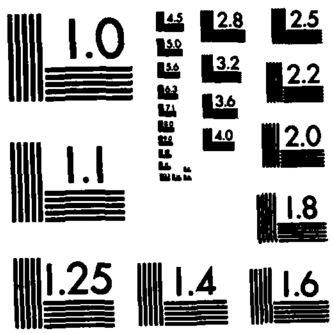
AD-A123 438 ASYMPTOTIC SPECTRAL ANALYSIS OF CROSS-PRODUCT MATRICES (U) PRINCETON UNIV NJ DEPT OF STATISTICS G S WATSON NOV 82 7 N00014-81-K-0146 1/1

UNCLASSIFIED F/G 12/1 NL

END
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

(12)

AD A123438

Report Number 7

ASYMPTOTIC SPECTRAL ANALYSIS OF CROSS-PRODUCT MATRICES

G.S. Watson
Department of Statistics
Princeton University
Princeton, N.J. 08544

NOVEMBER 1982

DTIC
ELECTE
JAN 17 1983
A

Prepared for

OFFICE OF NAVAL RESEARCH (Code 411SP)
Statistics and Probability Branch
Arlington, Virginia 22217
under Contract N00014-81-K-0146
SRO(103) Program in Non-Gaussian Signal Processing

S.C. Schwartz, Principal Investigator

DTIC FILE COPY

Approved for public release; distribution unlimited

83 01 17 049

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 7	2. GOVT ACCESSION NO. AD-A123438	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ASYMPTOTIC SPECTRAL ANALYSIS OF CROSS-PRODUCT MATRICES		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) G.S. Watson		8. CONTRACT OR GRANT NUMBER(s) N00014-81-K0146
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Princeton University Princeton, NJ 08544		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR SRO-103
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research (Code 411SP) Department of the Navy Arlington, Virginia 22217		12. REPORT DATE November 1982
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Spectral analysis
Asymptotic distribution
of covariance matrix

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let x be a random vector in IR^q and $M_n = n^{-1}\sum_1^n x_i x_i'$ be an estimator of M=Exx' with spectral form $\Sigma\lambda_j P_j$. An expository account is given of the estimation of λ_j and P_j from the eigenvalues and vectors of M_n when n is large. This includes a derivation of the basic formulae using a complex variable method in the book by
(over)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(Abstract con't.)

Kato (1980) and a contrasting matrix method. Several extensions are indicated.

DTIC COPY INSPECTED 2

Accession For
NTIS GRA&I
DTIC TAB

A

ASYMPTOTIC SPECTRAL ANALYSIS OF CROSS-PRODUCT MATRICES

by

G. S. Watson
Department of Statistics
PRINCETON UNIVERSITY

ABSTRACT

Let x be a random vector in $\mathbb{R}^q$ and $M_n = n^{-1} \sum_{1}^{n} x_i x_i'$ be an estimator of $M = Exx'$ with spectral form $\Sigma\lambda_j P_j$. An expository account is given of the estimation of λ_j and P_j from the eigenvalues and and vectors of M_n when n is large. This includes a derivation of the basic formulae using a complex variable method in the book by Kato (1980) and a contrasting matrix method. Several extensions are indicated.

ASYMPTOTIC SPECTRAL ANALYSIS OF CROSS-PRODUCT MATRICES

G. S. Watson

Princeton University

1. Introduction

T. W. Anderson (1963) derived the asymptotic distribution of the eigenvalues and vectors of the covariance matrix of a sample from a Gaussian distribution. Davis (1977) took his basic method and used it to get some results for the non-Gaussian case. The non-Gaussian case is of interest either because one wants to study the sensitivity of methods to deviations from Gaussianity - see e.g. Muirhead (1982) - or because one has to deal with other distributions. For example the distribution of the random vector might be entirely restricted to some manifold embedded in $\mathbb{R}^q$ like the surface of the unit sphere or an hyperboloid of rotation; the cases of interest to us.

Kim (1978) at the suggestion of R. J. W. Beran, used results from the book by Kato (1976, 1980) on the perturbation theory of linear operators to find the asymptotic distribution of the eigenvalues of the matrix $M_n = n^{-1} \sum_1^n x_i x_i'$ where the x_i's are independently drawn from a certain distribution on the surface Ω_q of the unit sphere in $\mathbb{R}^q$. Tyler (1979, 1981) also used Kato's method to get results in classical multivariate analysis. But the technique is not well-known, nor immediately evident from Kato's book.

Kato's method calls upon Cauchy's Theorem in Complex Variable Theory. Specifically consider the integral of $(z - z_0)^p$ anti-clockwise around a simple closed curve C in the complex plane which does not go through z_0.

$$\int_C (z - z_0)^p \, dz$$

where p is an integer. Unless $p = -1$, it is always zero. When $p = -1$, it is zero if z_0 is outside C and $2\pi i$ where z_0 is inside C.

The techniques and formulae below have many possible applications. Some are given in Watson (1982a) but most remain to be exploited. In the next section Kato's method is explained for symmetric non-random matrices and then applied in Section 3 to covariance matrices.

The key formulae in Section 2 and the results of Section 3 have of course been obtained before by direct matrix methods -

- though they are hard to justify. The Kato method not only gives a better insight but is easier to do and to extend, e.g. to provide asymptotic expansions.

2. The key to Kato

If T_0 and T_1 are real symmetric $q \times q$ matrices and x is a small real number

$$T(x) = T_0 + xT_1 \tag{2.1}$$

can be thought of as a linear perturbation of T_0. Let the spectral representation of the matrix T_0 be

$$T_0 = \sum_1^r \lambda_j P_j \; , \; r \le q \tag{2.2}$$

where

$$\left.\begin{array}{l} \lambda_1, \ldots, \lambda_r \text{ are distinct real numbers,} \\ P_j' = P_j, \;\; P_j P_k = \delta_{jk} P_j, \\ \text{rank } P_j = \text{trace } P_j = q_j, \;\; \sum_1^r q_j = q \end{array}\right\} \tag{2.3}$$

Thus λ_j is an eigenvalue of T_0 that is repeated q_j times. The invariant subspace V_j associated with λ_j has dimension q_j and P_j projects orthogonally onto V_j whose direct sum is $\mathbb{R}^q$.

The matrix $T(x)$ may have q distinct eigenvalues but we would expect these to fall into r clusters about $\lambda_1, \ldots, \lambda_r$ and to condense on $\lambda_1, \ldots, \lambda_r$ as $x \to 0$. Equally the eigenvectors of $T(x)$ should lead us to the eigen subspaces V_j as $x \to 0$. To show how this happens, define the resolvent of T_0, $R_0(\zeta)$ as

$$R_0(\zeta) = (T_0 - \zeta I_q)^{-1} \tag{2.4}$$

where ζ is a complex number. By (2.2) we may write

$$R_0(\zeta) = \sum_1^r (\lambda_j - \zeta)^{-1} P_j \tag{2.5}$$

Observe that T_0 and $R_0(\zeta)$ commute.

If C is any contour in the complex plane which does not go through any λ_j, which are points on the real axis, Cauchy's Theorem and (2.5) imply that

$$\frac{1}{2\pi i} \int_C R_0(\zeta) d\zeta = \sum_{j \in C} P_j \tag{2.6}$$

where the sum is over the projectors P_j associated with eigenvalues λ_j inside C. The integral of a matrix is the matrix of integrals. Similarly

$$\frac{1}{2\pi i} \int_C T_0 R_0(\zeta)d\zeta = \sum_{j\epsilon C} \lambda_j P_j \tag{2.7}$$

We observe that the trace of (2.6) gives the sum of the dimensions of the eigen subspaces associated with λ_j within C. Similarly the trace of (2.7) gives the sum of the eigen values (times their multiplicities) within C.

We now consider the resolvent of T(x).

$$R(x,\zeta) = (T(x) - \zeta I_q)^{-1} = R_0(\zeta)(I_q + xT_1R_0(\zeta))^{-1} \tag{2.8}$$

If we apply the results of the previous paragraph to $R(x,\zeta)$, we will get information about the eigen values $\lambda(T(x))$ and projectors $P(T(x))$, of $T(x)$. As $x \to 0$, we would expect the values of $\lambda(T(x))$ to condense on the eigen values λ_j of T_0

To obtain the required formulae, we need to expand (2.8) as a power series. For a $q \times q$ matrix A,

$$(I_q + xA)^{-1} = I_q - xA + x^2A^2 - \ldots \tag{2.9}$$

where the series is absolutely convergent provided $|x| \, \|A\| < 1$, where $\|A\|$ is a norm of A. Thus we can say that for x sufficiently small,

$$(I_q + xA)^{-1} = I_q - xA + O(x^2) . \tag{2.10}$$

Applying (2.10) to (2.8), we have, as $|x| \to 0$,

$$R(x,\zeta) = R_0(\zeta) - x\, R_0(\zeta)T_1R_0(\zeta) + O(x^2). \tag{2.11}$$

Consider now the analogue of (2.6) when C_j is a contour which encloses only the eigen value λ_j. Then

$$\frac{1}{2\pi i}\int_{C_j} R(x,\zeta)d\zeta = \frac{1}{2\pi i}\int_{C_j} R_0(\zeta)d\zeta + \frac{x}{2\pi i}\int_{C_j} R_0(\zeta)T_1R_0(\zeta)d\zeta + 0(x^2) \qquad (2.12)$$

The first term on the right hand side (r.h.s.) of (2.12) is P_j. To find the second term, we observe that, if we use (2.5) twice,

$$R_0(\zeta)T_1R_0(\zeta) = \sum_{k=1}^{r}\sum_{\ell=1}^{r} \frac{P_kT_1P_\ell}{(\lambda_k-\zeta)(\lambda_\ell-\zeta)}$$

$$= \sum_{k=1}^{r} \frac{P_kT_1P_k}{(\lambda_k-\zeta)^2} + \sum_{k<\ell} \frac{(P_kT_1P_\ell + P_\ell T_1P_k)}{\lambda_\ell - \lambda_k}\left(\frac{1}{\lambda_k-\zeta} - \frac{1}{\lambda_\ell-\zeta}\right) \qquad (2.13)$$

The contour integral of the first term on the r.h.s. of (2.13) is zero. We get contributions from the second term when k or ℓ equal j and they add to the symmetric matrix

$$\sum_{k\neq j} \frac{P_kT_1P_j + P_jT_1P_k}{\lambda_k - \lambda_j} \qquad (2.14)$$

Observe, for later use, that this matrix has a zero trace because P_jP_k is null. Thus

$$\frac{1}{2\pi i}\int_{C_j} R(x,\zeta)d\zeta = P_j + x\sum_{k\neq j}\frac{P_kT_1P_j+P_jT_1P_k}{\lambda_j - \lambda_k} + 0(x^2) \qquad (2.15)$$

is the analogue of (2.6).

The analogue of (2.7) is obtained by integrating $T(x)R(x,\zeta)$ which may, using (2.1) and (2.11), be written as

$$T(x)R(x,\zeta) = T_0R_0(\zeta) + x(T_1R_0(\zeta) - T_0R_0(\zeta)T_1R_0(\zeta)) + 0(x^2). \qquad (2.16)$$

The integral of the first term on the r.h.s. of (2.16) is that in (2.7).

To find the second and third terms we note that

$$T_1R_0(\zeta) = \sum_{k=1}^{r}(\lambda_k-\zeta)^{-1}T_1P_k \,,$$

$$T_0R_0(\zeta)T_1R_0(\zeta) = \sum_{k=1}^{r}\frac{\lambda_kP_kT_1P_k}{(\lambda_k-\zeta)^2} \tag{2.17}$$

$$+ \sum_{k<\ell}\frac{\lambda_kP_kT_1P_\ell+\lambda_\ell P_\ell T_1P_k}{(\lambda_\ell-\lambda_k)}\left(\frac{1}{\lambda_k-\zeta}-\frac{1}{\lambda_\ell-\zeta}\right), \tag{2.18}$$

where we have used (2.2) and (2.13). Thus we find that

$$\frac{1}{2\pi i}\int_{C_j} T(x)R(x,\zeta)d\zeta = \lambda_jP_j+x(T_1P_j + \sum_{k\neq j}\frac{\lambda_kP_kT_1P_j+\lambda_jP_jT_1P_k}{\lambda_j-\lambda_k})$$

$$+ O(x^2) \,. \tag{2.19}$$

Observe that the trace of the second term in the coefficient of x is zero. (2.19) is the analogue of (2.7).

In the applications we have in mind $T(x)$ will have q distinct eigenvalues $\lambda_1(x), \ldots, \lambda_q(x)$ and (orthonormal) eigenvectors $v_1(x), \ldots, v_q(x)$ so that

$$T(x) = \sum_{1}^{q}\lambda_i(x)v_i(x)v_i(x)' \tag{2.20}$$

is the spectral form for $T(x)$. By using the reasoning that led to (2.6) and (2.7) we may then evaluate the l.h.s.'s of (2.15) and (2.19). Thus

$$\frac{1}{2\pi i}\int_{C_j} R(x,\zeta)d\zeta = \Sigma\, v_i(x)v_i(x)' \tag{2.21}$$

and

$$\frac{1}{2\pi i}\int_{C_j} T(x)R(x,\zeta)d\zeta = \Sigma\, \lambda_i(x)v_i(x)v_i(x)' \tag{2.22}$$

where both sums are over i such that $\lambda_i(x)$ are points inside the contour C_j.

Since trace $v_i(x)v_i(x)' = v_i'(x)v_i(x) = 1$, taking the trace of both sides of (2.15) and using (2.21) yields

$$\#\, \lambda_i(x) \text{ inside } C_j = q_j + O(x^2) \tag{2.23}$$

for any contour C_j enclosing λ_j. As $x \to 0$, one could use smaller and smaller contours. Hence as $x \to 0$, the eigenvalues of $T(x)$ form <u>clusters</u> of q_j roots about λ_j $(j = 1, \ldots, r)$ which condense upon λ_j. If we do not take the trace of (2.15) and write

$$\hat{P}_j = \sum_{\lambda_i(x)\subset C_j} v_i(x)v_i(x)' \tag{2.24}$$

then (2.15) may be written as

$$\hat{P}_j = P_j + x \sum_{k\neq j} \frac{P_kT_1P_j + P_jT_1P_k}{\lambda_j - \lambda_k} + O(x^2) \tag{2.25}$$

Taking the trace of (2.19) yields

$$\Sigma\, \lambda_i(x) \text{ within } C_j = q_j\lambda_j + x \text{ trace } T_1P_j + O(x^2) \tag{2.26}$$

so dividing through by q_j and calling the l.h.s. $\overline{\lambda}_j$, the arithmetic mean of the j^{th} cluster, we have

$$\bar{\lambda}_j = \lambda_j + \frac{x}{q_j} \text{ trace } T_1 P_j + 0(x^2) \tag{2.27}$$

The formulae (2.25) and (2.27) are ideal for statistical applications, as will be seen in the next section. We close this section by observing that there is no problem except complexity in getting higher order approximations - one merely takes higher order terms in (2.10). For example, the coefficient of x^2 in $R(x_1\ )$ is $R_0T_1R_0T_1R_0$ so using (2.5) and partial fraction expansions the contour integral may be evaluated to give a lengthy formula. One then finds that (2.23) may be improved to

$$\# \lambda_i(x) \text{ inside } C_j = q_j + 0(x^3) \tag{2.28}$$

In Watson (1982a) explicit results are given when r = 2.

3. Large sample theory of symmetric cross-product matrices

Let x be a random vector in $\mathbb{R}^q$ with components $x^1, x^2, .., x^q$ and suppose that $Ex^ix^jx^kx^\ell$ exists for all $i, j, k, \ell = 1, .., q$. Let x' denote the transpose of x. Call $Exx' = E[x^ix^j] = M$, a symmetric q × q matrix with spectral form

$$M = \sum_1^r \lambda_j P_j \tag{3.1}$$

If $x_1, .., x_n$ are independent copies of x, define

$$M_n = n^{-1} \sum_1^n x_i x_i' \tag{3.2}$$

Then $M_n \to M$ by the law of large numbers and by the multivariate central limit theorem

$$n^{1/2}(M_n - M) \xrightarrow{d} G. \tag{3.3}$$

The $q(q+1)/2$ functionally independent elements of the symmetric matrix G are jointly Gaussian with zero means and a covariance matrix V whose elements are

$$E\, x^i x^j x^k x^\ell - E(x^i x^j) E(x^k x^\ell), \quad i \leq j, \ k \leq \ell. \tag{3.4}$$

To use the results of Section 2, we may write

$$M_n = M + n^{-1/2} \{n^{1/2}(M_n - M)\} \tag{3.5}$$

instead of

$$T(x) = T_0 + x\, T_1$$

From (3.3), T_1 corresponds to G, x to $n^{-1/2}$, and M to T_0. Provided no λ_j in (3.1) is zero, the matrix M_n will, with probability one, have distinct eigenvalues - Okamoto (1973). If say $\lambda_1 = 0$, $E(P_1 x)(P_1 x)'$ is a matrix of zeros so that, taking the trace, $E\,\|P_1 x\|^2 = 0$. Thus $P_1 x$ is a null vector and M_n will have q_1 zero roots and the data will determine the eigen subspace V_1 exactly. This case has little interest so we assume that all the $\lambda_j > 0$.

The matrix M_n will be used to estimate the λ_j and P_j, $j=1, \ldots, r$. Combining (3.5) with (2.24), (2.25) and (2.27), we have the key results:

for $j = 1, \ldots, r$,

$$n^{1/2}\,(\hat{P}_j - P_j) \xrightarrow{d} \sum_{k \neq j} \frac{P_k G P_j + P_j G P_k}{\lambda_j - \lambda_k} \tag{3.6}$$

$$n^{1/2}\,(\bar{\lambda}_j - \lambda_j) \xrightarrow{d} \frac{1}{q_j} \operatorname{trace} G P_j\,. \tag{3.7}$$

The r.h.s.'s of (3.6) and (3.7) are linear in the Gaussian matrix G so that the l.h.s.'s have asymptotically Gaussian distributions with zero means and variances and covariances that depend upon the covariance matrix V of G.

(3.7) is univariate and so easy to understand, e.g. it leads to a normal confidence interval for λ_j, although we will see that one will do better with a transformation. (3.6) describes the difference between estimated and true projectors and needs further simplification. Using the Euclidean matrix norm ($\|A\|^2 = \text{trace } AA'$),

$$n\|\hat{P}_j - P_j\|^2 \xrightarrow{d} 2\sum_{k\neq j} \frac{\text{trace } P_j G P_k G}{(\lambda_j - \lambda_k)^2} \tag{3.8}$$

Again, one might examine the different effects of $\hat{P}_j$ and P_j on vectors. For example, if $v \in V_j$,

$$n^{1/2}(\hat{P}_j v - P_j v) \xrightarrow{d} \sum_{k\neq j} \frac{P_k G v}{\lambda_j - \lambda_k} \tag{3.9}$$

so

$$n\|\hat{P}_j v - P_j v\|^2 \xrightarrow{d} \sum_{k\neq j} \frac{v' G P_k G v}{\lambda_j - \lambda_k} \tag{3.10}$$

More fundamentally if $\hat{V}_j$ is the subspace onto which $\hat{P}_j$ projects, $\hat{V}_j$ will be "close" to V_j if $\cos\theta = v'\hat{v}$ is always large when $v \in V_j$ and $\hat{v} \in \hat{V}_j$, $\|v\| = 1$, $\|\hat{v}\| = 1$. Thus we should seek the stationary values of $(P_j u)'(\hat{P}_j w)$, given $\|P_j u\| = \|\hat{P}_j w\| = 1$, i.e., we should consider

$$2u'P_j\hat{P}_j w - \theta u'P_j u - \phi w'\hat{P}_j w$$

where θ and ϕ are Lagrangian multipliers. Hence

$$\left.\begin{aligned} P_j\hat{P}_j w - \theta P_j u &= 0 , \\ \hat{P}_j P_j u - \phi \hat{P}_j w &= 0 , \end{aligned}\right\} \qquad (3.11)$$

so that

$$\begin{aligned} \theta = \phi &= \text{stationary value of } (P_j u)'(\hat{P}_j w) \\ &= C, \text{ say} . \end{aligned}$$

Hence the equations (3.11) will only have a solution if

$$\begin{vmatrix} -C\,P_j & P_j\,\hat{P}_j \\ \hat{P}_j\,P_j & -C\,\hat{P}_j \end{vmatrix} = 0 \qquad (3.12)$$

This equation for C may be reduced to

$$\left| P_j\,\hat{P}_j\,P_j - C^2\,P_j \right| = 0 \qquad (3.13)$$

which has q_j non-zero roots $C_\ell^{\,2}$. If, however, (3.6) is used, one finds eventually that all the $C_\ell^{\,2}$ are unity. Watson (1982a) deals with the case where $r = 2$ and shows, by taking the next term in the expansions in Section 2, that $n(1 - C_\ell^{\,2})$ have asymptotic distributions. It is conjectured that for any r the asymptotic joint distribution of $n(1-C_1)$, ..., $n(1 - C_{q_j})$ is the joint distribution of the non-zero eigenvalues of

$$\sum_{k \neq j} \frac{P_j G P_k G P_j}{(\lambda_j - \lambda_k)^2} \qquad (3.14)$$

Some of the above results become easier to understand if we write, since $I_q = P_1 + \dots + P_r$,

$$y_j = P_j x, \quad x = y_1 + \dots + y_r \tag{3.15}$$

One of the reasons results become simpler for the Gaussian is that there $y_1, \dots, y_r$ are independent. Since $Exx' = M = \lambda_j P_j$,

$$\left.\begin{array}{l} Ey_k y_\ell' = 0,\ \ell \neq k,\ Ey_j y_j' = \lambda_j P_j \\ Ey_\ell' y_k = 0,\ (\ell \neq k),\ Ey_j' y_j = \lambda_j q_j \end{array}\right\} \tag{3.16}$$

Thus (3.7) may be rewritten as

$$n^{1/2} (\bar{\lambda}_j - \lambda_j) \sim \frac{n^{1/2}}{q_j} \operatorname{trace} \left(\frac{1}{n} \sum_{i=1}^{n} y_{ji} y_{ji}' - \lambda_j P_j \right)$$

$$= \frac{n^{1/2}}{q_j} \left(\frac{1}{n} \sum_{i=1}^{n} y_{ji}' \, y_{ji} - \lambda_j q_j \right) \tag{3.17}$$

so that by (3.16) and the Central Limit Theorem

$$\mathcal{L}\ n^{1/2} (\bar{\lambda}_j - \lambda_j) \longrightarrow G_1(0, \operatorname{var}(y_j' y_j)\ q_j^{-2}) \tag{3.18}$$

where $G_q(\mu, \Sigma)$ stands for the Gaussian distribution in q dimensions with mean vector μ and covariance matrix Σ. Similarly (3.6) can be written as

$$n^{1/2} (\hat{P}_j - P_j) \sim n^{-1/2} \sum_{i=1}^{n} \sum_{k \neq j} \frac{y_{ki} y_{ji}' + y_{ji} y_{ki}'}{\lambda_j - \lambda_k} \tag{3.19}$$

If $\mathcal{L} x = G_q(0,M)$, $y_1, y_2, \ldots, y_r$ are independent and $\mathcal{L}\, y_j' y_j \lambda_j^{-1} = \chi^2_{q_j}$ so that $\mathrm{var}(y_j' y_j) = \lambda_j^2 2q_j$. Then (3.18) reads:

$$\mathcal{L}\, n^{1/2} (\bar{\lambda}_j - \lambda_j) \longrightarrow G_1(0,\, 2\lambda_j^2/q_j)$$

Hence

$$\mathcal{L}\, n^{1/2} (\log (\bar{\lambda}_j/\lambda_j) - 1) \longrightarrow G_1(0,\, 2/q_j) \tag{3.20}$$

giving the variance stabilizing transformation. Moreover in this Gaussian case the $n^{1/2} (\bar{\lambda}_j - \lambda_j)$ or $n^{1/2} (\log(\bar{\lambda}_j/\lambda_j) - 1)$ are asymptotically independent, a simplifying result which is not true in general. Under no circumstances could it be expected that the $\hat{P}_j$ would be independent since $\sum_{j=1}^{r} \hat{P}_j = I_q$.

With this introduction, the compact paper by Tyler (1981) may be read for more details on $\hat{P}_j$. He also gives tests. special case of $r = 2$ and distributions restricted to Ω_q, Watson (1982a). If an additional assumption is made that the distribution of x depends only upon $\|y_1\|, \ldots, \|y_r\|$ more results may be derived - see Watson 1982b).

4. Direct approach to large sample theory of cross-product matrices.

The eigenvalues of M_n are the roots $\hat{\lambda}$ of

$$|M_n - \lambda I| = |M - \lambda I + \frac{1}{\sqrt{n}} T_1| = 0 \tag{4.1}$$

where, as in Section 3, $M = \sum_1^r \lambda_j P_j$, $T_1 = \sqrt{n}(M_n - M)$. Suppose orthonormal eigenvectors are selected to span each of the invariant subspaces V_j and arranged as column vectors to form a $q \times q$ orthogonal matrix H. Let the first q_1 columns correspond to V_1, the next q_2 columns to V_2, etc., and write it in partitioned form

$$H = [H_1, \ldots, H_r] \tag{4.2}$$

Then since $H'H = HH' = I_q$, we have

$$H'_a H_b = 0 (a \neq b) \ , \ H'_a H_a = I_{q_a} \ ,$$

$$H_1 H'_1 + \ldots + H_r H'_r = I_q \ , \tag{4.3}$$

$$H_a H'_a = P_a \ , \ a = 1, \ldots, r \ ,$$

and

$$H'MH = D(\lambda_j I_{q_j}) \ , \tag{4.4}$$

a matrix partitioned so all r^2 submatrices are zero except for the multiples of identity matrices on the diagonal.

If H is applied to (4.1), it takes the partitioned form.

$$\left| (\lambda_i - \lambda) \ I_{qi} \delta_{ij} + n^{-1/2} \ H'_i T_1 H_j \right| = 0 \tag{4.5}$$

Since $n \to \infty$, we seek the $0(1)$ and $0(n^{-1/2})$ terms only in the expansion of (4.5). Applying the formula

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} = |A| \, |D - BA^{-1}C|$$

when A is the leading submatrix of (4.5), it is seen that $BA^{-1}C$ is $O(n^{-1})$ and so negligible. Hence we may repeat the procedure to find that equation (4.5) and hence (4.1) is, to this order

$$\prod_{j=1}^{r} \left|(\lambda_j - \lambda)\, I_{q_j} + n^{-1/2}\, H_j' T_1 H_j\right| = 0 \tag{4.6}$$

This shows that the eigenvalues of M_n, for large n, form clusters about the r distinct roots λ_j of M. Expanding the j^{th} factor in (4.6) to $O(n^{-1/2})$, we only need the product of the diagonal terms and find the equation

$$(\lambda_j - \lambda)^{q_j}\left(1 + n^{-1/2}\,\frac{\text{trace } H_j' T_1 H_j}{\lambda_j - \lambda}\right) = 0 \tag{4.7}$$

Since trace $H_j' T_1 H_j$ = trace $H_j H_j' T_1$ = trace $P_j T_1$ by (4.3) the q_j roots of (4.7) tend to λ_j as $n \to \infty$ and the leading terms of the polynomial (degree q_j) equation for λ are

$$\lambda^q - \{q\lambda_j + n^{-1/2} \text{ trace } P_j T_1\}\lambda^{q-1} + \ldots = 0 \tag{4.8}$$

so that if the roots of this equation are denoted by $\hat{\lambda}$,

$$\Sigma\hat{\lambda} = q\lambda_j + n^{-1/2} \text{ trace } P_j T_1 \text{ ,} \tag{4.9}$$

as we found in (3.27). But one cannot expect to obtain the roots in the cluster for λ_j from (4.7) (it gives them to be λ_j(q-1 times), $\lambda_j + n^{-1/2}$ trace $P_j T_1$ (once)) because when λ is within $n^{-1/2}$ of λ <u>all</u> the terms in the matrix in (4.6) are of order $n^{-1/2}$. However, (4.8) does give the correct coefficient for λ^{q-1} in (4.6).

Since the eigenvalues of M_n will in general be distinct, the approximations made above are inadequate to discuss e.g. the joint distribution of the eigenvalues in a cluster.

The following direct derivation of the analogue of (2.25) or (3.6) is harder to justify. Write

$$\hat{P}_j = P_j + n^{-1/2}\Delta, \quad \Delta = n^{1/2}(\hat{P}_j - P_j) \tag{4.10}$$

and, because the roots in the cluster are within n^{-1} of λ_j, set

$$M_n\hat{P}_j = (\lambda_j + n^{-1/2}\delta)\hat{P}_j \tag{4.11}$$

i.e.

$$(M + n^{-1/2}T_1)(P_j + n^{-1/2}\Delta) = (\lambda_j + n^{-1/2}\delta)(P_j + n^{-1/2}\Delta)$$

so that the terms in $n^{-1/2}$ yield the equation

$$T_1P_j + M\Delta = \lambda_j\Delta + \delta P_j$$

or

$$(M - \lambda_jI)\Delta = -T_1P_j + \delta P_j \tag{4.12}$$

But $M - \lambda_jI = \sum_{k\neq j}(\lambda_k - \lambda_j)P_k$ so that we could replace Δ by $\Delta - P_jX$ and still satisfy (4.12). Thus (4.12) is solved by multiplying (4.12) by $\Sigma(\lambda_k - \lambda_j)^{-1}P_k$ and adding P_jX so that

$$\Delta = \sum_{k\neq j}(\lambda_j - \lambda_k)^{-1}P_kT_1P_j + P_jX.$$

However, from (4.10) we see that Δ must be symmetric and this determines X. Thus

$$\Delta = n^{1/2}(\hat{P}_j - P_j) = \sum_{k\neq j} \frac{P_k T_1 P_j + P_j T_1 P_k}{\lambda_j - \lambda_k} \tag{4.13}$$

which is the desired result. However, without the results of Section 3, (4.10) and (4.11) are merely intuitions.

5. Additional Remarks

(i) Suppose (x_i, y_i) for $i = 1, \ldots, n$ are independent copies of a pair of random vectors (x, y), $x \in \mathbb{R}^q$, $y \in \mathbb{R}^p$. Define the estimator of $N = Exy'$

$$N_n = n^{-1} \sum_1^n x_i y_i' \tag{5.1}$$

and assume that the Central Limit Theorem holds so that

$$n^{1/2} (N_n - N) \xrightarrow{d} F \tag{5.2}$$

One may wish to estimate the singular values of N so one would find the non-zero eigenvalues of $N_n N_n'$ or $N_n' N_n$ whichever is the smaller. But using (5.2),

$$N_n' N_n \sim N'N + n^{-1/2}(F'N + N'F) \tag{5.3}$$

so that the previous theory is applicable.

(ii) Suppose that $x_1, \ldots, x_n$ is a sample from one q dimensional distribution, $x_1^*, \ldots, x_m^*$ a sample from another distribution in $\mathbb{R}^q$ and let $M_n = n^{-1} \Sigma x_i x_i'$, $M_m^* = m^{-1} \Sigma x_i^*, x_i^{*\prime}$. Then we often need to study the solutions of

$$(M_n - \lambda M_m^*)v = 0 \tag{5.4}$$

Anderson (1958) gives examples and calls λ and v the eigen values and vectors of M_n in the metric of M_m. To study them in large samples we suppose that the Central Limit Theorem applies in both cases so that as $m, n \to \infty$

$$\left.\begin{aligned} M_n &\sim M + n^{-1/2} G , \\ M_m^* &\sim M^* + m^{-1/2} G^* . \end{aligned}\right\} \tag{5.5}$$

The eigenvalues are those of $M_m{}^{*-1/2} M_n M_m{}^{*-1/2}$ which by (5.5) are those of

$$\begin{aligned} & M^{*-1/2} (I_q - \tfrac{m}{2}^{-1/2} G^* M^{*-1})(M + n^{-1/2} G)(I_q - \tfrac{m}{2}^{-1/2} M^{*-1} G^*) M^{*-1/2} \\ = \; & M^{*-1/2} M M^{*-1/2} + M^{*-1/2} G M^{*-1/2} \\ & - \tfrac{m}{2}^{-1/2} M^{*-1/2} G^* M^{*-1} M M^{*-1/2} \\ & - \tfrac{m}{2}^{-1/2} M^{*-1/2} M M^{*-1} G^* M^{*-1/2} \end{aligned} \tag{5.6}$$

If we set $n = \alpha \ell$, $m = \beta \ell$ with $\alpha, \beta > 0$ and $\ell \to \infty$ (5.6) has the form of a symmetric fixed matrix plus $\ell^{-1/2}$ times a symmetric Gaussian matrix so that the earlier thoery is applicable.

References

Anderson, T.W. (1958), Introduction to Multivariate Analysis, John Wiley & Sons, New York.

Anderson, T.W. (1963), Asymptotic Theory for Principal Components, Ann.Math.Stat. 34, 122-148.

Davis, A.W. (1977), Asymptotic Theory for Principal Components, Austral.J.Statist. 19 (3), 206-212.

Kim, K.M. (1978), Orientation Shift Model on the Sphere. Ph.D. Thesis, Univ. of California, Berkeley.

Kato, T. (1980), Perturbation Theory for Linear Operators, 2nd Edition, Springer-Verlag, New York.

Muirhead, Robb J. (1982), Aspects of Multivariate Statistical Theory, John Wiley & Sons, New York.

Tyler, D.E. (1979), Redundancy Analysis and Associated Asymptotic Distribution Theory, Ph.D. Thesis, Princeton University.

Tyler, D.E. (1981), Asymptotic Inference for Eigenvectors, Ann.Statistics 9 (4), 725-736.

Watson, G.S. (1982a), Large Sample Theory for Distributions on the Hypersphere with Rotational Symmetries. Technical Report, Dept. of Statistics, Princeton University.

Watson, G.S. (1982b), Distributions in $\mathbb{R}^q$ with Rotational Symmetries. Technical Report, Dept. of Statistics, Princeton University.

OFFICE OF NAVAL RESEARCH
STATISTICS AND PROBABILITY PROGRAM

BASIC DISTRIBUTION LIST
FOR
UNCLASSIFIED TECHNICAL REPORTS

FEBRUARY 1982

	Copies
Statistics and Probability Program (Code 411(SP)) Office of Naval Research Arlington, VA 22217	3
Defense Technical Information Center Cameron Station Alexandria, VA 22314	12
Commanding Officer Office of Naval Research Eastern/Central Regional Office Attn: Director for Science Barnes Building 495 Summer Street Boston, MA 02210	1
Commanding Officer Office of Naval Research Western Regional Office Attn: Dr. Richard Lau 1030 East Green Street Pasadena, CA 91101	1
U. S. ONR Liaison Office - Far East Attn: Scientific Director APO San Francisco 96503	1
Applied Mathematics Laboratory David Taylor Naval Ship Research and Development Center Attn: Mr. G. H. Gleissner Bethesda, Maryland 20084	1
Commandant of the Marine Coprs (Code AX) Attn: Dr. A. L. Slafkosky Scientific Advisor Washington, DC 20380	1

	Copies
Navy Library National Space Technology Laboratory Attn: Navy Librarian Bay St. Louis, MS 39522	1
U. S. Army Research Office P.O. Box 12211 Attn: Dr. J. Chandra Research Triangle Park, NC 27706	1
Director National Security Agency Attn: R51, Dr. Maar Fort Meade, MD 20755	1
ATAA-SL, Library U.S. Army TRADOC Systems Analysis Activity Department of the Army White Sands Missile Range, NM 88002	1
ARI Field Unit-USAREUR Attn: Library c/o ODCSPER HQ USAEREUR & 7th Army APO New York 09403	1
Library, Code 1424 Naval Postgraduate School Monterey, CA 93940	1
Technical Information Division Naval Research Laboratory Washington, DC 20375	1
OASD (I&L), Pentagon Attn: Mr. Charles S. Smith Washington, DC 20301	1

Copies

Director
AMSAA
Attn: DRXSY-MP, H. Cohen
Aberdeen Proving Ground, MD 21005 — 1

Dr. Gerhard Heiche
Naval Air Systems Command (NAIR 03)
Jefferson Plaza No. 1
Arlington, VA 20360 — 1

Dr. Barbara Bailar
Associate Director, Statistical Standards
Bureau of Census
Washington, DC 20233 — 1

Leon Slavin
Naval Sea Systems Command (NSEA 05H)
Crystal Mall #4, Rm. 129
Washington, DC 20036 — 1

B. E. Clark
RR #2, Box 647-B
Graham, NC 27253 — 1

Naval Underwater Systems Center
Attn: Dr. Derrill J. Bordelon
Code 601
Newport, Rhode Island 02840 — 1

Naval Coastal Systems Center
Code 741
Attn: Mr. C. M. Bennett
Panama City, FL 32401 — 1

Naval Electronic Systems Command (NELEX 612)
Attn: John Schuster
National Center No. 1
Arlington, VA 20360 — 1

Defense Logistics Studies Information Exchange
Army Logistics Management Center
Attn: Mr. J. Dowling
Fort Lee, VA 23801 — 1

Copies

Reliability Analysis Center (RAC)
RADC/RBRAC
Attn: I. L. Krulac
Data Coordinator/
Government Programs
Griffiss AFB, New York 13441 — 1

Technical Library
Naval Ordnance Station
Indian Head, MD 20640 — 1

Library
Naval Ocean Systems Center
San Diego, CA 92152 — 1

Technical Library
Bureau of Naval Personnel
Department of the Navy
Washington, DC 20370 — 1

Mr. Dan Leonard
Code 8105
Naval Ocean Systems Center
San Diego, CA 92152 — 1

Dr. Alan F. Petty
Code 7930
Naval Research Laboratory
Washington, DC 20375 — 1

Dr. M. J. Fischer
Defense Communications Agency
Defense Communications Engineering Center
1860 Wiehle Avenue
Reston, VA 22090 — 1

Mr. Jim Gates
Code 9211
Fleet Material Support Office
U. S. Navy Supply Center
Mechanicsburg, PA 17055 — 1

Mr. Ted Tupper
Code M-311C
Military Sealift Command
Department of the Navy
Washington, DC 20390 — 1

	Copies	Copies
Mr. F. R. Del Priori Code 224 Operational Test and Evaluation Force (OPTEVFOR) Norfolk, VA 23511	1	

END

FILMED

2-83

DTIC