{C} = \mathbb{C}_0 + \delta\psi \mathbb{C}_1 + \delta\epsilon \mathbb{C}_2 \quad (19)$$

where  $\mathbb{R}_0$  and  $\mathbb{C}_0$  are the "errorless" transformation matrices given by Equations (9) and (10) with  $\Lambda$ ,  $\Delta\psi$ , and  $\epsilon$  replaced with  $\Lambda_0$ ,  $\Delta\psi_0$ , and  $\epsilon_0$ , respectively. The error matrices have the following forms:

$$\mathbb{R}_1 = \begin{bmatrix} -\sin \Lambda_0 & \cos \Lambda_0 & 0 \\ -\cos \Lambda_0 & -\sin \Lambda_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (20)$$

$$\mathbb{C}_1 = \begin{bmatrix} -\sin \Delta\psi_0 & \cos \Delta\psi_0 \cos \bar{\epsilon} & \cos \Delta\psi_0 \sin \bar{\epsilon} \\ \cos \epsilon_0 \cos \Delta\psi_0 & -\sin \Delta\psi_0 \cos \epsilon_0 \cos \bar{\epsilon} & -\sin \Delta\psi_0 \cos \epsilon_0 \sin \bar{\epsilon} \\ \sin \epsilon_0 \cos \Delta\psi_0 & -\sin \Delta\psi_0 \sin \epsilon_0 \cos \bar{\epsilon} & -\sin \Delta\psi_0 \sin \epsilon_0 \sin \bar{\epsilon} \end{bmatrix}, \quad (21)$$

and

$$\mathbb{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ -\sin \epsilon_0 \sin \Delta\psi_0 & -\sin \epsilon_0 \cos \Delta\psi_0 \cos \bar{\epsilon} + \cos \epsilon_0 \sin \bar{\epsilon} & -\sin \epsilon_0 \cos \Delta\psi_0 \sin \bar{\epsilon} - \cos \epsilon_0 \cos \bar{\epsilon} \\ \cos \epsilon_0 \sin \Delta\psi_0 & \cos \epsilon_0 \cos \Delta\psi_0 \cos \bar{\epsilon} + \sin \epsilon_0 \sin \bar{\epsilon} & \cos \epsilon_0 \cos \Delta\psi_0 \sin \bar{\epsilon} - \sin \epsilon_0 \cos \bar{\epsilon} \end{bmatrix}. \quad (22)$$

Equations (18) and (19) may be used in Equation (3) to give the following first-order result:

$$\vec{X}_E = \left( \mathbb{R}_0 \mathbb{C}_0 + \delta\psi \mathbb{R}_0 \mathbb{C}_1 + \delta\epsilon \mathbb{R}_0 \mathbb{C}_2 + \delta H \mathbb{R}_1 \mathbb{C}_0 \right) \vec{X}_M \quad (23)$$

Assume that  $\vec{X}_E$  can be decomposed into the sum of an "errorless" part,  $\vec{X}_{E0}$ , and a part corresponding to the interpolation induced error,  $\delta\vec{X}_E$ . Since

$$\vec{X}_{E0} = \mathbb{R}_0 \mathbb{C}_0 \vec{X}_M, \quad (24)$$

then

$$\delta\vec{X}_E = \left( \delta\psi \mathbb{R}_0 \mathbb{C}_1 + \delta\epsilon \mathbb{R}_0 \mathbb{C}_2 + \delta H \mathbb{R}_1 \mathbb{C}_0 \right) \vec{X}_M. \quad (25)$$

Taking the norm of both sides of Equation (25) gives

$$|\delta\vec{X}_E| = \left| \left( \delta\psi \mathbb{R}_0 \mathbb{C}_1 + \delta\epsilon \mathbb{R}_0 \mathbb{C}_2 + \delta H \mathbb{R}_1 \mathbb{C}_0 \right) \vec{X}_M \right|. \quad (26)$$

Upon application of the Schwartz and triangle inequalities, the last equation may be rewritten as the following inequality:

$$\frac{|\delta\vec{X}_E|}{|\vec{X}_M|} \leq |\delta\psi| |\mathbb{R}_0 \mathbb{C}_1| + |\delta\epsilon| |\mathbb{R}_0 \mathbb{C}_2| + |\delta H| |\mathbb{R}_1 \mathbb{C}_0|. \quad (27)$$

The matrix norms  $|\mathbb{R}_i \mathbb{C}_j|$  have the values given by<sup>1</sup>:

<sup>1</sup>Noble, B., Applied Linear Algebra (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1969) pp. 427-431.

$$|\mathbb{B}_i \mathbb{C}_j| = \lambda_{ij \max}^{1/2}, \quad (ij) = (01, 02, 10), \quad (28)$$

where

$$\lambda_{ij \max} = \text{maximum eigenvalue of } (\mathbb{B}_i \mathbb{C}_j)^h (\mathbb{B}_i \mathbb{C}_j). \quad (29)$$

The superscript "h" in the last equation means hermitian transpose. Since  $\mathbb{B}_i$  and  $\mathbb{C}_j$  are real matrices, the hermitian transpose is simply the matrix transpose, so that

$$\begin{aligned} (\mathbb{B}_i \mathbb{C}_j)^h (\mathbb{B}_i \mathbb{C}_j) &= (\mathbb{B}_i \mathbb{C}_j)^t (\mathbb{B}_i \mathbb{C}_j) \\ &= \mathbb{C}_j^t \mathbb{B}_i^t \mathbb{B}_i \mathbb{C}_j. \end{aligned} \quad (30)$$

For the case  $i = 0$ , Equation (4) applies, and Equation (30) becomes

$$(\mathbb{B}_0 \mathbb{C}_j)^h (\mathbb{B}_0 \mathbb{C}_j) = \mathbb{C}_j^t \mathbb{C}_j \quad (j = 1, 2). \quad (31)$$

The maximum eigenvalue  $\lambda_{0j \max}$  is then obtained from the solution of the characteristic equation

$$\det \{ \mathbb{C}_j^t \mathbb{C}_j - \lambda_{0j} \mathbb{I} \} = 0, \quad (j = 1, 2), \quad (32)$$

where "det  $\{ \mathbb{A} \}$ " means the determinant of  $\mathbb{A}$ , and  $\mathbb{I}$  is the identity matrix. Using Equations (21) and (22) to form  $\mathbb{C}_j^t \mathbb{C}_j$  ( $j = 1, 2$ ) gives

$$\mathbb{C}_1^t \mathbb{C}_1 = \begin{bmatrix} 1 & -2 \sin \Delta\psi_0 \cos \Delta\psi_0 \cos \bar{\epsilon} & -2 \sin \Delta\psi_0 \cos \Delta\psi_0 \sin \bar{\epsilon} \\ -2 \sin \Delta\psi_0 \cos \Delta\psi_0 \cos \bar{\epsilon} & \cos^2 \bar{\epsilon} & \sin \bar{\epsilon} \cos \bar{\epsilon} \\ -2 \sin \Delta\psi_0 \cos \Delta\psi_0 \sin \bar{\epsilon} & \sin \bar{\epsilon} \cos \bar{\epsilon} & \sin^2 \bar{\epsilon} \end{bmatrix} \quad (33)$$

and

$$\mathbb{C}_2^t \mathbb{C}_2 = \begin{bmatrix} \sin^2 \Delta\psi_0 & \sin \Delta\psi_0 \cos \Delta\psi_0 \cos \bar{\epsilon} & \sin \Delta\psi_0 \cos \Delta\psi_0 \sin \bar{\epsilon} \\ \sin \Delta\psi_0 \cos \Delta\psi_0 \cos \bar{\epsilon} & \cos^2 \Delta\psi_0 \cos^2 \bar{\epsilon} + \sin^2 \bar{\epsilon} & -\sin^2 \Delta\psi_0 \sin \bar{\epsilon} \cos \bar{\epsilon} \\ \sin \Delta\psi_0 \cos \Delta\psi_0 \sin \bar{\epsilon} & -\sin^2 \Delta\psi_0 \sin \bar{\epsilon} \cos \bar{\epsilon} & \cos^2 \Delta\psi_0 \sin^2 \bar{\epsilon} + \cos^2 \bar{\epsilon} \end{bmatrix} \quad (34)$$

Upon substitution of the last two expressions into Equation (32) and after much algebraic manipulation, it is found that the associated characteristic equations reduce to the simple forms:

$$\lambda_{01} \left[ \lambda_{01}^2 - 2\lambda_{01} - (4 \sin^2 \Delta\psi_0 \cos^2 \Delta\psi_0 - 1) \right] = 0 \quad (35)$$

and

$$\lambda_{02} \left[ \lambda_{02}^2 - 2\lambda_{02} + 1 \right] = 0 , \quad (36)$$

which yield

$$\lambda_{01} = \{ 0, 1 \pm 2 \sin \Delta\psi_0 \cos \Delta\psi_0 \} , \quad (37)$$

and

$$\lambda_{02} = \{ 0, 1, 1 \} . \quad (38)$$

Thus

$$\lambda_{01 \max} = 1 + 2 | \sin \Delta\psi_0 \cos \Delta\psi_0 | , \quad (39)$$

and

$$\lambda_{02 \max} = 1 . \quad (40)$$

Using Equation (30) it is seen that for  $i = 1, j = 0$ , the maximum eigenvalue  $\lambda_{10 \max}$  is obtained from the solution of the characteristic equation

$$\det \{ \mathcal{C}_0^t \mathcal{R}_1^t \mathcal{R}_1 \mathcal{C}_0 - \lambda_{10} \mathcal{L} \} = 0 . \quad (41)$$

However, by inspection, it is found that

$$\mathcal{R}_1^t \mathcal{R}_1 = \mathcal{L} - \mathcal{L} , \quad (42)$$

where

$$(\mathcal{L})_{\ell k} = \delta_{\ell 3} \delta_{k 3} . \quad (43)$$

In the last expression,  $\delta_{mn}$  is the Kronecker delta. Using this result, the properties of the identity matrix and the relation (see Equation (5))

$$\mathcal{C}_0^t \mathcal{C}_0 = \mathcal{L} \quad (44)$$

allow Equation (41) to be rewritten as

$$\det \left\{ \mathcal{C}_0^t \left[ (1 - \lambda_{10}) \mathcal{L} - \mathcal{L} \right] \mathcal{C}_0 \right\} = 0 . \quad (45)$$

Application of the properties of determinants to this expression allows it to be recast into the form

$$\det \{ \mathcal{C}_0^t \} \det \{ (1 - \lambda_{10}) \mathcal{L} - \mathcal{L} \} \det \{ \mathcal{C}_0 \} = 0 . \quad (46)$$

Since

$$\det \{ \mathcal{C}_0^t \} = \det \{ \mathcal{C}_0^{-1} \} = \frac{1}{\det \{ \mathcal{C}_0 \}} , \quad (47)$$

then Equation (46) reduces to

$$\det \{ (1 - \lambda_{10}) \mathcal{L} - \mathcal{L} \} = 0 . \quad (48)$$

The solution to this equation is straightforward, and it is found that the characteristic equation reduces to

$$\lambda_{10} (\lambda_{10} - 1) (\lambda_{10} - 1) = 0 ; \quad (49)$$

yielding

$$\lambda_{10} = \{ 0, 1, 1 \} . \quad (50)$$

Thus

$$\lambda_{10 \max} = 1 . \quad (51)$$

Use of Equations (28), (39), (40), and (51) in Equation (27) gives the final form of the error equation:

$$\frac{|\delta \vec{X}_E|}{|\vec{X}_M|} \leq (1 + 2 |\sin \Delta \psi_0 \cos \Delta \psi_0|)^{1/2} |\delta \psi| + |\delta \epsilon| + |\delta H| . \quad (52)$$

Consider now the error associated with the inverse transformation, i.e., that given by Equation (8). Inserting Equations (18) and (19) into Equation (8), retaining only first-order terms, using the inverse of Equation (24), and applying the Schwartz and triangle inequalities to the resulting norms provide the inequality

$$\frac{|\delta \vec{X}_M|}{|\vec{X}_E|} \leq |\delta \psi| |\mathcal{C}_1^t \mathcal{B}_0^t| + |\delta \epsilon| |\mathcal{C}_2^t \mathcal{B}_0^t| + |\delta H| |\mathcal{C}_0^t \mathcal{B}_1^t| . \quad (53)$$

Analogous to Equation (28), the matrix norms  $|\mathcal{C}_j^t \mathcal{B}_i^t|$  have the values given by:

$$|\mathcal{C}_j^t \mathcal{B}_i^t| = \gamma_{ji \max}^{1/2}, \quad (ji) = (10, 20, 01) , \quad (54)$$

where

$$\gamma_{ji_{\max}} = \text{maximum eigenvalue of } (C_j^t B_i^t)^h (C_j^t B_i^t), \quad (55)$$

i.e., the  $\gamma_{ji_{\max}}$  is the maximum eigenvalue obtained from the solution of the characteristic equation

$$\det \{ B_i C_j C_j^t B_i^t - \gamma_{ji} I \} = 0. \quad (56)$$

For the case when  $(ji) = (01)$  the transpose of Equation (44) applies and Equation (56) reduces to

$$\det \{ B_1 B_1^t - \gamma_{01} I \} = 0. \quad (57)$$

However, it is seen from Equation (20) that

$$B_1 B_1^t = B_1^t B_1, \quad (58)$$

so that Equation (42) applies and Equation (57) reduces to Equation (48). Thus

$$\gamma_{01_{\max}} = \gamma_{10_{\max}} = 1. \quad (59)$$

For the cases when  $(ji) = (10, 20)$  use may be made of the fact that

$$B_0 B_0^t = I \quad (60)$$

to rewrite Equation (56) as

$$\det \{ B_0 [C_j C_j^t - \gamma_{j0} I] B_0^t \} = 0, \quad (j = 1, 2) \quad (61)$$

or

$$\det \{ B_0 \} \det \{ C_j C_j^t - \gamma_{j0} I \} \det \{ B_0^t \} = 0, \quad (j = 1, 2). \quad (62)$$

Since

$$\det \{ B_0^t \} = \frac{1}{\det \{ B_0 \}}, \quad (63)$$

then Equation (62) reduces to

$$\det \{ C_j C_j^t - \gamma_{j0} I \} = 0. \quad (64)$$

Using Equation (21) it is found that  $\mathcal{C}_1 \mathcal{C}_1^t$  is given by Equation (33) with  $\bar{\epsilon}$  replaced everywhere with  $\epsilon_0$  and Equation (64) reduces to an equation of the same form as Equation (35) since it is independent of  $\bar{\epsilon}$  and  $\epsilon_0$ . Therefore

$$\gamma_{10 \max} = \lambda_{01 \max} = 1 + 2 | \sin \Delta\psi_0 \cos \Delta\psi_0 | . \quad (65)$$

Similarly, using Equation (22), it is found that

$$\mathcal{C}_2 \mathcal{C}_2^t = \mathcal{I} - \mathcal{K} , \quad (66)$$

where

$$(\mathcal{K})_{lk} = \delta_{l1} \delta_{k1} . \quad (67)$$

As before,  $\delta_{mn}$  is the Kronecker delta. Since this matrix is diagonal, the eigenvalues lie along the diagonal and are given by Equation (38) so that

$$\gamma_{20 \max} = \lambda_{02 \max} = 1 . \quad (68)$$

Therefore the right-hand side of inequality (53) is the same as that of inequality (52), i.e., the upper error bounds are identical for the transformation and its inverse, and inequality (52) is valid as written or with the 'E' and 'M' subscripts interchanged.

### III. NUMERICAL APPLICATION: THE SIX-POINT LAGRANGE INTERPOLATOR

In order to examine the utility of the error equation that is derived in the previous section, a numerical test case is discussed in this section. Since the six-point Lagrange interpolator is frequently used as the method for computing interpolated nutation angles, it has been chosen for use in this application. This method involves fitting a polynomial of degree five to six of the data points of the type being fitted (i.e.  $\Delta\psi$ ,  $\Delta\epsilon$ ,  $\Delta H$ ) and centering the fit such that three of the points lie on either side of the interpolation point. The spacing between all data points, i.e., the interpolation interval, is assumed to be equal and uniform. The polynomial  $f(t)$  used to interpolate the nutation angle data has the following form:

$$f(t) = \sum_{i=0}^5 \mathcal{L}_i(t) f(t_i) , \quad t_2 < t < t_3 , \quad (69)$$

where

$$\mathcal{L}_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^5 \left( \frac{t - t_j}{t_i - t_j} \right) , \quad (70)$$

and  $f(t)$  represents  $\Delta\psi(t)$ ,  $\Delta\epsilon(t)$ , or  $\Delta H(t)$ .

The primary objectives to be satisfied by the analysis of this numerical application are

- (i) Numerically determine the variation of the maximum interpolation errors  $|\delta\psi|_{\max}$ ,  $|\delta\epsilon|_{\max}$ , and  $|\delta H|_{\max}$  with interpolation interval for the six-point Lagrange interpolator
- (ii) Numerically validate inequality (52)
- (iii) Quantify the maximum errors induced by this interpolation method over a range of Earth satellite position vector moduli

To generate data for this analysis, values for the errors  $\delta\psi(t)$ ,  $\delta\epsilon(t)$ , and  $\delta H(t)$  resulting from the six-point Lagrange interpolation algorithm were computed every 15 min using a 1-day interpolation interval over the 30-day span from 5 January 1984 through 3 February 1984. The variations of  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  during this time are shown in Figure 1. The maximum errors  $|\delta\psi|_{\max}$ ,  $|\delta\epsilon|_{\max}$ ,  $|\delta H|_{\max}$  for this timespan were isolated and subjected to recomputation using 0.75-, 0.50-, and 0.25-day interpolation intervals. These error computations were performed using sections of A. R. Darnell's ROTATE computer program<sup>2</sup> (i.e., subroutines SMTRAN, NOD, and COMPHO) to generate the "errorless" values  $\Delta\psi_0$ ,  $\Delta\epsilon_0$ , and  $\Delta H_0$  at the discrete time points required by the interpolator, as well as the values of these quantities at the interpolation times  $t$ . The "errorless" and interpolated  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  values were subtracted to obtain the  $\delta\psi$ ,  $\delta\epsilon$ , and  $\delta H$  errors, respectively, at each interpolation time. It should be noted that all computations performed by the ROTATE program are done in a fashion which is consistent with the new J2000 fundamental reference system and associated transformations that are planned for use in 1984. Figure 2 shows the resulting variation of  $|\delta\psi|_{\max}$ ,  $|\delta\epsilon|_{\max}$ , and  $|\delta H|_{\max}$  with interpolation interval.

The ROTATE program was also used to evaluate Equation (3) for an arbitrary  $\vec{X}_M$  of (1000,1000,1000) over an 18-day span from 17 January 1984 through 3 February 1984. The  $\mathbb{R}$  and  $\mathbb{C}$  matrices were evaluated using both "errorless" and interpolated  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  values, and the respective  $\vec{X}_E$  vectors were subtracted to obtain the error vector  $\delta\vec{X}_E$  induced by the interpolation errors  $\delta\psi$ ,  $\delta\epsilon$ , and  $\delta H$ . This process was repeated every 15 min using a 1-day interpolation interval. The maximum daily errors  $|\delta\psi|_{\text{Dayi}_{\max}}$ ,  $|\delta\epsilon|_{\text{Dayi}_{\max}}$ ,  $|\delta H|_{\text{Dayi}_{\max}}$ , and  $|\delta\vec{X}_E|_{\text{Dayi}_{\max}}$  ( $i = 17, 18, \dots, 34$ ) were isolated and used to validate inequality (52) written in the form

$$|\delta\vec{X}_E|_{\text{Dayi}_{\max}} \lesssim \left\{ |\delta\psi|_{\text{Dayi}_{\max}} + |\delta\epsilon|_{\text{Dayi}_{\max}} + |\delta H|_{\text{Dayi}_{\max}} \right\} |\vec{X}_M|, \quad (71)$$

where use has been made of the fact that  $\Delta\psi_0$  is small so that

$$(1 + 2|\sin \Delta\psi_0 \cos \Delta\psi_0|)^{1/2} \approx 1. \quad (72)$$

These results are shown in Figure 3. There the dots represent the common logarithm of the daily maximum error of  $|\delta\vec{X}_E|$  computed, using the ROTATE program; the X's represent the common logarithm of the right-hand side of inequality (71), using the daily maximum values for  $|\delta\psi|$ ,  $|\delta\epsilon|$ , and  $|\delta H|$  computed by ROTATE; and

$$|\vec{X}_M| = |(1000, 1000, 1000)| = 1732.05. \quad (73)$$

<sup>2</sup>Darnell, A. R., Transformation Between the ECI and ECEF Coordinate Systems, NSWC TN (Dahlgren, Virginia, 1982) In preparation.

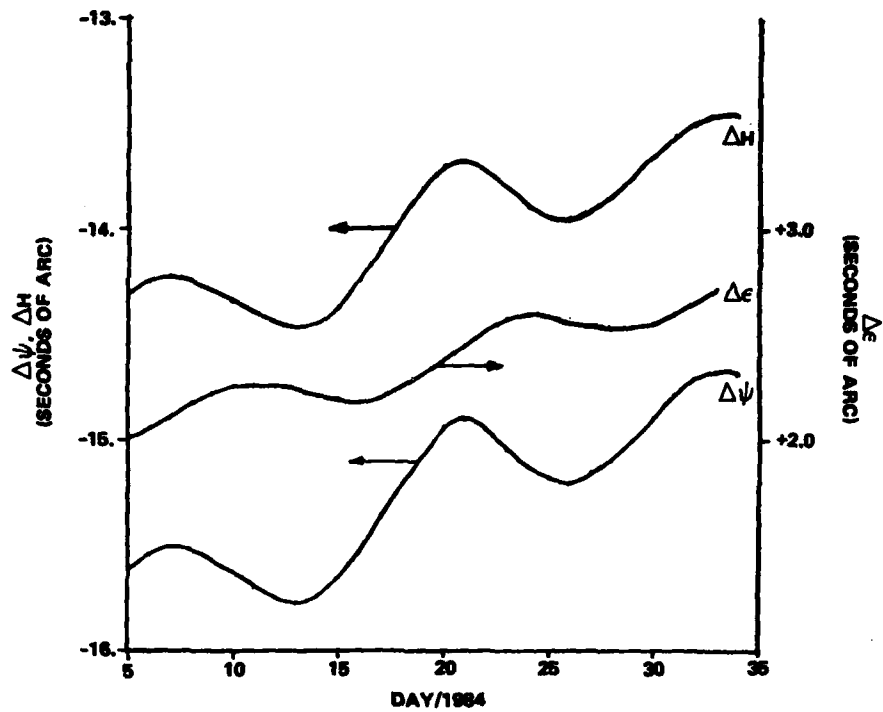


Figure 1. Variation of  $\Delta\psi$ ,  $\Delta\epsilon$ , and  $\Delta H$  From 5 January 1984 Through 3 February 1984

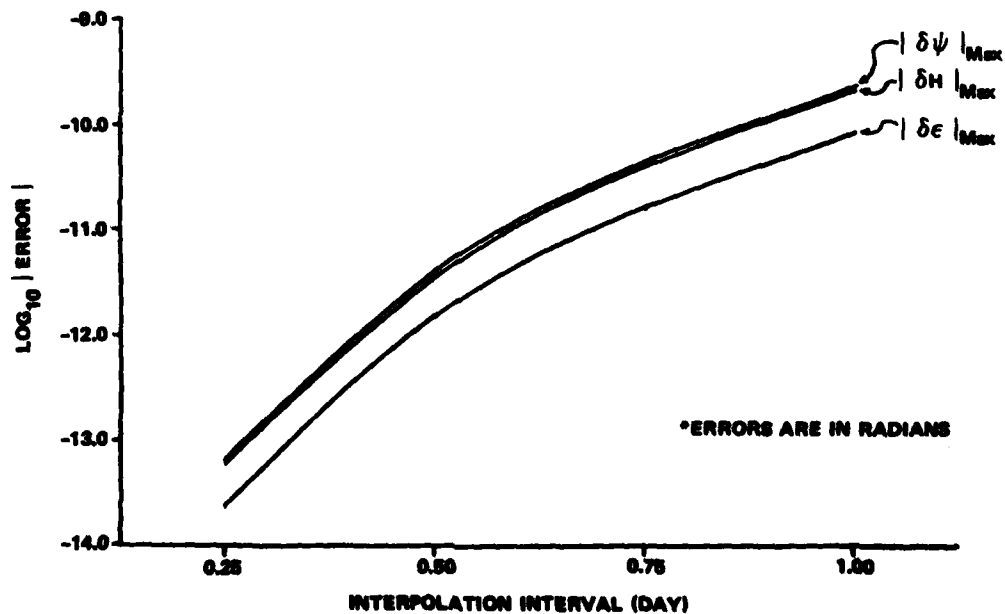


Figure 2. Variation of Six-Point Lagrange Interpolation Errors\* With Interpolation Interval

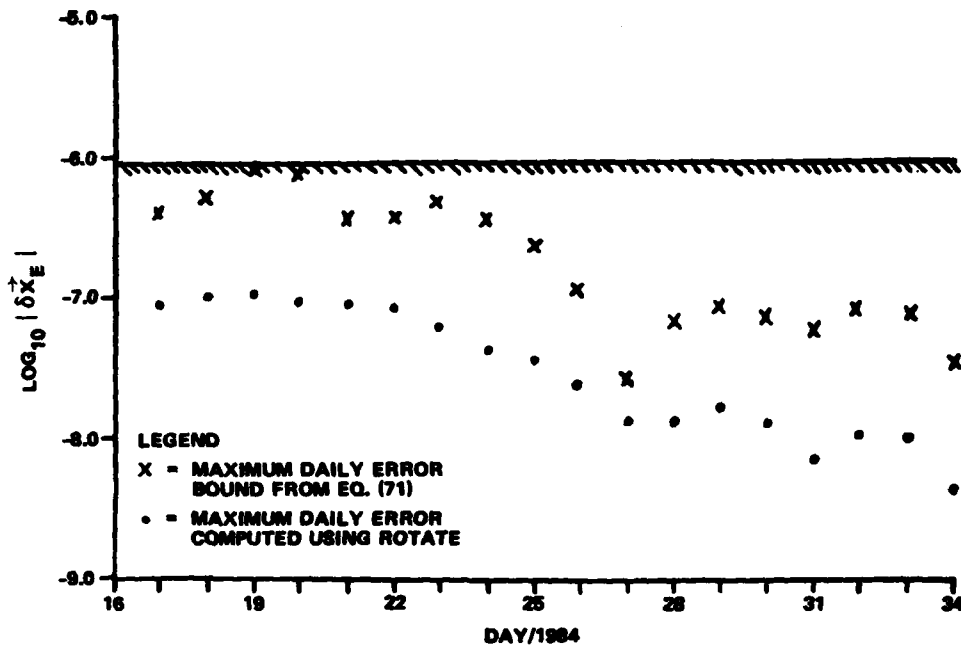


Figure 3. Maximum Daily Error Induced in  $\vec{X}_E$  by Using Interpolated Nutation Angles in the  $\mathcal{B} \mathcal{C}$  Transformation Matrices for  $|\vec{X}_M| = 1732.05$

As can be seen from this figure, the computed error is bounded in every case by the right-hand side of inequality (71). Thus inequality (71) is valid over this timespan.

To satisfy objective (iii), the following assumption is made: the maximum interpolation errors  $|\delta\psi|_{\max}$ ,  $|\delta\epsilon|_{\max}$ , and  $|\delta H|_{\max}$  obtained during the 5 January 1984 through 3 February 1984 timespan are typical of those obtained from a six-point Lagrange interpolator for any timespan when using that nutation model, which is consistent with the J2000 fundamental reference system and associated transformations. From this assumption, inequality (52) may be rewritten to represent the maximum error bound on  $|\vec{X}_E|$  induced by the use of nutation angles computed from a six-point Lagrange interpolator. This inequality is given by

$$|\delta\vec{X}_E|_{\max} \leq \left\{ |\delta\psi|_{\max} + |\delta\epsilon|_{\max} + |\delta H|_{\max} \right\} |\vec{X}_M|, \quad (74)$$

where use has been made of approximation (72). The results for the maximum interpolation errors shown on Figure 2 can be used to evaluate the factor in curly braces in the last expression for any interpolation interval between 1.00 and 0.25 days. The result obtained for a 1-day interpolation interval when  $|\vec{X}_M| = 1732.05$  is shown by the hatched line on Figure 3. Note that this result bounds all the daily computed errors and daily error bounds computed from inequality (71).

If  $\vec{X}$  is assumed to represent a satellite position vector  $\vec{r}$ , then inequality (74) can be used to construct Figure 4. There the common logarithm of the upper error bound on the earth-fixed position vector modulus in centimeters induced by the use of interpolated nutation angles in the mean-of-date/instantaneous earth-fixed transformation is plotted against radial distance in kilometers for interpolation intervals of 1.00, 0.75, 0.50, and 0.25 days. It can be inferred from this figure that the induced positional error modulus for most satellite operational altitudes is insignificant when transformation nutation angles are interpolated using an interpolation interval of a day or less.

#### IV. SUMMARY AND CONCLUSIONS

A first-order analytic expression for the upper bound on the error induced into a vector modulus by the use of interpolated nutation angles in the coordinate transformations between the mean-of-date reference system and the instantaneous earth-fixed reference system has been developed. This expression was used in conjunction with computer-derived numerical data to quantify the upper error bounds associated with the six-point Lagrange interpolator when interpolation intervals between 1.00 and 0.25 days are used and when reference system transformations and nutation angle computations are performed in a fashion consistent with the new J2000 fundamental reference system. These numerical results indicate that the use of the six-point Lagrange interpolator with an interpolation interval of 1.00 day to compute interpolated nutation angles introduces insignificant mean-of-date/earth-fixed transformation errors.

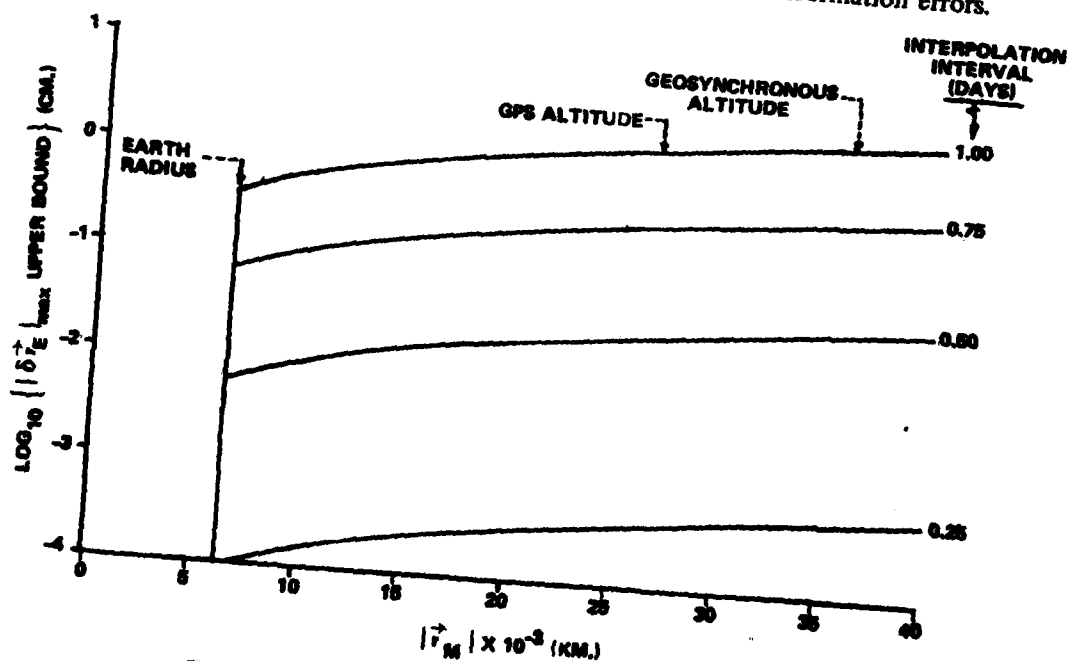


Figure 4. Upper Error Bounds on Satellite Earth-Fixed Position Vector Moduli

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