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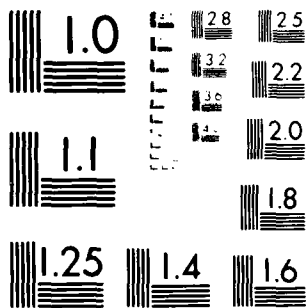
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CONSTRAINED AND UNCONSTRAINED VARIATIONAL FINITE  
ELEMENT FORMULATION OF SOLUTIONS TO A STRESS  
WAVE PROBLEM - A NUMERICAL COMPARISON

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October 1982



US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND  
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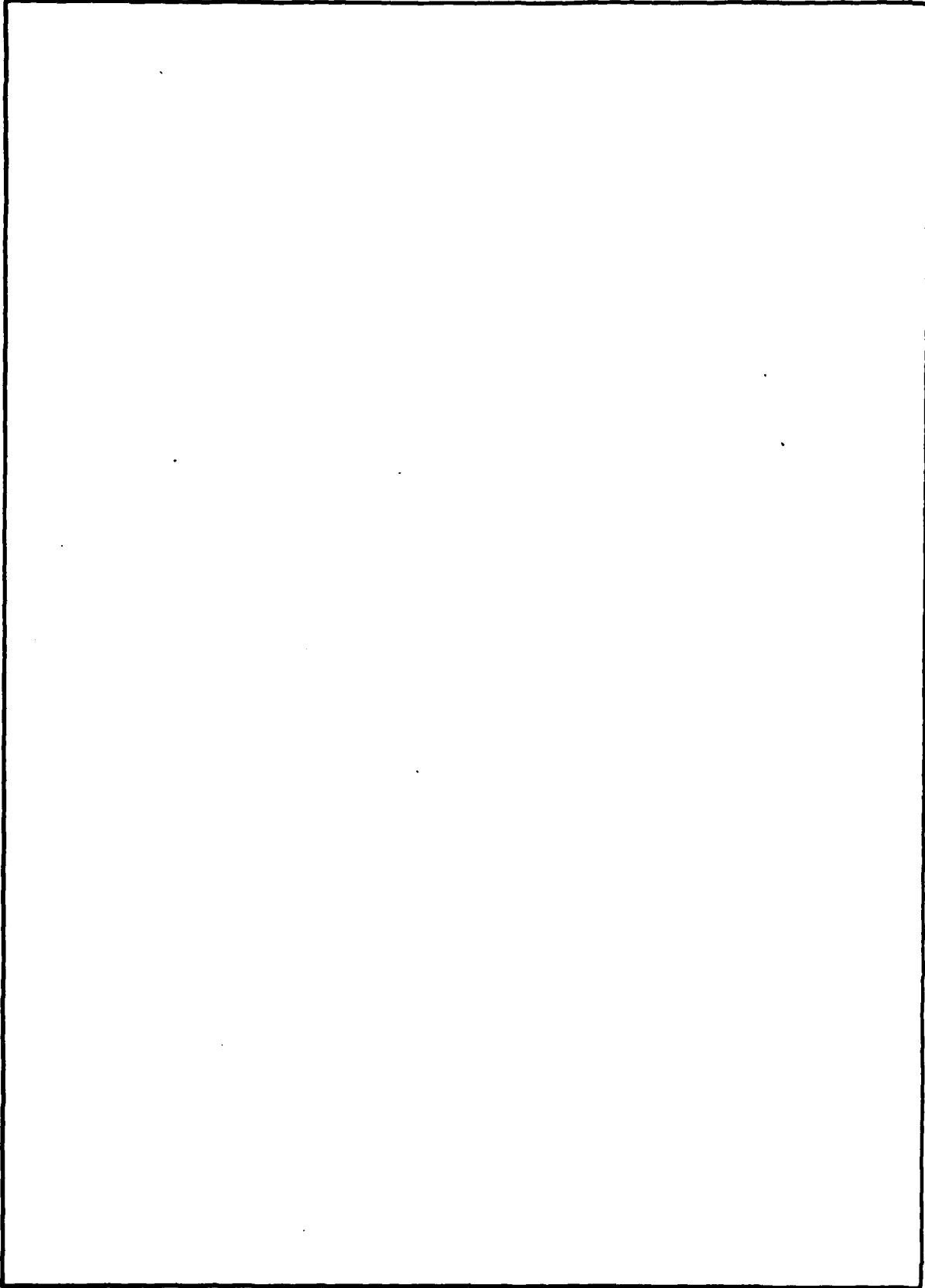
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## 1. INTRODUCTION

This note presents the solution formulation and finite element discretization of a stress wave problem with discontinuous data in two variational schemes. The first is in a sense a generalized Galerkin's approach in that it works for non-self adjoint problem and that all the end conditions are made to be natural ones and hence none of them are required to be satisfied by the trial functions. This unconstrained variational finite element formulation has been applied to initial/boundary value problems other than wave equations previously.<sup>1,2</sup> More recently, an adjoint bilinear variational principle has been developed for initial and initial/boundary value problems which requires that the initial conditions be satisfied exactly and the variations of the adjoint variable be set to vanish.<sup>3</sup> It is consequently a constrained variational formulation. This note compares one formulation and numerical results with those of the other.

First, in Section 2, the physical problem of a longitudinal stress wave in an elastic rod is stated. The rod is fixed at one end and free at the other end. The discontinuity data arises from the initial linear displacement, corresponding to a constant stress, due to a force applied at the "free"

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<sup>1</sup>J. J. Wu, "The Initial Boundary Value of Gun Dynamics Solved by Finite Element Unconstrained Variational Formulations," Innovative Numerical Analysis For the Applied Engineering Science, R. P. Shaw, et al, Editors, University Press of Virginia, Charlottesville, pp. 733-741, 1980.

<sup>2</sup>J. J. Wu, "Solutions to Initial Value Problems by Use of Finite-Elements-Unconstrained Variational Formulations," 1977 Journal of Sound and Vibration, 53, p. 341-356.

<sup>3</sup>C. N. Shen and J. J. Wu, "A New Variational Method for Initial Value Problems, Using Piecewise Hermite Polynomial Spline Functions," presented at the 1981 Army Numerical Analysis & Computers Conference, Huntsville, AL, February 1981.

end. This force suddenly disappears at time zero causing a stress discontinuity at the free end. The two variational formulations for the stated problem are introduced in Section 3. Finite element discretization and shape functions are introduced in Section 4. Finally, numerical results and comparisons are made in Section 5.

## 2. STATEMENT OF THE PROBLEM

The problem considered here is that of a longitudinal stress in a rod. The differential equation can be written as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} \quad ; \quad \begin{matrix} 0 \leq x \leq \ell \\ 0 \leq t \leq T \end{matrix} \quad (1)$$

with

$$a^2 = E/\rho \quad (2)$$

where  $u = u(x,t)$  is the axial displacement

$x,t$  are the coordinates in axial direction and in time, respectively

$\rho, E$  are density and Young's modulus, respectively, of the rod material

$\ell$  = length of the rod

$T$  = some finite time of interest

For the boundary conditions, we will consider a rod fixed at one end and not restrained at the other end. Hence

$$\begin{aligned} u(0,t) &= 0 \\ \frac{\partial u}{\partial x}(\ell,t) &= 0 \end{aligned} \quad (3)$$

The dynamics of the problem is due to the initial conditions. It is assumed that the rod is stretched to a linear displacement by a force  $P$  which vanishes at time  $t > 0$  (see Figure 1). The initial velocity of the rod is

assumed to be zero. Thus

$$u(x,0) = \frac{P}{AE} x$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad (4)$$

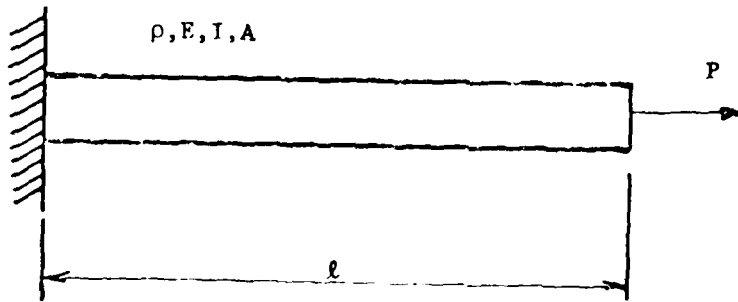


Figure 1. Problem Configuration and Applied Load at Zero Time (i.e.,  $P = 0$  for  $t > 0$ ).

It is convenient to use dimensionless parameters. Let

$$\bar{u} = u/l, \quad \bar{x} = x/l, \quad \bar{t} = t/T \quad (5)$$

Then, Eq. (1) in dimensionless form is

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = b^2 \frac{\partial^2 \bar{u}}{\partial \bar{t}^2}, \quad \begin{matrix} 0 \leq \bar{x} \leq 1 \\ 0 \leq \bar{t} \leq 1 \end{matrix} \quad (6)$$

where

$$b^2 = \frac{1}{a^2} \left( \frac{l}{T} \right) \quad (7)$$

The boundary conditions become

$$\bar{u}(0, \bar{t}) = 0, \quad \frac{\partial \bar{u}}{\partial \bar{x}}(1, \bar{t}) = 0 \quad (8)$$

and

$$\bar{u}(x,0) = \bar{P}x, \quad \frac{\partial \bar{u}}{\partial t}(x,0) = 0 \quad (9)$$

where

$$\bar{P} = \frac{P}{AE} \quad (10)$$

is the force in dimensionless form.

The stated problem in dimensionless form are Eqs. (6), (8), and (9) with the new dimensionless parameters related to physical counterparts by Eqs. (5), (6), and (10). To simplify writing, we shall drop the bars in Eqs. (6), (8), and (9), and rewrite them as

$$u'' - b^2 u = 0; \quad \begin{array}{l} 0 < x < 1 \\ 0 < t < 1 \end{array} \quad (6')$$

$$u(0,t) = 0; \quad u'(1,t) = 0 \quad (8')$$

$$u(x,t) = Px; \quad \dot{u}(x,0) = 0 \quad (9')$$

where a prime (') indicates differentiation with respect to  $x$  and a dot ( $\dot{\phantom{x}}$ ), with respect to  $t$ .

### 3. TWO VARIATIONAL FORMULATIONS OF SOLUTIONS

Consider a variational problem

$$\delta I_0 = 0 \quad (11a)$$

with

$$I_0 = I_0(u,v) = \int_0^1 \int_0^1 (-u'v' + b^2 uv) dx dt \quad (11b)$$

where  $u(x,t)$  and  $v(x,t)$  are said to be adjoint to each other. It is a simple matter to see that this problem is an indeterminate one. However, the functional of Eq. (11b) can be modified to a variational problem which is equivalent to the boundary/initial problem of Eqs. (6'), (8'), and (9').

Thus, consider

$$\delta I = 0 \quad (12a)$$

with

$$\begin{aligned} I = I(u,v) = & \int_0^1 \int_0^1 (-u'v' + b^2 \ddot{u}v) dx dt \\ & + k_1 \int_0^1 u(0,t)v(0,t) dt \\ & + k_2 b^2 \int_0^1 [u(x,0) - u_0(x)]v(x,1) dx + b^2 \int_0^1 u_1(x)v(x,0) dx \end{aligned} \quad (12b)$$

We shall take the first variation of the function  $I(u,v)$  of Eq. (12b) in such a manner that  $\delta v$  is completely arbitrary while  $\delta u$  is set to zero identically. Hence, by means of integrations-by-parts, one has

$$\begin{aligned} (\delta I)_{\delta u=0} = & \int_0^1 \int_0^1 (u'' - b^2 \ddot{u}) \delta v dx dt \\ & - \int_0^1 u'(1,t) \delta v(1,t) dt \\ & + \int_0^1 [u(0,t) + k_1 u(0,t)] \delta v(0,t) dt \\ & + b^2 \int_0^1 [\dot{u}(x,1) + k_2 [u(x,0) - u_0(x)]] \delta v(x,1) dx \\ & - b^2 \int_0^1 [\dot{u}(x,0) - u_1(x)] \delta v(x,0) dx = 0 \end{aligned} \quad (13)$$

The fact that  $\delta v(x,t)$  is completely arbitrary enables us to conclude from Eq. (13) that

$$\begin{aligned} u'' - b^2 u &= 0 ; & 0 < x < 1 \\ & & 0 < t < 1 \\ u'(1,t) &= 0 \\ u'(0,t) + k_1 u(0,t) &= 0 \\ u(x,1) + k_2 [u(x,0) - u_0(x)] &= 0 \end{aligned} \quad (14)$$

and

$$u(x,0) - u_1(x) = 0$$

It is then observed that the initial/boundary value problem defined by Eq. (14) reduces to that of Eqs. (6'), (8'), and (9') if one lets  $k_1$  and  $k_2$  go to infinity (and with  $u_0(x) = Px$  and  $u_1(x) = 0$ ). This fact suggests that the variational problem of Eqs. (12) can be used as a basis of a finite element discretization for the approximate solutions to the original initial/boundary problem. It should be noted that all the auxiliary conditions in Eqs. (14) are the so called natural boundary conditions. They are the consequence of the variational problem - just like the differential equation itself. For this reason, the above solution is referred to as an unconstrained variational formulation.

Another approach begins from Eq. (11b). With  $\delta u = 0$  once again, one has

$$\begin{aligned} \delta I_0 &= \int_0^1 \int_0^1 (-u' \delta v' + b^2 \dot{u} \delta \dot{v}) dx dt \\ &+ \int_0^1 u'(1,t) \delta v(1,t) dt - \int_0^1 u'(0,t) \delta v(0,t) dt \\ &- b^2 \int_0^1 \dot{u}(x,1) \delta v(x,1) dx + b^2 \int_0^1 \dot{u}(x,0) \delta v(x,0) dx = 0 \end{aligned} \quad (15)$$

with the constrained conditions

$$\begin{aligned} u(0,t) &= 0 \quad ; \quad u(1,t) = 0 \quad \text{for } 0 < t < 1 \\ u(x,0) &= u_0(x) \quad ; \quad \dot{u}(x,0) = 0 \quad \text{for } 0 < x < 1 \end{aligned} \quad (16)$$

It was shown in another paper<sup>4</sup> that the variations of the adjoint variable

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<sup>4</sup>C. N. Shen, "Method of Solution for Variational Principle Using Bicubic Hermite Polynomial," presented at the 17th Conference of Army Mathematicians, West Point, NY, June 1981.

must be constrained as follows

$$\begin{aligned} \delta v(1,t) = 0 & \quad ; \quad \delta v'(0,t) = 0 \quad \text{for } 0 \leq t \leq 1 \\ \delta v(x,1) = 0 & \quad ; \quad \delta v'(x,1) = 0 \quad \text{for } 0 \leq x \leq 1 \end{aligned} \quad (17)$$

#### 4. FINITE ELEMENT DISCRETIZATION AND SHAPE FUNCTIONS

Through nondimensionalization, the region of interest always remains to be a unit square:  $0 \leq x \leq 1$  and  $0 \leq t \leq 1$ . The finite element discretization is a subdivision of this unit square into smaller rectangles, the elements. A typical element scheme is shown in Figure 2 where a typical  $(i,j)^{\text{th}}$  element is also shown. In terms of the element variables Eq. (12) is now written as

$$\delta I = \sum_{i,j} \delta I(i,j) = 0 \quad (18)$$

Variables  $u(x,t)$  and  $v(x,t)$  become  $u_{(i,j)}(\xi, \eta)$  and  $v_{(i,j)}(\xi, \eta)$  respectively where  $\xi, \eta$  are local independent variables in spatial and temporal axis also shown in Figure 2.

Relations between global and local coordinates are given as follows

$$\begin{aligned} \xi &= \xi(i) = Kx - i + 1 \\ \eta &= \eta(j) = Lt - j + 1 \end{aligned} \quad (19)$$

where  $K$  and  $L$  are the number of segments in  $x$  and  $t$  directions, respectively (see Figure 2).

Shape functions are introduced as follows. Let

$$u_{(i,j)}(\xi, \eta) = \underline{a}^T(\xi, \eta) \underline{U}(i,j) \quad (20)$$

where  $\underline{a}(\xi, \eta)$  is the shape function vector and  $\underline{U}(i,j)$  is the discretized unknown vector. In this paper,  $\underline{a}(\xi, \eta)$  is selected as the following.

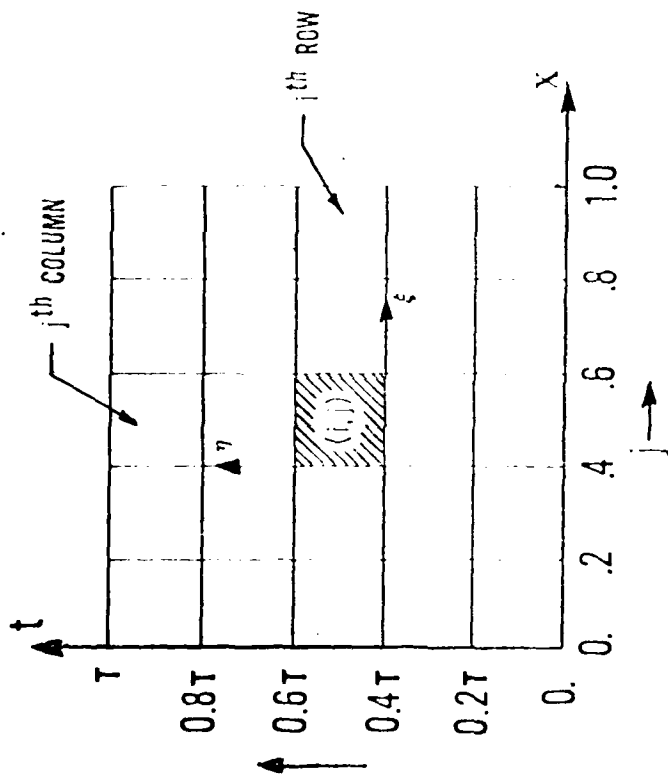


Figure 2. A Typical Finite Element Grid Scheme Showing the  $(i,j)$ <sup>th</sup> Element and the Global, Local Coordinates.

Let  $a_k(\xi, n)$  be a component of vector  $a(\xi, n)$   $k = 1, 2, \dots, 16$ , and

$$a_k(\xi, n) = b_1(\xi)b_j(n) \quad ; \quad \begin{array}{l} k = 1, 2, \dots, 16 \\ i, j = 1, 2, 3, 4 \end{array} \quad (21)$$

with

$$\begin{aligned} b_1(\xi) &= 1 - 3\xi^2 + 2\xi^3 \\ b_2(\xi) &= \xi - 2\xi^2 + \xi^3 \\ b_3(\xi) &= 3\xi^2 - 2\xi^3 \\ b_4(\xi) &= -\xi^2 + \xi^3 \end{aligned} \quad (22)$$

The correspondence between the index  $k$  and the pair  $(i, j)$  in Eq. (20) is given in Table I.

TABLE I. CORRESPONDENCE BETWEEN  $k$  AND  $(i, j)$  IN EQ. (20)

$k$	$(i, j)$	$k$	$(i, j)$
1	(1,1)	9	(1,3)
2	(2,1)	10	(2,3)
3	(1,2)	11	(1,4)
4	(2,2)	12	(2,4)
5	(3,1)	13	(3,3)
6	(4,1)	14	(4,3)
7	(3,2)	15	(3,4)
8	(4,2)	16	(4,4)

With the conventions as stated above, the meaning of the unknowns  $U_k(i,j)$  in the vector  $U(i,j)$  is as follows

$$\begin{aligned}
 U_1 &= u(0,0) ; U_2 = \frac{\partial u}{\partial \xi}(0,0) ; U_3 = \frac{\partial u}{\partial \eta}(0,0) ; U_4 = \frac{\partial^2 u}{\partial \xi \partial \eta}(0,0) \\
 U_5 &= u(1,0) ; U_6 = \frac{\partial u}{\partial \xi}(1,0) ; U_7 = \frac{\partial u}{\partial \eta}(1,0) ; U_8 = \frac{\partial^2 u}{\partial \xi \partial \eta}(1,0) \\
 U_9 &= u(0,1) ; U_{10} = \frac{\partial u}{\partial \xi}(0,1) ; U_{11} = \frac{\partial u}{\partial \eta}(0,1) ; U_{12} = \frac{\partial^2 u}{\partial \xi \partial \eta}(0,1) \\
 U_{13} &= u(1,1) ; U_{14} = \frac{\partial u}{\partial \xi}(1,1) ; U_{15} = \frac{\partial u}{\partial \eta}(1,1) ; U_{16} = \frac{\partial^2 u}{\partial \xi \partial \eta}(1,1)
 \end{aligned} \tag{23}$$

for each element  $(i,j)$ .

For the unconstrained formulation, Eq. (20) is used in Eq. (12). The result is

$$\begin{aligned}
 \sum_{i=1}^K \sum_{j=1}^L \delta V(i,j)^T \left[ -\frac{K}{L} \underline{A} + b^2 \frac{L}{K} \underline{B} \right] U(i,j) + \sum_{j=1}^L \delta V(i,j)^T \left( \frac{k_1}{L} \right) \underline{C} U(i,j) \\
 + \sum_{i=1}^L \delta V(i,1)^T \left( \frac{k_2 b^2}{K} \right) \underline{D} U(i,1) = \sum_{i=1}^K \delta V(i,L)^T \left( \frac{b^2 k_2}{K} \right) \underline{E}(i) \\
 \sum_{i=1}^K \delta V(i,1)^T \left( \frac{b^2}{K} \right) \underline{G}(i)
 \end{aligned} \tag{24}$$

where

$$\begin{aligned}
 \underline{A} &= \int_0^1 \int_0^1 \underline{a}_{,\xi} \underline{a}^T_{,\xi} d\xi d\eta ; \underline{B} = \int_0^1 \int_0^1 \underline{a}_{,\eta} \underline{a}^T_{,\eta} d\xi d\eta \\
 \underline{C} &= \int_0^1 \underline{a}(0,\eta) \underline{a}^T(0,\eta) d\eta ; \underline{D} = \int_0^1 \underline{a}(\xi,1) \underline{a}^T(\xi,0) d\eta \\
 \underline{E}(i) &= \int_0^1 u_0(i) \underline{a}(\xi,1) d\xi ; \underline{G}(i) = \int_0^1 u_1(i) \underline{a}(\xi,0) d\xi
 \end{aligned} \tag{25}$$

The expression of  $F(1)$  and  $G(1)$  can be further reduced into a form more readily computed. Write

$$u_0^{(1)}(\xi) = \underline{a}^T(\xi, 0) \underline{U}_0^{(1)} = \sum_{k=1}^{16} a_k(\xi, 0) U_{0k}^{(1)} \quad (26)$$

$$u_1^{(1)}(\xi) = \underline{a}^T_{,\eta}(\xi, 0) \underline{U}_0^{(1)} = \sum_{k=1}^{16} a_{k,\eta}(\xi, 0) U_{0k}^{(1)}$$

Since

$$a_k(\xi, 0) = b_1(\xi) b_j(0)$$

$$a_{k,\eta}(\xi, 0) = b_1(\xi) b'_j(0)$$

and

$$b_1(0) = 1, \quad b_j(0) = 0 \quad \text{for } j = 2, 3, 4$$

$$b'_2(0) = 1, \quad b'_j(0) = 0 \quad \text{for } j = 1, 3, 4$$

From Table I, one then observes that

$$a_k(\xi, 0) = 0 \quad \text{for all } k \text{ except } k = 1, 2, 5, 6$$

$$a_{k,\eta}(\xi, 0) = 0 \quad \text{for all } k \text{ except } k = 3, 4, 7, 8$$

Hence, in Eq. (26), only  $U_{01}^{(1)}$ ,  $U_{02}^{(1)}$ ,  $U_{05}^{(1)}$ , and  $U_{06}^{(1)}$  are used in expressing  $u_0^{(1)}$  and only  $U_{03}^{(1)}$ ,  $U_{04}^{(1)}$ ,  $U_{07}^{(1)}$ , and  $U_{08}^{(1)}$  are used in expressing  $u_1^{(1)}(\xi)$ . Thus we shall write

$$\underline{F}(1) = \int_0^1 \underline{a}(\xi, 1) \underline{a}^T(\xi, 0) d\xi \underline{U}_0^{(1)} = \underline{F} \underline{U}_0^{(1)} \quad (27)$$

$$\underline{G}(1) = \int_0^1 \underline{a}(\xi, 0) \underline{a}^T_{,\eta}(\xi, 0) d\xi \underline{U}_0^{(1)} = \underline{G} \underline{U}_0^{(1)}$$

with

$$\underline{F} = \int_0^1 \underline{a}(\xi, 0) \underline{a}^T(\xi, 0) d\xi \quad (28)$$

$$\underline{G} = \int_0^1 \underline{a}(\xi, 0) \underline{a}^T_{,\eta}(\xi, 0) d\xi$$

The way to set up  $U_0^{(1)}$  is that first set all  $U_{ok}^{(1)}$  to zero for all  $k = 1, 2, \dots, 16$ . Then set  $U_{ok}^{(1)}$  for  $k = 1, 2, 3, 4, 5, 6, 7$  and  $8$  as follows.

$$\begin{aligned} U_{01}^{(1)} &= u_0^{(1)}(0) & ; & & U_{02}^{(1)} &= u_{0,\xi}^{(1)}(0) & ; & & U_{03}^{(1)} &= u_1^{(1)}(0) & ; \\ U_{04}^{(1)} &= u_{1,\xi}(0) & ; & & U_{05} &= u_0^{(1)}(1) & ; & & U_{06}^{(1)} &= u_{0,\xi}^{(1)}(1) & ; \\ U_{07}^{(1)} &= u_1^{(1)}(1) & ; & & U_{08}^{(1)} &= u_{1,\xi}^{(1)}(1) & & & & & \end{aligned} \quad (29)$$

With vectors  $\underline{U}_0^{(1)}$ ,  $\underline{F}$ , and  $\underline{G}$  completely defined above, Eq. (24) can be rewritten as

$$\begin{aligned} & \sum_{i=1}^K \sum_{j=1}^L \delta V_{(i,j)}^T \left\{ -\frac{K}{L} \underline{A} + b^2 \frac{L}{K} \underline{B} \right\} \underline{U}_{(i,j)} + \sum_{j=1}^L \delta V_{(1,j)}^T \left( \frac{k_1}{L} \right) \underline{C} \underline{U}_{(1,j)} \\ & + \sum_{i=1}^K \delta V_{(i,j)}^T \left( \frac{b^2 k_2}{K} \right) \underline{D} \underline{U}_{(i,1)} = \sum_{i=1}^K \left[ \delta V_{(i,L)}^T \left( \frac{b^2 k_2}{K} \right) \underline{F} - \delta V_{(i,1)}^T \left( \frac{b^2}{K} \right) \underline{G} \right] \underline{U}_0^{(1)} \end{aligned} \quad (30)$$

Now Eq. (30) is readily assembled into a global matrix equation in a standard manner.

$$\delta \underline{V}^T \underline{K} \underline{U} = \delta \underline{V}^T \underline{P} \quad (31)$$

or

$$\underline{K} \underline{U} = \underline{P} \quad (31)$$

due to the fact  $\delta \underline{V}$  is completely arbitrary. Thus Eq. (31) is solved for  $\underline{U}$ .

## 5. NUMERICAL RESULTS AND COMPARISONS

Some preliminary results of computation are presented here. We shall set  $b = 1$  in the differential equation (6') for simplicity. Thus,

$$b^2 = \frac{\rho \ell^2}{E T} = \frac{\ell^2}{a^2 T^2} = 1 \quad (32)$$

or,

$$T = \frac{\ell}{a} \quad (33)$$

The exact solution for  $t = 0, 0.2T$ , and  $0.4T$  are given in Figures 3 and 4.

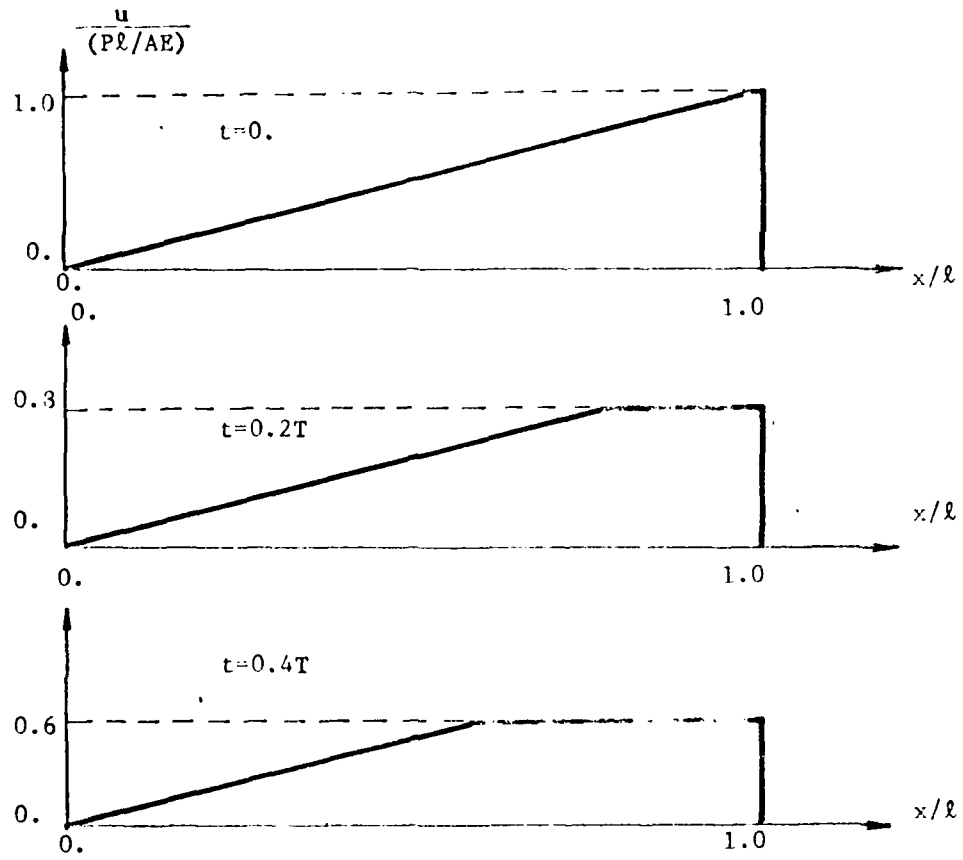


Figure 3. Exact Solution to the Problem: Longitudinal Displacement at  $t = 0, 0.2T$ , and  $0.4T$ .

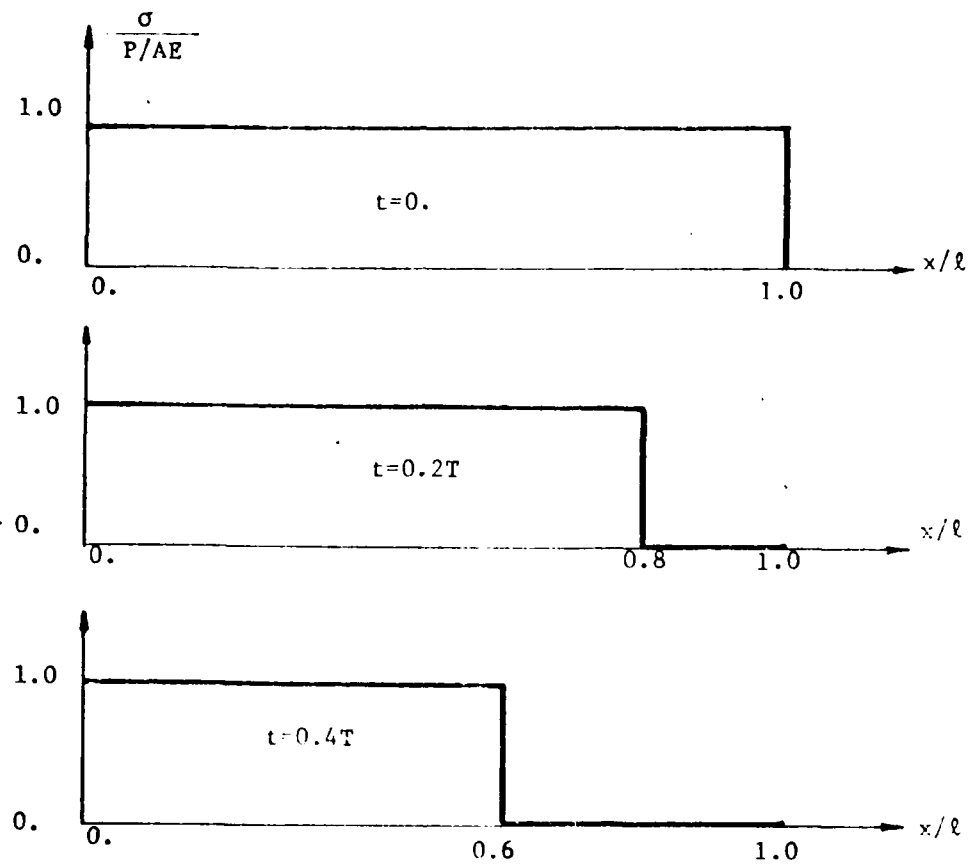


Figure 4. Exact Solution to the Problem: Axial Stress at  $t = 0, 0.2T,$  and  $0.4T$ .

First, the results from the unconstrained variational formulation. Using a grid scheme of 5x1, the numerical results for displacements and axial stresses are tabulated in Tables II and III where the exact solutions are also given for comparison. The graphic comparisons are shown in Figures 5 and 6 where the calculated solutions are indicated by crosses (x) and the exact solution is plotted in solid lines. It is clear in these figures that the computed results generally agree with the exact analytical solution. As a further evidence of convergence, a finer grid scheme of 10x1 is taken and the improved solution is shown in Figures 7 and 8.

TABLE II. SOLUTIONS TO THE STRESS WAVE PROBLEM USING UNCONSTRAINED VARIATIONAL FORMULATION

$$t = 0.2T = 0.2\left(\frac{l}{a}\right), b = 1.0; \text{Grid: } 5 \times 1$$

x	u		$\partial u / \partial x$	
	Computed	Exact	Computed	Exact
0	0.000	0.0	0.994	1.0
0.2	0.199	0.2	0.989	1.0
0.4	0.399	0.4	0.965	1.0
0.6	0.598	0.6	0.896	1.0
0.8	0.789	0.8*	0.550	0.0*
1.0	0.806	0.8	0.403	0.0

\*Point of discontinuity.

TABLE III. SOLUTIONS TO THE STRESS WAVE PROBLEM USING  
UNCONSTRAINED VARIATIONAL FORMULATION

$$t = 0.4T = 0.4\left(\frac{l}{a}\right), b = 1.0; \text{Grid: } 5 \times 1$$

x	u		$\partial u / \partial x$	
	Computed	Exact	Computed	Exact
0	0.000	0.0	0.988	1.0
0.2	0.200	0.2	0.976	1.0
0.4	0.399	0.4	0.927	1.0
0.6	0.594	0.6	0.714	0.0*
0.8	0.698	0.6	0.467	0.0
1.0	0.791	0.6	0.151	0.0

\*Point of discontinuity.

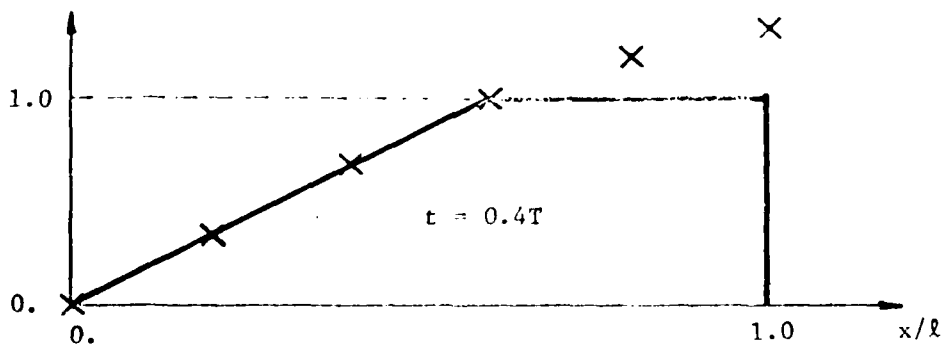
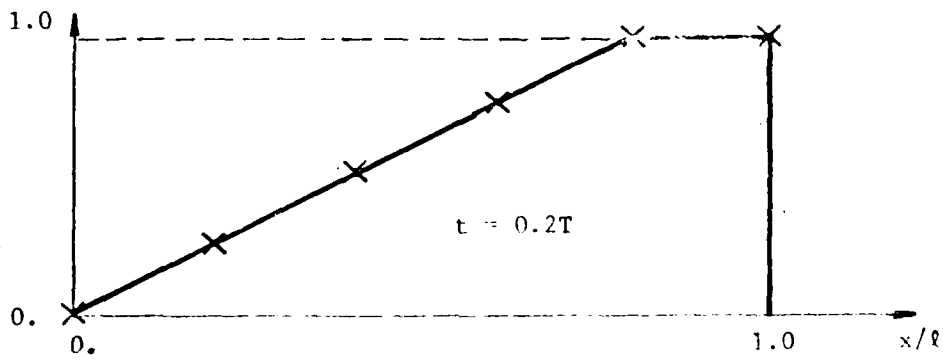
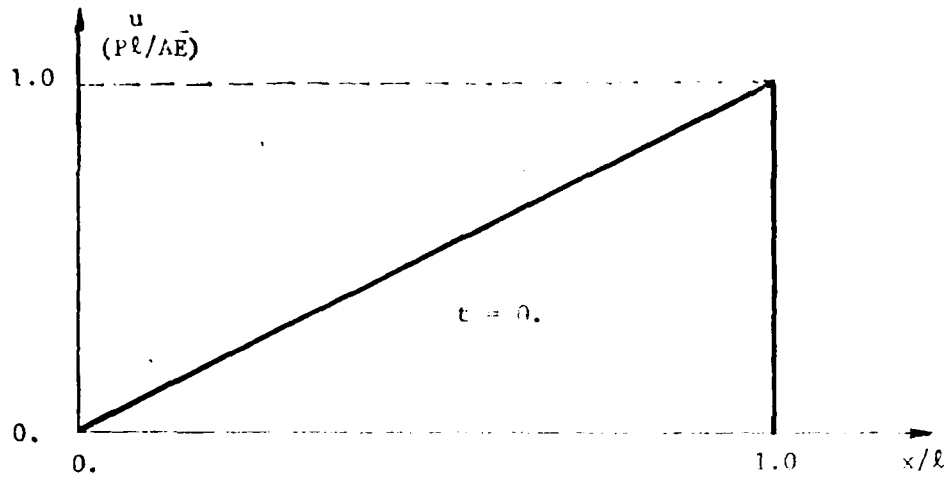


Figure 5. Displacement Solutions by the Unconstrained Variational Formulation ( $t = 0, 0.2T$ , and  $0.4T$ ), and Comparison with Exact Solutions. Grid:  $5 \times 1$ .

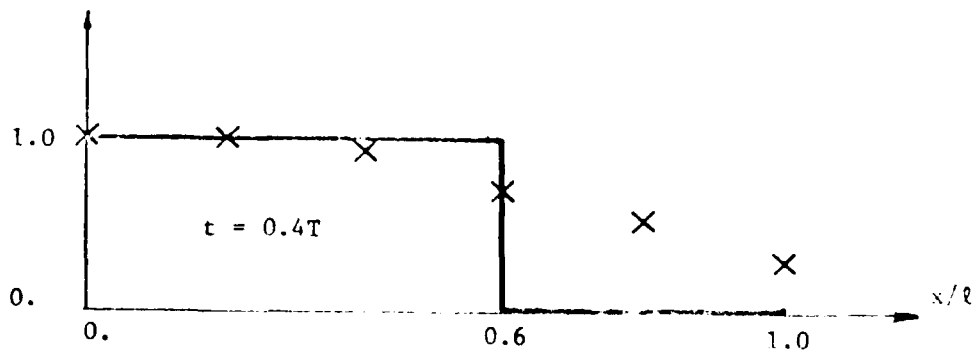
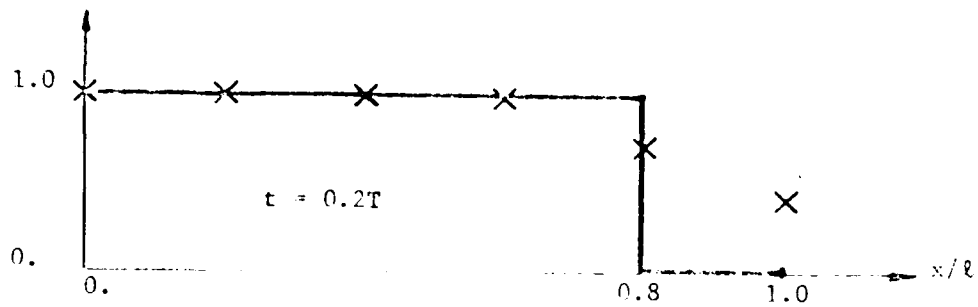
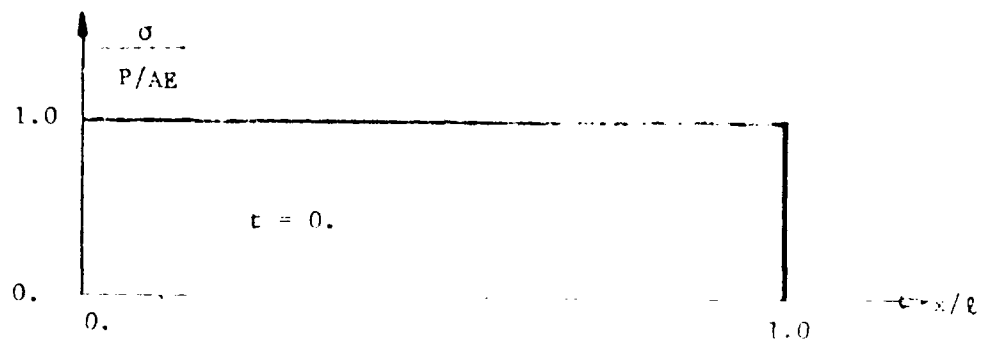


Figure 6. Stress Solutions by the Unconstrained Variational Formulation ( $t = 0, 0.2T$ , and  $0.4T$ ) and Comparison with Exact Solutions. Grid:  $5 \times 1$ .

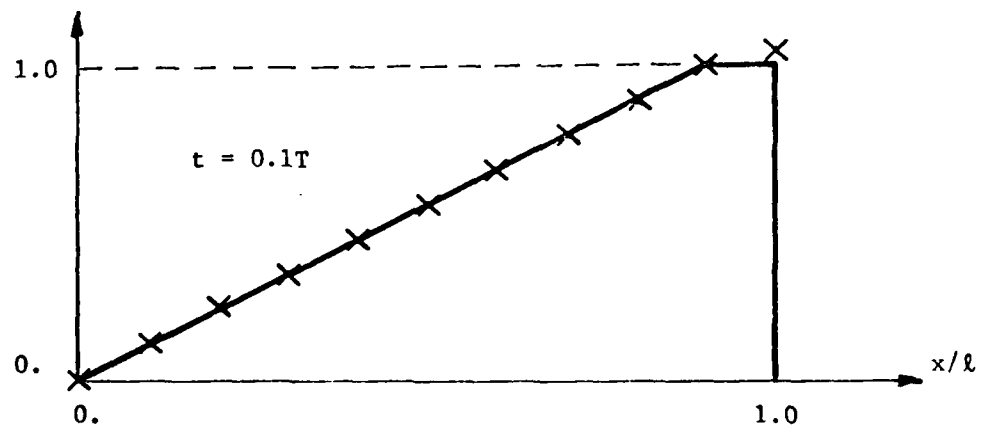
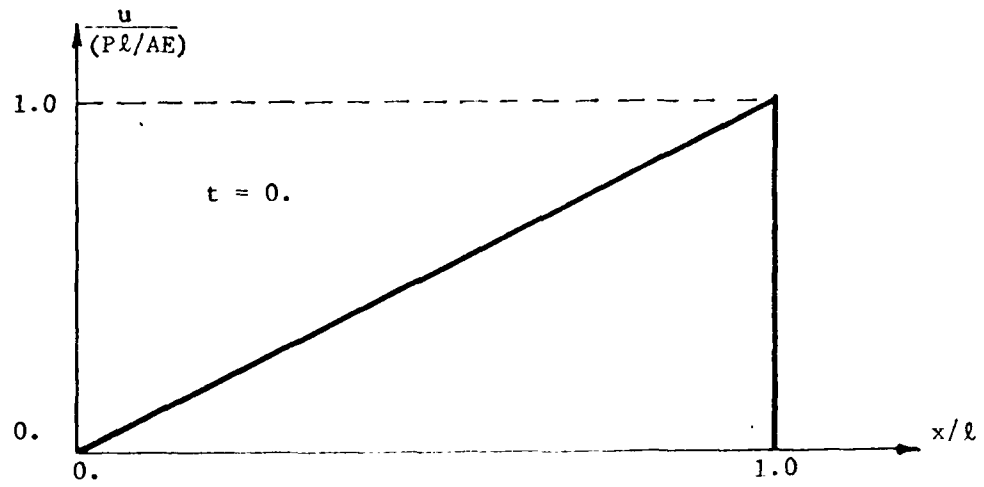


Figure 7. Displacement Solutions by the Unconstrained Variational Formulations ( $t = 0.1T$ ) and Comparison with Exact Solutions. Grid: 10x1.

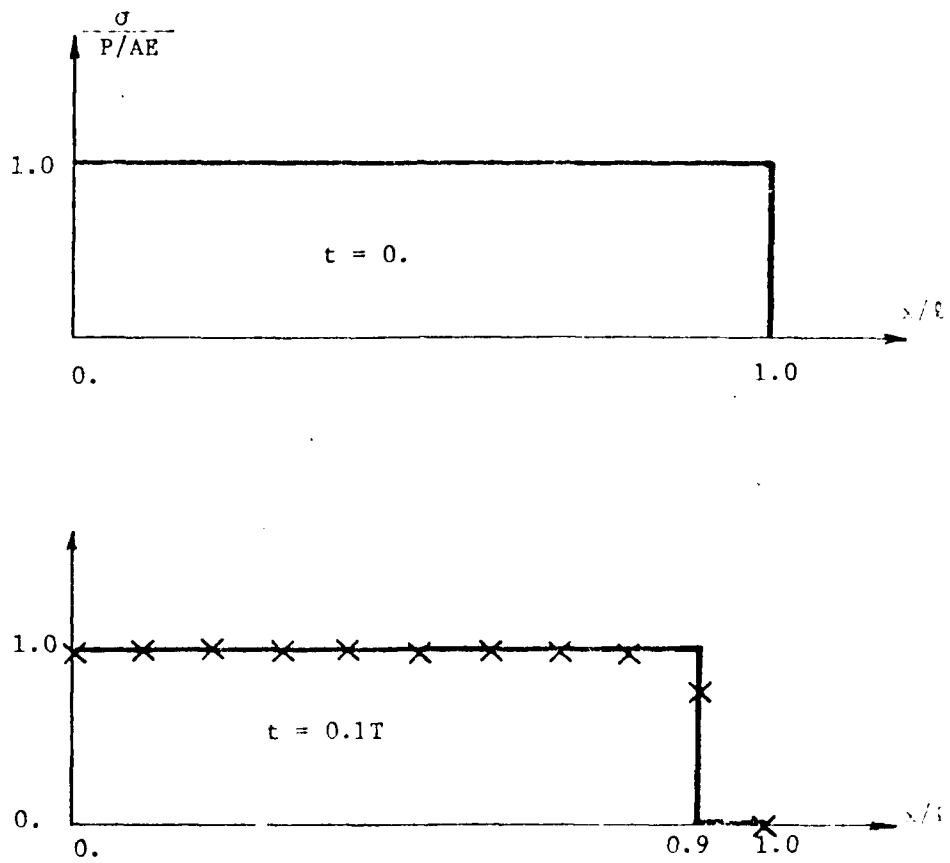


Figure 8. Stress Solution by the Unconstrained Variational Formulations ( $t = 0.1T$ ) and Comparison with Exact Solutions. Grid: 10x1.

Numerical results from the constrained formulation follow the general trend as the unconstrained one as indicated in Figures 9 and 10, as well as the tabulated comparison with the exact solution in Tables IV and V.

TABLE IV. SOLUTIONS TO THE STRESS WAVE PROBLEM USING CONSTRAINED VARIATIONAL FORMULATION

$c = 0.1T$ ,  $b = 1$ ; Grid:  $5 \times 1$

x	u		$\partial u / \partial x$	
	Computed	Exact	Computed	Exact
0	0.0	0.0	0.986	1.0
0.2	0.200	0.2	0.984	1.0
0.4	0.399	0.4	0.944	1.0
0.6	0.596	0.6	0.784	1.0
0.8	0.797	0.8*	- 0.049	0.0*
1.0	0.874	0.8	0.0	0.0

\*Point of discontinuity.

TABLE V. SOLUTIONS TO THE STRESS WAVE PROBLEM USING  
CONSTRAINED VARIATIONAL FORMULATION

$t = 0.1T$ ,  $b = 1.0$ ; Grid:  $5 \times 1$

x	u		$\partial u / \partial x$	
	Computed	Exact	Computed	Exact
0	0.	0.0	1.000	1.0
0.1	0.100	0.1	1.000	1.0
0.2	0.200	0.2	1.000	1.0
0.2	0.300	0.3	1.000	1.0
0.4	0.400	0.4	0.999	1.0
0.5	0.500	0.5	0.997	1.0
0.6	0.600	0.6	0.986	1.0
0.7	0.700	0.7	0.945	1.0
0.8	0.798	0.8	0.787	1.0
0.9	0.898	0.9*	- 0.038	0.0*
1.0	0.936	0.9	0.0	0.0

\*Point of discontinuity.

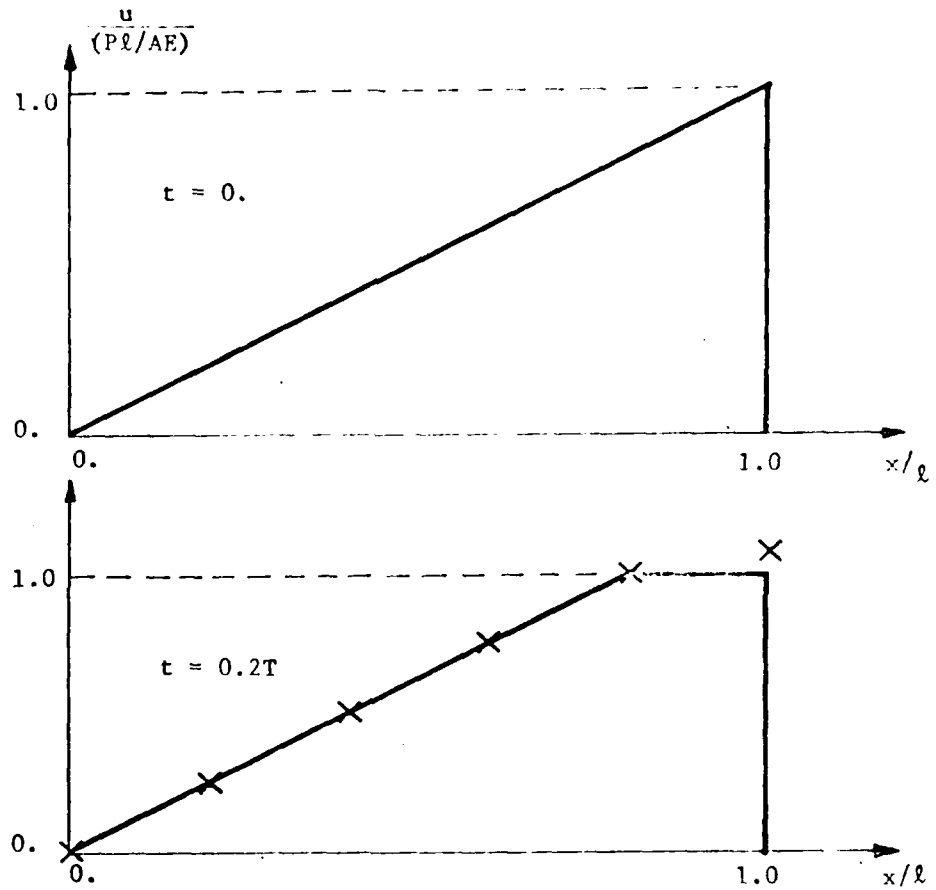


Figure 9. Displacement Solutions by the Constrained Variational Formulations ( $t = 0.2T$ ) and Comparison with Exact Solutions. Grid: 5x1.

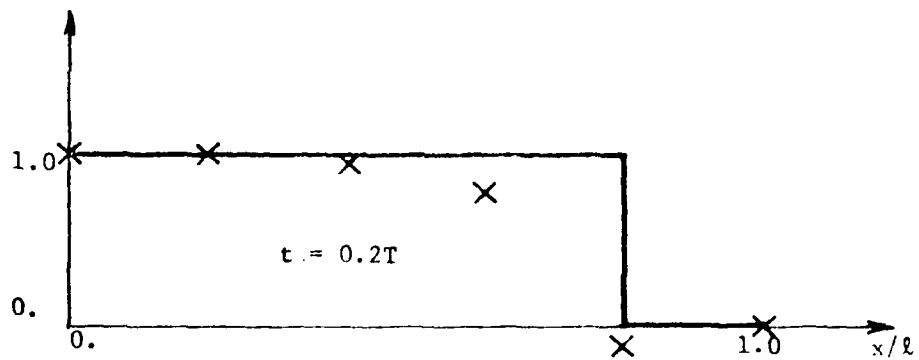
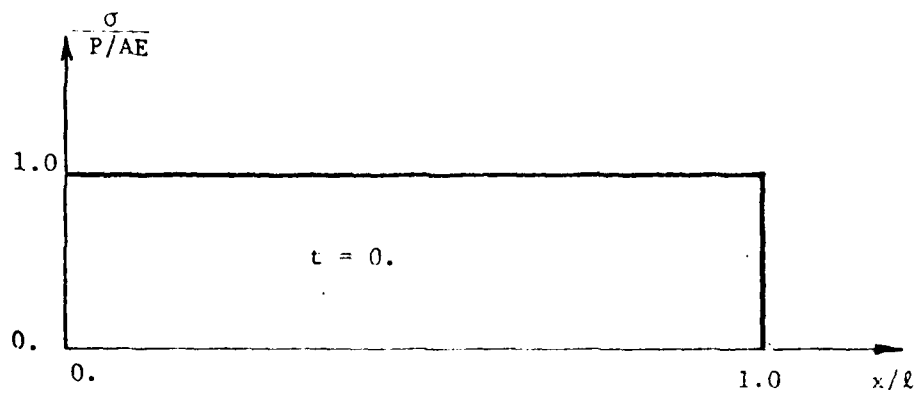


Figure 10. Stress Solution by the Constrained Variational Formulations ( $t = 0.2T$ ) and Comparison with Exact Solutions. Grid: 5x1.

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