

AD A122275

**the
ohio
state
university**

research foundation

1314 kinnear road
columbus, ohio
43212

OPTICAL TRANSFORMATION DURING MOVEMENT:
REVIEW OF THE OPTICAL CONCOMITANTS OF EGOMOTION

Rik Warren
Department of Psychology
Aviation Psychology Laboratory

For the Period
March 1, 1981 - November 30, 1981

DEPARTMENT OF THE AIR FORCE
Air Force Office of Scientific Research
Boiling Air Force Base, D.C. 20332

Grant No. AFOSR-81-0108

Approved for public release,
distribution unlimited.

October, 1982

FILE COPY

10 0213

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 82 - 1028	2. GOVT ACCESSION NO. AD-A122275	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) OPTICAL TRANSFORMATION DURING MOVEMENT: REVIEW OF THE OPTICAL CONCOMITANTS OF EGOMOTION		5. TYPE OF REPORT & PERIOD COVERED Final 3/1/81-11/30/81
		6. PERFORMING ORG. REPORT NUMBER 762591/713616
7. AUTHOR(s) Rik Warren		8. CONTRACT OR GRANT NUMBER(s) Grant No. AFOSR-81-0108
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Ohio State University Research Foundation, 1314 Kinnear Road Columbus, Ohio 43212		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2313109
11. CONTROLLING OFFICE NAME AND ADDRESS DEPARTMENT OF THE AIR FORCE Air Force Office of Scientific Research/NL Bolling Air Force Base, D.C. 20332		12. REPORT DATE October, 1982
		13. NUMBER OF PAGES 77
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) motion perception self-motion perception egomotion perception optical flow pattern		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The primary goals of this report are to make the formal mathematical descriptions of the optical concomitants of rectilinear self-motion through the environment more useful by consolidating, clarifying, and extending them. The report includes: (1) a critical review of the literature on the optical bases for the perception of rectilinear self-motion, (2) an outline of a comprehensive framework for the study of self-motion perception based on J. J. Gibson's ecological approach, and (3) an introduction to a unified mathematical (continued)		

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

Block 20 (Abstract) - Continued

treatment of the global optical effects of rectilinear self-motion. A careful distinction is maintained between geometrical facts and perceptual or psychological aspects of self-motion. The report is written in a tutorial style.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

OPTICAL TRANSFORMATION DURING MOVEMENT:
REVIEW OF THE OPTICAL CONCOMITANTS OF EGOMOTION

by

Rik Warren

Department of Psychology
Aviation Psychology Laboratory
The Ohio State University
404C W. 17th Ave.
Columbus, OH 43210

AFOSS Grant Number 81-0108
Final Technical Report
October 1982

Controlling Office:
USAF Office of Scientific Research
Bolling Air Force Base, DC 20332

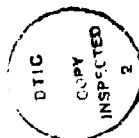
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TECHNICAL INFORMATION
This technical report has been reviewed and is
approved for distribution under AFM 190-12.
Distribution is unlimited.
MATTHEW J. KEMMER
Chief, Technical Information Division

ACKNOWLEDGEMENTS

The support of the Air Force Office of Scientific Research is gratefully acknowledged for funding this effort under Grant AFOSR-81-0108.

The author would also like to thank the staff of the Ohio State University Aviation Psychology Laboratory especially Dr. Dean H. Owen, Director.

The views and conclusions in this document are those of the author and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or of the U.S. Government.



Accession For	
NEWS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
1A	

✓

ABSTRACT

↑

The primary goals of this report are to make the formal mathematical descriptions of the optical concomitants of rectilinear self-motion through the environment more useful by consolidating, clarifying, and extending them. The report includes: (1) a critical review of the literature on the optical bases for the perception of rectilinear self-motion, (2) an outline of a comprehensive framework for the study of self-motion perception based on J. J. Gibson's ecological approach, and (3) an introduction to a unified mathematical treatment of the global optical effects of rectilinear self-motion. A careful distinction is maintained between geometrical facts and perceptual or psychological aspects of self-motion. The report is written in a tutorial style.

CONTENTS

ACKNOWLEDGEMENTS	ii
ABSTRACT	iii
	<u>page</u>
GOALS, SCOPE, AND EXCLUSIONS	1
Goals	1
Scope	1
Exclusions	1
PLAN OF THE REPORT	2
1. Reasons for limited usefulness of previous studies	2
2. Terms and distinctions	2
3. Literature review	2
4. Aspects of a comprehensive ecological analysis	2
5. Optic array vs. retinal description	2
6. Unified optical description	2
REASONS FOR LIMITED USEFULNESS OF PREVIOUS STUDIES	3
1. Incompleteness	3
2. Multiplicity of notations and analytic strategies	3
3. Distinction between generating vs. describing displays	3
4. Undetected errors	3
5. Inconsistencies and conceptual confusions	4
TERMS AND DISTINCTIONS	5
LITERATURE REVIEW	7
The 1950s	7
Gibson's contribution	7
Literary Descriptions	10
The 1960s	10
The 1960's Studies	10
Havron (1962)	10
Snyder (1964)	11
Gordon (1965)	11
Biggs (1966)	13

Egomotion Flow Pattern v

Whiteside and Samuel (1970)	14
Summary of Angular Speed Findings	15
1. Minimum angular speed	16
2. Maximal angular speed	16
3. Iso-angular speed contours: A conjecture	16
4. Alleged perceptually significant zones	17
a. Zone of nonperceptible flow	17
b. Blur zone	17
c. Zone of perceptible flow	17
Critique of the 1960's Studies	18
Independence of studies	18
Confusion of speed and velocity	18
Confusion of retinal and optic array flow	18
Uncritical acceptance of the threshold concept	18
Early 1970s Descriptions	19
Description of Local Scene Element Effects	19
Explicit Vector Description	21
Comments on the analysis of Nakayama and Loomis	22
A Cylindrical Coordinate Analysis	24
Reasons for primacy of spherical description	27
1. Mathematical reason	27
2. Biological reason	28
3. Phenomenological reason	28
A Non-Technical Description	28
Information Analyses	28
Problems and Strategies in Information Analysis	29
Information from Invariants in Flow	30
Extraction of Optical Flow Values	32
Information From Angular Relationships	32
Later 1970s and Recent Descriptions	34
Descriptions for Optimal Control Modeling	34
Lee and Visually Controlled Activity	35
Depth Perception During Egomotion	36
Information for the Perception of Egospeed	36
The problem of the perception of egospeed	36
Global optical flow rate	37
Global optical density	38
Edge rate	38
Conclusions	38
GENERAL COMMENTS ON OPTICAL DESCRIPTION	39
ASPECTS OF A COMPREHENSIVE ECOLOGICAL ANALYSIS	40
1. The Ego-Environment States to be Perceived	41
2. Geometric Description	41
3. Optical Information Analysis	41
4. Perceptual Utilization	42
5. Egomotion Display Generation for Research	42
(1) Possible inherent confounds	42

(2) Equations for description versus generation	43
OPTIC ARRAY VS. RETINAL DESCRIPTION	44
UNIFIED OPTICAL DESCRIPTION	46
Conventions	46
The Environment	46
Environmental Position	46
Static Optical Position	47
1. Sphere with "North" Pole Up	48
2. Sphere with "North" Pole Aimed at Horizon	49
3. Sphere with "North" Pole Aimed "Ahead"	50
4. Flat Frontal Plane	50
Rectilinear Egomotion	51
Speed of Optical Motion	52
Optical Speed by Differentiation: Cartesian Form	52
Azimuth and elevation rate components	52
Total angular speed from AZ and EL	53
Optical Speed by Differentiation: Optical Forms	53
Path speed and path angle	54
Path angle as an optical angle	55
Optical speed from path speed and path angle	55
Optical Speed by Geometrical Considerations	56
1. Assumption of constant meridian	57
2. Cartesian form of Eq. 30	57
3. Optical form of Eq. 30	58
4. Iso-angular speed surface	58
Vector Description of Optical Motion	58
The 3-D Vectors	59
The radius position vector	59
The environmental velocity vector	59
The observer's own velocity vector	59
The aim vector	59
The axial velocity vector	59
The tangential velocity vector	59
The angular velocity vector	60
Tangential velocity vector, Part 2	61
Comments on the Vector Description	61
Conceptual vs. computational forms	61
Significance of fractional rates of change	61
Alternate computational forms	62
Consequences of alternate forms on depiction	62
Angular vs. planar representation	62
The description of Nakayama and Loomis	62

REFERENCES	64
----------------------	----

GOALS, SCOPE, AND EXCLUSIONS

For the vast majority of animals, the control of self-locomotion (egomotion) is in general dominated by the visual system. For the control of flight in particular, visual guidance is definitely preferred and often crucial. Hence, an exact formulation of the optical information for the perception of egomotion would be of great theoretical interest and practical value. Accordingly, several mathematical analyses have been performed. But, impressive as some are, they are of limited usefulness for (1) practical applications (such as simulator design), (2) empirical testing, and (3) further theoretical development.

Goals

The primary goals of this report are to make the formal mathematical analyses of the optical concomitants of egomotion more useful and accessible by consolidating, clarifying, and correcting them. The emphasis on accessibility and usefulness means that a somewhat tutorial style is used.

Scope

Some of the analyses presented apply to all types of egomotion through all possible environments, but the scope of this report is in general limited to rectilinear egomotion over an endless flat terrain. However, in contradistinction to some studies which confine themselves to level egomotion, this study emphasizes paths at any angle to the ground.

Exclusions

Non-visual sources of information that may contribute to or contradict the visual experience are not considered. Also, the theories and literature on rotary egomotion are deliberately excluded.

PLAN OF THE REPORT

This report is organized into five major sections:

1. Reasons for limited usefulness of previous studies. This section contains the justifications for this report.

2. Terms and distinctions. This section is presented as an aid to understanding and organizing the literature.

3. Literature review. The literature review is concerned with mathematical optical analyses although a few empirical studies are discussed. The insistence on optical studies means that all references are post-WWII. Although the review is extensive, it is not intended to be exhaustive.

4. Aspects of a comprehensive ecological analysis. This section attempts to outline a comprehensive framework for the study of the perception of egomotion. An ecological approach based on Gibson (1979) is used.

5. Optic array vs. retinal description. This section attempts to clarify the distinction between optical description referred to the optic array and to the retina.

6. Unified optical description. This section presents the known mathematics of the optical concomitants of egomotion in a convenient form.

REASONS FOR LIMITED USEFULNESS OF PREVIOUS STUDIES

The usefulness of previous analytic studies is limited for five principal reasons:

1. Incompleteness. No analyses or set of analyses presents a complete list or exhaustive description of all the optical information available about egomotion.

2. Multiplicity of notations and analytic strategies. The problem of egomotion perception has attracted a broad spectrum of researchers ranging from philosophers, psychologists, physicists, engineers, and simulator designers to highway, harbor, and air safety personnel. This diversity has led to a growing but scattered and inchoate literature on egomotion perception. Because of the insular nature of the literatures of largely independent disciplines, the great wheel of the egomotion optical flow pattern has been rediscovered several times. These rediscoveries have employed a multiplicity of notations and analytic strategies. Thus it is not always easy to tell when two apparently different analyses are either formally equivalent or fundamentally different either because of errors or because different aspects of the flow pattern are being studied.

3. Distinction between generating vs. describing displays. Tests or applications require displays which exhibit a feature of interest in a controlled and possibly isolated manner. Controlled displays today generally mean computer graphics simulations. But, the equations which describe egomotion displays are unwieldy and even irrelevant for generating such displays. Hence, there exists the curious situation that people who design and build computer generated simulations use an entirely different set of equations and procedures than the people who describe and study the perceptual utilization of the resulting displays.

4. Undetected errors. One advantage of a computer generated egomotion display is that its appearance serves as a check on the correctness of the underlying procedure: Subtly flawed equations can produce bizarre and blatantly

flawed displays. This advantage is, of course, lost when different equations are used for generation and description/analyses. Indeed, some of the formal analyses cited by users of dynamic displays do contain errors. These errors have remained undetected, in part, because static figures or graphs made from flawed equations do not appear wrong as easily as dynamic displays do. Descriptive errors limit the usefulness of correctly generated dynamic scenes by misleading developers of perceptual hypotheses.

5. Inconsistencies and conceptual confusions. In addition to errors in the mathematical analyses, there are a number of inconsistencies and conceptual confusions with respect to the psychological aspects of egomotion.

TERMS AND DISTINCTIONS

The beginning of a comprehensive conceptual framework for understanding egomotion perception will be outlined after the descriptive literature has been reviewed. This review makes some distinctions and utilizes a terminology somewhat different from that found in the original articles since a greater level of precision and consistency of usage is needed. Some of those terms and distinctions are briefly presented here to aid the review.

Distinctions are made among (1) the description of the optical concomitants of egomotion, (2) the optical information available for egomotion, and (3) the perception or utilization of available information. A further distinction is made between optical descriptions made with respect to the human retina and those with respect to the ambient optic array.

The optical concomitants of egomotion refer to the projective geometric consequences of that egomotion at a moving point of observation. Although optical descriptions of those geometric effects may be with respect to the "retinal image" or the "proximal stimuli", description is best made with respect to the ambient optic array which may be conceived of in several ways such as (1) a generalized non-rotating panoptic retina, or as (2) a full spherical projection surface centered on a moving point of observation. Optic array descriptions and analyses are independent of any particular visual system.

If optical concomitant refers to the dynamic projective mapping of the environment onto the optic array during egomotion, then optical information refers to the inverse projective mapping back from the optic array to the observer/environment state. Optical information, or synonymously, optical specification, is necessarily mathematically completely adequate and sufficient to unambiguously determine the observer/environment state. Hence, the terms "optical information", "optical specifiers", and "optical bases" contrast sharply with the terms "clues" and "cues" which suggest an indeterminacy or inadequacy in the inverse geometric sense.

Another important distinction is that between the availability of information and the pickup or psychological utilization of that information. "Information pickup" and

"perception" are used synonymously. The point here is that the study of the optical concomitants and optical information is geometric and analytic whereas the study of information pickup is psychological and empirical.

A standard set of important kinematic distinctions is that among velocity, speed, angular velocity, and angular speed. Velocity and speed refer to translation. Velocity is a vector quantity having both direction and magnitude whereas speed is a scalar quantity equal to the magnitude of the velocity. Angular velocity and angular speed refer to rotation. Angular velocity is a vector oriented along the axis of rotation and pointing in the direction of advance of a right hand screw. Angular speed is the magnitude of the angular velocity.

LITERATURE REVIEW

The experience of one's own movement contrasts sharply with the perception of object motion and possesses both a "curiosity" or phenomenological quality and a survival value. The curiosity value is especially apparent due to the ubiquity of naturally occurring illusions of egomotion such as can be experienced when standing in a lake (Johansson, 1977). These illusions can be quite compelling and vivid and caught the attention of many lay people as well as pre-1950 psychologists. The pre-1950 psychological literature is thus peppered with many references to the phenomenological character of egomotion. The fewer pre-1950 attempts at explanation of the experience reduce to invocations of the particular theorist's regular theory of object motion perception coupled with an added factor such as attention or fixation. Hence, the pre-1950 references to egomotion will not be reviewed here.

The 1950s

Gibson's contribution

The modern treatment of egomotion stems from J. J. Gibson's insight that the survival value of egomotion, rather than the phenomenological quality, is the more important problem to be explained. This insight was forced on Gibson during World War II while he was attached to an Army Air Force research unit developing procedures for pilot selection (Gibson, 1947). Gibson reasoned, in effect, that in order to efficiently select and train people for complex egomotion guidance tasks, we ought to begin with the theoretical question of "What gives rise to the perception of egomotion?". His answer was that there must be an optical effect on the projective structure of light coming to a moving point of observation due to egomotion and that, in turn, this optic effect can serve as information about egomotion.

The optical effect or concomitant of egomotion is a particular total transformation of the structure of the ambient optic array at a moving point of observation. Gibson (1947, 1950) first pictorially described the optical

transformation or egomotion optic flow pattern. He used a set of schematic illustrations which essentially depict the optical traces that would be left on a photograph during a short time exposure. The depictions included lateral as well as frontal window views during level flight and a frontal view during a landing approach (Gibson, 1950, Figs. 53, 54, 58). Figures 9.8-9.12 from Gibson (1947, U.S. Government Printing Office) are reproduced here in Figures 1 and 2 for their historical value and as the principle illustrations of the egomotion optical flow pattern.

The first major mathematical analysis of the flow pattern was by Gibson, Olum, and Rosenblatt (1955). They provided a set of abstract analytically oriented illustrations of the flow pattern. These illustrations are abstract in that they do not correspond to tracings on a window but rather are a map of the differential optical velocities over the entire subhorizon hemisphere of the optic array corresponding to an infinite ground plain. These abstract illustrations, along with those from Gibson (1947, 1950), are intuitively clear, informative, and sufficient to enable the reader to grasp almost all qualitative aspects of the flow pattern. For example, the figures, make clear why Gibson et al. sometimes reserve the term "flow pattern" for level egomotion and use "expansion pattern" for egomotion toward a surface.

The quantitative aspects, of course, require formal analysis. The analysis of Gibson et al. is significant, technically correct, and yet leaves much to be desired. The analysis is significant for two reasons. First, it is very general in scope. Their Eqs. 1 and 3 give the magnitude of the optical (angular) velocities of ground points during egomotion along any rectilinear path at any angle of approach to the flat surface. In fact, in Footnote 13, they generalize the analysis to any points in space.

The second reason their analysis is significant is that they provide two alternate but equivalent analyses. The first analysis (Eq. 1) gives angular speed as a function of optical position referred to a meridian and eccentricity system of spherical coordinates. As such, Eq. 1 predates in generalized form (since it allows for any angle of approach) the analysis of Nakayama and Loomis (1974). In fact, the special case of Nakayama and Loomis is given in an unnumbered equation on p. 382. The second analysis (Eq. 3) gives angular speed as a function of optical position referred to an azimuth and elevation system of spherical coordinates. As such, Eq. 3 predates the analysis of Gordon (1965, 1966) and does so in a way that presents optical position in truly optical terms.

Despite its significance, the analysis of Gibson et al. does leave much to be desired. Although they give correct equations for the magnitude of the angular velocity vectors, they do not provide expressions for the actual vectors or their directions. Instead, they verbally define the "expansion vector" at a ground point to be a vector tangent to a line of constant meridian and having the magnitude indicated (p. 378). There is nothing wrong with defining such a vector. An explicit expression can be unambiguously derived and Gibson et al.'s omission results in nothing more than an inconvenience to readers who wish to make drawings, computer simulations, or further mathematically study such a vector field. However, Gibson et al.'s use and terminology are somewhat at variance with more sophisticated treatments such as Koenderink and van Doorn's (1981) and this creates problems.

A formal vector analysis (such as in this report) associates several different vectors with each ground point, each vector having its own direction, magnitude, and meaning. In particular, the angular velocity vector does have the same magnitude as Gibson et al.'s expansion vector but a different direction. However, this does not invalidate their findings and conclusions. Rather, readers must be sensitive to the core contribution under the terminology and formalisms used. This is not an easy task and Gibson et al. may be justly faulted for the obtuseness of their presentation. Their derivations and definitions tend to be unclear, the optic array significances of key variables are not developed, and most regrettable, many assertions are made about the existence of optical information for several egomotion parameters, but without mathematical support. In fairness, the task has subsequently proven quite difficult and more recent analyses are slowly confirming Gibson et al.'s intuitions about the existence of optical information.

A theoretical discussion of the implications of the flow pattern for aviation followed shortly (Gibson, 1955). This was in turn followed by a curious exchange in the same journal. Calvert (1956) took issue with some of Gibson's terms in favor of his own term "streamer pattern" put forth in an earlier paper (Calvert, 1954). He also objected to abstract diagrams that "merely tabulate angular velocities" (p. 478). Gibson's (1957) reply is still worth reading.

Although nonmathematical, Gibson's next theoretical paper is important because it emphasized the visual control of locomotion in all animals, not just humans. He explicitly presented verbal formulas couched in optic array terms for specific egomotion tasks such as aiming, steering, approach, braking, and pursuit and avoidance (Gibson, 1958).

Literary Descriptions

Other verbal and nonmathematical descriptions of the flow pattern have appeared in the science fiction literature (e.g., Asimov, 1953, chap. 2, pp. 26-27; Clarke, 1968, chap. 41). Attention is called to the descriptions by these famous authors, not because of any impact on scientific developments (there is no evidence of any such impact), but rather because they do show the appeal the flow pattern has for observing people.

The 1960s

One basic characterization of the flow pattern is that it is a field of angular velocities. After determining an angular speed formula, naturally early questions are those of determining maximum and minimum angular speeds and iso-angular speed contours. Surprisingly, only Schreiber (Havron, 1962) seems to have been concerned with determining maximal values, whereas almost all 1960's studies discuss iso-angular speed contours.

The 1960's Studies

Havron (1962). Havron (1962) reports an especially interesting analysis of the flow pattern which he credits to A. L. Schreiber as a ghost co-author. Schreiber derived two expressions for the optical (angular) speeds of ground points during rectilinear egomotion as a function of (1) path speed, (2) path angle, (3) path distance to the impact point, and (4) the ground coordinates of the ground points. The two expressions differ only in the use of Cartesian or polar ground coordinates and the origin of both systems is defined by the impact point itself. In fact, the analysis is not actually an optic array description since angular speed is given as a function of ground position rather than optical position. But, although no vector expression was used, Schreiber succeeded in making two major contributions concerning the ground loci yielding maximum angular speeds and equal angular speeds.

First, Schreiber determined the ridgeline of maximum angular speeds. Optical flow paths or meridian lines correspond to a set of straight ground lines all centered on the impact point and extending to the horizon. Along a ground line, optical flow is zero at the impact point, increases to a maximum value, and then slows to zero at the infinitely far horizon. Schreiber determined the distance

along each radial ground line at which optical flow is maximal for that ground line. No matter which radial ground line out from the impact point, and no matter what the angle of approach, the maximum angular speed on each ground line obtains at a distance from the impact point exactly equal to the egomotion path distance to the impact point. Hence, the ridgeline of maximum angular speeds is a circle on the ground centered on the impact point and having a radius equal to the path distance to the impact point. Each point on this circle yields a maximum optical velocity for its line but not all these maxima are equal. The ground point yielding the greatest flow for the whole optic array occurs at a point almost directly under the observer. This point is optically set back from the vertical by exactly one half the angle of approach.

Schreiber's second contribution was to determine the ground loci for which the angular speeds are the same. He determined that there are exactly three types of iso-angular speed curves. All descriptions here refer to the ground and not to a projection plane. Type I curves are very thin elongated curves which "look like ellipses" (p. 14). All have the impact point in their interior and all are contained within the circular ridgeline. Type II curves all contain the absolute maximum point within them and cross the ridgeline at two points. Type III curves completely surround the ridgeline. See Havron (1962) for more details.

Snyder (1964). Independently of Schreiber, Snyder (1964), in a study of visual factors in low altitude flight, derived an expression for the angular speeds during level flight as a function of ground speed and the three dimensional coordinates of ground points. An interesting feature is that he also presented an expression in which the forward coordinate is replaced by the time necessary to traverse the forward, but not the total, distance to a point. Snyder's equation can be used to determine iso-speed contours but only for level flight and is thus not as general as Schreiber's. Level flight is of sufficient interest that Snyder and also Harker and Jones (1980) used Snyder's equations to present a series of iso-speed contours to delimit the so-called (or better, alleged) "blur zone".

Gordon (1965). Thus far, all analyses have assumed rectilinear egomotion at a constant velocity and the main optical variable was angular speed. Gordon (1965, 1966) greatly extended the scope to include egomotion of varying speed along horizontal and vertical curved paths and also to include angular acceleration as an optical variable. For analysis, he used the field concept, a very powerful tool that is also used in this report. Simply put, a "field" is

used to represent a physical quantity that is a function of position in a given region. Gordon examined three fields: those of optical position, velocity, and acceleration.

Position as a function of position sounds odd. Gordon's positional field refers to optical position in terms of the azimuth and elevation spherical coordinates associated with a ground point. His optical position analysis is very clear. He gives explicit formulas for azimuth and elevation as functions of the Cartesian coordinates of a ground point, and moreover, precisely defines his coordinate system. This is an improvement over Gibson et al.

The velocity field analysis is more troublesome. He again is clear and explicit in determining the magnitude of the velocity vectors. Unfortunately, he is nowhere explicit in an expression for the vectors themselves, and worse, his expression for the vector magnitudes is wrong.

His method for determining total angular speed is to first determine the azimuthal and elevation velocity components, a totally acceptable and informative procedure. The error is the method of combining the components. Because he is using spherical coordinates, the components do not combine according to the simple Cartesian formula for distance which he used (Gordon, 1966, Eq. 46). Due to this error, his iso-angular speed contours (1965, Figs. 5 & 6; 1966, Figs. 12 & 13) are wrong. But even if the correct combination rule were used, his procedure for deriving the iso-velocity contours from this formulation in terms of Cartesian coordinates is very indirect and cumbersome. For Cartesian coordinates, the procedures of Schreiber or Snyder are more convenient.

His vectors themselves seem to be directed along the flow path optically "etched" on a fronto-parallel window during egomotion. Thus, they correspond to Gibson et al.'s velocity vectors and not the angular velocity vectors of modern analysis.

Gordon's study is not without merit. It attempted to greatly expand the scope of the flow pattern analysis and the problems are not irreparable. His velocity components are expressed in Cartesian rather than optic array terms, but this can be remedied. His method of combination is wrong, but again, that can be remedied. Finally, his discussion of curvilinear egomotion is interesting, and although there are some problems, it does anticipate some of David Lee's contributions.

Biggs (1966). The correct method of combining separate spherical angular rate components into total angular speed was exhibited by Biggs (1966, his Eqs. 4.2 & 4.5). He used a meridian and eccentricity system with the "north" pole oriented along the direction of egomotion. Thus his angular speed analysis is similar to that of Nakayama and Loomis (1974) but the flow vectors themselves are not specified and no projected flow diagrams are depicted. An unusual feature of Biggs' analysis is that in addition to describing the optical effects of egomotion for a zero-dimensional point in the environment, he also described the effects for a one-dimensional line and for a small two-dimensional surface element.

Consider the case of a two-dimensional surface element first:

Points and lines cannot expand, yet the flow pattern has sometimes been referred to as an expansion pattern. But the term is appropriate because the optical sizes of two-dimensional environmental features do increase with time as long as they remain in the "frontal" hemisphere of the optic array. (In the hind hemisphere, optical contours contract or negatively expand.) Biggs borrowed the engineering concept of "areal strain" and transformed it into an index, Y , of "rate of local areal expansion", which for rectilinear egomotion is given by

$$Y = 2(\dot{S}/r) \cos EC$$

where $\dot{S}$ is the egospeed, r is the distance to the environmental feature, and EC is the angle between the heading point and the feature. Notice that $\cos EC$, and hence Y , is positive in the frontal hemisphere of expansion and negative in the rear hemisphere of contraction. The local expansion, Y , can also be determined for curvilinear egomotion (Biggs' Eq. 4.6).

The one-dimensional case is here taken to include both straight lines and constantly curved lines such as the curbs of straight roads and uniform bends. Biggs noted that, when an observer maintains a constant distance to either type of curb, the optical projection of the curb maintains a constant optical position. This invariance of optical position for the line taken as a whole exists in spite of the shifting optical position of the individual points comprising the line. Biggs described the angular position of a straight line as the

$$\text{arc tan (signed lateral distance / height)}$$

This is the inverse of the projected slope and corresponds to what Warren (1932) calls the "optical splay" of the line.

In discussing the problem of controlling heading by tracking the optical position of specific elements such as curb lines, Biggs anticipated key elements of control-theoretic approaches to the perception of egomotion. He further anticipated, by differentiating between the effects of flow on specific scene elements and general global effects without reference to specific scene elements, two emergent "traditions" or approaches to flow pattern description. One approach, favored by psychologists and computer vision scientists, concentrates on the flow pattern as a whole by focusing on the behavior of single but arbitrary points. The other approach, favored by control-theory oriented engineers, concentrates on the optical behavior of specific greater-than-point-size scene elements such as the outline of a runway.

Biggs' study is rich in concepts and hypotheses which would have enriched research on the perception of egomotion. Unfortunately, he received virtually no citations in subsequent studies (possibly due to isolation in a British Mathematics Department and to expressing himself in terms of automobiles rather than using the more dominant vehicle of aircraft favored in the egomotion literature). His concept of rate of local areal expansion deserves further study.

Whiteside and Samuel (1970). The iso-angular speed discussions so far have been concerned with identifying the ground loci which yield a common optical speed. In a brief but noteworthy paper, Whiteside and Samuel (1970) extended the environmental domain to include all possible three-dimensional loci which yield a common optical speed. They argued that such a set of points forms a three-dimensional surface called a torus. This torus is a surface of revolution formed as follows: At the egocenter, position a circle tangent to the egomotion path and oriented so that the circle would roll forward along the path. Let the diameter of the circle be equal to the path speed divided by the desired or common angular speed in radians per second. Revolve the circle about the egomotion path. The resulting torus is a travelling doughnut, but it is a doughnut without a hole.

The ground iso-angular speed contours are formed by the intersection of the ground and the torus. In particular, Whiteside and Samuel found that (1) at zero altitude, the ground contour is a figure eight formed by two circles tangent at the egocenter and oriented as a pair of wings, (2) at an altitude equal to the radius of the generating circle, the ground contour (appropriate to the angular speed defined by the circle) has its greatest fore to aft extent, and (3) at an altitude equal to the diameter of the generating circle, the contour reduces to a point.

A few words of caution are in order. The above three contours of Whiteside and Samuel correspond to only one particular angular speed. There is not one torus but a family of tori, one for each ratio of path speed and intended angular speed. At zero altitude, all contours will be figures eight, but at a particular non-zero altitude, only one torus will intersect the ground plane at a point, all others generate a family of iso-angular speed contours.

Whiteside and Samuel do not give the formal equations for the toroidal analysis, but they do give a diagrammatic justification for their angular speed expression. Their main interest was in discussing the "blur zone" which is an alleged region in which angular speeds are too great for objects to be seen clearly. They discuss the problem of determining blur thresholds but do not give any values. Presumably some blur threshold value would be used to calculate a threshold torus which would then demarcate between regions of blur and non-blur. They acknowledged that actual boundaries may have to be somewhat empirically adjusted. Whatever the merit or lack of merit of their blur zone discussion, the introduction of the toroidal angular speed surface is a major contribution to the study of the flow pattern.

Summary of Angular Speed Findings

The concept of iso-angular speed contour and the allied concepts of blur zone and zone of nonperceptible motion emerged as the dominant themes of the 1960's studies of the flow pattern. Although the studies of the 1970's show a marked change of emphasis and attention, these three concepts remain alive today and numerous theoretical and empirical questions remain unanswered. As a flow pattern descriptive concept, angular speed is very germane to this report and it is appropriate to take stock now of what has been learned.

The allied zone concepts are not actually flow pattern descriptive but rather are psychological hypotheses about motion perception. As such, they are not germane to this report, but since they motivated the descriptive studies, a brief treatment is not totally irrelevant. The treatment has two facets. One facet examines what the 1960's studies said, and the other facet, in the next section, what they, and later studies, did not say. That is, several hidden assumptions will be brought to light.

1. Minimum angular speed. The minimum angular speed possible is zero. The inventory given here is from the 1950's (Gibson, 1950) but is included here for completeness. The ground point toward which the observer is heading and the point from which the observer is directly headed away both have zero angular speeds. In addition, all ground points infinitely far from the observer have associated optical speeds of zero. These points constitute the horizon.

2. Maximal angular speed. Optically, angular speed is zero at the focus of expansion, increases to some maximal value along each flow line, and then decreases back to zero at the horizon. Environmentally, these flow lines correspond to straight lines emanating from the aim point and continuing to infinity. These facts were pictorially illustrated in Figure 3 of Gibson, Olum and Rosenblatt and Figure 58 of Gibson (1950). Schreiber (Havron, 1962) determined that the set of ground points which correspond to the relative maximum angular speed on each flow line form a circle (the ridgeline) on the ground. The center of this circle is at the aim point and its radius is equal to the path distance between the observer and the aim point. In particular, the fastest angular speed of all is set back from the directly below of the observer by half the path angle.

3. Iso-angular speed contours: A conjecture. Although Whiteside and Samuel only discuss the contours resulting from the intersection of a flat ground plane and the iso-angular speed tori generated during level flight, it is reasonable to hypothesize that all iso-angular speed contours are special subsets of iso-angular speed surfaces. These are probably all zero-hole tori with their orientation determined by the direction of egomotion. However, the intersecting surfaces can be surfaces other than a flat ground plane such as mountain slopes, rolling hills, mesas, and winding valleys. If the ground surface is mathematically specified and if the path direction is given, then the exact iso-angular speed contours can be determined both in the environment and on any projection surface, fixed or rotating. Hence, Whiteside and Samuel's toroidal insight can be used to more exactly determine Schreiber's three types of contours for flat grounds and to generalize the descriptive analysis to more general situations.

The toroidal approach can also be used to determine maximal flow rates since the diameters of the generating circles are inversely related to flow rate.

4. Alleged perceptually significant zones. The concept of iso-angular speed is a geometric concept and does not depend on psychology for its definition and geometric significance. However, some researchers have ascribed a layer of perceptual significance to particular iso-angular speeds.

a. Zone of nonperceptible flow. Geometrically, only the foci of optical expansion and contraction and all horizon points have zero angular speeds. Ground points "near" these points have small but non-zero angular speeds. Hence, some researchers have argued that, due to threshold limitations, these small speeds are perceptually indistinguishable from zero flow and an early concern was the determination of a zone of nonperceptible flow. In particular, Havron devoted considerable effort to the problem of selecting the nonperceptible motion thresholds from the existing (object motion) literature. He then presented (1962, Figure 3) a typical pilot's eye view of a runway with iso-flow curves superimposed on it. An interesting feature is that he provides the zone demarcation for both daytime (10 min/s) and nighttime (30 min/s) vision.

b. Blur zone. Besides being too slow to be seen, motion may be too fast and induce blur. Although Calvert's "streamers" might be associated with blur effects, Snyder was the first to formally analyze the iso-angular speed contours demarcating blur and non-blur zones during level flight. He used a "good" blur threshold value of 30 deg/s and also a possible blur threshold value of 15 deg/s. Whiteside and Samuel in their paper entitled "Blur Zone" simply asserted that there must be a blur zone due to threshold considerations but they did not provide any values.

c. Zone of perceptible flow. If the zone concept is valid, in between the zones of nonperceptible flow and blur, there is presumably a zone of perceptible flow. The point here is that there should be three classes of zone, not three zones. For a very steep angle of approach, there might be two no-flow zones: one around the horizon and one around the aim point. For shallow angles of approach, these two zone might meld into one.

Critique of the 1960's Studies

The studies of the 1960's resulted in major advances in the description of the egomotion flow pattern. Nevertheless, there are a number of points a modern reader should be aware of:

Independence of studies. Several of the studies were conducted independently of those that came before them. This is not surprising since Havron's and Snyder's work appeared in Technical Reports and hence were not as readily available nor as widely disseminated as journal articles might have been. The effects, however, are a duplication of effort and a lack of continuity in the literature.

Confusion of speed and velocity. Many writers refer to iso-(angular)-velocity contours when in fact they are really referring to iso-speed contours. Failure to properly distinguish between velocity and speed is, unfortunately, very characteristic of the early flow pattern literature.

Confusion of retinal and optic array flow. The ascription of perceptual significance to angular speeds as so far described makes several untenable assumptions. Perceptual motion thresholds are generally referred to the retina. Implicitly at least, it has been assumed that the retinal flow pattern is equivalent to the optic array flow pattern. But this is true only when the eye is not rotating relative to the environment. Any eye rotation, slow or fast, will alter the pattern of retinal angular speeds (see the discussion later in this report). In particular, any fixation on a ground point results in a slow eye rotation and produces a zero retinal flow for that point. Hence, no fixated object will blur due to egomotion. Only when the eye fixates on the horizon is there a chance that the two flow patterns will be equal. But, in general, eye movements of all kinds will destroy the equivalence and angular speed values in the array will not be the same as on the retina.

Uncritical acceptance of the threshold concept. Even if angular speed is referred to the retina, asking whether a particular speed is above a threshold value presupposes the validity of the threshold concept. There are three problems with this supposition: (1) The most general problems about thresholds are those discussed by Signal Detection Theorists. These are well known and will not be developed here. (2) The second problem with the application of the threshold concept, assuming it has any utility at all, is with the assumption that threshold values obtained for

object motion are relevant for egomotion purposes. Threshold values are extremely context sensitive and the flow pattern is essentially a set of complex motion interdependencies among a large aggregate of points. If threshold values are desired, they should be obtained using the very flow pattern for which they are to be used. (3) The third problem with the threshold concept is the assumption of a sharp demarcation line between the perceptual zones. It is more reasonable to assume a gradual transition from no perceptible flow to perceptible flow or from non-blur to blur. Remember that any eyemovement or eye fixation directed at the zone or zone boundary immediately alters the retinal flow values. This, of course, undermines the original situation.

Early 1970s Descriptions

The 1970s brought some rediscoveries but also some new refinements and directions in the study of the optical concomitants of egomotion. The early 1970s were concerned primarily with optical description whereas the later 1970s saw a growing concern with demonstrating optical information. However, the distinction between optical description in the sense of an environment to optic array mapping and optical information in the sense of the inverse optic array to environment/observer state mapping was not yet realized.

Description of Local Scene Element Effects

This lack of distinction is seen in a descriptive and perceptually speculative study by Naish (1971) entitled "Control information in visual flight". Naish's study is noteworthy in two respects: First, the descriptive aspects are markedly different from those of the studies reviewed so far, and second, despite the confusion of description and information, the perceptually speculative aspects contain many interesting hypotheses about pilot's actual use of optical information.

Naish differed from previous researchers in that he was not primarily interested in either the geometry or perception of egomotion as such, but rather in manual control problems encountered by pilots making a landing approach. For a pilot to make a controlled landing, he must be able to detect vehicle movement. Hence Naish was interested in the relationship of egomotion induced visual scene changes to the "known" visual sensitivity of humans. He reasoned that not all visual scene changes proceed at the same rate. At a

given instant, some changes might be below, others just above, and still others greatly above threshold. Thus, he concluded that all sources of information are not equally perceptible, and hence might be rank ordered on this basis into a hierarchy of usefulness for manual control tasks.

In order to help isolate sources of information he restricted the visual scene to a fronto-parallel view of a runway outline and a horizon line. He then identified four classes of "visual effects" of descent and lateral egomotion on the simplified scene: (1) The perspectival inclination of the sides of the runway change depending on the pilot's height and lateral offset from the runway. (2) The outline of the runway undergoes expansion during egomotion. He does give an expression for the rate of angular expansion but limits it to apply only to the optical width of the runway rather than as an expression for the angular speed of any ground element. In general, the greater the rate of expansion, the closer the runway. (3) The visual size of the runway changes as a result of the expansion. In general, the greater the angular size, the closer the runway. (4) Each runway line undergoes apparent rotation as a result of the changes in inclination due to the lateral and height changes during egomotion.

Naish provided quantitative expressions for these effects as a function of a pilot's lateral and vertical position and egomotion. He evaluated these expressions using typical landing values, and then compared the results against typical threshold values. In this way, he proposed a speculative rank ordering of detectability and, hence, presumed usefulness for both lateral and vertical control tasks. In particular, he speculated that for lateral control, inclination effects are best, whereas rotation effects are worst. And that for vertical control, size effects are best whereas expansion and rotation effects provide no useable information.

Several points must be kept in mind when considering Naish's speculation about the relative usefulness of different sources of information:

1. The sources of information and the visual effects are discussed only for a highly simplified and schematic outline scene. Such a scene can resemble a real nighttime landing. That day landings are easier indicates there are usable sources of information in the richer scenes not considered by Naish.

2. Although other researchers have identified several tasks for the guidance of egomotion (e.g., Gibson's, 1958, tasks of steering, stopping, etc.), Naish concentrated on one task, steering, and analyzed it into separate subtasks of lateral and vertical control.

3. Whereas others have identified some optical concomitant for a particular egomotion aspect, Naish identified several concomitants for both lateral and vertical egomotion.

4. That the different sources of information for the same aspect may not be equally useful is an important psychological problem and traces back to attempts to rank order the classical cues for size and depth perception. Naish's attempt to do this for egomotion is interesting but needs discussion: (a) It is not clear that threshold values taken from the non-egomotion literature may be legitimately applied to the perception of egomotion. (b) Even if new thresholds were obtained using his schematic scene, the new values still may not be appropriate to rich natural scene egomotion. (c) Lastly, it is further not clear that detectability thresholds are an appropriate rank ordering device for utility or usage especially in dealing with greatly suprathreshold scenes. However, Naish's point that not all scene aspects are detectable until late in a landing approach is important.

5. Determinations of detectability and utility are empirical rather than geometrical. That such questions are not easily answered is evidenced by our current incomplete understanding of heading perception inspite of several empirical studies (Llewellyn, 1971; Johnston, White, & Cumming, 1973; Warren, 1976; Regan & Beverley, 1979, 1981; Riemersma, 1981).

In conclusion, Naish's analysis raises several important issues and that is one reason for the space given it here. Unfortunately, his analysis appeared in a control-theory proceedings and was not cited in the egomotion literature until recently.

Explicit Vector Description

Naish's analysis of the optical concomitants of egomotion is atypical and represents a departure from the "mainstream" descriptive studies. The mainstream is characterized by concern with describing the global optical flow pattern induced by rectilinear travel towards or along any arbitrarily textured endless flat surface. From the start (Gibson, 1947), the flow pattern was deemed to be a velocity vector field, and geometrical techniques were used to schematically depict both the vector directions and magnitudes.

As shown in the review of the studies of the 1950s and 1960s, there was great interest and progress in

mathematically specifying the magnitude of the angular velocity vector associated with each environmental point. Surprisingly, no studies reviewed here were concerned with mathematically specifying the vectors themselves until those of Nakayama and Loomis (1974) and Lee (1974).

Nakayama and Loomis reviewed and capitalized on several previous insights in preparing their description of the flow pattern. One insight was Gibson's distinction between optic array versus human retina referred flow description. Optic array description is gaze line independent whereas retinal flow description depends on the gaze line and changes of gaze line. The value of this distinction is best seen in descriptions of retinal flow under eyemovements and curvilinear egomotion. Nakayama and Loomis did include a major analysis of rotation effects in their vector description but that is not reviewed here.

The second insight they discussed is Gordon's (1965) use of the concepts of "instantaneous positional field" and "instantaneous velocity field" for description. Gordon himself did not provide total vector specifications although he did provide the azimuth and elevation rate components individually.

Nakayama and Loomis did not pursue Gordon's azimuth and elevation system but rather capitalized on a very simplifying (but not simplistic) insight of Whiteside and Samuel (1970): Specifically, all optical elements flow along constant meridian or great circle paths on a unit projection sphere during rectilinear egomotion. This means that the total change of position of an element is describable solely as a change in the element's eccentricity along the constant meridian path. Thus, Nakayama and Loomis were led to use a meridian and eccentricity spherical coordinate system, and further, to use Whiteside and Samuel's compact expression for total angular speed ($(\dot{s}/r) \sin EC$, in the notation of this paper).

Comments on the analysis of Nakayama and Loomis. Nakayama and Loomis's derivation of the vector directions and vector magnitudes is clear and explicit. They provide both a specification in terms of meridian and eccentricity and also a transformation to Cartesian coordinates. Theirs is a highly significant analysis in the history of research on egomotion and the following points should be noted:

1. Their choice of a meridian and eccentricity coordinate system was motivated by the simplicity of the description of optical effects during rectilinear egomotion. However, a major part of their study was concerned with rotation effects and it is interesting to note that Gordon's

stated reason for preferring to describe azimuth and elevation effects separately, rather than total effects, was because of the advantages for describing the effects of rotations.

2. Nakayama and Loomis's choice, however, meant that there was no problem as to how to combine the meridian and eccentricity speed components to yield total angular speed during rectilinear egomotion: Since here the meridian speed component is zero, total angular speed is due entirely to the eccentricity angular speed. Thus, Nakayama and Loomis's (actually, Whiteside and Samuel's) expression for total angular speed is correct and they avoided Gordon's mistake in determining total angular speed. Recall that total angular speed is not determined by the Cartesian version of the distance formula if the components are given in spherical coordinates. One implication is that total angular speed in the presence of rotation is not given by Whiteside and Samuel's expression since then both meridian and eccentricity angular velocity components are non-zero, and further, caution must be taken in combining these spherical coordinate components.

3. Although Nakayama and Loomis give credit to Whiteside and Samuel for the use of meridian and eccentricity coordinates and the simple expression for total angular speed, the same results may be found in Gibson, Olum and Rosenblatt (1955) but in a highly obfuscated form. Whiteside and Samuel do deserve credit for their clarity of exposition.

4. The expression, $\dot{\theta} = (\dot{s}/r) \sin EC$, although quite simple in form, is not really an optic array description since angular speed is not given as a function of purely optical variables. Specifically, neither r , the distance to a ground element, or $(\dot{s}/r)$, path speed divided by point distance, are optic array parameters. Further, this expression, although correct, can be misleading in that a cursory reading might suggest that angular speed depends on eccentricity but not meridian. Actually, angular speed depends on both and this can be seen in an all optical variable expression as presented in a following section.

5. The simple expression for angular speed is correct for travel at any angle to a surface. However, much of Nakayama and Loomis' development and discussion is appropriate to level travel or travel at a constant altitude. Specifically, in developing their expression for the angular velocity vector (their Eq. A1), they take advantage of a simplification resulting from aligning the direction of travel vector with the X-axis. Earlier they established the X-axis as parallel to the ground. It is straightforward, albeit cumbersome, to replace their

direction of travel vector, $(1,0,0)$, with a more general unit direction vector. The resulting more general expression is also more complex.

6. The "omega-T" vector they identify as the "angular velocity vector" is tangent to a unit sphere at the optical location of a point in the environment, and thus, their omega-T vector points in the instantaneous direction of the optical flow and has a magnitude equal to the total angular speed of the point. This particular specification seems very reasonable for describing the flow field. However, in standard kinematic analysis, angular velocity vectors are oriented along the axis of rotation and not the instantaneous direction of movement. Hence, the standard angular velocity vector would be oriented 90 degrees from Nakayama and Loomis' T and omega-T vectors in their Figure A1 and, instead, would be oriented along their N vector.

7. If conveying instantaneous direction of optical movement is primary, the the direction of their T and omega-T vectors is correct. The magnitude generally ascribed to a tangent vector in kinematic analysis is the tangential component of speed which is equal to the angular speed times the distance to the point of tangency. On a unit sphere, this distance is one so the length of such a tangent vector is indeed equal to the angular speed. But this is not true for tangent vectors at other than unit distance from the egocenter.

8. Nakayama and Loomis' study is clear, explicit, and contains many more ideas than are discussed here. With respect to description, they also analyze the effects of pure rotation and rotation combined with translation. Going beyond descriptive analysis, they provide an expression for relative distance information throughout the terrain as a function of only optical parameters (their Eq. 4). They further propose a neuronal model for extracting depth information from the flow pattern. In conclusion, their study is one of the most significant in the study of egomotion. The discussion here is meant to build upon their foundation.

A Cylindrical Coordinate Analysis

Almost all models of the optic array use a spherical projection surface. This is partially because of the shape of the retina, but a better reason is that lines and regions on a sphere directly and naturally map into angles, and the optic array may be defined as a set of angular relationships. However, since what ultimately matters is the angular relationships, any analytic system that can be transformed into those relationships may be useful.

As demonstrated by Lee (1974), a particularly useful system for analyzing rectilinear egomotion is based on a cylindrical projection surface. This is because the great circle meridian flow paths on a sphere become a set of parallel straight lines on the cylinder if the axis of the cylinder is aligned with the rectilinear egomotion path.

In order to describe the optical effects of egomotion, Lee also used the concepts of positional and velocity vector fields. Simplifying his notation somewhat: let Z and z be environmental and projected forward distances respectively, $\dot{Z}$ be path velocity of the observer, $\dot{z}$ be the projected velocity of a point, θ be the cylindrical polar angle corresponding to spherical meridian, and $\dot{\theta}$ be the rate of change of θ . Then, for an optic element, the optical position vector is represented as (θ, z) and the projected velocity vector as $(\dot{\theta}, \dot{z})$. (The usual third cylindrical coordinate, R , is "lost" in projecting onto a cylinder of radius $R = 1$.)

Lee defines the optical flow pattern as the "set of velocities of optic elements past positions on the cylindrical optic projection surface" (p. 253), and this is symbolized by the set $(\dot{\theta}, \dot{z})$. Since Lee restricts his analysis to rectilinear egomotion and, moreover, to level egomotion at a constant altitude, $\dot{\theta}$ equals zero, and the projected velocity vector becomes simply $(0, \dot{z})$. Notice that no value for $\dot{z}$ has yet been given.

For purposes of description, this cylindrical system does not seem immediately preferable to a spherical one. On the positive side, it is interesting to see that the optical flow paths form straight lines on a cylinder during rectilinear egomotion. For more complex egomotion, however, the simplicity is lost, and even for rectilinear egomotion it is desirable to translate back to a set of angular relationships.

But Lee's study was not concerned only with describing the flow pattern. Instead, he was primarily concerned with demonstrating the existence of information for rectilinear egomotion through a rigid environment. For that purpose, the cylindrical system does have considerable power. Although it is not the purpose of this review to discuss the information in the flow pattern, some of his information analysis is presented because it does have a descriptive component.

Lee begins by "determining three general 'rigidity' properties of the optic-velocity field that necessarily result from ... rectilinear locomotion through any rigid environment." (p. 256, emphasis his):

Rigidity Property I. On a cylindrical projection surface, all optic elements move in a common direction along flow lines parallel to the egomotion path. The common direction is opposite to that of the egomotion. This pattern, Lee points out, is equivalent to the radial flow on a sphere.

Rigidity Property II. If the observer moves with environmental velocity $\dot{Z}$ and acceleration $\ddot{Z}$, and the corresponding projected velocity and acceleration is $\dot{z}$ and $\ddot{z}$, then

$$\ddot{z}/\dot{z} = \ddot{Z}/\dot{Z} = \text{a constant}$$

for all optical positions. This expression is his Eq. 8 but in a simplified notation. The ratio is a constant because $\dot{Z}$ and $\ddot{Z}$ refer to observer motion and must have unique values at any given instant. It follows that if the egomotion is constant, then all projected velocities are constant but not necessarily equal at each optical position. That is, the projected velocity at a given optical position never changes.

Rigidity Property III. This property is concerned with the accretion/deletion of optical texture elements due to occlusion of farther environmental points by nearer points if the environment is not flat. Lee notes that, along a given flow line, faster optical elements will temporarily overtake and hide slower elements.

These three rigidity properties are actually descriptions of necessary optical effects of rectilinear egomotion through a rigid environment and as such are relevant here. Lee turned the descriptive analysis into an information analysis by claiming that the converse is also true. He argued that an optic flow pattern having both rigidity properties I and III specifies rectilinear egomotion through a rigid environment and that a pattern having rigidity property I by itself specifies rectilinear egomotion.

He also considered the problem of information about the environment as such. His demonstration (p. 260) that for rectilinear egomotion at a constant altitude, H , over a flat ground, $\theta = \theta$, and

$$\dot{z} = (\dot{Z}/H) \cos \theta$$

is of interest here. Descriptively, this equation indicates that flow is a joint function of two factors: One is projected (here cylindrical) position and the other is the ratio $(\dot{Z}/H)$ which is simply ego velocity scaled in eyeheight units. The significance of this will be discussed later.

This equation also indicates that projected velocity on the surface of a cylinder varies across flow lines (determined by the θ) but that all flow along a particular flow line is constant. This contrasts sharply with the radial flow pattern on a sphere in which flow varies with both the flow line (meridian) and position on the flow line (eccentricity).

Returning to the information analysis, Lee then demonstrates that the environmental cylindrical coordinates of a point may be determined from the projected or optical relationships but rescaled in units of eyeheight H . He notes that, for a particular animal whose eyeheight is constant during locomotion, the flow pattern makes available body scaled information about the environment. Of course, the body scaling is no longer true if eyeheight changes as with birds, but that all is scaled relative to the instantaneous eyeheight remains true.

Lee also discussed the information available for such diverse topics as the velocity and acceleration of the observer, object motion, obstacles in the path, and controlled braking (also in Lee, 1976). The range of these and the previous topics makes Lee's analysis a most significant one both in providing fresh insights into old problems and in opening new territory for the study of egomotion. The worth and power of a cylindrical analysis for both description and information analytic purposes is thus well demonstrated. However, it should be remembered that the "final form" of an optic array analysis should be in terms of angular relationships. For example, demonstrating that flow lines on a cylinder are parallel and that flow along a line is constant is useful and insightful. But, the fact remains that optic array angular activity is of varying speed along nonparallel radial lines.

Reasons for primacy of spherical description. The difference is neither trivial nor dismissible by arguing that one system may be transformed to the other. The spherical system is at least first among equals mathematically, biologically, and phenomenologically:

1. Mathematical reason. The use of a cylindrical projection surface is justifiable for the case of rectilinear egomotion with no rotation. If there is any need to describe rotation effects along a rectilinear path, as in shifting the description from the optic array to a rotating retina, or if the path is curvilinear, then a spherical projection surface is generally easier to use.

2. Biological reason. Biologically, eyes, including compound eyes, are spherical in structure and this influences the structure and operation of various pickup mechanisms such as the center-surround mechanism hypothesized by Nakayama and Loomis. These can, of course, operate on only spherically realized patterns.

3. Phenomenological reason. The pickup of information about ego-environment states is accomplished by all manner of species. The phenomenological awareness and introspective attitude occasionally and voluntarily realizable by humans is not a necessary condition for information pickup. Nevertheless, phenomenology can be a powerful tool. The flow pattern, when noticed, appears radial with elements varying in apparent speed along the flow lines and this is consistent with a spherical description.

A Non-Technical Description

Descriptions of the flow pattern have tended to increase in technical complexity. An exception is a nontechnical but still informative article by Hasbrook (1975) subtitled "Clues and cues to a safe touchdown" and intended for pilots. He discussed the relative merits of several features of the flow pattern that might be of use on a final approach. Of special interest are two diagrams composed of the superposition of several perspective views of a runway during a final approach. One diagram corresponds to a straight-line three-degree flight path and shows the superimposed views to form a family of nested, progressively larger, trapezoids. The other corresponds to a downward curved flight path and shows the superimposed views to form a set of nonuniformly proportioned, overlapping, trapezoids. Diagrams of this type are useful but they would be more useful if the horizon were included and left invariant. It should be noted that if an approach is aligned to the center-line of the runway, then all trapezoidal perspective views of a runway should be such that the long sides of the runway converge to the same vanishing point and that vanishing point is always on the horizon no matter what the flight angle, or distance out.

Information Analyses

Beginning with Lee (1974), the mid 1970s saw a decided shift from descriptive analysis to information analysis. Descriptive analyses are concerned with the effects of ego-environment states on the optic array. Information

analyses are concerned with the specification of ego-environment states given only optic array states and are thus generally more complex than descriptive analyses since they presume a certain degree of previous description.

Information analyses are briefly discussed here for two reasons: First, not only does information analysis presuppose some description, in certain respects it extends and amplifies old descriptions and also may reveal new projective effects and relations, that is, it can yield new descriptions. This is well illustrated by Lee (1974). The second reason is that it is a natural next-step after description.

Problems and Strategies in Information Analysis

The determination of information in the flow pattern may be approached as two successive subproblems. The first is concerned with ascertaining the existence of information under the most favorable mathematical conditions: The environment is idealized, projection is perfect, noise is non-existent, and texture elements exist in whatever quantity, density, and distribution as is mathematically convenient. The second is concerned with determining ego-environment states under the added realities of actually occurring and therefore mathematically non-ideal optic arrays. Such optic arrays may have scarce and "inconveniently" located optical elements and, further, the values of optical parameters may be affected by unknown amounts of noise from various sources (Zacharias, 1982).

Two strategies may be used to demonstrate the existence of information in idealized optic arrays. One strategy is to start with optical variables such as angular extents and angular rates of change and then to show that some functional combination of these parameters yields a value indicative of an ego-environment state (even if only to a scale factor). Such an approach was utilized by Lee (1974). It is especially important in using this strategy to make explicit any assumptions made about the environment.

A second strategy is based on the mathematical argument that a system is characterized by its invariants under transformations. Gibson (e.g., 1966) used this principle as a basis for his perceptual theory. We perceive a stable environment despite a highly transforming retinal image because the perceptual system is adapted for extracting invariants under the transformations. The invariants are assumed to be the basis for the optical specification of ego-environment states. Gibson's analysis, however seminal, was informal and intuitive. His examples of invariants were

never precise mathematical expressions nor was the optical specification formally demonstrated.

Both of these strategies for mathematically determining the information in the flow pattern take as their starting point a set of values of optic array parameters such as optical positions, extents, and velocities. How these values are "supplied" to the derivation and analysis of those "operators" is not relevant to the derivation and analysis of those operators. However, it is natural to also consider how a real biological or computer vision system could determine ego-environment states from the light coming to a moving receptor surface. Accordingly, a number of studies primarily concerned with computer vision, artificial intelligence, and biological extraction mechanisms have studied the problem of determining the values of the optical parameters from the flow field. Hence, much of the literature, especially that on computer vision, is at least of collateral interest.

The studies of invariants have generally antedated the studies on flow value realization which, in turn, generally antedate those on information in angular relationships. The order is reflected in the following discussion.

Information from Invariants in Flow

A rigorous analysis of the invariants and singularities of the flow pattern and their information significance has been provided by Koenderink and van Doorn (1975, 1976 a, 1976 b, 1981). Their treatment is general, highly sophisticated, and as an information analysis, beyond the scope of this review. However, several features are of interest:

1. The scope of their analysis is very general and applies to the flow pattern (or "motion parallax field" as they call it) arising from the movement of rigid bodies as well as of an observer.

2. Since the flow pattern arising from the relative movement of an object and observer need affect only a small region of the optic array, they carefully develop an analysis appropriate to the local as opposed to the global structure of the flow pattern. They are apparently the first to do this.

3. Another novel feature of their analysis is that they decompose the motion parallax field into elementary component fields and separately analyze them. In particular, they identify a lamellar or curl-free component field as the

specifier of the translation of the observer. (This is the "exterospecific component" of the motion parallax field discussed by Koenderink and van Doorn, 1981.) Another component field, the solenoidal or source-free field, is indicative of rotation. The importance of the decomposition is that effects due to rotation, such as eye movements, can always be subtracted out and the information for translatory egomotion be separately mathematically appraised.

4. The appraisal can also be done, they suggest, by elementary biological detector mechanisms as are now thought to exist. If this is true, the information for egomotion as such can be biologically extracted simply and the information for eye movements partialled out. It is worth emphasizing that simplicity of complexity of biological realization in perception is independent of the simplicity or complexity of the mathematical analysis related to the process.

5. Their mathematical analysis used such vector calculus concepts as the gradient of a scalar field and the divergence and curl of a vector field. One very important property of the gradient, divergence, and curl (and their functional combinations) is that they are invariant over transformation of coordinates. Hence, the coordinate system and the origin of the system may be chosen for convenience without affecting the results. The invariance of the gradient, divergence, and curl means that they are powerful concepts for description as well as for information analysis.

6. The terms "gradient", "divergence", and "curl" are in standard mathematical usage and have precise mathematical significance. Koenderink and van Doorn (1976 a, 1981) point out that the usage of the terms "gradient" by Gibson and "parallax curl" by Gordon is intuitive and informal, but without unambiguous mathematical meaning. The informality of psychological language need not pose any serious problems when compared with the rigor of mathematical language as long as the intended referents are essentially the same. If, however, the same term has different meaning for each discipline then unnecessary misunderstandings may arise. For example, Koenderink and van Doorn (1976 a, 1981) point out that Gibson's "focus of optical expansion" does not coincide with the locus of maximum divergence. The discrepancy is one of nomenclature only since each referent has its own behavior and significance. All problems would disappear if two separate terms were mutually agreed to.

7. But there is a deeper issue here: Although mathematics may have more precise definitions and a more ancient claim on particular terms, it does not follow that the phenomenon being studied by a psychologist is

necessarily illegitimate simply because of its naming or misnaming. Psychological significance is determined by psychological inquiry. A recent example, also involving the term "divergence", is an interesting perceptual study of the "blur" of the flow pattern by Harrington and Harrington (1981) and Harrington, Harrington, Wilkins, and Koh (1980). The blur they refer to is supposedly due to retinal resolution problems in processing optical flow. A better term might be optical traces induced by short periods of optical flow. They define "divergence" as the angle between the extreme left-most and right-most blur lines in a display. One problem with this definition is that a larger display means larger divergences. Other problems due to their terms are discussed by Prazdny (1982). Whatever, the terminological problems, their study has definite perceptual merit and does broaden our understanding of perception involving the egomotion flow pattern.

Extraction of Optical Flow Values

The basic raw input available to a biological or computer vision system is a structures, time-varying, two-dimensional distribution of light energy falling across its receptor surface. For an extensive review of the literature on how such systems analyze motion from time-varying energy distributions see Ullman (1981). Of particular interest with respect to the mathematical determination of optical flow are the works of Ballard and Brown (1982), Horn and Schunck (1980), Marr (1981), and Ullman (1979).

Information From Angular Relationships

The references cited here tend to be fairly recent. This is so because throughout the history of perception, it was generally assumed that the retinal image underdetermined the environment. That there was a possibility of, indeed a theoretical and practical demand for, demonstrating a mathematical specification of ego-environment states from the set of angular relationships in the optic array was realized first in the perceptual literature with the rise in interest in ecological optics (e.g., Gibson, 1950; Hay, 1966), and later, in the computer vision literature with the rise in interest in automatic image processing (see Ballard & Brown, 1982). In general, computer vision oriented studies have been most detailed and explicit because of the necessity of writing operational algorithms. Psychological research has tended to be content with "in principal" arguments.

Studies utilizing optical flow fields and primarily interested in demonstrating information about the environment include those by Clocksin (1980), Hoffman (1980), S. Prazdny (1981), and Ullman (1979). Studies utilizing flow fields and showing an additional significant concern with egomotion are rarer. Longuet-Higgins and K. Prazdny (1980) have show that "an observer can in principle determine the structure of a rigid scene and his direction of motion relative to it from the instantaneous retinal velocity field" (p. 394, emphasis theirs). A realization based on their equations requires sensitivity to the first and second spatial derivatives of the retinal velocity field.

K. Prazdny (1980) provides a clear, explicit, and useful study which has the virtue of bridging the perceptual and computer vision literatures. Although his primary concern is with demonstrating optical specification in a form suitable for computer implementation, his descriptive system is noteworthy. It is essentially an extension on Nakayama and Loomis' but further distinguishes between "retinal" velocity and angular velocity vectors. The angular velocity vectors are 3-dimensional and are those axial vectors of standard kinematic analysis. The "retina" is a flat projection plane tangent to a unit sphere so his "retinal" velocity vectors are 2-dimensional and describe motion on a flat projection surface or window. The development incorporates both translatory and rotary egomotion.

One nice feature is the inclusion of several "instantaneous positional velocity fields" on the planar retina representing various types of egomotion through different environments. These flow field depictions, and also those in Horn and Schunck (1980) and K. Prazdny (1982), look very different from those in Gibson (1947, 1950), Gordon (1965, 1966), and Lee and Lishman (1977). The difference is that the psychologists select an array of point equispaced in the environment which then become non-equispaced on a "window" according to the laws of perspective, whereas the computer scientists tend to select points which are equispaced on the window or computer CRT surface. Thus, psychologists depictions look "familiar" and intuitive, whereas those of the computer scientists look unfamiliar since they depict an environment which increases in texture density away from the observer. Both depictions are, however, correct on an individual element basis.

Later 1970s and Recent Descriptions

The shift in interest to information analysis does not mean that descriptive analysis was totally abandoned nor that there is no further need for description. What is adequate description and information analysis for a computer implementation may not be adequate or appropriate for human or insect vision. Just as one may experiment with a computer program to see how it responds to a variety of inputs, one may also experiment with humans and insects. Descriptions must be tailored not only to geometry but also to working systems and the tailoring is achieved by empirical research.

What follows is a discussion of several recent studies. The subdivisions are tentative and are mainly to provide some structure, however arbitrary.

Descriptions for Optimal Control Modeling

Certain elements of controlling an airplane, especially during a landing approach, may be considered analogous to a tracking task. At any given moment, the apparent visual scene (e.g., the perspective view of a runway) ought to have a certain configuration (e.g., the perspectival outline of the runway ought to be symmetric). Any deviation indicates an error from a desired course (e.g., a skewed, nonsymmetric perspectival outline indicates that the approach is not straight in). Both the degree and direction of deviation from the ideal perspectival shape and the rate of change of deviation can be used as sources of information for course control adjustments. This was the basic concern of Naish's (1971) descriptive analysis. Two other control-information oriented studies are by Grunwald and Merhav (1976) and Wewerinke (1978). Grunwald and Merhav provided a fronto-parallel illustration of the flow pattern arising from a curvilinear path. Wewerinke provided a detailed description of the effects of lateral and vertical motion on the projective appearance of a runway. Both studies developed optimal control theoretic models and found experimental support for them.

Description for purposes of optimal control tends to emphasize changes in particular small but coherent non-pointalistic scene elements such as the perspectival inclination of a runway edge and further emphasizes the future consequences of maintaining the current egomotion. In contrast, information analyses, such as those by Koenderink and van Doorn and also by K. Prazdny, are cast in terms of arbitrary environmental points or visual directions and not in terms of specific scene elements such as runway contours. The term "local" structure of the flow field, as used by the

later authors, refers to abstract points or directions and their equally abstract geometric "neighborhoods".

Lee and Visually Controlled Activity

Several recent studies by David Lee (1980 a & b; Lee & Reddish, 1981) are closely related to the optimal control oriented analyses. Lee's emphasis is on identifying sources of information useful for the visual guidance of locomotion, but no particular control model is offered.

Since all interesting egomotion ultimately involves an approach to some thing or some surface, a successful animal must be able to control its speed in order to achieve a desired result such as stopping just in front of, gently contacting, or violently ramming the "target" depending on its purpose. Lee argues that a successfully controlled approach involves an animal's sensitivity to time-to-contact information which is optically available by means of an optic variable he calls "tau" (Lee, 1976, 1980 b; Lee & Reddish, 1981). Time-to-collision is not new with Lee. For example, Sir Fred Hoyle, the astronomer, derived the basic relationship in a footnote to a science fiction book (Hoyle, 1973, pp. 15-17).

Successfully controlled locomotion also entails the ability to follow routes and in particular to control steering on curved paths. Lee points out that, whereas the focus of expansion does optically indicate the instantaneous direction of travel, there is currently a question about humans' sensitivity to this source of heading information. He further argues that "knowing" one's instantaneous heading is not as important as staying on a curved road. Taking the ability to stay on a curved path as a generalized ability to control heading, Lee identified a new non-focus-of-expansion source of optical heading information, namely, the degree to which the perspectival shape of a road conforms to the curvilinear flow of optic elements about the projection of the road (Lee & Lishman, 1977). This source of information is much larger in terms of optical expanse than a mere single point (such as the focus of expansion), and thus may be more perceptually useful.

In contrast to his earlier use of a cylindrical projection surface to describe optic array and retinal activity, Lee's (1980 a & b) more recent descriptive and information analytic studies now utilize a flat projection plane. Thus, Lee's retinal velocity vectors are the same as K. Prazdny's.

Depth Perception During Egomotion

The optical flow field is, in general, the main optical concomitant of egomotion. Particular aspects of the flow field are also optical concomitants of egomotion. Some particular aspects may take the form of invariants or non-change in certain ratios such as those studied by Koenderink and van Doorn, or systematic variation or change in other ratios such as those studied by Lee. So far, these unchanging or changing ratios have been regarded as sources of information for the more "ego" or egomotion side of the ego-environment state giving rise to the flow pattern.

But other ratios can be sources of information for the more environmental side of the ego-environmental state. One such ratio is the "subtense ratio" (ratio of the depression angle between the horizon and the top of an object and the angle subtended by the object) which Harker and Jones suggest can be used to control low-level flight over a non-flat terrain by providing a height metric. This ratio is not strictly invariant especially at near distances and so further analytic and psychophysical study is needed.

Their subtense ratio is a variant of the horizon ratio discussed by Sedgwick (1973) as a means of determining which objects in a frozen field of view (a static scene or picture) are higher or lower than an observer. Thus, an interesting aspect of Harker and Jones' paper is that they discuss the effects of egomotion on optical ratios and relationships which also exist in static optic arrays. Most other flow pattern aspects, including the flow pattern itself, do not exist in a frozen optic array.

Finding other useful sources of altitude information is still an important and even urgent task as accidents do occur when height is underestimated (Kraft, 1978).

Information for the Perception of Egospeed

The problem of the perception of egospeed. It was argued earlier that the description of the flow pattern is not yet complete since description must be relevant to perceptual experience as well as to Euclidian geometry. This section is motivated by the problem of determining the information for the perception of egospeed. For a regularly textured flat terrain, there is in fact a local source of information everywhere in the lower hemisphere of the optic array, namely, the local edge rate: To determine ground egospeed, simply pick any reference position in the optic array (e.g., a smudge mark on a windscreen, or edge of a

wing) and count the number of ground texture elements, or equivalently, edges, that optically pass the referent in one second. True ground egospeed scaled in edges per second is thus available no matter what the altitude, direction of travel, or path angle.

The problem is that in spite of the above information, the same physical ground speed may seem radically different depending on one's altitude as evidenced by the fact that travel over the same road at the same speed seems faster from a low sports car than from a high truck (see Figure 3). At much faster speeds, travel in a high flying jet can seem excruciatingly slow.

Global optical flow rate. To explain this, Warren (1982) introduced the concept of global optical flow rate. As is demonstrated in a later section, in any all-optical-variable expression for angular speed, angular speed is a function of just three factors: (1) a value wholly determined by optical position, (2) a value determined by the path angle, and (3) a scaling factor

$$\dot{S}/h$$

which is simply the egospeed scaled in eyeheights per second.

Since $(\dot{S}/h)$ is applied equally at every optical locus, Warren proposed calling it the "global optical flow rate". The global optical flow rate may be used to compare activity between different optic arrays such as in Figure 3 in which the array for the car is flowing, as a whole, twice as fast as that for the truck. Global flow rate contrasts with the local flow rate which varies with optical position within an array. Warren suggested that perceived egospeed is determined, at least in part, by the global activity in an optic array.

As a convenient reference, the global flow rate during a brisk walk is about one eyeheight/sec. Cars and trucks typically travel between 10 to 20 eyeheights/sec whereas aircraft cruise between .16 and .03 eyeheights/sec. It is instructive for the reader to verify that high speed, low level flight increases the global optical flow only to that of a fast car. Slow flow also arises in supertankers and may be useful in understanding the perceptual problems in maneuvering these mammoths such as discussed by Wagenaar (1978). Control of fast jets and slow supertankers obviously depends on their response dynamics, but the global flow rate may provide a basis for a unified theory of perceptual guidance and experience.

Global optical density. Inspection of Figure 3 shows that, at a particular level of environmental grain, the optical density for the lower (car) array is sparser on the whole than that for the higher (truck) array. Warren (1982) proposed another global optic array descriptor, namely the "global optical density" index given by

$$\text{altitude} / \text{ground element size}$$

which is simply the number of unit ground elements required to span one eyeheight distance. Its units are thus "ground units per eyeheight". The global density index is optical position independent and may be used to compare the global densities between two arrays as a whole. It thus contrasts with local optical density which varies with position within an optic array.

For level travel, the product of global optical flow rate and global density yield ground speed since the altitude factors for each index cancel.

Edge rate. The high correlation between global optical flow rate and perceived egospeed suggested by the examples does not mean that global flow is the sole determinant of perceived egospeed. Strong support for the influence of edge rate (the number of optical edges crossing a given optical locus in unit time) is provided by a study in which the spacing between lines across a road was progressively decreased (Denton, 1980). Constant speed travel over such a road would keep the global flow rate constant while the edge rate (here the rate at which stripes are being traversed) would progressively accelerate. The accelerating edge rate was dramatically effective in inducing drivers to reduce their speed.

Conclusions. The purpose here is not to decide between possible optical bases for the perception of egospeed, but rather to introduce the concepts of global optical flow rate, global optical density, and optical edge rate as new and possibly useful descriptors of the egomotion optic array. Attention is called to the contrast between these more global concepts and the local nature of a single flow vector and its magnitude. Even edge rate is a global descriptor since it depends not on one edge or optical margin, but rather on the spacing relationships across a number of edges.

GENERAL COMMENTS ON OPTICAL DESCRIPTION

The major conclusion after reviewing the literature is that optical description for egomotion is not yet complete. The specification of optical velocity vectors and magnitudes is not enough. Once the describing scalar and vector fields are determined, useful techniques such as those of the vector calculus may be applied as demonstrated by Koenderink and van Doorn.

Beyond this, another level of psychophysically relevant description is needed to understand specific tasks in the visual guidance of egomotion such the time-to-contact problem studied by Lee. That yet another level of psychophysically relevant description is needed is illustrated by the problem of the perception of egospeed. This later problem shows the need for developing global, as opposed to local, descriptors. Lastly, the relationship between certain already existing concepts needs to be further elucidated. For example, just what are the perceptual roles played by the mathematical divergence, Harrington's "divergence", and Biggs' "areal expansion"?

ASPECTS OF A COMPREHENSIVE ECOLOGICAL ANALYSIS

The purpose of this section is to provide a conceptual framework for understanding the achievements, mistakes, inconsistencies, and omissions of previous studies and also for guiding future studies. The distinctions drawn here have not always been made but these distinctions may clarify some of the apparent inconsistencies and contradictions in the literature.

A comprehensive analysis of the optical bases for the perception of egomotion may be usefully partitioned into five major aspects or subanalyses. This partitioning is patterned after Gibson's (1979) "ecological approach to visual perception" in which he differentiates between (1) the environment to be perceived, (2) the information for perception, and (3) perception proper. The modifications made here are that the environmental analysis is expanded with respect to the ego-environmental relationships (something Gibson was very concerned with), and that what Gibson placed under the rubric of information is here more sharply differentiated as optical description and optical information.

The fourth part of Gibson's book is about "depiction" and the fifth aspect here is display generation, but despite the similarities of the titles, the problems discussed are very different. As an ecological theorist, Gibson was concerned with problems posed by perception via surrogates -- very subtle and profound epistemic problems. However, for the purpose of this report, no distinction is drawn between real and simulated egomotion. But that is not to imply that such a distinction is not worth while. A very real practical consequence is the problem of the transfer of training in transitioning from a simulator to an operational craft. This is true not only for airplanes but also for supertankers (Wagenaar, 1978).

Interestingly, research within the five chosen aspects may proceed fairly independently of the others. These aspect are:

1. The Ego-Environment States to be Perceived

The optical concomitants of egomotion are a function of (1) the layout of the environment, (2) the egomotion path shape, and (3) the speed of travel including changes of speed. A fourth factor not discussed so far is concerned with effects due to independently moving others, but that is beyond the current scope. So far only simple and idealized environments and paths have been analyzed, but obviously more realistic scenarios must be studied. For example, consider the ego-environment relationships encountered by a pedestrian in crossing a busy city street with other pedestrians, or by a small calf in a moving herd. An analysis of event types is needed.

2. Geometric Description

Once a particular environment-egomotion situation has been selected, a logical next step is to provide a complete (i.e., ambient optic array) description of the optical concomitants of that egomotion.

The optical description may be referred to a variety of models of the ambient optic array. The two most convenient are the optic array considered as a set of nested plane and solid angles at a moving point of observation and, the optic array considered as a unit sphere projection surface centered at the moving point of observation. Other projection surfaces may be used such as variously oriented flat planes or complexly curved surfaces such as windscreens. Whatever model of the optic array is used, it is important to distinguish between descriptions of a full ambient optic array versus descriptions of a restricted sample of the optic array such as those corresponding to "frontal", "lateral", or "bottom" views. It is especially important to further distinguish between optic array descriptions and retinal (usually, human) descriptions. The distinction between the optic array and any animal's ocular organs is fundamental at all levels of analysis.

3. Optical Information Analysis

Optical description considers how an environment geometrically effects the structure of light at a moving point of observation. It is basically a study of geometric projection. Optical information analysis is basically the study of the inverse process: Given only the projective activity in the optic array, what can be mathematically ascertained about the nature of the egomotion and the nature

of the environment being traversed? In an ecological optics sense, "information" means mathematical specificity, that is, optical information is geometrically deterministic, not probabilistic (Gibson, 1979).

Thus the synonymous terms "optical information", "optical specifier", and "optical basis" stand in sharp contradistinction to the terms "clue" and "cue" since "clue" and "cue" imply indeterminism or underdeterminism at the inverse projective geometric level. The distinction between information-specification and clue-cue is important since the problem of the ultimate adequacy (or inadequacy) of the underlying optical support for perception has radical implications for the nature of tenable perceptual and epistemic theories.

4. Perceptual Utilization

Geometric determinism, if it exists, does not imply psychological or perceptual determinism. Information permits or enables perception, it does not force perception. Questions about the pickup or utilization of information define another broad aspect of a comprehensive egomotion analysis. Only this aspect is, strictly speaking, psychological or perceptual in nature. Although empirical perceptual studies are, by design, excluded from this report, the aim of this report is to help guide the formulation of perceptual hypotheses and the selection of the contents of egomotion displays.

5. Egomotion Display Generation for Research

Once an experimenter has selected an information variable and formulated a perceptual hypothesis suitable for empirical testing, he must generate displays that provide the desired information. Egomotion display generation has two interesting features that researchers concerned with assessing information use should be aware of:

(1) Possible inherent confounds. Whether an information variable is selected as a result of phenomenological introspection on egomotion scenes, or by analytic consideration of the parameters of descriptive and inverse-geometric equations, a researcher ideally would employ displays which present the particular information of interest in pure form. All other sources of information would be rigorously controlled by counterbalancing, eliminating, or holding them constant. Unfortunately, this

ideal experimental design technique may often, if not always, be unattainable in egomotion research. There may always be inherent, confounding, alternate sources of information present in egomotion displays of any complexity. This is probably due to a shortage of display degrees of freedom compared to the number of parameters in the descriptive and inverse-geometric equations (Warren & Owen, in press).

The significance of the inherent confounding of alternative information sources is that the assessment of information utilization may prove to be an unexpectedly formidable, but interesting, problem. It may be a long time before a final empirical verdict is rendered on the various optical concomitants of egomotion.

(2) Equations for description versus generation. Although there are several methods possible for generating egomotion displays, the technique of choice today is that of computer generated imagery. Computer generation requires precise equations, and a natural set of generation equations would be those which describe the optical effects of egomotion. Computer programs could be written which directly act on the descriptive parameters. Unfortunately, such a procedure could not be done in real time on most computers today due to the numerical complexity of the equations. Fortunately, a radically different and numerically simpler set of generation equations exist that result in the same desired displays albeit in real time. The generation equations are not reviewed in this report but may be found in computer graphics books such as Newman and Sproull (1979).

The computer generation equations and techniques are not mere transformations of the descriptive equations. The differences are fundamental, although the results are the same. The significance for researchers is that it is possible (and common) to provide an observer with an effective computer display but whose exact mathematical description and information content are not known. But if the exact description and information content are not known, how can a satisfying theory of egomotion perception be constructed?

OPTIC ARRAY VS. RETINAL DESCRIPTION

Optical description in this report generally refers to the optic array. Optical speeds and iso-angular speed contours may refer to the optic array or, without undue confusion, to a set of environmental loci. Once optic array locations and activities are specified, they may be projected onto any possible projection surface. For depiction and analysis, a flat fronto-parallel or side window is typically used. But, the surface may be curved, and moreover, may assume any orientation including one that is constantly changing. The fixating or tracking retina is such a surface. Three issues arise in considering the flow pattern referred to a rotating (and specifically fixating or tracking) projection surface:

1. Optically, the effect of fixation is to null the optical flow of the fixated feature and to zero its angular velocity. But the angular velocity (not just optical speed) of all other points is also affected. The effect due to rotation/fixation is geometrically well specified and is essentially the addition of a rotational component to the existing translational flow field. See Gibson (1950, Figure 57) for a pictorial depiction and Nakayama and Loomis (1974) and Prazdny (1980) for the mathematics. Hence, descriptively there is no problem.

2. Analytically, the problem of decomposing the flow field back into rotational and translational components is more complex but, in principle, can be done (Nakayama & Loomis, 1974; Koenderink & van Doorn, 1981; Prazdny, 1980). Hence, analytically at least, the presence of eye rotation does not deteriorate or confound the information available in the flow pattern.

3. Psychologically, or perceptually, the issue is whether or not the rotationally altered flow pattern poses a problem for the extraction or utilization of the optically available information. This is a complex issue and involves both theoretical and empirical aspects. Much of the problem is due to confusion resulting from failure to keep optic array flow description separate from retinal flow description.

For example, Gibson (1947, 1950) hypothesized that the focus of expansion, since it is optically coincident with the ground point being approached, could be used to guide a

landing approach. But others (Regan & Beverley, 1982) have noted that the focus of expansion is not coincident with the aim point and hence cannot be used for heading guidance. There is no real contradiction here. Gibson was speaking about the optic array; Regan and Beverley about the retina. Geometrically, it is true that the retinal focus of expansion and iso-angular speed contours differ in environmental reference from those of the optic array. The problem, then, is not one of descriptive geometry or information availability. Rather, the perceptual problem is how a visual system extracts information from a retinal or other receptor surface flow pattern and with what accuracy. How does the visual system separate the rotational and translational components in the receptor flow pattern?

UNIFIED OPTICAL DESCRIPTION

The purpose of this section is to present a convenient summary of the procedures for determining an egomotion flow pattern for translatory egomotion. For rotation effects, see Gordon (1965), Nakayama and Loomis (1974), and Prazdny (1980).

Conventions. Since many angles are used, a double capital is used for mnemonic value. Vectors are symbolized by either underscored letters or as pairs or triplets of numbers in parenthesis. The magnitude of a vector is generally represented by the same letter without the underscore.

The Environment

The environment is assumed to be given or known as a set of points in a Cartesian coordinate framework. In general, no restriction is placed on the "sculpting" of the environment although certain equations do describe the special case of an infinite flat terrain.

Environmental Position

The environmental position of a point is given in egocentric Cartesian coordinates. The axes form a right-handed system in which the positive Z-axis is aligned with gravity and always points up. Hence, the X- and Y-axes always define a horizontal plane which would be parallel to a flat ground surface. Since ground points are generally below the observer, they tend to have negative Z values, and further, all points on a flat ground have the same negative Z value. Whatever the egomotion path angle, PA, to the ground, positive X represents forward distance relative to the observer, and positive Y represents left-lateral distance. "Forward" and "lateral" directions can change with time, but all further discussion here assumes rectilinear forward egomotion. The ego is always at (0,0,0). The radius vector from the egocenter to the point (x,y,z) is represented

$$\underline{r} = (x,y,z) = x\underline{i} + y\underline{j} + z\underline{k} \quad (1)$$

The length of the radius vector is the total distance to the point and is given by

$$r = \text{SQRT} (x^2 + y^2 + z^2) \quad (2)$$

The flat ground distance to a point is

$$g = \text{SQRT} (x^2 + y^2) \quad (3)$$

Another useful auxillary distance is the distance from the ego to the projection of the point (x,y,z) on the YZ plane which is the point $(0,y,z)$:

$$Q = \text{SQRT} (y^2 + z^2) \quad (4)$$

Static Optical Position

The static optical position of a point depends on the particular model of the optic array that is used. Four models are considered depending on the projection surface used: (1) sphere with "north" pole up, (2) sphere with "north" pole aimed at the horizon, (3) sphere with "north" pole aimed "ahead", and (4) flat fronto-parallel plane. Unit spheres centered on the ego form convenient models of the optic array because spherical extents and relations directly correspond to the plane or solid angular relations that describe an optic array. The three spherical systems discussed here are all variants on the same theme -- only the directions of their "north" poles differ. The fourth model, a flat frontal plane actually captures only one hemisphere of the optic array but is popular and convenient. "Horizon" here always refers to the optical horizon and not the "edge" of the earth. The "ground" is, unless qualified,

always assumed flat, horizontal, and below the observer.

Note on Arc Functions and Quadrants

If the environmental position of a point is known or given in Cartesian coordinates (x,y,z), the optical position is found using inverse trigonometric functions. Typically a computer or calculator is used for the calculations, and since computers generally return only the principal values of the arc functions, all returned angles are in a 180° range. For example, most calculators return the same angle for the points (2,5) and (-2,-5) if $\text{arc tan } (y/x)$ is used. Thus some angular positions will be misrepresented. The solution is to check for proper quadrant assignment or to use "four-quadrant" arc functions instead of the typical "two-quadrant" arc functions.

1. Sphere with "North" Pole Up

This section is a modification of Gordon (1965). When the "north" pole is aimed upwards or aligned with gravity, optical position is given in terms of azimuth, AZ, and either elevation, EL, or declination, DC, angles. Azimuth (also called "longitude") is simply angular position along the horizon where

$$AZ = \text{arc tan } (y/x) \quad (5)$$

There are two usual choices for the second spherical coordinate. Elevation (also called "latitude") indicates a point's angular position above or below the horizon where

$$EL = \text{arc sin } (z/r) \quad (6)$$

EL is +90° at the zenith and -90° at the nadir. Notice that the elevation of all ground points is negative since they are below the observer and hence optically below the horizon.

The alternate second spherical coordinate is that of angular declination (also called "co-latitude") from the "north" pole and is given by

$$DC = \text{arc cos } (z/r) \quad (7)$$

DC is zero at the zenith, 90° at the horizon and 180° at the nadir. Care must be taken not to confuse this usage with that in astronomy.

Which system to use? Gordon used an azimuth and elevation system and it is natural to consider points below the horizon as negative. For this same reason declination seems unnatural for ground points. However, most mathematical treatments and applications of spherical coordinates use the azimuth and declination (or longitude and co-latitude) system. Since one purpose of this report is to clarify usage and encourage easy communication between psychologists and other scientists, the declination system will be favored, but often the elevation version of an equation is given. Declination and elevation are related by

$$EL = 90^\circ - DC \quad (8)$$

and

$$\sin EL = \cos DC \quad (9)$$

A third three-dimensional spherical coordinate is simply the total distance, r (or sometimes ρ), to the point. This third coordinate is "lost" in projection or always taken as $r = 1$ for a unit sphere.

2. Sphere with "North" Pole Aimed at Horizon

This system is based on Nakayama and Loomis (1974) and Whiteside and Samuel (1970). The key to this system is the fact that projected points flow along great circle paths on a sphere during rectilinear egomotion, and further, that all great circle paths intersect at an optically stationary focus of expansion. Thus the great circles can be used as a set of meridians (or lines of longitude) for which the "north" pole points in the direction of the aim point or focus of expansion. Position along a given meridian is located by the eccentricity or angular distance from the focus of expansion. Hence eccentricity corresponds to co-latitude and ranges from 0° to 180° . If the egomotion is level, then the aim point is toward the horizon. To emphasize this the meridian and eccentricity angles are symbolized HM and HE which correspond to Nakayama and Loomis's alpha and beta. If egomotion is along the X-axis, then

$$HM = \arctan (z/y) \quad (10)$$

and

$$HE = \arccos (x/r) \quad (11)$$

This system anticipates rectilinear egomotion and has certain advantages over an azimuth and elevation system for such egomotion. Although the system may also be used for curvilinear egomotion, the advantages would be lost.

3. Sphere with "North" Pole Aimed "Ahead"

If the egomotion is not aimed toward the horizon, then the focus of expansion and the "north" pole will also be off the horizon. The equation for HM and HE would have to be modified to take the non-horizon "north" pole into account. Expressions for focal meridian, FM, and focal eccentricity, FE, are unwieldy and are best obtained by applying suitable axis rotation matrices to the expressions for HM and HE. The reader may check the unwieldiness, for example, by substituting an arbitrary direction of travel vector, $(x_{aim}, y_{aim}, z_{aim})$ for the $(1, 0, 0)$ aim vector used by Nakayama and Loomis in their Eq. A1 and then carrying out the indicated algebra.

The symbol "EC" is generally used here for eccentricity when there is no special need to distinguish between horizon and focal eccentricity, or when a general property is being discussed.

4. Flat Frontal Plane

A popular model of both the retina and the optic array is a flat projection surface or window. This model is favored by computer vision workers and by Lee in his more recent works. Most treatments use is a fronto-parallel surface toward which the observer is travelling. K. Prazdny (1980) specifically allows for a more generally oriented projection surface. Computer implementations for arbitrary motions and rotations is best done by the matrix techniques given in Newman and Sproull (1979).

Table 1 presents a summary of the optical angles described here. This table is useful in converting from one system to another.

TABLE 1

Summary of Optical Positions

	AZ	EL	HM	HE	SP
sin	y/g	z/r	z/Q	Q/r	y/Q
cos	x/g	g/r	y/Q	x/r	z/Q
tan	y/x	z/g	z/y	Q/x	y/z

where: $r = \text{SQRT} (x^2 + y^2 + z^2)$
 $g = \text{SQRT} (x^2 + y^2)$
 $Q = \text{SQRT} (y^2 + z^2)$

Rectilinear Egomotion

Rectilinear egomotion would normally be represented by a velocity vector originating from the observer and pointing in the direction of travel. However, under this egocentric analysis, the ego has no velocity vector since it always remains at the origin. Instead, it is the environment that is considered to displace relative to the ego. Hence, each point in the environment is assigned a velocity vector which is simply the negative of the usual observer velocity vector. This environmental velocity vector describes what is happening to each environmental vector, $\underline{r} = (x, y, z)$, and so may be symbolized as

$$\dot{\underline{r}} = \underline{v} = (\dot{x}, \dot{y}, \dot{z}) \quad (12)$$

It is important to note that there are an infinity of values for the environmental position vectors, $\underline{r}$, but only one value for the environmental velocity $\dot{\underline{r}}$.

That single value of $(\dot{x}, \dot{y}, \dot{z})$ is determined by the negative of the observer's relative velocity, but the components directly describe the motion of environmental points. Hence,

$\dot{x}$ negative is forward observer motion,

$\dot{y}$ negative is leftward observer motion, and

$\dot{z}$ positive is downward observer motion.

The path speed either of the observer or a point is

$$\dot{s} = \text{SQRT} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \dot{r} = v \quad (13)$$

Since the derivatives, $\dot{r}$, of all the positional radius vectors, r , are all equal to the negative of the observer's velocity vector, the environmental vector velocity field looks like a field (two or three dimensional) of parallel, equi-length needles. The needle at the point toward which the observer is heading points directly at the observer.

Speed of Optical Motion

Optical motion or instantaneous change of optical (i.e., angular) position is best described by a vector since it has both speed and direction associated with it. Historically, optical speed was described first, and only later was the vector as such described. This sequence is followed here.

Two approaches to determining optical or angular speed are: (1) Take the time derivatives of any two angles defining angular position and then combine the components into total angular speed, or (2) capitalize on certain geometric properties of the radius (position) and velocity vectors.

Optical Speed by Differentiation: Cartesian Form

Formulas in this section express optical speed and its components as functions of x , y , and z . As such, they are well-suited for computer work. Equations 5, 6, 14, 15, 16, and 17, are taken from Gordon (1965).

Azimuth and elevation rate components. Change in optical position may be resolved into separate azimuth and elevation rate components. Straightforward differentiation of Eqs. 5 and 6:

$$AZ = \text{arc tan } (y/x) \quad (5)$$

and

$$EL = \text{arc sin } (z/r) \quad (6)$$

yields

$$\dot{AZ} = (1/gr^2) (-y\dot{x} + x\dot{y}) \quad (14)$$

and

$$\dot{EL} = (1/gr^2) (-xz\dot{x} - yz\dot{y} + g^2\dot{z}) \quad (15)$$

Lateral velocity may be simply eliminated during rectilinear egomotion by orienting the X-axis in the forward direction. All further discussion assumes this has been done and that $\dot{y} = 0$ always. Accordingly, Eqs. 14 and 15 simplify to

$$\dot{AZ} = (-\dot{x}y/g^2) \quad (16)$$

and

$$\dot{EL} = (1/gr^2) (-xz\dot{x} + g^2\dot{z}) \quad (17)$$

Total angular speed from AZ and EL. The total magnitude of the rate of change of optical position, $\dot{OP}$, is

$$\dot{OP} = \text{SQRT} (\dot{AZ}^2 \cos^2 EL + \dot{EL}^2) \quad (18)$$

The $\cos^2 EL$ factor is a scaling adjustment to make distance along azimuth and elevation lines comparable. (Recall that 1° of longitude at the equator delimits a greater distance than 1° of longitude near the North Pole.) Gordon (1965, 1966) omitted the scaling factor, whereas Biggs (1966) did include it.

If Cartesian forms of $\dot{AZ}$ and $\dot{EL}$, such as Eqs 16 and 17, are substituted into Eq. 18, it may be desirable to use the Cartesian equivalent of $\cos^2 EL$ which is $(g/r)^2$.

Optical Speed by Differentiation: Optical Forms

The Cartesian expressions may be transformed into optical forms by multiplying them by unity in the guise of (z/z) , (r/r) , or (g/g) , and then arranging the terms so that $\dot{x}$ and $\dot{z}$ are divided by z and all other factors form ratios which appear in Table 1:

$$\dot{AZ} = (-\dot{x}/z) (y/g) (z/g) \quad (19)$$

and

$$\begin{aligned} \dot{EL} = & (-\dot{x}/z) (x/g) (z/r) (z/r) \\ & + (\dot{z}/z) (g/r) (g/g) (z/r) \end{aligned} \quad (20)$$

which become

$$\dot{AZ} = (-\dot{x}/z) \sin AZ \tan EL \quad (21)$$

and

$$\begin{aligned} \dot{EL} = & (-\dot{y}/z) \cos AZ \sin^2 EL \\ & + (\dot{z}/z) \sin EL \cos EL \end{aligned} \quad (22)$$

Each expression teaches us something about the flow pattern, but care must be taken in interpretation. At first glance, Eq. 21 appears to indicate a dependency of azimuthal position on altitude, but this is not so as evidenced by Eqs. 5, 14, and 16. The "z" factor in Eq. 21 was artificially introduced by the (z/z) multiplication needed to obtain Eq. 19. Its effects are offset by the $\tan EL$ factor.

It is argued that Eqs. 21 and 22 are in all-optical form at least for cases in which z is everywhere equal as in level terrain. Specifically, the $(\dot{x}/z)$ and $(\dot{z}/z)$ factors are considered global optical scaling factors since they are applied equally to each point. They are optically available in that they are measures of the global activity in an optic array: some arrays flow quickly, others slowly. The factor $(\dot{x}/z)$ is the forward velocity scaled in altitude units, and $(\dot{z}/z)$ is the fractional rate of change in the up/down dimension.

Path speed and path angle. Equation 18 for total angular speed may be expanded using other Cartesian or optical expressions for AZ and EL . No expression given so far has directly incorporated path speed, $\dot{s}$, and path angle, PA , and yet these are two common parameters of egomotion. For the special case of no lateral motion, $\dot{y} = 0$, path speed is

$$\dot{s} = \text{SQRT} (\dot{x}^2 + \dot{z}^2) \quad (23)$$

Equation 23 is thus a simplified form of Eq. 13 and equally represents the observer's own speed or the speed of an environmental point in an egocentric system.

Path slope is generally the ratio of descent rate to directed ground speed, thus

$$\text{path slope} = \dot{z}/\dot{x} = \tan PA \quad (24)$$

or

$$PA = \arctan (\dot{z}/\dot{x}) \quad (25)$$

Since the origin is egocentric, the ego never moves and only environmental points displace. If the observer is moving

forward and downward simultaneously, $\dot{x}$ is negative and $\dot{z}$ is positive for all environmental points. Thus, environmental points move at an angle 180° different from that of the observer in a geocentric system. One further cautionary point: Path angles are generally given in absolute values. If a landing approach is said to be at a 30° path angle, then the observer is headed 30° below the horizon and the environmental points travel with a path angle PA of 177° (not at 183°).

Equations 23 and 25 together imply the relationships summarized in Table 2.

Path angle as an optical angle. Path angle is argued to be an optical angle in that it is available from optic array relationships. Specifically, in a geocentric system, the angular separation of the focus of expansion from the horizon is equal to the (signed) path angle. In an egocentric system, the value of PA is just $180^\circ +$ the (signed) angular separation of the focus of expansion and the horizon. For example, if the focus of expansion is 30° below the horizon, then PA is 177° . Optical availability implies only geometric existence and not necessarily perceptual or biological effectiveness.

TABLE 2

Path Speed and Path Angles

$\dot{z}/\dot{s} = \sin PA$	$\dot{z} = \dot{s} \sin PA$
$\dot{x}/\dot{s} = \cos PA$	$\dot{x} = \dot{s} \cos PA$
$\dot{z}/\dot{x} = \tan PA$	$\dot{s} = \text{SQRT} (\dot{x}^2 + \dot{z}^2)$

Optical speed from path speed and path angle. The relations in Table 2 may be used to re-express the previous equations in terms of path speed and path angle. Many different versions are possible. Equation 26 is expressed solely in terms of global flow rate ($\dot{s}/h$), path angle, azimuth, and elevation. Letting

$$h = \text{SQRT} (\dot{z}^2) = \text{ABS} (\dot{z}) \quad (25)$$

represent altitude, global flow rate ($\dot{S}/h$) is always positive. Total optical speed is

$$\begin{aligned} OP = (\dot{S}/h) \sin^2 EL \text{ SQRT } (\\ (\sin^2 AZ \csc^2 EL + \cos^2 AZ) \cos^2 PA \\ + \cot^2 EL \sin^2 PA \\ - \cos AZ \cot EL \sin PA \cos PA) \end{aligned} \quad (26)$$

Equation 26 is equivalent to Eq. 3 in Gibson, Olum, and Rosenblatt (1955) but the demonstration is tedious.

Optical Speed by Geometrical Considerations

An alternate method for determining total angular speed is based on geometric and kinematic considerations (e.g., Biggs, 1966; Whiteside & Samuel, 1970). The effect of the velocity vector $\dot{\mathbf{r}}$ or $\mathbf{v}$ is to turn the radius vector $\mathbf{r}$ at some angular speed. But, the turning effect of $\dot{\mathbf{r}}$ on $\mathbf{r}$ depends on their relative orientation: Assume that the tail of the turning vector $\dot{\mathbf{r}}$ is attached to the head of the radius vector $\mathbf{r}$. The turning vector $\dot{\mathbf{r}}$ can be resolved into two component vectors: one aligned with the radius vector and one 90° to it. The aligned or axial component of $\dot{\mathbf{r}}$ can have absolutely no turning effect on $\mathbf{r}$. The entire turning effect is due to the tangential component of $\dot{\mathbf{r}}$ relative to $\mathbf{r}$.

The magnitude of the tangential component of $\dot{\mathbf{r}}$ is well known to be equal to the length of $\dot{\mathbf{r}}$, $\dot{S}$, times the sine of the (smaller) angle between them (when both originate at the origin). Now, $\dot{\mathbf{r}}$ is oriented exactly 180° opposite to the aim vector which points at the focus of expansion. The angle between the focus of expansion and the radius vector $\mathbf{r} = (x, y, x)$ is, by definition, the focal eccentricity angle FE or EC. Since EC and the angle between $\dot{\mathbf{r}}$ and $\mathbf{r}$ must form 180° or a straight line, the angle between $\dot{\mathbf{r}}$ and $\mathbf{r}$ is $(180^\circ - EC)$. Thus the magnitude of the tangential component of $\dot{\mathbf{r}}$ is

$$\dot{r}' = \dot{r} \sin (180^\circ - EC) \quad (27)$$

or

$$\dot{r}' = \dot{S} \sin EC = v' \quad (28)$$

since $\dot{r} = \dot{S}$ and $\sin (180^\circ - EC) = \sin EC$.

Angular speed is now easily determined since angular speed in radians per second is simply tangential speed scaled in turning radius units. Thus

$$\dot{EC} = (\dot{S} \sin EC) / r = v'/r \quad (29)$$

or

$$\dot{EC} = (\dot{S}/r) \sin EC \quad (30)$$

since r is the length of the radius vector (or positional) vector (x, y, z) , and $\dot{S} \sin EC$ is the tangential speed of the point (x, y, z) relative to the egocenter.

If egomotion is rectilinear, then Eq. 30 is an elegantly simple expression for the total angular speed, $\dot{OP}$, since here

$$\dot{OP} = \dot{EC} \quad (31)$$

Due to the simplicity of Eqs. 30 and 31, some comments are in order:

1. Assumption of constant meridian. The equivalence of the eccentricity rate, $\dot{EC}$, and the total angular rate, $\dot{OP}$, is true only if flow is along a line of constant meridian. This was recognized by Biggs (1966), Whiteside and Samuel (1970), and Nakayama and Loomis (1974). The simplicity thus introduced explains why an eccentricity and meridian system is preferred for describing rectilinear egomotion. If there is any change in meridian value with egomotion, then the total angular speed is not equal to $\dot{EC}$ and an expression similar to Eq. 18 must be used. Biggs, explicitly provided such an expression. Meridian will change during curvilinear egomotion and also during rectilinear egomotion when the "north" pole defining the meridians is not aligned with the heading point.

2. Cartesian form of Eq. 30. Equation 30 is neither in an all-Cartesian form nor in an all-optical form. It is not in all-optical form because the factor $(\dot{S}/r)$ is not an optical variable. Global optical variables such as $(\dot{S}/z)$, assuming a flat terrain, apply equally at each optical position. But, $(\dot{S}/r)$ varies with position since r varies with position.

Assume that the egomotion is level over a flat terrain so that Eqs. 10 and 11 apply. Then horizon referred eccentricity, HE , is the same as focal eccentricity, EC , and by Table 1

$$\sin EC = \sin HE = Q/r$$

thus

$$\dot{EC} = (\dot{S}/r) (Q/r) = (\dot{SQ}/r^2) \quad (32)$$

which is in an all-Cartesian form. If the egomotion is not level, Eq. 32 does not apply since $\sin EC$ would not equal (Q/r) .

3. Optical form of Eq. 30. Equation 30 may be recast into several all-optical forms by multiplying by unity in various guises such as (z/z) and using Table 1. For example,

$$\dot{EC} = (\dot{SQ}z/zrr)$$

or

$$\dot{EC} = (\dot{S}/z) \sin EC \sin EL \quad (33)$$

which is in an all-optical form. It should not matter to the visual system that eccentricity and elevation are not usually used together. For another version let

$$\dot{EC} = (\dot{SQ}Qz/zrrQ)$$

then

$$\dot{HE} = (\dot{S}/z) \sin^2 HE \sin HM \quad (34)$$

which is all-optical and exclusively within a horizon meridian and horizon eccentricity system.

4. Iso-angular speed surface. Whiteside and Samuel (1970) argued that a set of three-dimensional points, all of which have the same angular speed, form a torus, but gave no equations. No derivation is provided here either, but it is possible to demonstrate that Eq. 30 set to a constant is indeed the equation of a torus. Use Eq. 32, assume $\dot{OP}$ is a constant, expand Q and r into their x , y , and z forms, and after much rearranging of terms you will have the equation of a torus. For non-level egomotion, apply a rotation matrix.

Vector Description of Optical Motion

This section is an introduction to the vector description of the flow pattern corresponding to rectilinear egomotion. Several different vectors play a role in the flow pattern. The tasks of this section are to (1) identify

the vectors, (2) provide expressions for their directions and magnitudes, and (3) discuss their interrelationships.

The 3-D Vectors

Vectors useful for analysis and description include:

The radius position vector. The position or radius vector $\underline{r} = (x, y, z)$ is the vector from the egocenter to the point (x, y, z) in the environment. Equation 2 gives its magnitude.

The environmental velocity vector. The environmental velocity vector $\underline{\dot{r}}$ or $\underline{v} = (\dot{x}, \dot{y}, \dot{z})$ describes the motion of all environmental points relative to the egocenter. All vectors $\underline{v}$ are equal. The speed of the ego or the points is given by Eq. 13 and symbolized either as $\dot{r}$, v , or $\dot{s}$.

The observer's own velocity vector. The observer's own velocity vector relative to the environment is simply the exact opposite of the environmental velocity vector $\underline{v}$ and hence is $-\underline{v}$ or $-\underline{\dot{r}}$ and the egospeed is also given by Eq. 13.

The aim vector. The aim or heading vector is any vector pointing in the same direction as the observer's own velocity vector.

The axial velocity vector. As already discussed, the environmental velocity vector, $\underline{v}$ may be resolved into two components. The axial velocity component is aligned with the radius vector and hence can have no turning effect on it. Accordingly, this vector is of no immediate interest.

The tangential velocity vector. As already discussed, only the tangential component (here $\underline{v}'$ or $\underline{\dot{r}}'$) of the environmental velocity vector, $\underline{v}$ or $\underline{\dot{r}}$, has any turning effect on the position vector $\underline{r}$. This turning effect increases the angular separation, EC , between $\underline{r}$ and the aim vector with an angular speed $\dot{EC}$. As presented in Eqs. 27 and 28, the magnitude of $\underline{v}'$ or $\underline{\dot{r}}'$ is $\dot{s} \sin EC$ or $v \sin EC$. So far, the only thing said about the direction of $\underline{v}'$ is that it is at right angles to the position vector $\underline{r}$. Before further specifying $\underline{v}'$, it is helpful to examine the angular velocity vector.

The angular velocity vector. In general, an angular velocity vector, $\underline{\omega}$ (usually represented by an ω), is assigned a magnitude, ω , equal to the speed of the angular change and a direction aligned with the axis of rotation. It further points in the direction of the advance of a right-hand screw. The basic kinematic relationship between a radius vector, $\underline{r}$, a (tangential) velocity vector, $\underline{v}'$, and an angular velocity vector, $\underline{\omega}$, is that

$$\underline{v}' = \underline{\omega} \times \underline{r} \quad (35)$$

where "X" represents the vector cross product.

The length of $\underline{v}'$ is, by definition,

$$\omega r \sin WR$$

where WR is the angle between $\underline{\omega}$ and $\underline{r}$. Since $\underline{\omega}$ is aligned to the axis of rotation for $\underline{r}$, $\underline{\omega}$ and $\underline{r}$ are at right angles to each other and hence $\sin WR = 1$. Hence

$$v' = \omega r \quad (36)$$

and

$$\omega = v'/r \quad (37)$$

Equation 37 shows that the definition of the length of $\underline{v}'$ conforms to the general rule that angular speed is equal to the tangential speed scaled in turning radius units. Here, in particular, angular speed, ω , is EC which was shown to indeed equal v'/r in EQ. 29.

The direction of $\underline{\omega}$ is found by noting that $\underline{\omega}$, as the axis of rotation, must be perpendicular to both $\underline{r}$ and $\underline{v}$ since they determine the plane of rotation. The cross product, $\underline{r} \times \underline{v}$, is, by design, a vector mutually perpendicular to both $\underline{r}$ and $\underline{v}$, and a unit vector in the direction of $\underline{\omega}$ is

$$(\underline{r} \times \underline{v}) / |\underline{r} \times \underline{v}| \quad (38)$$

In the discussion of Eqs. 27 and 28, the angle between $\underline{r}$ and $\underline{v}$ was shown to be $(180^\circ - EC)$, hence

$$|\underline{r} \times \underline{v}| = rv \sin (180^\circ - EC) = rv \sin EC \quad (39)$$

Equation 38 becomes

$$(\underline{r} \times \underline{v}) / rv \sin EC \quad (40)$$

or

$$(1/\sin EC) ((\underline{r}/r) \times (\underline{v}/v)) \quad (41)$$

Thus, the unit vector in the direction of $\underline{w}$ is simply $(1/\sin EC)$ times the cross product of the unit vectors $(\underline{r}/r)$ and $(\underline{v}/v)$.

The vector $\underline{w}$ itself is the product of the length of $\underline{w}$, (v'/r) by Eq. 37) and the unit vector in the direction of $\underline{w}$:

$$\underline{w} = (v'/r) (1/\sin EC) ((\underline{r}/r) \times (\underline{v}/v)) \quad (42)$$

or, since $v' = v \sin EC$ by Eq. 28:

$$\underline{w} = (v/r) ((\underline{r}/r) \times (\underline{v}/v)) \quad (43)$$

or just

$$\underline{w} = (\underline{r} \times \underline{v}) / (r^2) \quad (44)$$

Tangential velocity vector, Part 2. An expression for the tangential velocity vector $\underline{v}'$ attached to the point (x,y,z) is now easily found by combining Eq. 35

$$\underline{v}' = \underline{w} \times \underline{r} \quad (35)$$

and Eq. 44:

$$\underline{v}' = (1/r^2) (\underline{r} \times \underline{v}) \times \underline{r} \quad (45)$$

Comments on the Vector Description

Conceptual vs. computational forms. Several different forms have been given for certain vectors. Some forms are better suited for calculation and others for conceptual development. For example, with respect to the angular velocity vector $\underline{w}$, Eq. 44 is simplest for calculation. On the other hand, Eqs. 42 and 43 are best conceptually since each factor has a ready interpretation.

Significance of fractional rates of change. Fractional rates of change of egomotion parameters appear to be playing significant roles in both the mathematical description of egomotion effects and also in the perceptual utilization of egomotion information (Owen, Warren, Jensen, Mangold, & Hettinger, 1981).

Alternate computational forms. The equations developed here have assumed that the position and velocity vectors are known in Cartesian coordinates. This is reasonable since a researcher will typically have such coordinates in mind. Other forms may be developed as needed by means of Tables 1 and 2 or by standard procedures for coordinate transformation.

Consequences of alternate forms on depiction. If equispaced environmental points are chosen, as would be likely using a Cartesian scheme, then a two-dimensional projection or depiction of the various points and flow vectors will exhibit linear perspective qualities. As has already been pointed out, it is possible to begin with points which are equispaced on a projection or receptor surface. This method is in use by computer vision scientists and results in depictions without a linear perspective character. However, picking points equispaced on a viewing or receptor surface is indeed a natural beginning for analyzing an operational visual system.

Angular vs. planer representation. Referring optical activity to a receptor surface raises the question of angular vs. planer models of the optic array. The equations here describe optical activity in terms of angular relationships. Such description may be transformed into flat projection plane relationships (e.g., Prazdny, 1980).

The description of Nakayama and Loomis. Much of the analysis here is indebted to that of Nakayama and Loomis (1974). There are some differences a reader should be aware of. The current report uses the vector $\underline{v}$ to represent the relative environmental motion. This vector is the opposite of the observer's own motion vector and this affects the appearance of some equations. Specifically, description with respect to the observer's own velocity vector, $\underline{a} = -\underline{v}$, entails using the cross product $\underline{a} \times \underline{r}$. Since

$$\underline{a} \times \underline{r} = -\underline{v} \times \underline{r} = \underline{r} \times \underline{v} \quad (46)$$

a different "pattern" of cross products emerges, but there is no difference in the results. Thus, Nakayama and Loomis's unit normal vector $\underline{N}$ is equivalent to the unit vector in the direction of $\underline{v}$ here. However, the tangential vector $\underline{v}'$ here and their omega-T vector do differ in magnitude although the directions are the same. The length $\underline{v}'$ here is $v \sin EC$, whereas the length of omega-T is $(v/r) \sin EC$. The difference is that omega-T is tangent to a unit sphere whereas $\underline{v}'$ is assumed to originate at the point (x, y, z) . Both lengths are consistent with the requirement that the

angular speed be equal to the tangential speed divided by the turning radius. In general, all tangential velocity vectors "attached" to a unit sphere will have the same magnitude as their associated angular velocity vectors. The magnitude of the angular velocity vector is, in turn, determined by the length of the tangential component of the environmental velocity vector "attached" to the environmental point divided by the distance to the point.

REFERENCES

- Asimov, I. Second foundation. New York: Avon, 1953.
- Ballard, D. H., & Brown, C. M. Computer vision. Englewood Cliffs, N. J.: Prentice-Hall, 1982.
- Biggs, N. L. Directional guidance of motor vehicles -- A preliminary survey and analysis. Ergonomics, 1966, 9, 193-202.
- Calvert, E. S. Visual judgments in motion. Journal of the Institute of Navigation, 1954, 7, 233-251.
- Calvert, E. S. Technical terms in science and technology. American Journal of Psychology, 1956, 69, 476-479.
- Clarke, A. C. 2001: A space odyssey. New York: New American Library, 1968.
- Clocksink, W. F. Perception of surface slant and edge labels from optical flow: a computational approach. Perception, 1980, 9, 253-269.
- Denton, G. G. The influence of visual pattern on perceived speed. Perception, 1980, 9, 393-402.
- Gibson, J. J. Motion picture testing and research. AAF Aviation Psychology Research Report No. 7. Washington, D.C.: Government Printing Office, 1947.
- Gibson, J. J. The perception of the visual world. Boston: Houghton Mifflin, 1950.
- Gibson, J. J. The optical expansion-pattern in aerial locomotion. American Journal of Psychology, 1955, 68, 480-484.
- Gibson, J. J. Technical and scientific communication: A reply to Calvert. American Journal of Psychology, 1957, 70, 129-131.
- Gibson, J. J. Visually controlled locomotion and visual orientation in animals. British Journal of Psychology, 1958, 49, 182-194.

- Gibson, J. J. The senses considered as perceptual systems. Boston: Houghton Mifflin, 1966.
- Gibson, J. J. The ecological approach to visual perception. Boston: Houghton Mifflin, 1979.
- Gibson, J. J., Olum, P., & Rosenblatt, F. Parallax and perspective during aircraft landings. American Journal of Psychology, 1955, 68, 372-385.
- Gordon, D. A. Static and dynamic visual fields in human space perception. Journal of the Optical Society of America, 1965, 55, 1296-1303.
- Gordon, D. A. Perceptual basis of vehicular guidance. Public Roads, 1966, 34, 53-68.
- Grunwald, A. J., & Merhav, S. J. Vehicular control by visual field cues -- Analytical model and experimental validation. IEEE Transactions on Systems, Man, and Cybernetics, 1976, SMC-6, 835-845.
- Harker, G. S., & Jones, P. D. Depth perception in visual simulation (AFHRL-TR-80-19). Brooks Air Force Base, Tex.: Air Force Human Resources Laboratory, August 1980.
- Harrington, T. L., & Harrington, M. K. Perception of motion using blur pattern information in the moderate and high-velocity domains of vision. Acta Psychologica, 1981, 48, 227-237.
- Harrington, T. L., Harrington, M. K., Wilkins, C. A., & Koh, Y. O. Visual orientation by motion-produced blur patterns: Detection of divergence. Perception & Psychophysics, 1980, 28, 293-305.
- Hasbrook, A.H. The approach and landing: Cues and clues to a safe touchdown. Business and Commercial Aviation, November 1975, pp. 39-43.
- Havron, M. D. Information available from natural cues during final approach and landing (Tech. Rep. HSR-RR-62/3-MK-X). Arlington, Va.: Human Sciences Research Inc., March 1962.
- Hay, J. C. Optical motions and space perception: An extension of Gibson's analysis. Psychological Review, 1966, 73, 550-565.
- Horn, B. K. P., & Schunck, B. G. Determining optical flow. A. I. Memo No. 572, Artificial Intelligence Laboratory, Massachusetts Institute of Technology, April 1980.

- Hoyle, F. The black cloud. New York: Signet - New American Library, 1973.
- Johansson, G. Studies on visual perception of locomotion. Perception, 1977, 6, 365-376.
- Johnston, I. R., White, G. B., & Cumming, R. W. The role of optical expansion patterns in locomotor control. American Journal of Psychology, 1973, 86, 311-324.
- Koenderink, J. J., & van Doorn, A. J. Invariant properties of the motion parallax field due to the movement of rigid bodies relative to an observer. Optica Acta, 1975, 22, 773-791.
- Koenderink, J. J., & van Doorn, A. J. Local structure of movement parallax of the plane. Journal of the Optical Society of America, 1976, 66, 717-723. (a)
- Koenderink, J. J., & van Doorn, A. J. The singularities of the visual mapping. Biological Cybernetics, 1976, 24, 51-59. (a)
- Koenderink, J. J., & van Doorn, A. J. Exteroscopic component of the motion parallax field. Journal of the Optical Society of America, 1981, 71, 953-957.
- Kraft, C. L. A psychophysical contribution to air safety: Simulator studies of visual illusions in night visual approaches. In H. L. Pick, Jr., H. W. Leibowitz, J. E. Singer, A. Steinschneider, & H. W. Stevenson (Eds.), Psychology: From research to practice. New York: Plenum, 1978.
- Lee, D. N. Visual information during locomotion. In R. B. MacLeod & H. L. Pick, Jr. (Eds.), Perception: Essays in honor of James J. Gibson. Ithaca, N. Y.: Cornell University Press, 1974.
- Lee, D. N. A theory of visual control of braking based on information about time to collision. Perception, 1976, 5, 437-459.
- Lee, D. N. The optic flow field: the foundation of vision. Philosophical Transactions of the Royal Society of London, 1980, B-290, 169-179. (a)
- Lee, D. N. Visual-motor coordination in space-time. In G. E. Stelmach & J. Requin (Eds.), Tutorials in motor behavior. Amsterdam: North-Holland Publishing, 1980. (b)
- Lee, D. N., & Lishman, R. Visual control of locomotion. Scandinavian Journal of Psychology, 1977, 18, 224-230.

- Lee, D. N., & Reddish, P. E. Plummeting gannets: A paradigm of ecological optics. Nature, 1981, 293, 293-294. (September 24, 1981)
- Llewellyn, K. R. Visual guidance of locomotion. Journal of Experimental Psychology, 1971, 91, 245-261.
- Longuet-Higgins, H. C., & Prazdny, K. The interpretation of a moving retinal image. Proceeding of the Royal Society of London, 1980, B-208, 385-397.
- Marr, D. VISION: A computational investigation in the human representation and processing of visual information. San Francisco: Freeman, 1981.
- Naish, J. M. Control information in visual flight. In Seventh Annual Conference on Manual Control (June 1971) (NASA SP-281). Washington, D.C.: U.S. Government Printing Office, 1972.
- Nakayama, K., & Loomis, J. M. Optical velocity patterns, velocity sensitive neurons, and space perception: a hypothesis. Perception 1974, 3, 63-80.
- Newman, W. M., & Sproull, R. F. Principles of interactive computer graphics (2nd ed.). New York: McGraw-Hill, 1979.
- Owen, D. H., Warren, R., Jensen, R. S., Mangold, S. J., & Hettinger, L. J. Optical information for detecting loss in one's own forward speed. Acta Psychologica, 1981, 48, 203-213.
- Prazdny, K. Egomotion and relative depth map from optical flow. Biological Cybernetics, 1980, 36, 87-102.
- Prazdny, K. Blur patterns: A comment. Perception & Psychophysics, 1982, 31, 190-191.
- Prazdny, S. A note on "Perception of surface slant and edge labels from optical flow". Perception, 1981, 10, 579-582.
- Regan, D., & Beverley, K. I. Visually guided locomotion: Psychophysical evidence for a neural mechanism sensitive to flow patterns. Science, 1979, 205, 311-313.
- Regan, D., & Beverley, K. I. How do we avoid confounding the direction we are looking and the direction we are moving? Science, 1982, 215, 194-196.
- Riemersma, J. E. J. Visual control during straight road driving. Acta Psychologica, 1981, 48,

- Sedgwick, H. A. The visible horizon: A potential source of visual information for the perception of size and distance. (Doctoral dissertation, Cornell University, 1973). Dissertation Abstracts International, 1973, 34, 1301B-1302B. (University Microfilms No. 73-22,530)
- Snyder, H. L. Visual aspects of low-level flight. In J. W. Miller (Ed.), Visual, display, and control problems related to flight at low altitude (ONR Symposium Report ACR-95). Washington, D.C.: Office of Naval Research, March 1964. (NTIS No. AD-602 823)
- Ullman, S. The interpretation of visual motion. Cambridge, Mass.: MIT Press, 1979.
- Ullman, S. Analysis of visual motion by biological and computer systems. Computer, August 1981, vol. 14(8), pp.57-69.
- Wagenaar, W. A. Maneuvering the mammoth. In H.L. Pick, Jr., H. W. Leinowitz, J. E. Singer, A. Steinschneider, & H. W. Stevenson (Eds.), Psychology: From research to practice. New York: Plenum, 1978.
- Warren, R. The perception of egomotion. Journal of Experimental Psychology: Human Perception and Performance, 1976, 3, 443-456.
- Warren, R. Global optical bases for the perception of egospeed. Manuscript in preparation, 1982.
- Warren, R., & Owen, D. H. Functional optical invariants: A new methodology for aviation research. Aviation, Space, and Environmental Medicine, in press.
- Wewerinke, P. H. A theoretical and experimental analysis of the outside world perception process. In Fourteenth annual conference on manual control (April 1978) (NASA Conference Publication 2060). Moffet Field, Ca.: NASA-Ames, November 1978.
- Whiteside, I. C. D., & Samuel, G. D. Blur zone, Nature, 1970, 225, 94-95.
- Zacharias, G. Personal communication, July 1982.

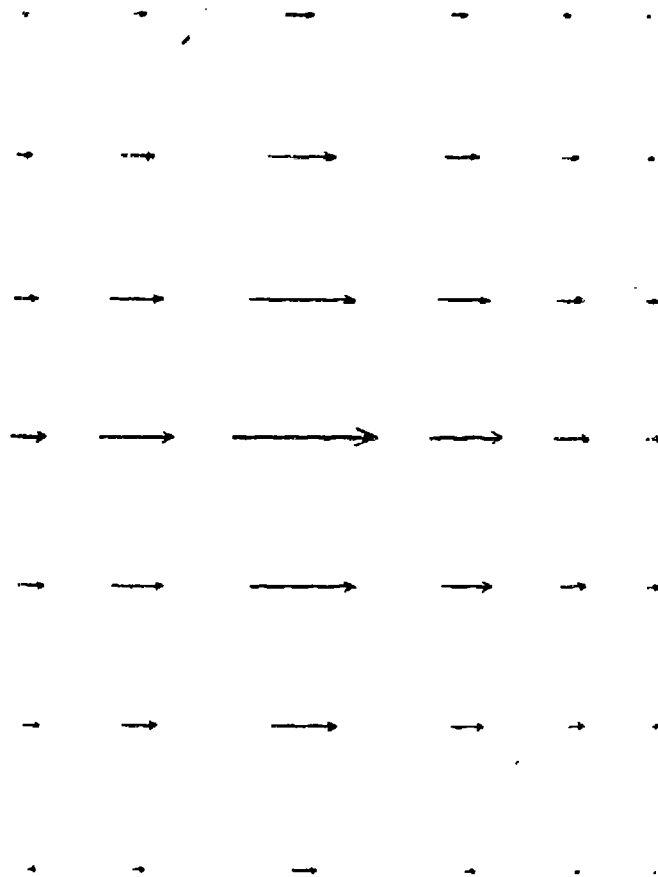


FIGURE 9.10.—Retinal Motion Perspective Looking Down.

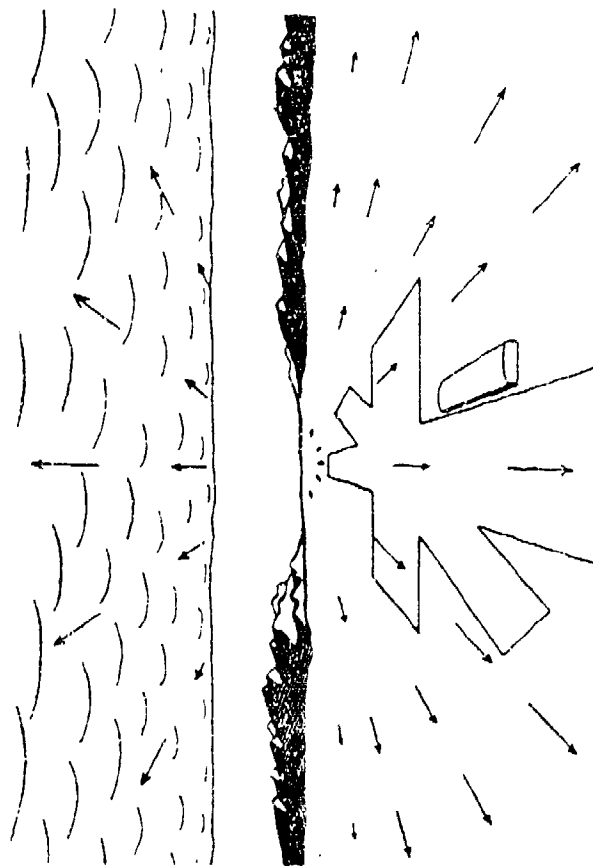


FIGURE 9.11.—Motion Perspective with a Ceiling.

Figure 1: Examples of the Flow Pattern from Gibson (1947)

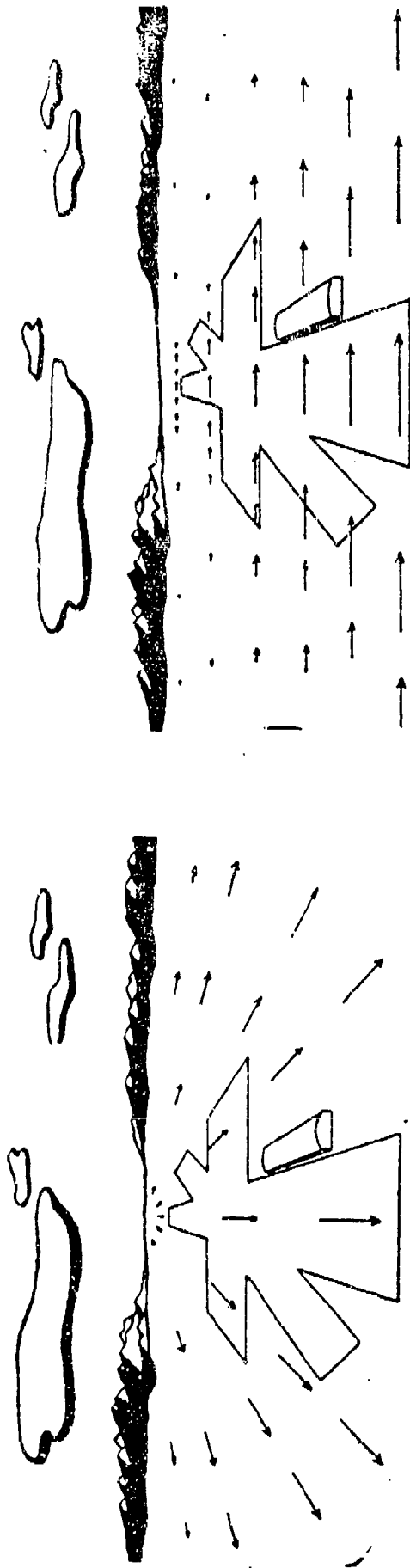


FIGURE 9.8.—Retinal Motion Perspective Looking Ahead.

FIGURE 9.9.—Retinal Motion Perspective Looking to the Right.

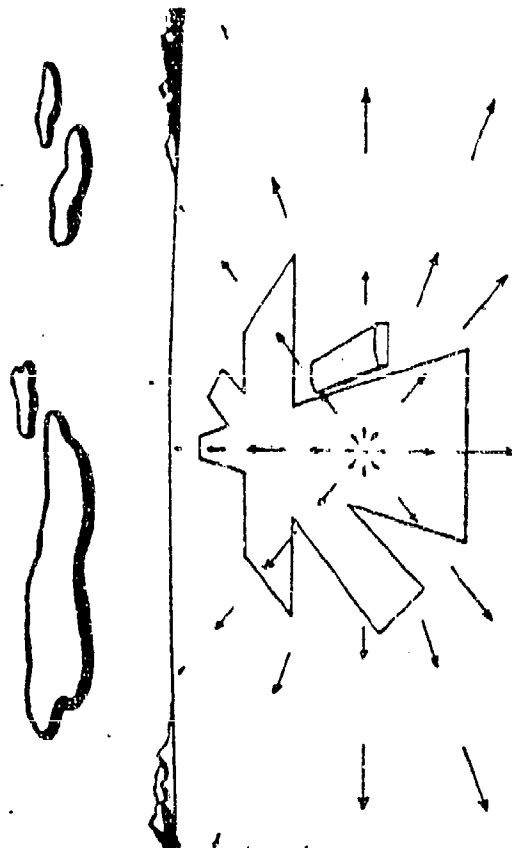


FIGURE 9.12.—Retinal Motion Gradients During a Landing Glide.

Figure 2: Examples of the Flow Pattern from Gibson (1947)

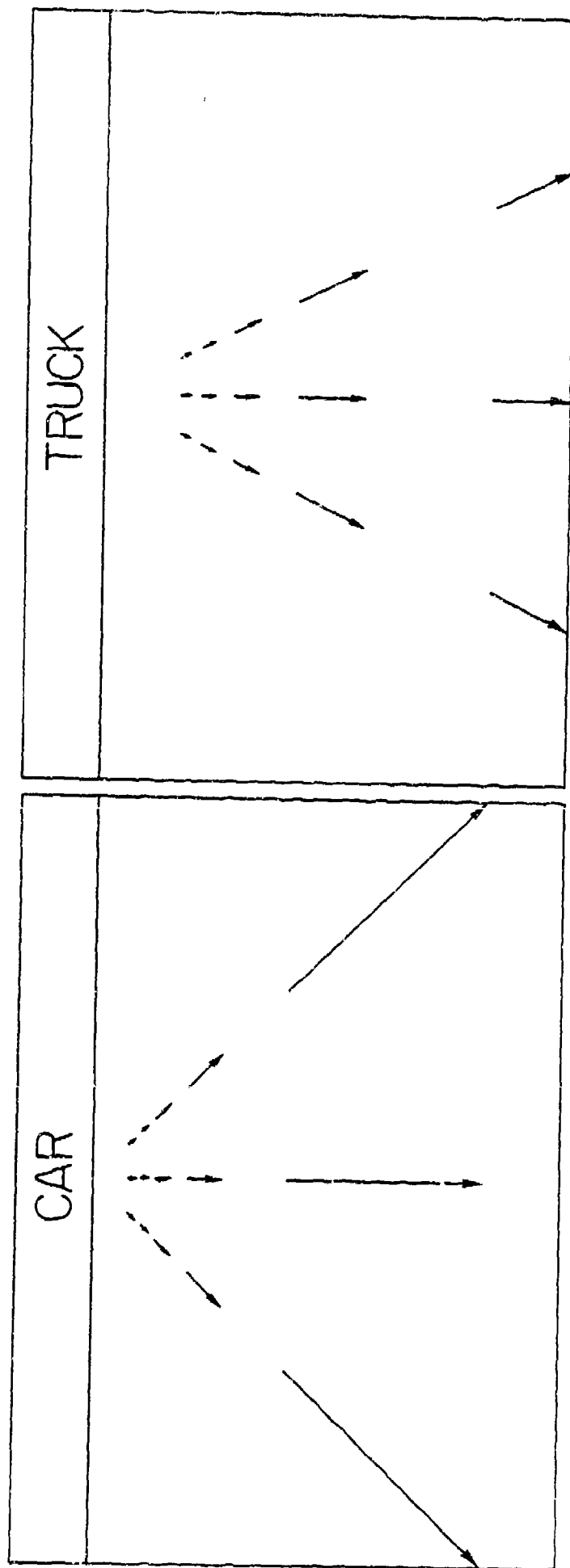


Figure 3: Flow Pattern for the Same Speed at Two Altitudes