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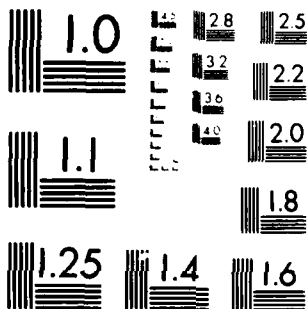
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TECHNICAL REPORT RD-82-16

AN APPROXIMATION FOR  $Z[G(s)F(s)]$  VIA THE SECOND MEAN VALUE THEOREM OF THE INTEGRAL CALCULUS: THE DIGITAL SIMULATION OF CONTINUOUS LINEAR SYSTEMS VIA Z-TRANSFORMS

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US Army Missile Laboratory

November 1982



**U.S. ARMY MISSILE COMMAND**

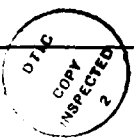
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  This report presents the approximation for Z[G(s)F(s)], which is based upon the second mean value theorem of the integral calculus.														

AN APPROXIMATION FOR  $Z[G(s)F(s)]$  VIA THE SECOND MEAN VALUE THEOREM OF THE INTEGRAL CALCULUS:  
THE DIGITAL SIMULATION OF CONTINUOUS LINEAR SYSTEMS VIA Z-TRANSFORMS

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Summary

The problem with applying z-transforms to the digital simulation of continuous linear systems is that the Raggazzini-Zadeh identity only applies to the initial conditions. This difficulty may be overcome by approximating  $Z[G(s)F(s)]$ , for example, Halijak's "Trapezoidal Convolution"<sup>1,2</sup> and its extensions.<sup>3</sup> These approximations did not yield perfect response for a unit step into a single pole filter unless adjusted by a residue.<sup>4</sup>

The approximation for  $Z[G(s)F(s)]$  presented here, which is based upon the second mean value theorem of the integral calculus, has the desired response for a unit step into a single pole filter and does not require adjustment.

Some Preliminaries

The Laplace transform is defined to be

$$F(s) = \int_{-\infty}^{\infty} f(t) u(t) e^{-st} dt \quad (1)$$

where  $u(t)$  is the Heaviside unit step,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \quad (2)$$

The z-transform will be defined as a "discrete" Laplace transform

$$F(z) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t) \delta(nT - t) u(t) e^{-st} dt \quad (3)$$

and from the "sifting" property of the Dirac delta distribution,

$$f(nT) = \int_{-\infty}^{\infty} f(t) \delta(nT - t) dt \quad (4)$$

equation (3) becomes

$$F(z) = \sum_{n=-\infty}^{\infty} f(nT) u(nT) e^{-snT} \quad (5)$$

and letting

$$z = e^{sT} \quad (6)$$

one has

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad (7)$$

the usual definition<sup>2</sup> of the z-transform.

The Raggazzini-Zadeh identity may be readily deduced from the convolution properties of the Laplace transform

$$Z[G(z)F(s)] = \sum_{n=-\infty}^{\infty} z^{-n} u(nT) \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g(t) \delta(kT - t) u(t) f(nT - t) u(nT - t) dt \quad (8)$$

The "sifting" property of the Dirac delta, equation (4), yields

$$Z[G(z)F(s)] = \sum_{n=0}^{\infty} \sum_{k=0}^n g(kT) z^{-k} f(nT - kT) z^{-(n-k)} \quad (9)$$

where the summation over  $k$  is discrete convolution. From the Cauchy product of power series

$$z G(z)F(s) = \left( \sum_{k=0}^{\infty} g(kT) z^{-k} \right) \left( \sum_{n=0}^{\infty} f(nT) z^{-n} \right) \quad (10)$$

and from equation (7)

$$Z[G(z)F(s)] = G(z)F(z) \quad (11)$$

the Raggazzini-Zadeh identity.

$Z[G(s)F(s)]$  Approximation

It is well known that, in general,  $Z[G(s)F(s)] \neq T G(z)F(z)$  and therein lies the difficulty in applying z-transforms to the digital simulation of continuous systems. A very useful relationship would be the solution of

$$Z[G(s)F(s)] = \sum_{n=-\infty}^{\infty} z^{-n} u(nT) \quad (12)$$

$$\times \int_{-\infty}^{\infty} g(t) u(t) f(nT - t) u(nT - t) dt$$

If both  $G(s)$  and  $F(s)$  are known *a priori*, there is no difficulty; but, in digital simulation, though the plant,  $G(s)$ , is known *a priori*, the input,  $f(nT)$ , is not.

By the second mean value theorem of the integral calculus, Bonnet's theorem, one may write

$$Z[G(s)F(s)] = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \left\{ f(nT - kT) \int_{kT}^{kT+\eta_k T} g(t) dt + f(nT - kT - T) \int_{kT+\eta_k T}^{kT+T} g(t) dt \right\} z^{-n} \quad (13)$$

In general  $\eta_k$  would be different for each  $k$ . Assume that  $\eta$  is the same for each  $k$ , that is,

$$\eta_k = \eta \quad (14)$$

Equation (13) is almost, but not quite, the form of the Cauchy product of power series. To obtain the

proper form

$$Z[G(s)F(s)] = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n-1} f(nT - kT) \int_{kT}^{kT+nT} g(t) dt + \sum_{k=1}^n f(nT - kT) \int_{kT+(n-1)T}^{kT} g(t) dt \right\} z^{-n} \quad (15)$$

$$Z[G(s)F(s)] = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^n f(nT - kT) z^{-(n-k)} \int_{kT}^{kT+nT} g(t) dt z^{-k} - f(0) \int_{nT}^{nT+nT} g(t) dt z^{-n} \right. \\ \left. + \sum_{k=0}^n f(nT - kT) z^{-(n-k)} \int_{kT+(n-1)T}^{kT} g(t) dt z^{-k} - f(nT) z^{-n} \int_{(n-1)T}^0 g(t) dt \right\} \quad (16)$$

and finally

$$Z[G(s)F(s)] = [F(z) - f(0)] \sum_{n=0}^{\infty} \int_{nT}^{nT+nT} g(t) dt z^{-n} + F(z) \left[ \sum_{n=0}^{\infty} \int_{nT+(n-1)T}^{nT} g(t) dt z^{-n} - \int_{(n-1)T}^0 g(t) dt \right] \quad (17)$$

It is interesting to compare this result with Halijak's "Trapezoidal Convolution:"<sup>1,2</sup>

$$Z[G(s)F(s)] = \frac{T}{2} \{ [F(z) - f(0)] G(z) + F(z) [G(z) - g(0)] \} \quad (18)$$

which may be derived from the first mean value theorem of the integral calculus.<sup>3</sup>

that is

$$f(t) = 1 \quad (24b)$$

#### Two Plants of Interest

Consider the case where the plant is a single integrator; that is:

$$G(s) = 1/s \quad (19a)$$

$$g(t) = 1 \quad (19b)$$

Substituting into equation (18) one has

$$Z[F(s)/s] = [F(z) - f(0)] \sum_{n=0}^{\infty} \int_{nT}^{nT+nT} dt z^{-n} \\ + F(z) \left[ \sum_{n=0}^{\infty} \int_{nT+(n-1)T}^{nT} dt z^{-n} - \int_{(n-1)T}^0 dt \right] \quad (20)$$

and integrating

$$Z[F(s)/s] = [F(z) - f(0)] \sum_{n=0}^{\infty} nT z^{-n} \\ + F(z) \left[ \sum_{n=0}^{\infty} (1-n)T z^{-n} - (1-n)T \right] \quad (21)$$

which may be written after summing

$$Z[F(s)/s] = [F(z) - f(0)] [nT/(1-z^{-1})] \\ + F(z) [(1-n)T/(1-z^{-1}) - (1-n)T] \quad (22)$$

and combining like terms

$$Z[F(s)/s] = \frac{[n + (1-n)z^{-1}] T F(z) - nT f(0)}{1 - z^{-1}} \quad (23)$$

For a unit step input,

$$F(s) = 1/s \quad (24a) \quad \text{that is}$$

and

$$F(z) = 1/(1 - z^{-1}) \quad (24c)$$

equation (23) becomes

$$z[(1/s)/s] = \frac{[n + (1-n)z^{-1}]T}{(1-z^{-1})^2} - \frac{nT}{1 - z^{-1}} \quad (25)$$

$$Z[(1/s)/s] = \frac{T z^{-1}}{(1 - z^{-1})^2} \quad (26)$$

which is exact for any  $n$ .

$$\text{For } F(s) = 1/s^2 \quad (27a)$$

$$\text{that is } f(t) = t \quad (27b)$$

$$\text{and } F(z) = Tz/(1 - z^{-1})^2 \quad (27c)$$

equation (23) becomes

$$Z[(1/s^2)/s] = \frac{[n + (1-n)z^{-1}]T}{1 - z^{-1}} \left( \frac{T z^{-1}}{(1 - z^{-1})^2} \right) \quad (28)$$

$$Z[(1/s^2)/s] = \frac{[n + (1-n)z^{-1}]T^2 z^{-1}}{(1 - z^{-1})^3} \quad (29)$$

which for  $n = 1/2$  is

$$Z[(1/s^2)/s] = \frac{T^2 z^{-1} (1 + z^{-1})}{2(1 - z^{-1})^3} \quad (30)$$

which is exact.

A more interesting plant is the single pole filter:

$$G(s) = a/(s + a) \quad (31a)$$

$$g(t) = a e^{-aT} \quad (31b)$$

In this case equation (17) becomes

$$Z[a F(s)/(s+a)] = [F(z) - f(0)] \sum_{n=0}^{\infty} \int_{nT}^{nT+nT} a e^{-at} dt z^{-n} + F(z) \left[ \sum_{n=0}^{\infty} \int_{nT+(n-1)T}^{nT} a e^{-at} dt z^{-n} - \int_{(n-1)T}^0 a e^{-at} dt \right] \quad (32)$$

and after performing the indicated integration and sums

$$Z[a F(s)/(s+a)] = [F(z) - f(0)] \frac{(1 - e^{-aT})}{(1 - e^{-aT} z^{-1})} + F(z) \left[ \frac{(e^{-a(n-1)T} - 1) e^{-at} z^{-1}}{1 - e^{-aT} z^{-1}} \right] \quad (33)$$

In this case, "Trapezoidal Convolution," equation (18) would yield

$$Z[a F(s)/(s+a)] = \left(\frac{aT}{2}\right) \left\{ [F(z) - f(0)] / (1 - e^{-aT} z^{-1}) + F(z) [e^{-aT} z^{-1} / (1 - e^{-aT} z^{-1})] \right\} \quad (34)$$

and with residue adjustment

$$Z[a F(s)/(s+a)] = \tanh\left(\frac{aT}{2}\right) \left\{ [F(z) - f(0)] / (1 - e^{-aT} z^{-1}) + F(z) [e^{-aT} z^{-1} / (1 - e^{-aT} z^{-1})] \right\} \quad (35)$$

$$\text{For a unit step input, } F(s) = 1/s \quad (24a)$$

that is

$$f(t) = 1 \quad (24b)$$

and

$$F(z) = 1/(1 - z^{-1}) \quad (24c)$$

equation (33) becomes

$$Z\left[\frac{1}{s} \left(\frac{a}{s+a}\right)\right] = \left[\frac{1}{1 - z^{-1}} - 1\right] \left(\frac{1 - e^{-aT}}{1 - e^{-aT} z^{-1}}\right) + \left(\frac{1}{1 - z^{-1}}\right) \left[\frac{(e^{-aT} - 1) e^{-aT} z^{-1}}{1 - e^{-aT} z^{-1}}\right] \quad (36)$$

and collecting terms

$$Z\left[\frac{1}{s} \left(\frac{a}{s+a}\right)\right] = \frac{(1 - e^{-aT}) z^{-1}}{(1 - z^{-1})(1 - e^{-aT} z^{-1})} \quad (37)$$

which is exact for any  $n$ !

Without residue adjustment "trapezoidal convolution" yields<sup>4</sup>

$$Z\left[\frac{1}{s} \left(\frac{a}{s+a}\right)\right] = \left(\frac{aT}{2}\right) \frac{(1 + e^{-aT}) z^{-1}}{(1 - z^{-1})(1 - e^{-aT} z^{-1})} \quad (38)$$

With residue adjustment one has<sup>4</sup>

$$Z\left[\frac{1}{s} \left(\frac{a}{s+a}\right)\right] = \tanh\left(\frac{aT}{2}\right) \frac{(1 + e^{-aT}) z^{-1}}{(1 - z^{-1})(1 - e^{-aT} z^{-1})} \quad (39a)$$

$$Z\left[\frac{1}{s} \left(\frac{a}{s+a}\right)\right] = \frac{(1 - e^{-aT}) z^{-1}}{(1 - z^{-1})(1 - e^{-aT} z^{-1})} \quad (39b)$$

In the sample data domain a second order Runge-Kutta integrator would yield:

$$Z\left[\frac{1}{s} \left(\frac{a}{s+a}\right)\right] = \frac{[1 - (1 - aT + a^2 T^2/2)] z^{-1}}{(1 - z^{-1})[1 - (1 - aT + a^2 T^2/2) z^{-1}]} \quad (40)$$

Note the (2/0) Pade' approximation for the exponential.

A unit step input and a Tustin's substitution<sup>2</sup> for the plant would lead to

$$Z\left[\frac{1}{s} \left(\frac{a}{s+a}\right)\right] = \frac{[1 - (1 - aT/2)/(1 + aT/2)] z^{-1}}{(1 - z^{-1})[1 - (1 - aT/2)z^{-1}]/(1 + aT/2)} \quad (41)$$

Note the (1/1) Pade' approximations for the exponential.

For the large time step,  $T$ , the approximation presented here will out perform "Trapezoidal Convolution," Tustin's substitution method and a second order Runge-Kutta integrator.

"Trapezoidal Convolution"<sup>1,2</sup> yields exact results for a damped exponential input into a single real pole

filter when the time constant of the input and the filter are the same<sup>4</sup>

$$Z\left[\frac{1}{s+a} \left(\frac{a}{s+a}\right)\right] = \frac{aT e^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2} \quad (42)$$

In this case, equation (17) yields

$$Z\left[\frac{1}{s+a} \left(\frac{a}{s+a}\right)\right] = \frac{e^{-aT} (1 - e^{-aT}) z^{-1}}{(1 - e^{-aT} z^{-1})^2} \quad (43)$$

and setting this numerator equal to the exact numerator and solving for  $n$ , one has

$$n = \frac{-1}{aT} \ln\left(\frac{aT e^{-aT}}{1 - e^{-aT}}\right) \quad (44)$$

For  $aT \ll 1$

$$n \approx 1/2 \quad (45)$$

#### Recurrence for A Single Pole Filter

To develop a recurrence for a single real pole filter with non-zero initial conditions, start with the differential equation

$$\dot{x}(t) + a x(t) = a f(t) \quad (46)$$

and transform to the Laplace domain using

$$L[f^{(n)}(t)] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \quad (47)$$

$$s X(s) - x(0) + a X(s) = a F(s) \quad (48)$$

solving for  $X(s)$  and then taking the  $z$ -transform,

$$X(z) = Z\left[\frac{aF(s)}{s+a}\right] + Z\left[\frac{x(0)}{s+a}\right] \quad (49)$$

Since the initial condition,  $x(0)$ , is a constant in the Laplace domain (an impulse in the time domain), the Raggazzini-Zadeh identity, Equation (11), would apply. The forcing function,  $F(s)$ , is another matter, and requires approximation. Using equation (33)

$$(1 - z^{-1} e^{-aT}) X(z) = \left\{ (1 - e^{-aT}) + (e^{-aT} - e^{-aT}) z^{-1} \right\} F(z) - (1 - e^{-aT}) f(0) + x(0) \quad (50)$$

and substituting equation (7) and equating coefficients of like powers of "z"

$$x(0) = x(0) \quad (51a)$$

$$x(nT) \approx e^{-aT} x(nT-T) + (1 - e^{-aT}) f(nT) + (e^{-aT} - e^{-aT}) f(nT - T), n > 0 \quad (51b)$$

For a second order damped filter with non-zero initial conditions<sup>4</sup>

$$X(s) = \frac{\omega_0^2 F(s) + (s + 2\sigma) x(0) + \dot{x}(0)}{s^2 + 2\sigma s + \omega_0^2} \quad (52)$$

where

$$0 < \frac{\sigma}{\omega_0} < 1$$

a partial fraction expansion yields

$$X(s) = \frac{1}{2\omega} \left[ \frac{i}{s + \sigma + i\omega} + \frac{-i}{s + \sigma - i\omega} \right] x \left[ \omega_0^2 F(s) + x(0) + \dot{x}(0) \right] + \frac{1}{2} \left[ \frac{1}{s + \sigma + i\omega} + \frac{1}{s + \sigma - i\omega} \right] x(0) \quad (53)$$

where

$$\omega = (\omega_0^2 - \sigma^2)^{1/2} \quad (54)$$

Let

$$X(s) = \frac{U(s) + V(s)}{2} \quad (55)$$

where

$$U(s) = \frac{i[\omega_0^2 F(s) + \sigma x(0) + \dot{x}(0)]/\omega + x(0)}{s + \sigma + i\omega} \quad (56)$$

and

$$V(s) = \frac{-i[\omega_0^2 F(s) + \sigma x(0) + \dot{x}(0)]/\omega + x(0)}{s + \sigma - i\omega} \quad (57)$$

Since  $V(s)$  is the complex conjugate of  $U(s)$ , only  $u(nT)$  need be found. The problem becomes that of finding the recurrence for a single complex pole filter. In this case, equation (51 a,b) would become:

$$u(0) = x(0) + i[\sigma x(0) + \dot{x}(0)]/\omega \quad (58a)$$

$$u(nT) = e^{-(\sigma+i\omega)T} u(nT - T) + i\omega \left(\frac{\omega_0}{\omega}\right)^2 \left[ 1 - e^{-\eta(\sigma+i\omega)T} \right] f(nT) + i\omega \left(\frac{\omega_0}{\omega}\right)^2 \left[ e^{-\eta(\sigma+i\omega)T} - e^{-(\sigma+i\omega)T} \right] f(nT - T) \quad (58b)$$

where

$$e^{-(\sigma+i\omega)T} = e^{-\sigma T} (\cos \omega T - i \sin \omega T) \quad (59)$$

and

$$x(nT) = \text{Real } u(nT) \quad (60a)$$

$$\dot{x}(nT) = \omega \text{Imaginary } [u(nT)] - \sigma \text{Real } [u(nT)] \quad (60b)$$

## Conclusions

The approximation proposed here is exact for a unit step input into a single pole filter. The pole may be either real, imaginary, or complex. The response to a unit step is unaffected by the choice of  $\eta$ ; and  $\eta$  may be chosen to adjust the response for some other input of interest, though a value of one half is generally appropriate.

Higher order problems should first be "simplified" by a partial fraction expansion, though the initial conditions should be incorporated before the expansion. The approach lends itself to parallel processing.

## References

1. C. A. Halijak, "Digital Approximations of Solutions of Differential Equations Using Trapezoidal Convolution," Bendix Corporation, Bendix Systems Division, Report No. ITM-64, 26 August 1960.
2. J. S. Rosko, *Digital Simulation of Physical Systems*, Reading, PA: Addison-Wesley Publishing Company, 1972.
3. R. E. Dickson, "Approximations for  $Z[G(s)F(s)]$ ," Proc. of the 1980 Army Numerical Analysis and Computers Conference, ARO Report 80-3, 1980, pp. 147-157.
4. R. E. Dickson, "Approximations for  $Z[G(s)F(s)]$  Adjusted for Unit Step Response Via Residues," Conf. Proc. IEEE Southeastcon '81, April 1981, pp. 240-244.

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