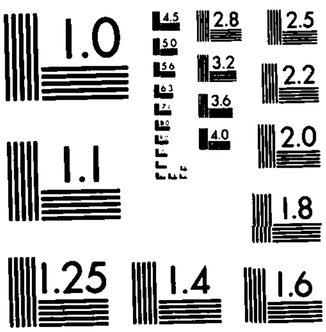
AD-A123 610 FACING AN UNCERTAIN FUTURE(U) STANFORD UNIV CA INST FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES W M GORMAN JAN 82 TR-359 N00014-79-C-0685 1/1

UNCLASSIFIED F/G 5/1 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

FACING AN UNCERTAIN FUTURE

by

William M. Gorman

Technical Report No. 359

January 1982

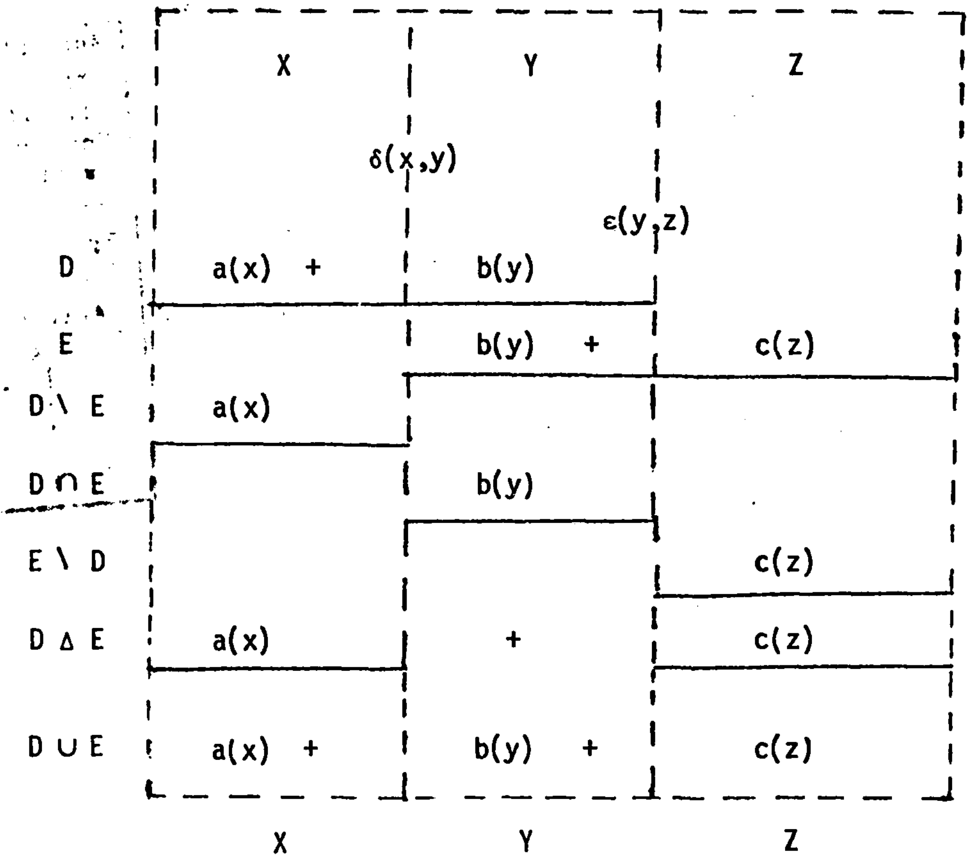
A REPORT OF THE
CENTER FOR RESEARCH ON ORGANIZATIONAL EFFICIENCY
STANFORD UNIVERSITY

Contract ONR-N00014-79-C-0685, United States Office of Naval Research

THE ECONOMICS SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Fourth Floor, Encina Hall
Stanford University
Stanford, California
94305

Figure 1: Illustrating overlapping separable spaces $X \times Y$, $Y \times Z$, or sets D, E, with their subutilities $\delta(x,y)$, $\varepsilon(y,z)$ and the associated separable spaces, or sets, with their normalized subutilities.



FACING AN UNCERTAIN FUTURE*

by

William M. Gorman

1. Introduction and motivation

Economists often assume that organizations use criteria of the form

$$(1.1)\qquad \sum_{t=0}^{T} c^t E[f(y_t)] \quad ,$$

in deciding on their actions, where $E(\cdot)$ is an expectations operator. If you like: that they seek to maximize the mathematical expectation of a discounted utility stream. This is a special case of an additive criterion

$$(1.2)\qquad \sum_{s,t} f^{st}(y_{st}) \quad ,$$

where y_{st} is a vector of flows which occur in period t if state s obtains.

The convenience of assuming (1.1) or (1.2) is clear. Is either justified: They require, after all, that utility is additive over time, and over states, and <u>that the same normalization does for each</u>.

In many problems, all three of these results arise as the joint products of single argument. This is because addition is, effectively, the only strictly increasing associative operation. Suppose for instance, that

---

*This research was supported by the Office of Naval Research Grant ONR-N00014-7-C-0685 at the Center for Research on Organizational Efficiency, Stanford University.

Figure 2: The Social Welfare Function - Bentham & Bernouilli at a stroke.

Consumption Vectors: $y_{hs} \in y_{hs}$

STATES OF THE WORLD	1	2	...	x	...	
HOUSEHOLDS						extended vector
1	y_{11}	y_{12}	...	y_{1s}	...	$x_1 \in X_1$
2	y_{21}	y_{22}	...	y_{2s}	...	$x_2 \in X_2$
...	...	...	...	...	...	
h	y_{h1}	y_{h2}	...	y_{hs}	...	
...	...	...	...	...	...	
extended vector	z_1	z_2	...	z_s	...	
space	Z_1	Z_2	...	Z_s	...	

Each space $X_h = \prod_{s \in S} Y_{hs}$ is separable by consumer sovereignty.

Each space $Z_s = \prod_{h \in H} Y_{hs}$ is separable by the weak independence axiom.

$$(1.3) \qquad f(w,x,y,z) = d(w,\delta(x,y),z) = e[w,x,\varepsilon(y,z)] \quad ,$$

where all functions are continuous, and $\uparrow$ [1] in the subutilities, δ, ε. We then say that $X \times X$, $X \times Z$ are both <u>separable</u>, where X is the space of x,.... This is possible iff

$$(1.4) \qquad f(w,x,y,z) = g[w,a(x) + b(y) + c(z)] \quad ,$$

where $g(\cdot)$ is continuous and $g(w,\cdot)\uparrow$. The secret is the <u>overlap</u> between (x,y) and (y,z), or, better, between $X \times Y$ and $Y \times Z$. To be able to tear y out of its association with x in $\delta(x,y)$, and put it in with z in $\varepsilon(y,z)$ instead, one effectively needs addition, or at least a strictly increasing transformation $g(w,\cdot)$ of it.

What one wants to generate a criterion like (1.2) is a considerable number of such overlaps.

Consider the following, related example. A society contains a finite set H of self regarding households, h, and faces a finite set S of possible states, s, of the world. h consumes y_{hs} in state s and is interested only in its own welfare

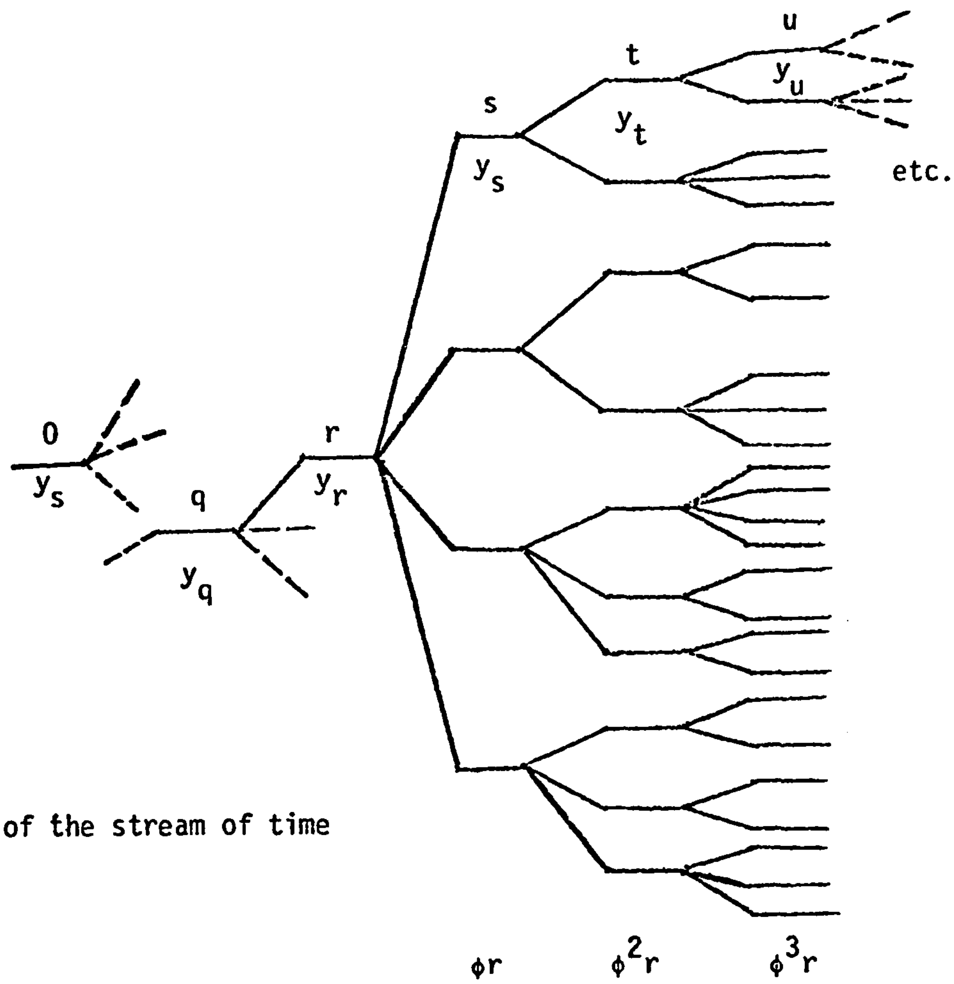
$$(1.5) \qquad \alpha^h(x_h) \quad ; \quad x_h = (y_{hs})_{s\in S} = h\text{'s consumption vector} \quad ,$$

while society as a whole is only interested in the welfare of its members as judged by themselves and wishes them all well. It is accordingly guided by a $\uparrow$ Social Welfare Function

$$(1.6) \qquad W = a[\alpha^h(x_h)|h \in H] = f(y_{hs}|h \in H, s \in S) \quad , \text{ say} \quad .$$

Figure 3: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r,\quad y_r,\quad z_r \rightarrow$

$\leftarrow x_s,\quad y_s,\quad z_s \rightarrow$

$\leftarrow x_t,\quad y_t,\quad z_t \rightarrow$

$\leftarrow x_u,\quad y_u,\quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t \phi^2 r$, $u \phi^3 r$, $u \phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

Society also satisfies Samuelson's weak independence axiom in the form

$$(1.7) \qquad f(x_{hs} | h \in H, s \in S) = b(\beta^{s}(z_s) | s \in S) \quad , \quad b(\cdot) \uparrow\uparrow ,$$

where all functions are assumed continuous. Both continuity and the very strict monotonicity needed can be derived from the Debreu [1960] approach followed by Gorman [1968], for instance.

In this case each $X_h = \prod_{s \in S} Y_{hs}$, and each $Z_s = \prod_{h \in H} Y_{hs}$ is separable. Each pair X_h, Z_s actually overlap, in Y_{hs}, so that it is not surprising that (1.6) and (1.7), common assumptions in welfare economics and the theory of choice under uncertainty respectively, jointly imply that we can write

$$(1.8) \qquad W = \sum_{\substack{h \in H \\ s \in S}} f^{hs}(y_{hs}) = \sum_{h \in H} \alpha^{h}(x_h) = \sum_{s \in S} \beta^{s}(z_s) \quad ,$$

where

$$(1.9) \qquad \alpha^{h}(x_h) = \sum_{s \in S} f^{hs}(y_{hs}) \quad ,$$

$$(1.10) \qquad \beta^{s}(z_s) = \sum_{h \in H} f^{hs}(y_{hs}) \quad ,$$

are the overall welfare, or utility of household h, and of society as a whole in state s: Bentham and Bernouilli at a stroke - and with the same normalization holding for each.

Why not apply a similar analysis to time and uncertainty in the case of an organization as to households and uncertainty here? Once one assumes Samuelson's weak axiom for uncertainty in the form (1.6), one has a considerable choice of assumptions about time to replace (1.5). One can assume the exact analogue, that each period is separable from society's point of view, or that the future from each period forward is, to take two simple possibilities, either of which yields (1.7)-(1.9) with h now standing for time.

Unhappily, time and uncertainty are not orthogonal in this pleasant manner. Instead, the world unfolds before us as we meander through time. ~~Contrast Figure 2, facing page 2, with Figure 3.~~ Here I go to the other extreme and assume that the world unfolds itself in a predetermined manner. We do not know who will be elected President in November, but we do know that someone will be.[2/] This is the model I will explore.

Before proceeding further with this model may I discuss the argument whereby Samuelson, in particular, derived additivity over states. He replaced his weak axiom (1.7) by a strong axiom which states that <u>each set of states</u> of the world is separable, not just the individual states themselves. The idea is, I suppose that once one knows that one will be in some one of a particular set of states it would be absurd to allow one's calculations to be affected by what one might have done, if in some quite different state. This yields innumerable separable spaces, which overlap most satisfactorily, and yield the required additively separable criterion

(1.11) $$\sum_{s \in S} f^s(y_s) \quad ,$$

as is proven by Debreu [1960].

The bother about that argument, as I see it, is that it assumes from the outset that we are all very bright, and especially so at computation. A similar approach involving time as well would be even more demanding.

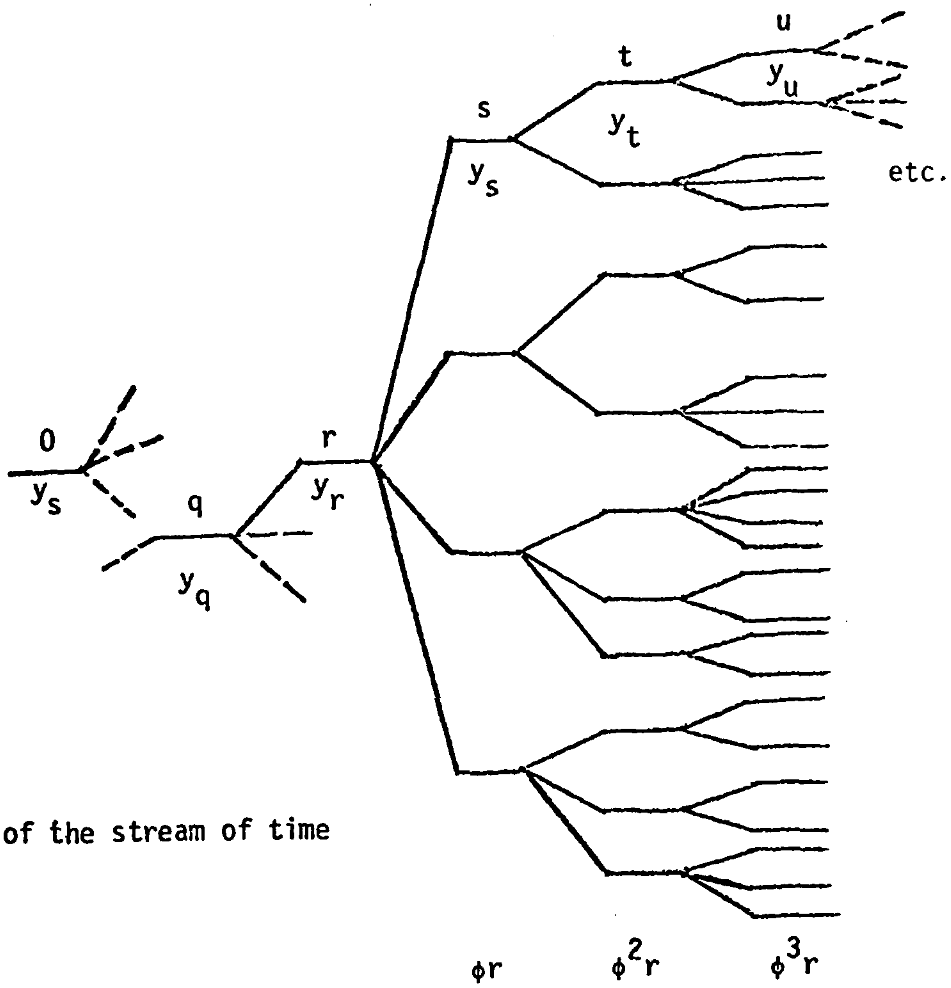
The approach I take in this paper is based on the contrary idea, that we are pretty limited beings, only able to hold a few ideas in our minds at a time, and accordingly unwilling to attempt difficult calculations until we really need to, and that organizations are collectively quite as limited as their members. It is interesting that, in Britain at least, it is the politicians, representing the biggest organization, who take pride in not crossing bridges until they come to them!

To be precise, I assume that we look ahead two periods in detail, summarizing the impact of our choices on more distant prospects in a single figure - "capital," if you like. Should you believe organizations to be more far-sighted, use a k-period rolling horizon, with $k > 2$. The results will be much the same, though the end effects will last longer.

The other notion underlying this discussion is that, dull though we are, we manage quite well. In particular, we do not spend much time bewailing missed opportunities. If so, this may be another piece of stupidity - we do not realize we missed them [3/] - or a piece of benign sociobiological economy which saves us from wasting our limited mental resources on lost causes. In this paper I explore another alternative: that our tastes are adapted to our abilities, so that we can really satisfy them quite well - to wit perfectly - by these limited procedures. Man grew as a social animal and his ability to organize developed as he did, so that I will use the same argument about organizations. I doubt myself

<u>Figure 3</u>: Illustrating A1 and A2

Time:	$0 \ldots$	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	$\ldots$	T
Parity:	$\theta(0)\ldots$	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	$\ldots$	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t \phi^2 r$, $u \phi^3 r$, $u \phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

whether we do look ahead as clear-headedly in organizations as we do as individuals - for one thing individual members may become politically committed to particular policies, and be unwilling to recognize unfavorable developments. One reason why British politicians are unwilling to cross bridges until they come to them is probably that they realize the danger of precommitting themselves in public. Nevertheless, I will assume that we are.

Of course, this is all rather dishonest. The motivating idea of my analysis is the need to economize in computing. Yet computing costs are nowhere to be seen in it. I claim that my assumptions are made more realistic than Samuelson's; since they imply and are implied by virtually the same results, implicitly I assume pretty well what he does.[4/]

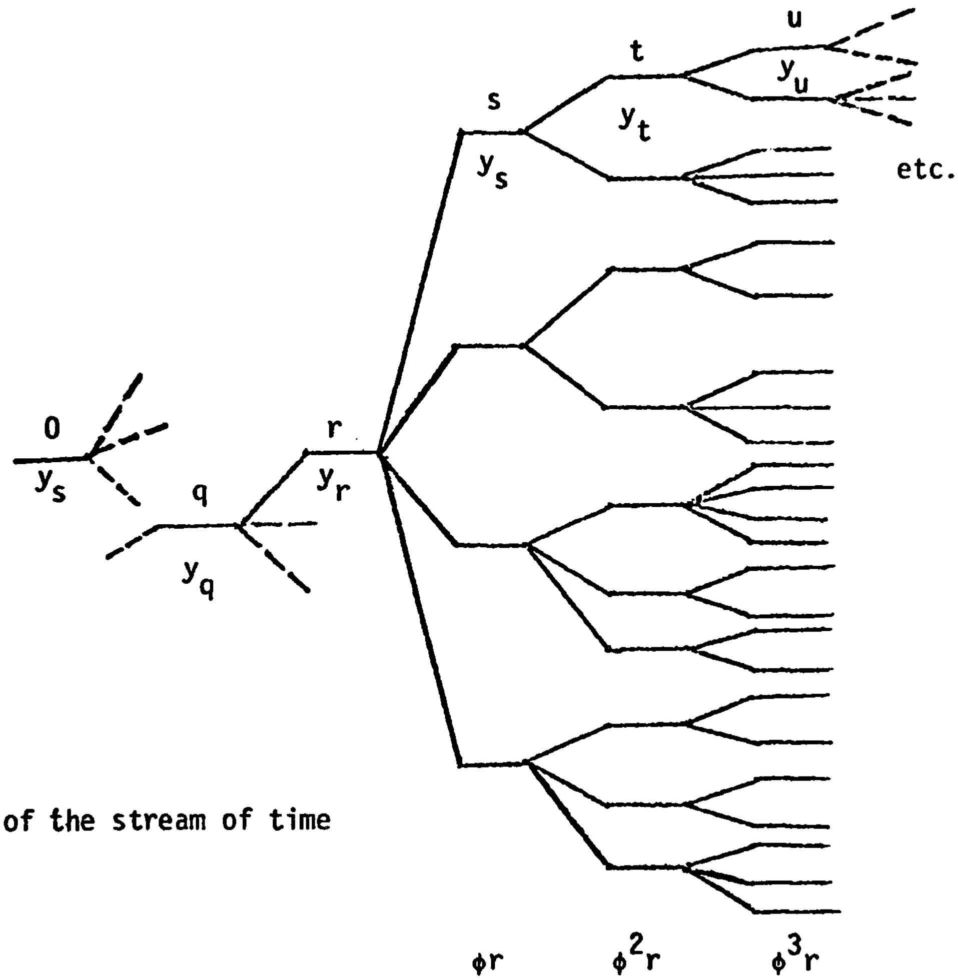
Perhaps, I should repeat that, despite the absence of probabilities, this paper is about uncertain futures. One can introduce probabilities into treatments of this type by, e.g., defining them as "the proportion of equally likely alternatives" which are favourable to an event. Frequently, they seem to me to obscure, rather than enlighten.

2. More about separability

Consider the world as seen from an arbitrary initial point 0, from which paths into the future radiate, as they do from r in Figure 3. I will deal with the set R of such nodes r, including 0 itself, and assume a continuous preference ordering $\succsim$ defined on the connected, topologically separable,[5/] product space $Y_R = \prod_{r \in R} Y_r$, where $y_r \in Y_r$ may be thought of as the output vector of the organization at node r.

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t \phi^2 r$, $u \phi^3 r$, $u \phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

I will assume that each sector Y_r is strictly essential: what happens on it matters, whatever is happening off it.

I will call all this Assumption A1. It implies that the preference ordering may be represented by a continuous criterion function

$$(2.1) \qquad f^0(y_R) \quad .$$

Consider $Y_S = \prod_{r \in S} Y_r$, $S \subseteq R$. I hold production $y_{\sim S} = \bar{y}_{\sim S}$ off Y_S constant and define the conditional preference ordering $(\succsim_S | \bar{y}_{\sim S})$ on Y_S in the obvious manner. It can be represented by the continuous criterion

$$(2.2) \qquad f^S(y_S | \bar{y}_{\sim S}) = f^0(y_S, \bar{y}_{\sim S}) \quad .$$

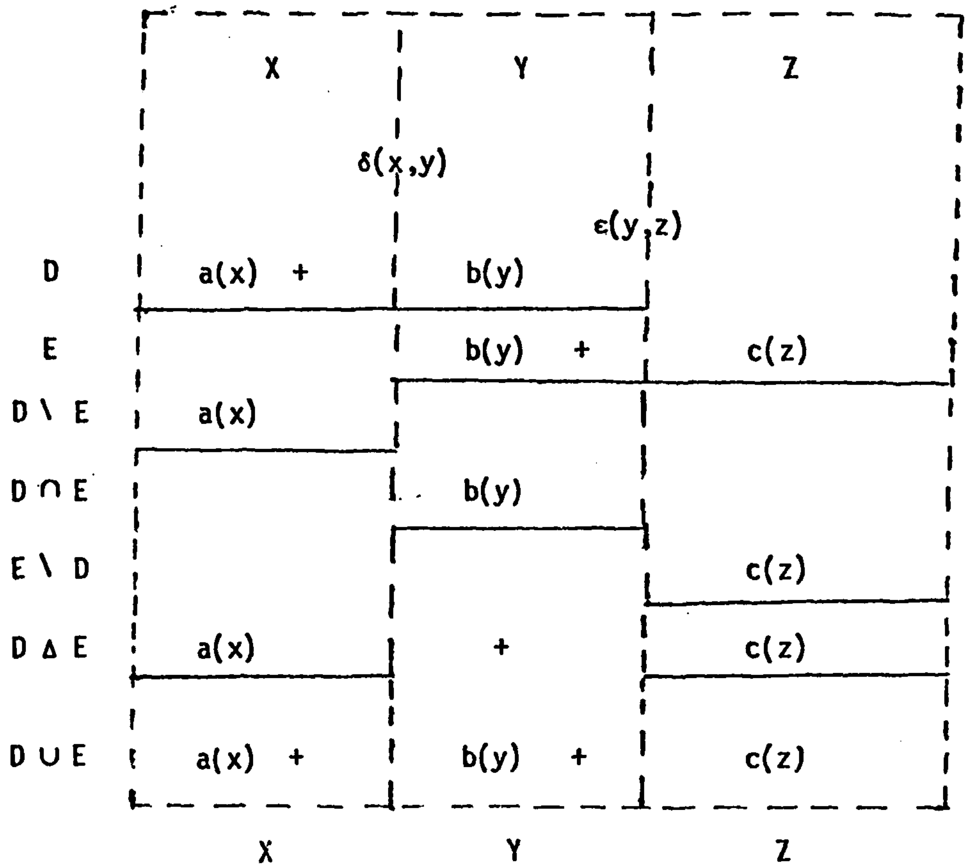
Iff this ordering is the same for each $\bar{y}_{\sim S} \in Y_{\sim S}$, we say that Y_S is separable. In that case the condition $\bar{y}_{\sim S}$ does not matter. Accordingly, let us take an arbitrary vector in Y_R as basis, call it 0 by the simple procedure of measuring y_R as a deviation from it. Then

$$(2.3) \qquad f^0(y_R) = g^S(f^S(y_S), y_{\sim S}) \quad , \quad g^S(\cdot)\text{cns} \quad , \quad g^S(\cdot, y_{\sim S}) \uparrow \quad ,$$

$$(2.4) \qquad f^S(y_S) := f^0(y_S, 0_{\sim S}) \quad .$$

I will make copious use of this normalizing device in Section 3. Note the strict monotonicity of the higher order utility function, $f^0(\cdot)$, in the lower, $f^S(\cdot)$. This is always so, given A1, as is the continuity of $g^S(\cdot)$. I will use these facts continually in Section 3 as well.

<u>Figure 1</u>: Illustrating overlapping separable spaces $X \times Y$, $Y \times Z$, or sets D, E, with their subutilities $\delta(x,y)$, $\varepsilon(y,z)$ and the associated separable spaces, or sets, with their normalized subutilities.



Given all this, (1.3) $\Rightarrow$ (1.4) is the basic theorem in this field. It allows us to determine functions for which given collections of spaces, $Y_S = \prod_{r \in S} Y_r$, $S \subseteq R$, are separable.

For the present let us talk of the sets $S \subseteq R$ as being separable, rather than the corresponding spaces Y_S.

Look at Figure 1 opposite again.

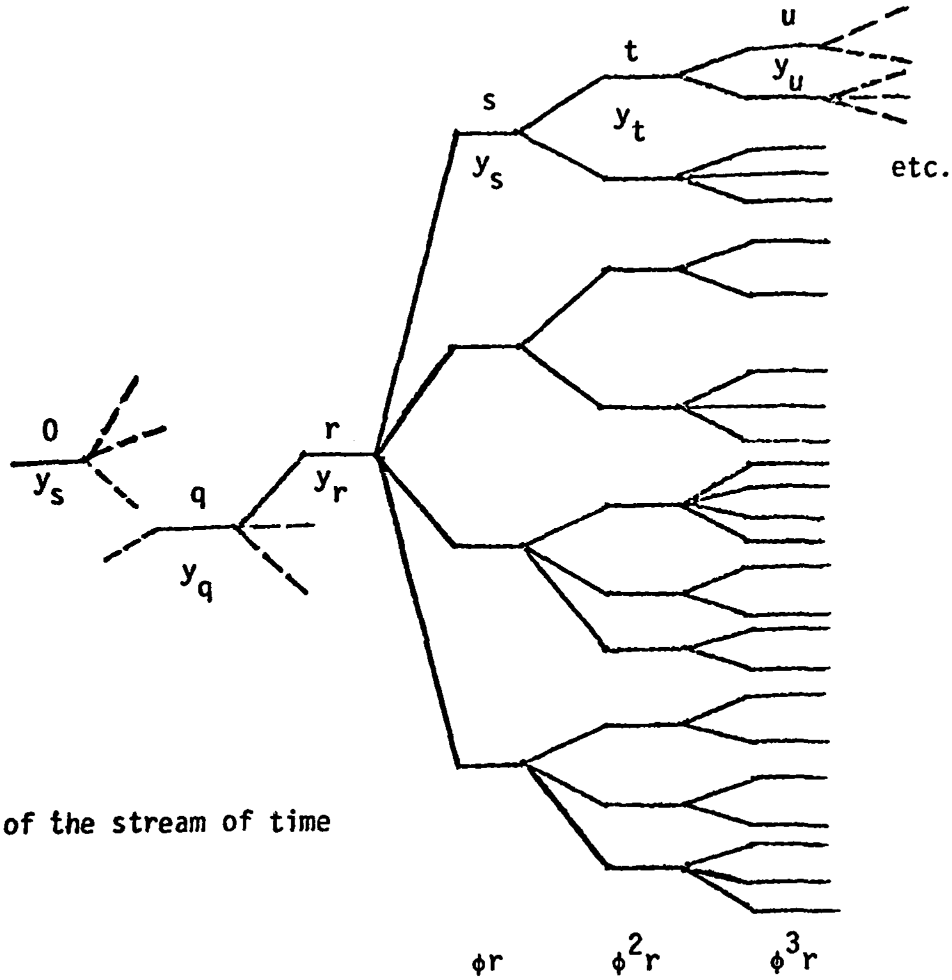
(1.3) $\Rightarrow$ (1.4) is equivalent to saying that, if D, E are separable and overlap, then $D \setminus E$, $D \cap E$, $E \setminus D$, $D \Delta E$, and $D \cup E$ are also separable. In (1.3) the spaces corresponding to D and E are $X \times Y$ and $Y \times Z$, and, to the other, X, Y, Z, $X \times Z$ and $X \times Y \times Z$ in turn, with subutilities $a(x) + b(y)$, $b(y) + c(z)$, $a(x)$, $b(y)$, $c(z)$, $a(x) + c(z)$, and $a(x) + b(y) + c(z)$ respectively. Call a collection $\mathcal{A}$ of subset $S \subseteq R$ <u>complete</u> if it contains R, the empty set $\emptyset$, and, with overlapping D, E, the list associated with them above, and the <u>completion</u> $\mathcal{A}(\mathcal{B})$ of any collection $\mathcal{B}$ of $S \subseteq R$, the intersection of all complete collections containing it. To know that the elements of $\mathcal{B}$ are separable is to know that those of $\mathcal{A}(\mathcal{B})$ are under A1.

As it happens, complete collections have a particularly simple ordering under set inclusion "$\supseteq$", and the structure of the utility function mirrors it. This is discussed at length in Gorman [1968], continuing Debreu [1960] and Leontief [1947]. All we need to know here is the following: take a continuous function $h(x_1, x_2, \ldots, x_n)$ defined on $X_1 \times X_2 \times, \ldots, \times X_n$. It can be put in the form

$$(2.5) \qquad \sum_{j=1}^{n} h^j(x_j) \quad ,$$

Figure 3: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

by a continuous $\ddagger$ transformation iff $V_i = \prod_{j \neq i} x_j$ is separable for each $i = 1,2,\ldots,n$. The corresponding subutilities are, of course, $\sum_{j \neq i} h^j(x_j)$, each i.

This is the result which I will bring to bear in Section 3. Before reading it, may I remind you of the tree, in Figure 3, which we will study there, and the notation defined below it.

In particular, at node r: x_r is past output, y_r current output, and z_r a vector of possible future outputs, along all the paths directed from r into the future.

Let me remind you that I will use normalizations such as (2.4) very frequently, and that A1 assures that all my functions will be continuous in all their arguments and $\ddagger$ in the "subutilities" among them, as in (2.3).

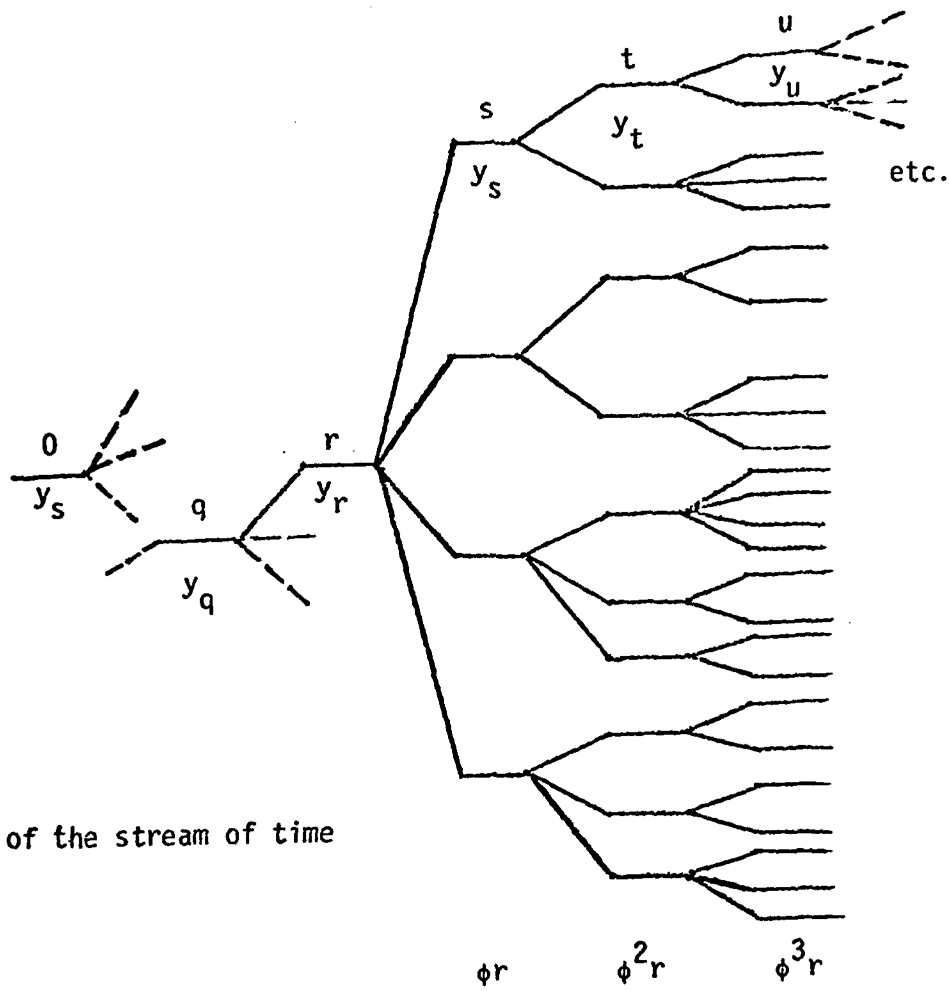
3. Main analysis

Look at Figure 3 again, opposite. It represents a particular type of unfolding future. At a given node r, one does not know what is about to happen, but does know the possibilities facing one; and, looking forward from r to some $v\psi r$, one knows what questions will be answered at v, should we ever reach it, though not what the answers will be.

Think of Figure 3 as representing the outward flow of blood from the heart 0, if you like. Major arteries divide into smaller and yet smaller ones, then into major capillaries, ... Think of the organization as a corpuscle carried along in the blood stream. Having reached r it does not know down which of the succeeding channels it will be carried, but it does know that it will be one of those leading directly to an $s\phi r$ -

Figure 3: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

that is to a direct successor of r - and that it will reach that s in the next period, if at all.

Now for some formal notation, spelled out under Figure 3. First successors:

(3.1) $t\phi^{\tau}r$ iff $t\phi^{\tau-1}s$, some $s\phi r$, $\tau = 2,3,\ldots$,

(3.2) $t\phi^{1}r$ iff $t\phi r$, $r\phi^{0}r$,

(3.3) $t\psi r$ iff $t\phi^{\tau}r$, some $\tau = (0,1,2,\ldots,T)$.

Next the period:

(3.4) $\theta(r) = \rho$,

in which r occurs if it ever does. If $t\phi^{\tau}r$,

(3.5) $\theta(t) = \rho + \tau$.

We choose

(3.6) $\theta(0) = 0$, so that $\theta(r) = \rho$ iff $r\phi^{\rho}0$.

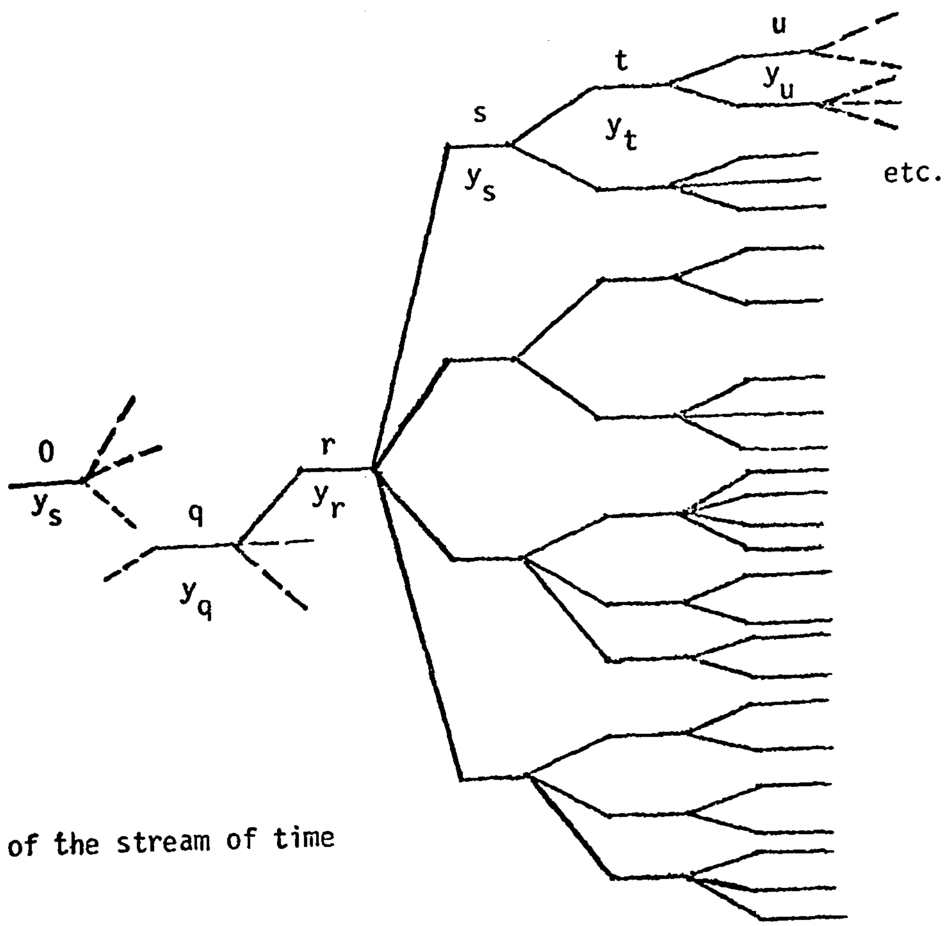
Next output:

(3.7) x_r , y_r , z_r ,

stand respectively for past, present, and potential future output vectors at r. Hence

Figure 3: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

ϕr $\quad \phi^2 r$ $\quad \phi^3 r$

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

$$x_s = (x_r, y_r) \quad , \quad \text{each} \quad s \phi r \quad , \tag{3.8}$$

$$z_r = (y_s, z_s)_{s \phi r} \quad . \tag{3.9}$$

May I now remind you that the aim of this analysis is to derive the common form

$$f^r(x_r, y_r, z_r) = \sum_{v \psi r} \alpha^v(y_v) \quad , \tag{3.10}$$

of the criterion function at r,

$$\text{each} \quad r \in R = \psi 0 = \{r | r \psi 0\} \quad ? \tag{3.11}$$

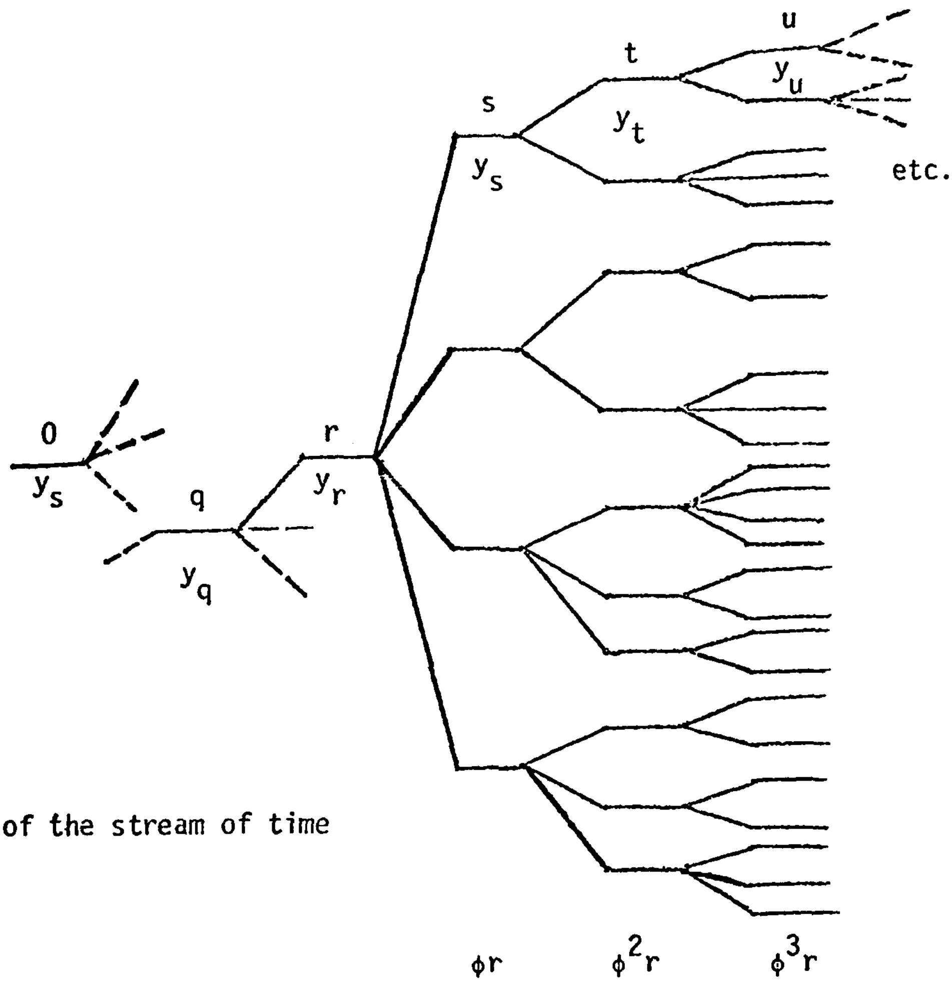
Remarks

i) Note the assumption, implicit in the form $f^r(x_r, y_r, z_r)$, that, given that it has reached r, it does not matter what the organization might have chosen, had the world evolved differently. The form $\sum_{v \psi r} \alpha^v(y_v)$, on the right-hand side, implies that the actual behaviour in the past, x_r, does not matter either. While this may be less repugnant for organizations than individuals, one would not like to assume it from the outset. Hence, the x_r in $f^r(x_r, y_r, z_r)$.

ii) Note, too, that we do not have columns, representing states of the world, overlapping rows, representing time periods, in the way which yielded additive separability in a related problem in Section 2. If overlaps are needed for additive separability, they will have to be introduced explicitly, as in the assumption, A3 below, about the existence of a two-period rolling horizon. This assumption does not give the necessary

Figure 3: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t \phi^2 r$, $u \phi^3 r$, $u \phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

overlaps at the very beginning and very end of the organization's life - so that we never quite get (3.10). Very similar results, with longer-lived end-effects, can be found with a k-period rolling horizon when $k > 2$.

If you think of T as at all large, and the distant future as not being of critical importance, these end effects scarcely matter. In any case, I will assume

(3.12) $$T > 3$$

as a minimum. Include it in A1. In addition, to A1, and the rolling horizon A3 just mentioned, I will need

A2 <u>Assumption 2: Very weak independence assumption</u>

(3.13) $$f^r(x_r, y_r, z_r) = g^r(x_r, y_r;\ f^s(x_s, y_s, z_s) \mid s \not= r)$$

(3.14) $$= g^r(x_s, f^s(x_s, y_s, z_s) \mid s \not= r) \quad ,$$

where

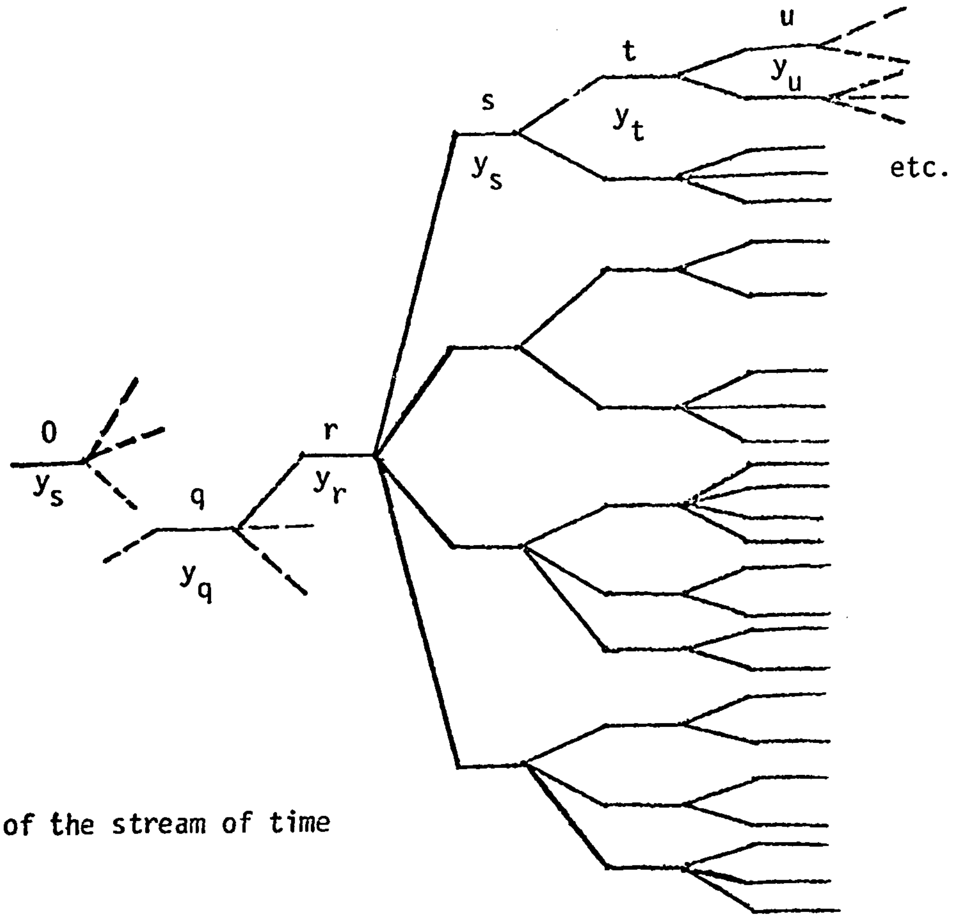
(3.15) $$g^r(x_r, y_r, \cdot) \text{ is } \uparrow .$$

<u>Remarks</u>

i) $(x_r, y_r) = x_s$ is repeated in (3.14) for each $s \not= r$, so that $g^r(\cdot)$ is not strictly the same there as in (3.13), nor even uniquely specified. The values are the same, and the notation is convenient.

Figure 3: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\phi r \qquad \phi^2 r \qquad \phi^3 r$

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

ii) I use '$\cdot | s \phi r$' instead of '$(\cdot)_{s \phi r}$' where it seems simpler. Its domain extends back to the previous ';' if there is one. In either case '$\cdot$' is taken for each $s \phi r$. I have dropped the quantifier to preserve the analogy with the normal notation.

iii) Why: very weak independence axiom?

Samuelson's weak axiom may be written in the form

$$(3.16) \qquad f(z) = g(f^{s}(y_s) | s \in S) \quad , \text{ say } \quad g(\cdot)^{\ddagger} \quad ,$$

all functions being continuous. It ignores both past and present output, x_r, y_r at the point of consideration r. (3.13) seems weak since $(x_r, y_r) = x_s$, appears through each of the future criteria $f^{s}(x_s, y_s, z_s), s \phi r$, and also as an additional argument in $g^{r}(\cdot)$.

Next to show that A2 implies consistency over time <u>in an obvious sense</u>.[6/]

Remember that I warned you in Section 2 that I would make copious use of normalizations based on the arbitrary reference vector.

$$(3.17) \qquad \bar{y} = (\bar{y}_r)_{r \in R} = 0 \quad ,$$

without loss of generality. In particular, I will set

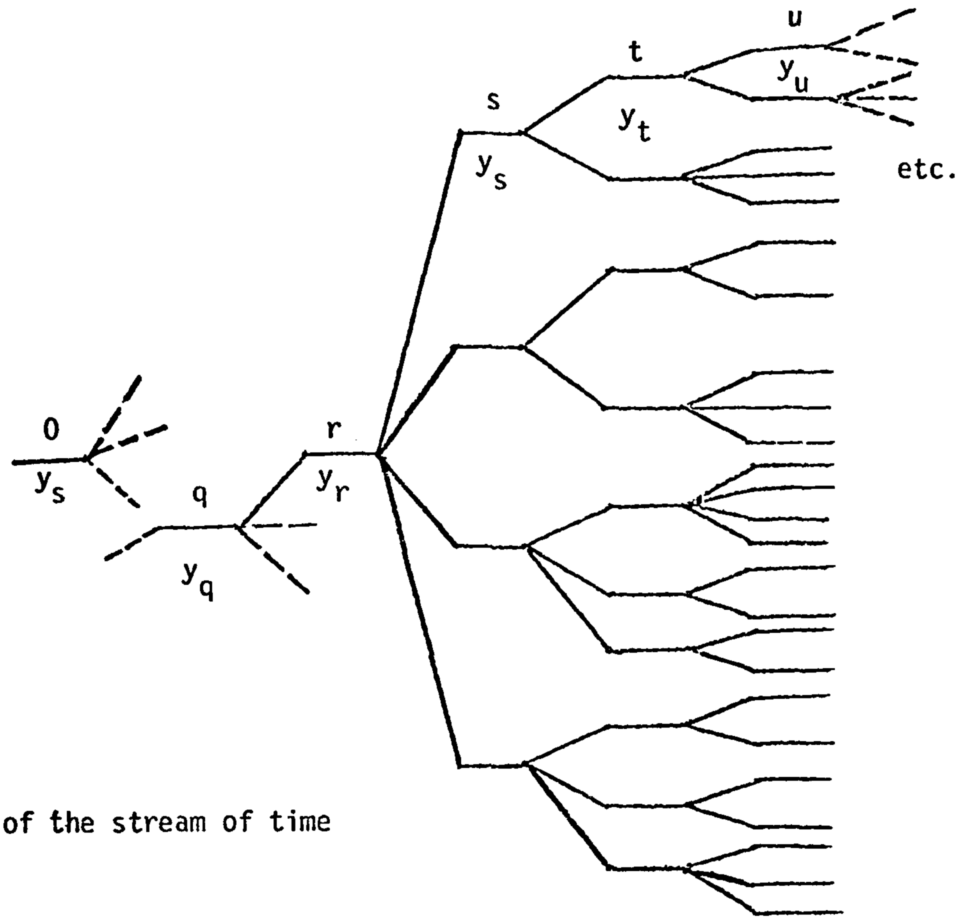
$$(3.18) \qquad f^{r}(0) = 0 \quad , \quad \forall r \in R \quad ,$$

again without loss of generality.

Note now that

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

ϕr $\quad \phi^2 r$ $\quad \phi^3 r$

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

$$(3.19) \qquad f^r(x_r,y_r,z_r) = f^r(x_s,y_s,z_s;\ (y_\sigma,z_\sigma)|\sigma\phi r,\sigma \neq s) \quad , \text{ each } s\phi r \quad ,$$

and define

$$(3.20) \qquad \tilde{f}^s(x_s,y_s,z_s) := f^r(x_s,y_s,z_s;\ (0_\sigma)|\sigma\phi r.\sigma \neq s) \quad ,$$

$$(3.21) \qquad =: f^r(x_s,y_s,z_s;\ X_s \times Y_s \times Z_s) \quad ,$$

$$(3.22) \qquad = g^r(x_s, f^s(x_s,y_s,z_s);\ f^\sigma(x_s,0,0)|\sigma\phi r,\sigma \neq s) \quad .$$

Remarks

i) Since g^r is $\uparrow$ in f^s given x_s in (3.22), and x_s is already past history at s, $\tilde{f}^s(\cdot)$ is just as good a criterion at s as is $f^s(\cdot)$.

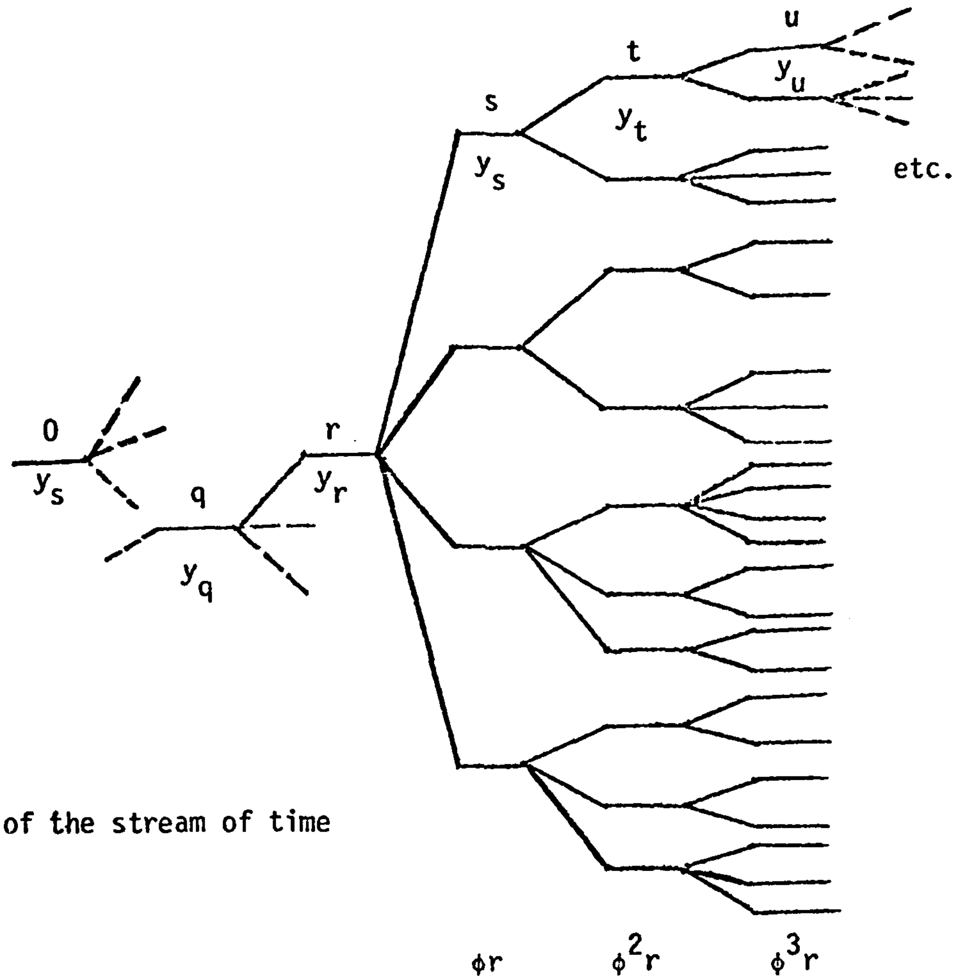
ii) Clearly (3.20)-(3.22) are calculated by ignoring irrelevant information--or, more precisely, setting it at its reference level 0. This is done explicitly in (3.20), and implicitly in (3.21) which lists the relevant space $X_s \times Y_s \times Z_s$, where information should not be written over in this manner. It is sometimes convenient to write

$$(3.23) \qquad f^r(x_s,y_s,z_s;\ X_s \times Y_s \times Z_s) =: \bar{f}^r(x_s,y_s,z_s) \quad ,$$

where the ellipsis is understood. This exploits the symmetry about ';' in $f^r(';')$ while the bar in $\bar{f}^r(x_s,y_s,z_s)$ reminds one that it is not merely a matter of replacing (x_r,y_r,z_r) in $f^r(x_r,y_r,z_r)$ by (x_s,y_s,z_s), which would imply an infinite time horizon, a great deal of extra structure, and essentially constant 'tastes' in the organization.

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r,\quad y_r,\quad z_r \rightarrow$

$\leftarrow x_s,\quad y_s,\quad z_s \rightarrow$

$\leftarrow x_t,\quad y_t,\quad z_t \rightarrow$

$\leftarrow x_u,\quad y_u,\quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

iv) One must imagine the normalization carried out with $r = 0$ first, then with the r's with $\theta(r) = 1,2,\ldots,T-1$ in turn.

v) Having done so we find

$$(3.24)\qquad f^r(x_r,y_r,z_r) = g^r(x_s, f^s(x_s,y_s,z_s)\,|\,s\phi r)\ ,$$

$$(3.25)\qquad = h^r(x_t, f^t(x_t,y_t,z_t)\,|\,t\phi^2 r)\ ,$$

$$(3.26)\qquad = k^r(x_u, f^u(x_u,y_u,z_u)\,|\,u\phi^3 r) = \ldots\ ,$$

say, from (3.14), and

$$(3.27)\qquad f^s(x_s,y_s,z_s) = f^r(x_s,y_s,z_s;\ X_s \times Y_s \times Z_s) = \bar{f}^r(x_s,y_s,z_s)\ ,\quad \forall s\phi r\ ,$$

$$(3.28)\qquad = f^q(x_s,y_s,z_s;\ X_s \times Y_s \times Z_s) = \bar{f}^q(x_s,y_s,z_s)\ ,\quad \forall s\phi^2 q\ ,$$

$$(3.29)\qquad = f^0(x_s,y_s,z_s;\ X_s \times Y_s \times Z_s) = \bar{f}^0(x_s,y_s,z_s)\ ,\quad \forall s \in R\ ,$$

from (3.20) and (3.23).

Remarks

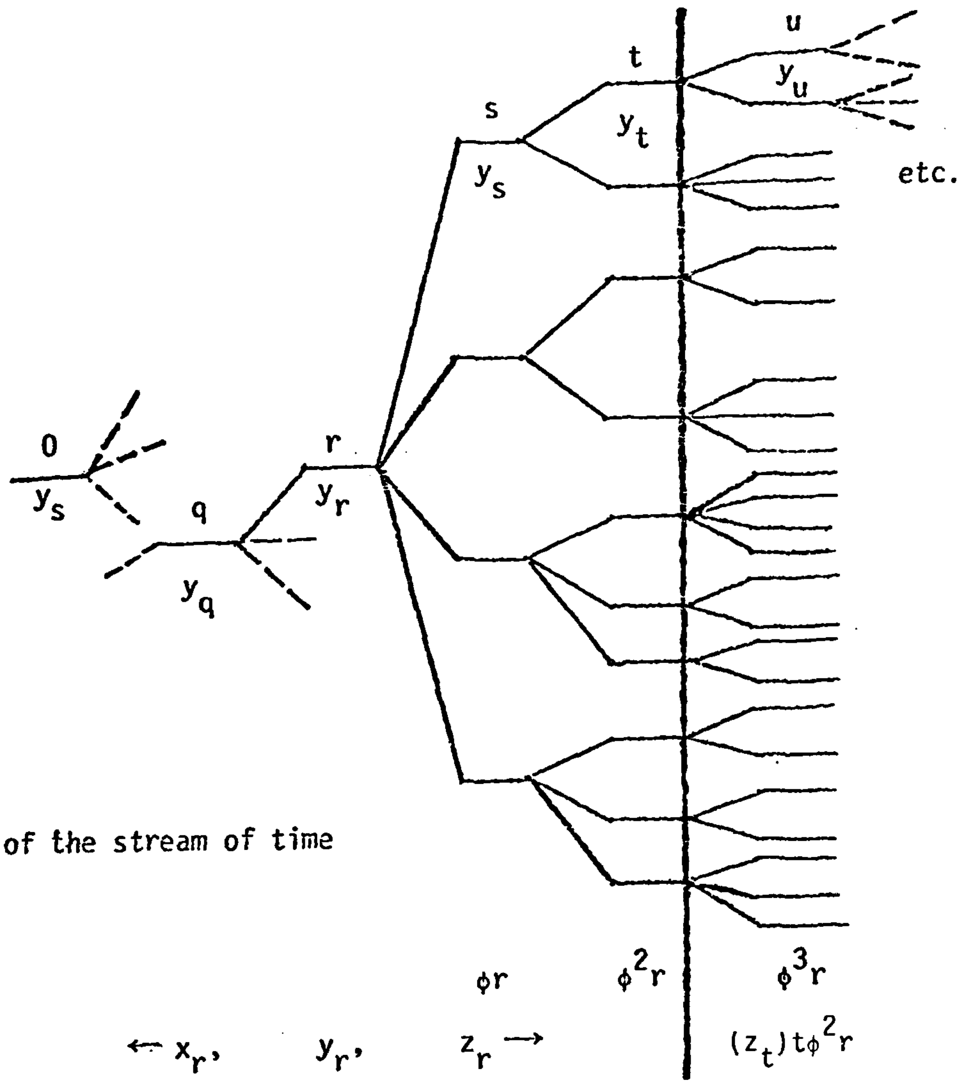
Because of A1 we were able to choose $f^0(\cdot)$ to be continuous. So therefore is each $f^r(\cdot)$. Because Y_r is strictly essential, each $r \in R$:

(3.30) The functions in (3.24)-(3.29) are continuous; and $\uparrow$ in their functional arguments.

Figure 4: Illustrating A 3

Time: 0 ... $\rho - 1$ ρ $\rho + 1$ $\rho + 2$ $\rho + 3$ ... T

Parity: $\theta(0)$.. $\theta(q)$ $\theta(r)$ $\theta(s)$ $\theta(t)$ $\theta(u)$ ... $\theta(w)$



$\leftarrow x_r$, y_r, $z_r \rightarrow$

$\leftarrow x_s$, y_s, $z_s \rightarrow$

$\leftarrow x_t$, y_t, $z_t \rightarrow$

$\leftarrow x_u$, y_u, $z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

I now turn to my main assumption

A3 Assumption 3: Two period rolling horizon

$$f^r(x_r,y_r,z_r) = c^r((x_t,y_t)_{t\phi^2 r}, \gamma^r(z_t|t\phi^2 r)) \quad , \tag{3.31}$$

which states that the organization looks into the next two periods in detail, but summarizes what may happen thereafter in the single number γ^r.

Since each period is strictly essential, A1 implies that we can normalize to get

$$\gamma^r(z_t|t\phi^2 r) = f^r((z_t)_{t\phi^2 r}; \prod_{t\phi^2 r} Z_t) \quad , \tag{3.32}$$

and

$$c^r(\cdot) \text{ continuous} \quad , \quad c^r((x_t,y_t)_{t\phi^2 r},\cdot) \uparrow \quad . \tag{3.33}$$

So far I have paid no attention to possible end effects. From now on I will have to take them seriously. Let

$$\theta(R) = \{\theta(r)|r \in R\} = (0,1,2,\ldots,T) \quad , \tag{3.34}$$

as in Figure 4. When

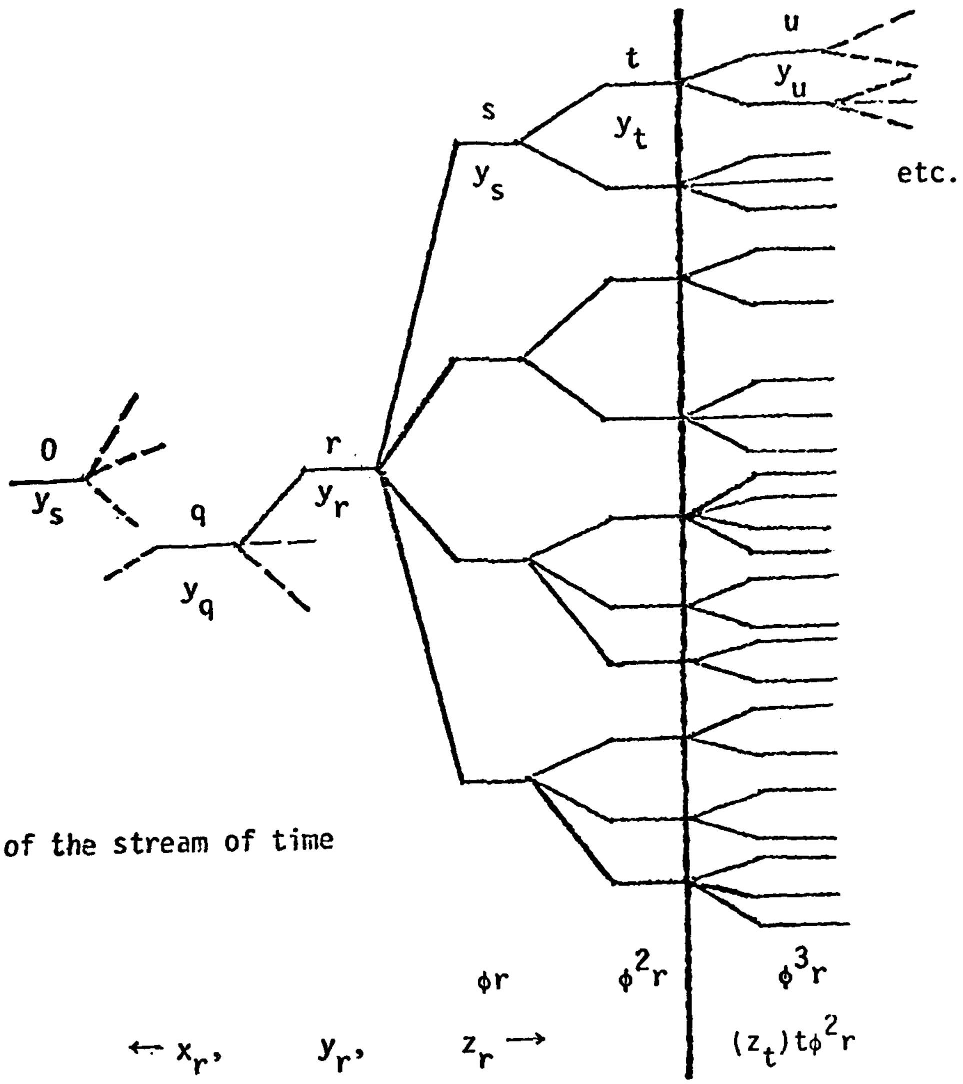
$$\theta(r) \geq T - 2 \quad , \tag{3.35}$$

$$\prod_{t\phi^2 r} Z_t \text{ is null,} \tag{3.36}$$

so that arguments based on A3 are empty. Hence, the need for care.

<u>Figure 4</u>: Illustrating A 3

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

A1-3 imply that we may take the future as separable at all nodes u such that

(3.37) $\theta(u) \geq 3$,

and that this implies that we can write the criterion at u as

(3.38) $f^u(y_u, z_u)$ when $\theta(u) \geq 2$,

in an appropriate normalization.

Since $T \geq 3$, there is an r such that

(3.39) $u\phi^3 r$,

and

(3.40) $f^u(x_u, y_u, z_u) = f^r(x_u, y_u, z_u; X_u \times Y_u \times Z_u)$, by (3.27) ,

(3.41) $\qquad = c^r(x_u, \gamma^r(y_u, z_u; Y_u \times Z_u); X_u)$,

where $c^r(\cdot)$ is continuous and $c^r(x_u, \cdot) \uparrow$. Hence,

(3.42) $\tilde{f}^u(y_u, z_u) := \gamma^r(y_u, z_u; Y_u \times Z_u)$, $u\phi^3 r$,

is a perfectly good criterion at u, when $\theta(u) \geq 3$.

It is easily verified that the $\tilde{f}^r(\cdot)$ satisfy (3.13), (3.27) and (3.31) when $\theta(r) \geq 3$, so that we can drop the tildes, and get

FIGURE 5

sets	1	3	5	7
	2	4	6	8

spaces	Y_1	Y_2	Y_3	Y_4
	Z_1	Z_2	Z_3	Z_4

Figure 5: Illustrating (2.5) & (3.44-5)

Given that $W_t = X_t \times Y_t$ is separable, each t, i.e. that each of the sets $A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $A_3 = \{5,6\}$, $A_4 = \{7,8\}$ is separable,

and that $Z = \prod_t Z_t$ is separable, i.e. that each of the sets $B = (2,4,6,8)$ in $h(y_t, Z_t | t = 1,2,3,4)$,

I will show that the complementary spaces

$$R_1 = \prod_{t-2} Y_t \times \prod_{t=1} Z_t; \; S_1 = \prod_{t=1} Y_t \times \prod_{t=2} Z_t$$

of Y_1 and Z_1 respectively are separable. Since the ordering (1,2,3,4) is immaterial this will show that the complement of each cell above is separable and hence $h = \sum \alpha^t(y_t) + \sum \beta^t(z_t)$, say, in an appropriate normalization.

Proof: in terms of sets $R_1 \sim ((B \cup A_2) \cup A_3) \cup A_4$

$$S_1 \sim (((B \cup A_1) \cup A_2) \cup A_3) \cup A_4$$

$$(3.25)^* \quad f^r(x_r,y_r,z_r) = h^r(x_t,f^t(y_t,z_t)|t\phi^2 r) \quad ,$$

$$(3.31)^* \quad = c^r((x_t,y_t)_{t\phi^2 r},\gamma^r(z_t|t\phi^2 r)) \quad ,$$

in particular, when

$$(3.43) \quad 1 \leq \theta(r) \leq T - 3 \quad ,$$

the functions being continuous; and $\updownarrow$ in their functional arguments. Hence,

$$(3.44) \quad \prod_{t\phi^2 r} Z_t \quad ; \text{ and } \quad Y_t \times Z_t \quad , \text{ each } \quad t\phi^2 r \quad ,$$

are separable in $f^r(\cdot)$, given (3.43), so that

$$(3.45) \quad f^r(x_r,y_r,z_r) = j^r((x_t)_{t\phi^2 r}, \sum_{t\phi^2 r} (\alpha^t(y_t) + \beta^t(z_t))) \quad , \text{ say} \quad ,$$

where

$$(3.46) \quad j^r(\cdot) \text{ continuous} \quad , \quad j^r((x_t)_{t\phi^2 r},\cdot) \updownarrow \quad ,$$

$$(3.47) \quad 1 \leq \theta(r) \leq T - 3 \quad ,$$

by the argument sketched in and below Figure 5 opposite.

Clearly we can normalize to get

$$(3.48) \quad \alpha^t(0) = \beta^t(0) = 0 \quad ,$$

FIGURE 5

sets	1	3	5	7
	2	4	6	8

spaces	Y_1	Y_2	Y_3	Y_4
	Z_1	Z_2	Z_3	Z_4

Figure 5: Illustrating (2.5) & (3.44-5)

Given that $W_t = X_t \times Y_t$ is separable, each t, i.e. that each of the sets $A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $A_3 = \{5,6\}$, $A_4 = \{7,8\}$ is separable,

and that $Z = \prod_t Z_t$ is separable, i.e. that each of the sets $B = (2,4,6,8)$ in $h(y_t, Z_t | t = 1,2,3,4)$,

I will show that the complementary spaces

$$R_1 = \prod_{t-2} Y_t \times \prod_{t=1} Z_t; \; S_1 = \prod_{t=1} Y_t \times \prod_{t=2} Z_t$$

of Y_1 and Z_1 respectively are separable. Since the ordering $(1,2,3,4)$ is immaterial this will show that the complement of each cell above is separable and hence $h = \sum \alpha^t(y_t) + \sum \beta^t(z_t)$, say, in an appropriate normalization.

Proof: in terms of sets $R_1 \sim ((B \cup A_2) \cup A_3) \cup A_4$

$$S_1 \sim (((B \cup A_1) \cup A_2) \cup A_3) \cup A_4$$

and therefore

$$(3.49)\qquad f^t(y_t,z_t) = f^r(y_t,z_t;\ Y_t \times Z_t) = j^r(0;\ \alpha^t(y_t) + \beta^t(z_t)) \quad ,$$

$$=: k^r(\alpha^t(y_t) + \beta^t(z_t));\ t\phi^2 r \quad , \quad 3 \leq \theta(t) \leq T - 1 \quad ,$$

where $k^r(\cdot)$ is continuous and $\uparrow$. Hence,

$$(3.50)\qquad \tilde{f}^t(y_t,z_t) = \ell^r(f^t(y_t,z_t)) = \alpha^t(y_t) + \beta^t(z_t) \quad ,$$

is a perfectly good alternative criterion at t, where

$$(3.51)\qquad \ell^r(k^r(\varepsilon)) = \varepsilon \quad , \quad 1 \leq \theta(r) \leq T - 3 \quad ,$$

and

$$(3.52)\qquad 3 \leq \theta(t) \leq T - 1 \quad .$$

The next step towards (3.10) is to show that, in a trivial renormalization,

$$(3.53)\qquad \beta^s(z_s) = \sum_{t\phi s} (\alpha^t(y_t) + \beta^t(z_t)) \quad , \quad 3 \leq \theta(s) \leq T - 2 \quad .$$

Assume for the moment that (3.53) holds. Then

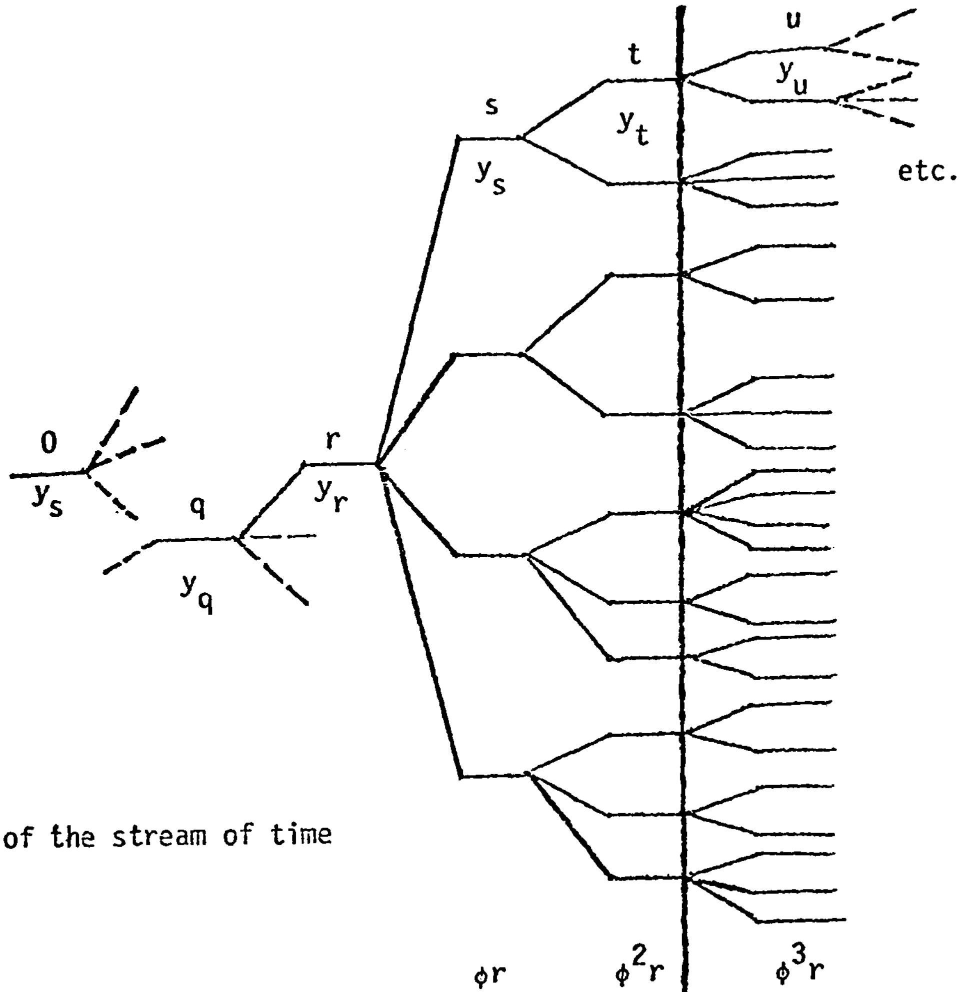
$$(3.54)\qquad \tilde{f}^s(y_s,z_s) = \alpha^s(y_s) + \beta^s(z_s) \quad ,$$

$$= \alpha^s(y_s) + \sum_{t\phi s} (\alpha^t(y_t) + \beta^t(z_t)) = \dots \quad ,$$

$$= \sum_{\substack{t\psi s \\ \theta(t)\leq T-1}} \alpha^t(y_t) + \sum_{\substack{t\phi s \\ \theta(t)=T-1}} \beta^t(z_t) \quad ,$$

Figure 4: Illustrating A 3

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow \qquad (z_t)t\phi^2 r$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

when

(3.55) $\theta(s) \geq 3$.

The proof of (3.53) is illustrated in Figure 4 opposite, where

(3.56) $t \phi s \phi r \phi q$,

to yield

(3.57) $k^q(\beta^s(z_s)) = f^s(0,z_s) = f^r(z_s;\ Z_s) = k^r(\sum_{t\phi s}(\alpha^t(y_t) + \beta^t(z_t)))$,

(3.58) $k^q(\sum_{s\phi r}\beta^s(z_s)) = f^r((z_s)_{s\phi r};\ \prod_{s\phi r} Z_s) = k^r(\sum_{t\phi^2 r}(\alpha^t(y_t) + \beta^t(z_t)))$,

where

(3.59) $1 \leq \theta(q) \leq T - 4$.

Define now

(3.60) $m^r(\varepsilon) = \ell^r(k^q(\varepsilon))$, $2 \leq \theta(r) \leq T - 3$,

to get

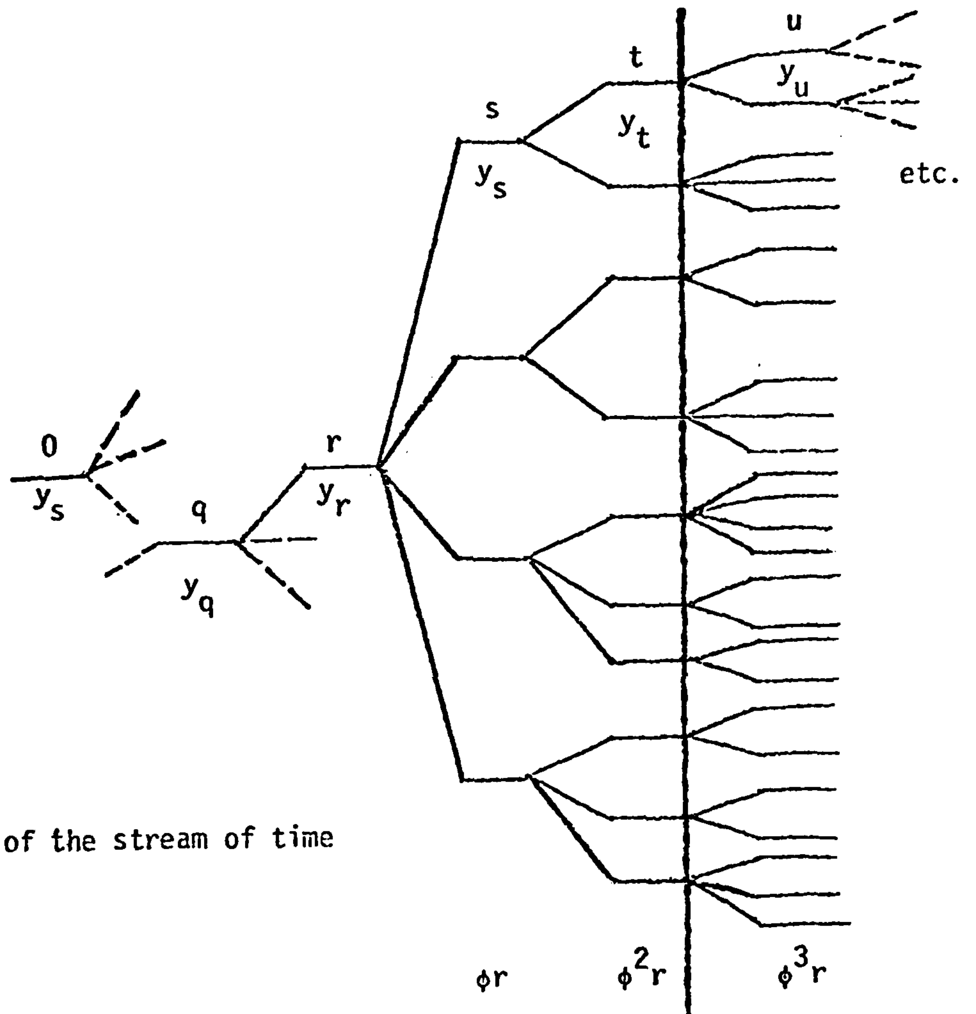
(3.61) $m^r(\sum_{s\phi r}\beta^s(z_s)) = \sum_{t\phi^2 r}(\alpha^t(y_t) + \beta^t(z_t))$,

(3.62) $= \sum_{s\phi r}\sum_{t\phi s}(\alpha^t(y_t) + \beta^t(z_t))$,

$= \sum_{s\phi r} m^r(\beta^s(z_s))$, $2 \leq \theta(r) \leq T - 3$.

Figure 4: Illustrating A 3

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow \qquad (z_t) t \phi^2 r$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t \phi^2 r$, $u \phi^3 r$, $u \phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

Since $m^r(\cdot)$ is cns and $\uparrow$ this implies

$$(3.63) \qquad m^r(\varepsilon) = a_r\varepsilon \quad , \quad \text{say,} \quad a_r \ \text{constant,} \quad 2 \leq \theta(r) \leq T - 3 \quad ,$$

so that

$$(3.64) \qquad a_r\beta^s(z_s) = \sum_{t\in s} (\alpha^t(y_t) + \beta^r(z_t)) \quad , \quad 3 \leq \theta(s) \leq T - 2 \quad ,$$

when we set $y_v = z_v = 0$, $v\phi^2 r$, $v{\sim}\phi s$, in (3.56).

Now introduce a set of positive constants b_t, one for each node with $3 \leq \theta(t) \leq T - 1$ such that

$$(3.65) \qquad a_r b_t = b_s \quad , \quad t\phi s\phi r \quad ,$$

as we clearly can, and define

$$(3.66) \qquad \tilde{\alpha}^t(y_t) = b_t\alpha^t(y_t) \quad , \quad \tilde{\beta}^t(z_t) = b_t\beta^t(z_t) \quad , \quad 3 \leq \theta(t) \leq T - 1 \quad ,$$

and hence

$$(3.67) \qquad \tilde{\beta}^r(z_r) = \sum_{s\phi r} (\tilde{\alpha}^s(y_s) + \tilde{\beta}^s(z_s)) \quad , \quad 3 \leq \theta(r) \leq T - 2 \quad .$$

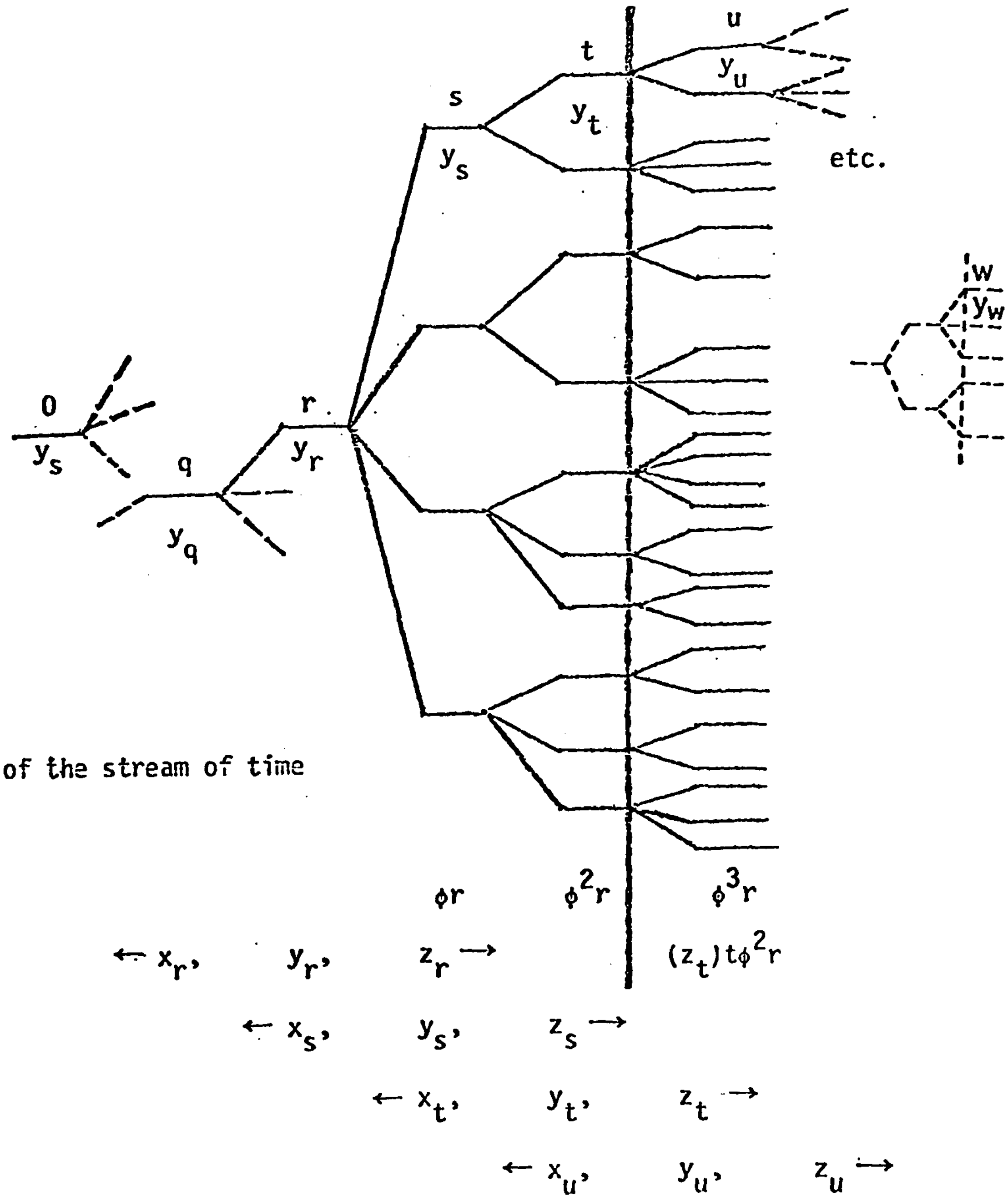
Drop the tildes to get (3.53) and (3.54).

<u>Corollary 1</u>

In this normalization, when $\theta(r) \geq 3$,

Figure 6: Illustrating (3.71)

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

$$f^r(y_r,z_r) = \alpha^r(y_r) + \beta^r(z_r) \quad , \tag{3.68}$$

$$= \alpha^r(y_r) + \sum_{s\phi r} \alpha^s(y_s) + \sum_{s\phi r} \beta^s(z_s) \quad , \tag{3.69}$$

$$= \ldots, = \sum_{\substack{v\phi r\\ \theta(v)\leq T-1}} \alpha^v(y_v) + \sum_{\substack{v\phi r\\ \theta(v)=T-1}} \beta^v(z_v) \quad , \tag{3.70}$$

$$= \sum_{\substack{v\psi r\\ \theta(v)\leq T-1}} \alpha^v(y_v) + \sum_{\substack{v\psi r\\ \theta(v)=T-1}} \delta^v(\varepsilon(y_w)|w\phi r) \quad . \tag{3.71}$$

(3.68)-(3.70) are trivial. To get (3.71) we note that the criterion function at a terminal point w, with $\theta(w) = T$, may be defined by

$$f^w(y_w) \;:=\; f^v(y_w;\, Y_w) = \beta^v(y_w;\, Y_w) \quad , \quad w\phi v \quad , \tag{3.72}$$

$$=: \; \varepsilon^w(y_w) \quad ,$$

implying (3.71) by (3.24) and (3.31) where

$$\delta^v(\varepsilon) = g^v(0,\varepsilon) \quad . \tag{3.73}$$

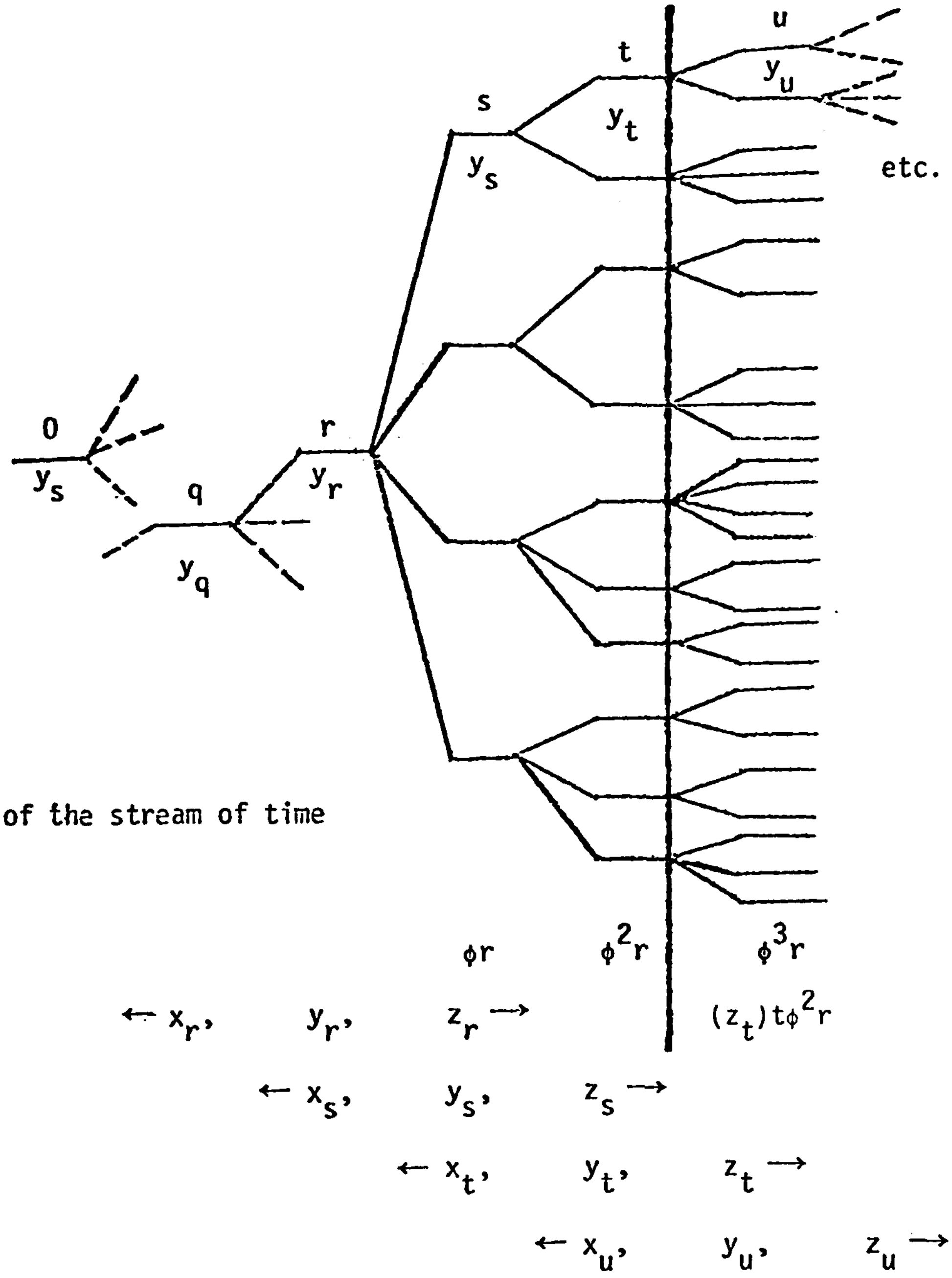
In a sense (3.73) advances the terminal point for (3.70) by one more hesitant step. One can push back the opening value, $\theta(r)$, from 3 to 2 to get

$$f^r(x_r,y_r,z_r) = \alpha^r(x_r,y_r) + \beta^r(z_r) \quad , \quad \text{say} \quad , \quad \theta(r) = 2 \quad , \tag{3.74}$$

where

<u>Figure 4</u>: Illustrating A3

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

$$\beta^r(z_r) = \sum_{s \phi r} (\alpha^s(y_s) + \beta^s(z_s)) \quad , \tag{3.75}$$

now for $\theta(r) = 2$, as well as $3 \leq \theta(r) \leq T - 2$ as in (3.67). I will not spell out the argument here. It is similar to that given above, and just as clumsy.

I suspect that such arguments are unnecessary, and that the results in this section can be derived directly from Theorem 1 in Gorman [1968]. I hope to have a shot at this later.7/

4. Additivity over streams

Tastes are learned from experience; organizations presumably "learn by doing" and may "improve" their criteria in consequence. Separability over time is doubtfully desirable.

What about additivity over streams, identified by their terminal nodes w, such that

$$\theta(w) = T \quad ? \tag{4.1}$$

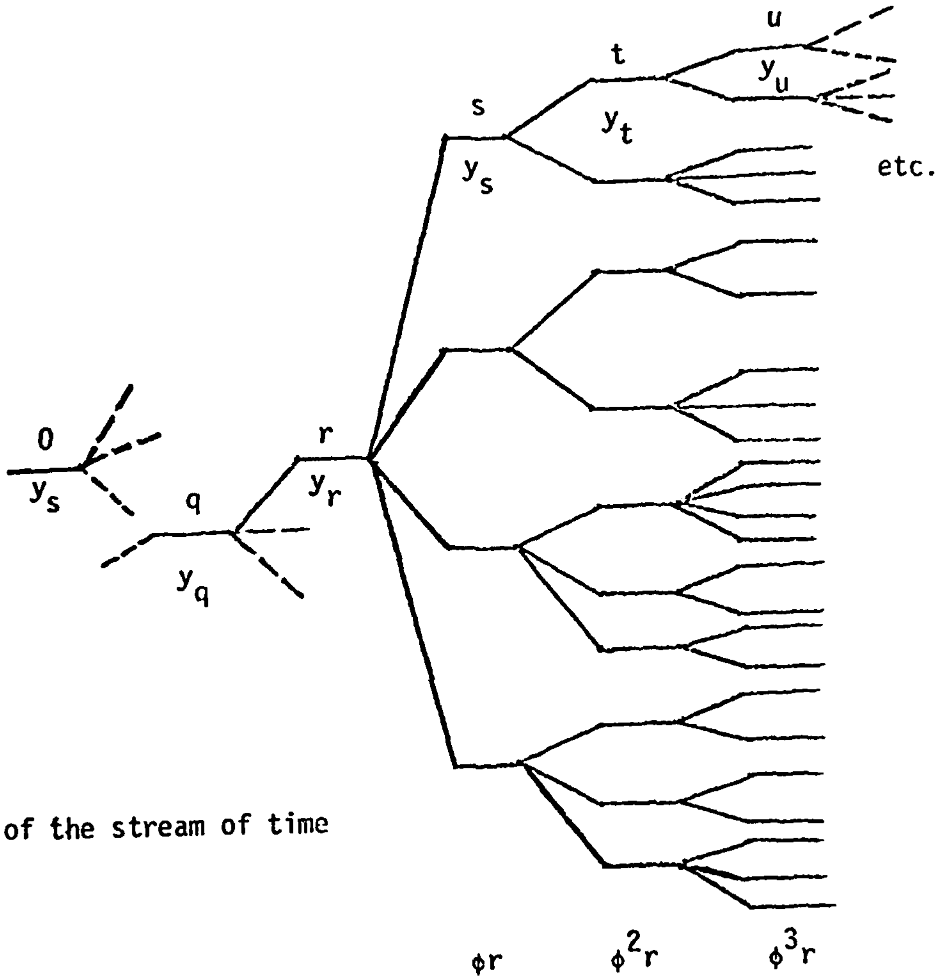
First, consider a timeless model. By tomorrow we will know in which of $2k \geq 4$ states we are in. For the usual reasons, the weak independence axiom

$$f(z) = g(f^1(y_1), f^2(y_2), \ldots, f^{2k}(y_{2k})), \; g(\cdot) \text{ continuous and } \uparrow \quad , \tag{4.2}$$

is acceptable. Now suppose that $1, 2, \ldots, 2k$ is a natural order - say by the rate of inflation, or the % of Democrats in the House. Assume that one will know ahead of time in which particular "broad set"

Figure 3: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

(4.3) $S_1 = \{1,2,\ldots,k\}, S_2 = \{2,3,\ldots,k+1\},\ldots,S_{k+1} = \{k+1,k+2,\ldots,2k\}$

the world will be in. These are the bridges, one of which the organization will soon be called upon to cross. Perhaps it is reasonable to assume each separable. Call this broad separability. Then

(4.4) $f(z) = h^j(v_{\sim j},k^j(v_j))$, each $h^j(\cdot)$ continuous, $h^j(v_{\sim j},\cdot) \uparrow$,

where $v_j = (y_j,y_{j+1},\ldots,y_{j+k-1})$, $v_{\sim j}$ its complement.

Call the complement of i

(4.5) $\sim i = \{1,2,\ldots,i-1,i+1,\ldots,2k\}$,

and take $i \leq k$ without loss of generality. When $i = 1$,

(4.6) $\sim 1 = (\ldots((S_2 \cup S_3) \cup S_4)\ldots) \cup S_{k+1}$.

When $1 < i \leq k$,

(4.7) $\sim i = (\ldots((S_1 \Delta S_i) \cup S_{i+1}) \cup S_{i+2}\ldots) \cup S_k$.

Each $\sim i$ is therefore separable and (4.4) yields

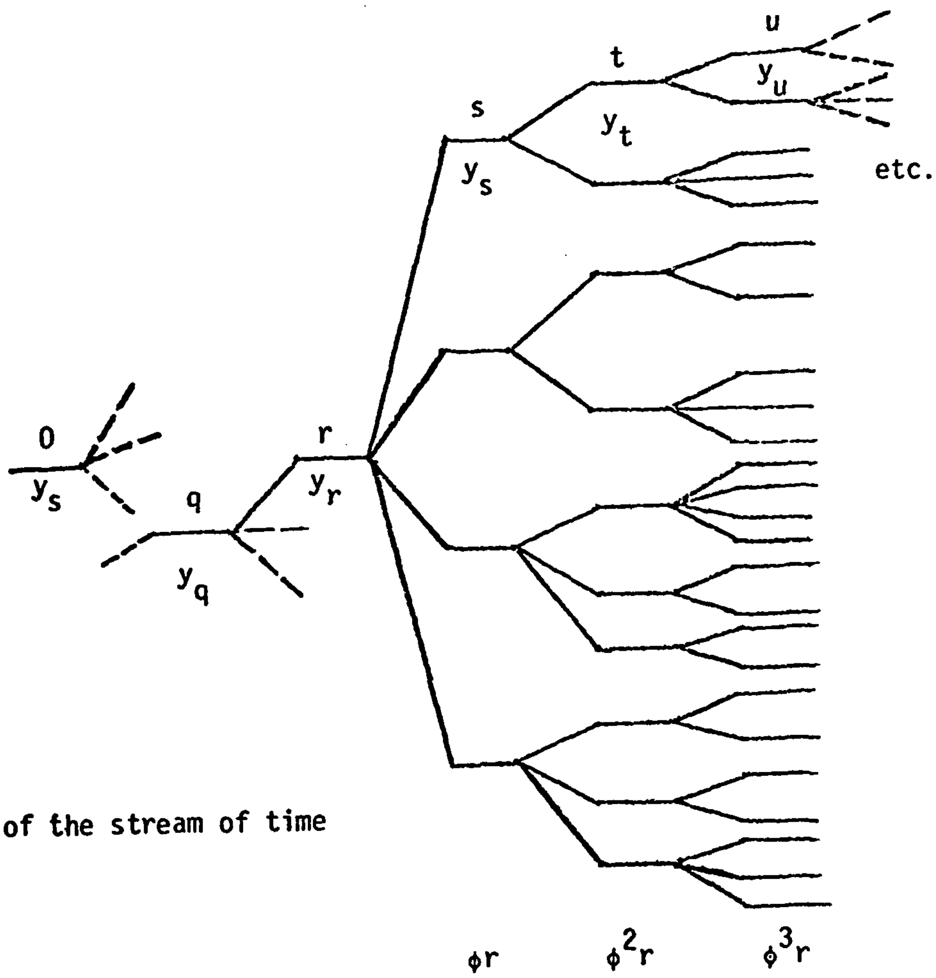
(4.8) $$f(z) = \sum_{i=1}^{2k} f^i(y_i) ,$$

in an appropriate normalization.

A similar treatment yields the same result when there are $2k - 1 \geq 3$ states.

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t \phi^2 r$, $u \phi^3 r$, $u \phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

Note that (4.2) is a result, not an assumption, in this case.

Can one get (4.8), or something like it when the world reveals itself as in Figure 3 opposite?

One can, but only by applying broad separability, based on rough foresight, at least two periods ahead - rather as in A3. In fact, the simplest assumptions which suggest themselves are A1, A2 and

A3* <u>Assumption 3*: Broad separability within a two period rolling plan</u>

$$(4.9) \qquad f^r(x_r,y_r,z_r) = d^S((x_t)_{t\phi^2 r},(y_t,z_t)_{t\in\sim S},\delta^S(x_t,y_t,z_t|t\in S))$$

say, for each of the broad sets S of $t\phi^2 r$ defined as in (4.3), $\sim S$ being its complement with respect to the set of $t\phi^2 r$. Given A1, this immediately yields

$$(4.10) \qquad f^r(x_r,y_r,z_r) = a^r((x_t)_{t\phi^2 r},\sum_{t\phi^2 r}\alpha^t(x_t,y_t,z_t)) \quad , \quad \text{say} \quad ,$$

where

$$(4.11) \qquad a^r((x_t)_{t\phi^2 r},\cdot) \updownarrow \quad ,$$

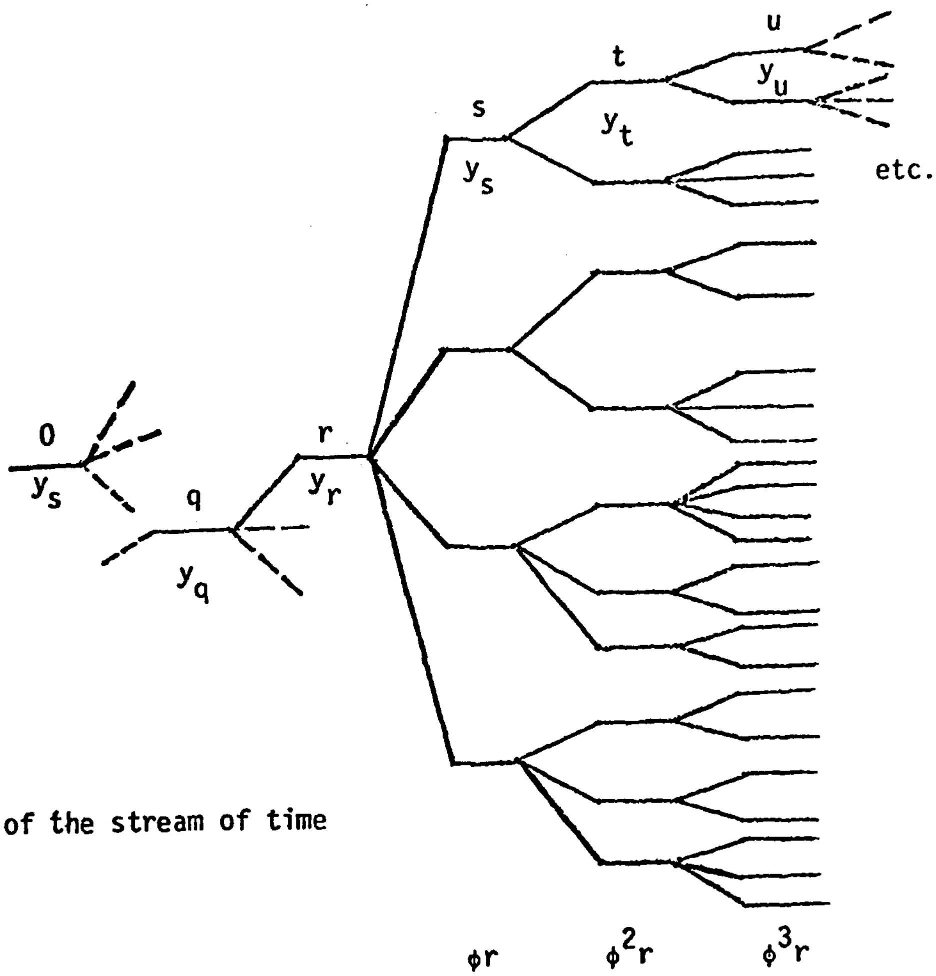
and we can normalize to get

$$(4.12) \qquad \alpha^t(x_t,0,0) = 0 \quad , \quad \text{when} \quad \theta(t) \geq 2,$$

by absorbing any such term into the blanket $(x_t)_{t\phi^2 r}$ argument in $a^r(\cdot)$. Do so.

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

Normalize the $f^r(\cdot)$ to satisfy (3.27-9) as in Section 3. (4.9) then yields

(4.13) $$f^t(x_t,y_t,z_t) = f^r(x_t,y_t,z_t;\ X_t \times Y_t \times Z_t) \quad , \quad t\phi^2 r \quad ,$$

(4.14) $$= c^t(x_t,\alpha^t(x_t,y_t,z_t)) \quad , \quad \text{say} \quad , \quad t\phi^2 r \quad ,$$

(4.15) $$f^s(x_s,y_s,z_s) = f^r(x_s,y_s,z_s;\ X_s \times Y_s \times Z_s) \quad , \quad \text{say} \quad , \quad s\phi r \quad ,$$

(4.16) $$= b^s(x_s,y_s,\sum_{t\phi s} \alpha^t(x_t,y_t,z_t)) \quad , \quad \text{say} \quad , \quad s\phi r \quad ,$$

where

(4.17) $$a^r((x_t)_{t\phi^2 r},\cdot) \quad , \quad b^s(x_s,y_s,\cdot) \quad , \quad c^t(x_t,\cdot) \quad \uparrow \quad ,$$

when

(4.18) $$\theta(r) \geq 0 \quad , \quad \theta(s) \geq 1 \quad , \quad \theta(t) \geq 2 \quad .$$

Clearly

(4.19) $$\alpha^t(x_t,y_t,z_t)$$

is a perfectly good criterion at t when

(4.20) $$\theta(t) \geq 2 \quad .$$

I will now prove

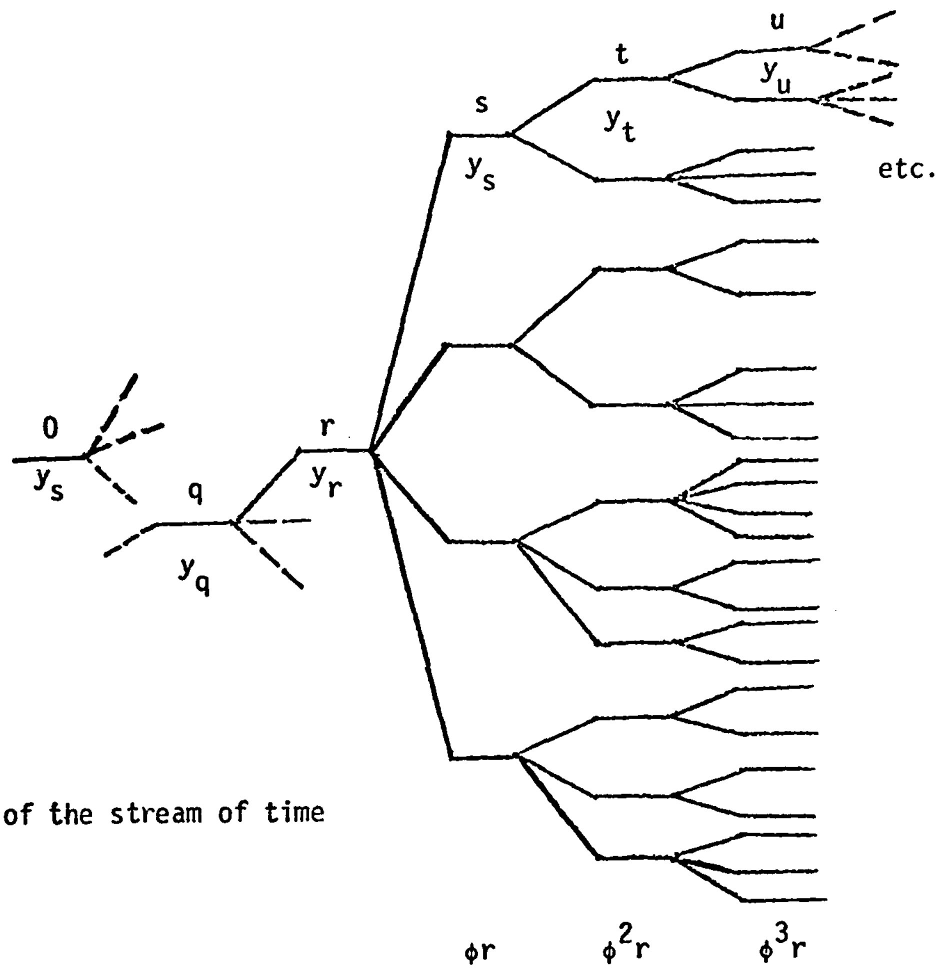
<u>Theorem</u>

<u>In a simple renormalization</u>

<u>Figure 3</u>: Illustrating A1 and A2

Time: 0 ... $\rho - 1$ ρ $\rho + 1$ $\rho + 2$ $\rho + 3$ ... T

Parity: $\theta(0)$.. $\theta(q)$ $\theta(r)$ $\theta(s)$ $\theta(t)$ $\theta(u)$ ... $\theta(w)$



Part of the stream of time

$\leftarrow x_r,\quad y_r,\quad z_r \rightarrow$

$\leftarrow x_s,\quad y_s,\quad z_s \rightarrow$

$\leftarrow x_t,\quad y_t,\quad z_t \rightarrow$

$\leftarrow x_u,\quad y_u,\quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

$$(4.21) \qquad \alpha^r(x_r,y_r,z_r) = \sum_{s\phi r} \alpha^s(x_s,y_s,z_s) = \sum_{t\phi^2 r} \alpha^t(x_t,y_t,z_t) = \dots ,$$

$$= \sum_{w\psi r}^{*} \alpha^w(x_w,y_w,z_w) \quad , \text{ when } \theta(r) \geq 2 \quad ,$$

where $\sum^*$ is taken over terminal nodes w, as foreshadowed in (4.1).

Remarks

This is virtually what I set out to get. Once again, there is an end - or rather initial - effect. (4.21) does not apply when $\theta(r) = 0$ or 1. In these cases we have

$$(4.22) \qquad f^0(y_0,z_0) = a^0((x_t)_{t\phi^2 0}, \sum{}^{*} \alpha^w(x_w,y_w,z_w)) \quad ,$$

$$(4.23) \qquad f^r(x_r,y_r,z_r) = b^r(x_r,y_r, \sum_{w\psi r}^{*} \alpha^w(x_w,y_w,z_w)) \quad \text{when} \quad \theta(r) = 1 \quad ,$$

by (4.10) and (4.16) respectively.

Proof

The argument is similar to that in Section 3. Once again we set out to derive a linear relationship via the linearity equation $f(\sum x_i) = \sum f(x_i)$.

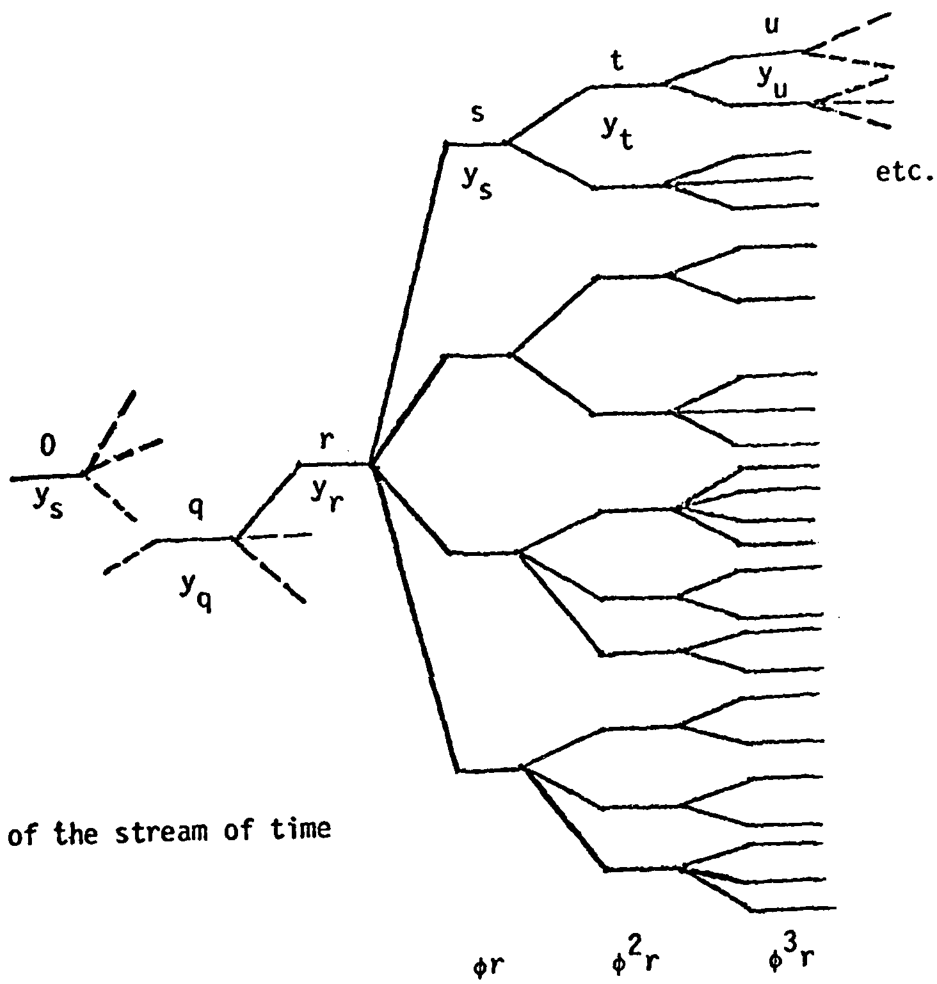
First solve (4.14) for α^t and substitute s for t in the result to get

$$(4.24) \qquad \alpha^s(x_s,y_s,z_s) = i^s(x_s,f^s(x_s,y_s,z_s)) \quad , \text{ say } , \quad i^s(x_s,\cdot) \uparrow \quad ,$$

$$(4.25) \qquad = j^s(x_s,y_s, \sum_{t\phi s} \alpha^t(x_t,y_t,z_t)) \quad , \text{ say } , \quad j^s(x_s,y_s,\cdot) \uparrow \quad ,$$

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r,\quad y_r,\quad z_r \rightarrow$

$\leftarrow x_s,\quad y_s,\quad z_s \rightarrow$

$\leftarrow x_t,\quad y_t,\quad z_t \rightarrow$

$\leftarrow x_u,\quad y_u,\quad z_u \rightarrow$

a) $u\phi t\phi s\phi r\phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s\phi r$

d) $z_r = (x_s, y_s)$, $s\phi r$

by (4.16). Then solve (4.16) for $\sum_{t\phi s} \alpha^t(x_t, y_t, z_t)$ and substitute s for t and r for s in the result to get

$$(4.26) \qquad \sum_{s\phi r} \alpha^s(x_s, y_s, z_s) = k^r(x_r, y_r, f^r(x_r, y_r, z_r)) \quad , \text{ say } , \quad k^r(x_r, y_r, \cdot) \uparrow \; ,$$

$$(4.27) \qquad = \ell^r((x_t)_{t\phi^2 r}, \sum_{t\phi^2 r} \alpha^t(x_t, y_t, z_t)) \quad , \text{ say } ,$$

$$\ell^r((x_t)_{t\phi^2 r}, \cdot) \uparrow \; ,$$

when

$$(4.28) \qquad \theta(s) \geq 2 \quad , \quad \theta(r) \geq 1 \quad \text{respectively} \; .$$

Hence,

$$(4.29) \qquad \sum_{s\phi r} j^s(x_s, y_s, \sum_{t\phi s} \alpha^t(x_t, y_t, z_t)) = \ell^r((x_s, y_s)_{s\phi r}, \sum_{t\phi^2 r} \alpha^t(x_t, y_t z_t)) \; .$$

Notice the linearity equation beginning to emerge. Take now $s\phi r$ and set

$$(4.30) \qquad y_t = z_t = 0 \quad , \quad t\phi^2 r \quad , \quad t\sim\phi s \quad ,$$

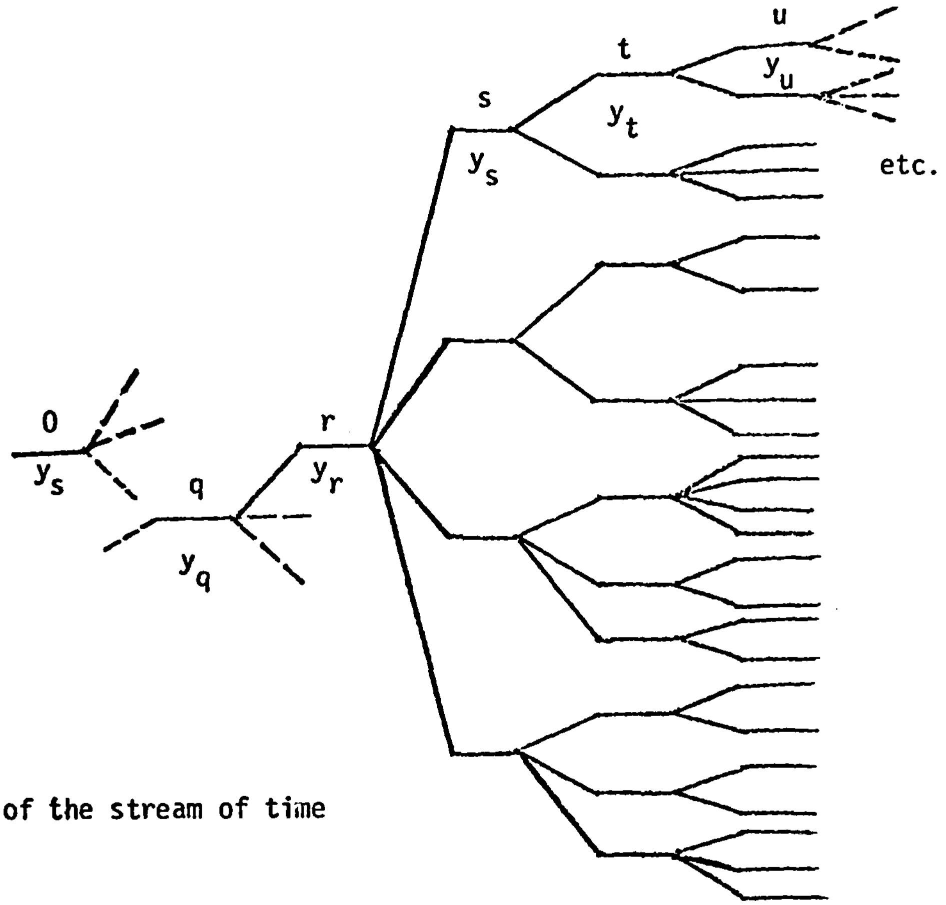
to get

$$(4.31) \qquad j^s(x_s, y_s, \sum_{t\phi s} \alpha^t(x_t, y_t, z_t)) = \ell^r((x_s, y_s)_{s\phi r}, \sum_{t\phi s} \alpha^t(x_t, y_t, z_t)) \quad ,$$

by (4.12) and (4.25), so that

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

ϕr $\quad \phi^2 r \quad \phi^3 r$

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

$$(4.32)\qquad j^{s}(x_s,y_s,\varepsilon) = \ell^{r}((x_s,y_s)_{s\phi r},\varepsilon) = m^{r}(x_r,y_r,\varepsilon) \quad , \quad \text{say} \quad ,$$

where, of course,

$$(4.33)\qquad (x_r,y_r) = x_s \quad , \quad \text{each} \quad s\phi r.$$

(4.29) therefore becomes

$$(4.34)\qquad \sum_{s\phi r} m^{r}(x_r,y_r,\varepsilon_s) = m^{r}(x_r,y_r,\sum_{s\phi r}\varepsilon_s) \quad , \quad \text{when} \quad \theta(r) \geq 1 \quad ,$$

where

$$(4.35)\qquad \varepsilon_s = \sum_{t\phi s}\alpha^{t}(x_t,y_t,z_t) \quad .$$

Since $m^{r}(x_r,y_r,\cdot)\ \updownarrow$ this implies

$$(4.36)\qquad m^{r}(x_r,y_r,\varepsilon) = e^{r}(x_r,y_r)\varepsilon \quad , \quad \text{say} \quad , \quad e^{r} > 0.$$

and hence,

$$(4.37)\qquad \alpha^{s}(x_s,y_s,z_s) = j^{s}(x_s,y_s,\sum_{t\phi s}\alpha^{t}(x_t,y_t,z_t)) \quad , \quad \text{by (4.25)} \quad ,$$

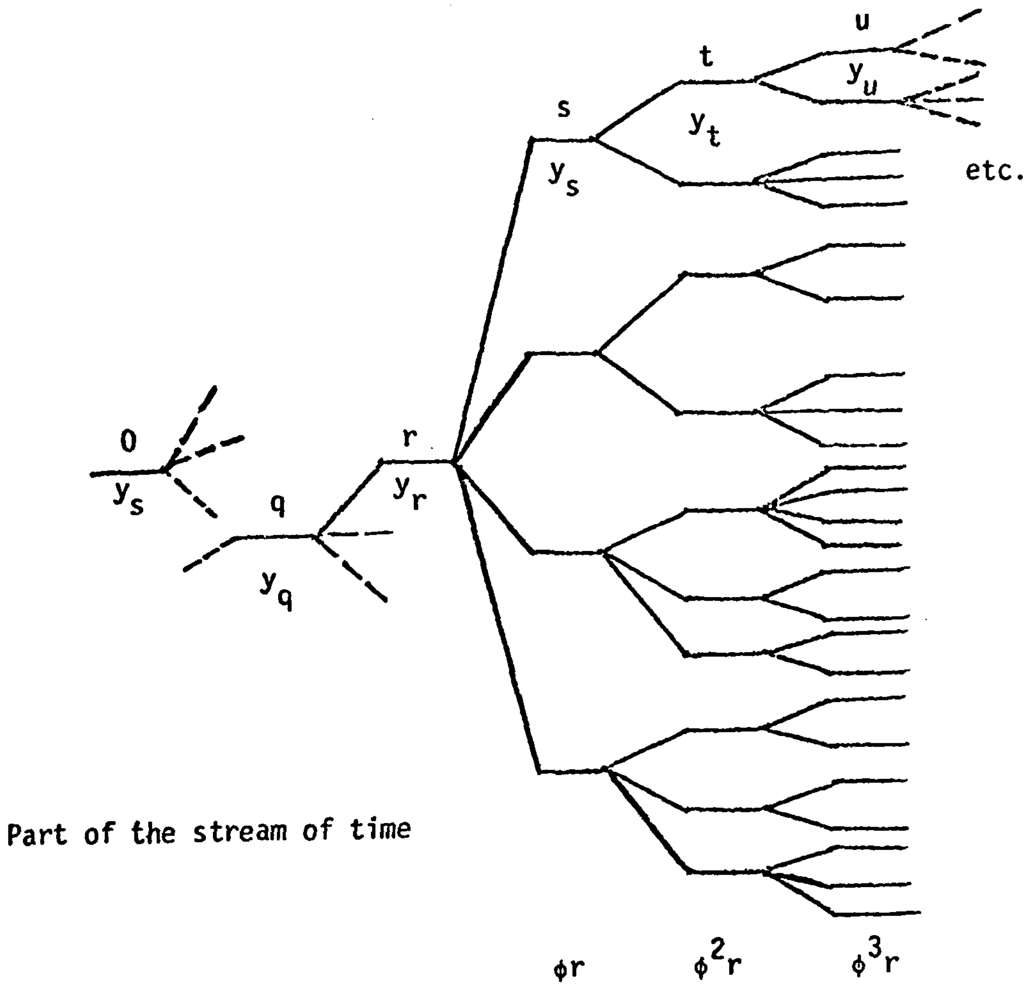
$$(4.38)\qquad = m^{r}(x_r,y_r,\sum_{t\phi s}\alpha^{t}(x_t,y_t,z_t)) \quad , \quad \text{by (4.32)} \quad ,$$

$$(4.39)\qquad = e^{r}(x_r,y_r)\sum_{t\phi s}\alpha^{t}(x_t,y_t,z_t) \quad .$$

Now, choose multipliers

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t\phi^2 r$, $u\phi^3 r$, $u\phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

$$(4.40) \qquad \lambda^r(x_r,y_r) > 0 \quad , \quad \text{each } r \text{ such that } \theta(r) \leq T - 2 \quad ,$$

with $\lambda^0(y_0) = 1$, such that

$$(4.41) \qquad \lambda^s(x_s,y_s) = e^s(x_s,y_s)\lambda^r(x_r,y_r) \quad , \quad \text{each } s\phi r \quad ,$$

and define

$$(4.42) \qquad \tilde{\alpha}^t(x_t,y_t,z_t) = \lambda^r(x_r,y_r)\alpha^t(x_t,y_t,z_t) \quad , \quad \text{each } t\phi^2 r \quad ,$$

to get

$$(4.43) \qquad \tilde{\alpha}^s(x_s,y_s,z_s) = \sum_{t\phi s} \tilde{\alpha}^t(x_t,y_t,z_t) \quad , \quad 2 \leq \theta(s) \leq T - 1 \quad .$$

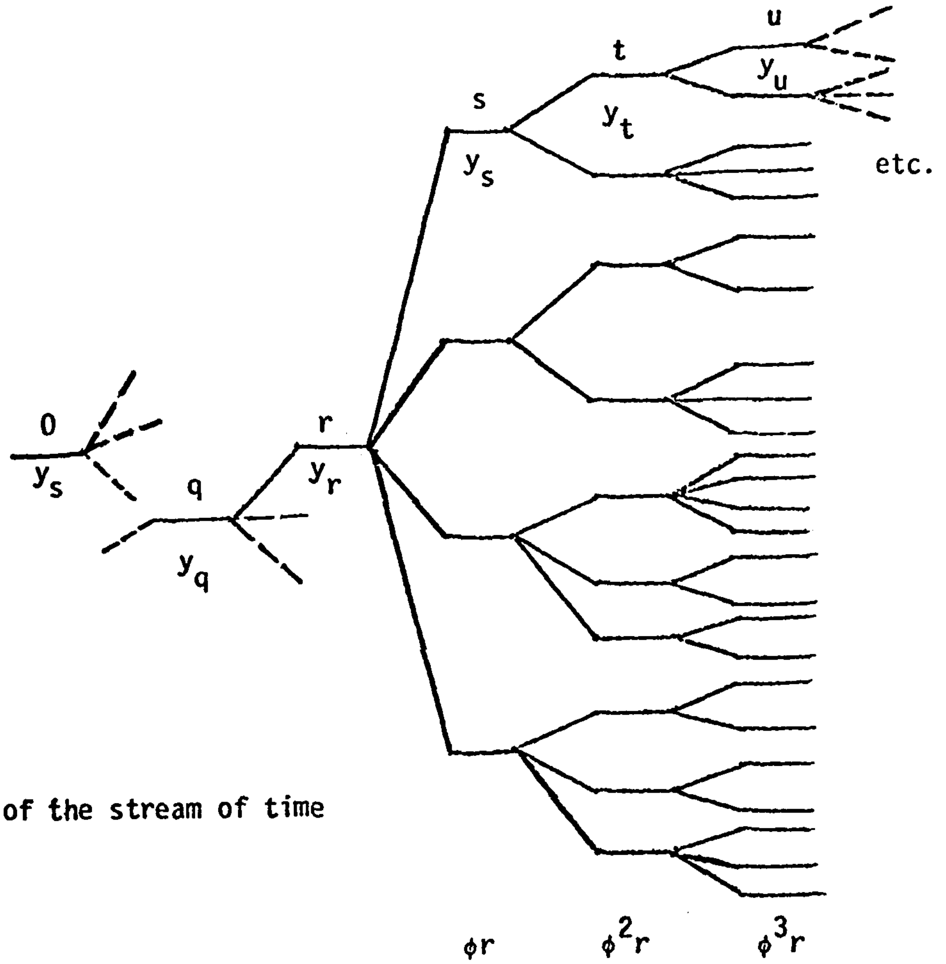
The construction is clearly simple in a model such as that depicted in Figure ? opposite, that is, given A1. Moreover, (x_r,y_r) is past history at t, so that $\tilde{\alpha}^t(x_t,y_t,z_t)$ is just as good a criterion there as is $\alpha^r(x_t,y_t,z_t)$. We may, therefore, drop the tildes in (4.43) and replace s by r to get the first equation in (4.21). The others follow trivially. We have proved the Theorem.

Remarks

"Proved" is a big word. To justify it I would have to discuss the range of the ε's in (4.34). That might be tricky. I do not think it would affect the result.

<u>Figure 3</u>: Illustrating A1 and A2

Time:	0 ...	$\rho - 1$	ρ	$\rho + 1$	$\rho + 2$	$\rho + 3$	...	T
Parity:	$\theta(0)$..	$\theta(q)$	$\theta(r)$	$\theta(s)$	$\theta(t)$	$\theta(u)$	...	$\theta(w)$



Part of the stream of time

$\leftarrow x_r, \quad y_r, \quad z_r \rightarrow$

$\leftarrow x_s, \quad y_s, \quad z_s \rightarrow$

$\leftarrow x_t, \quad y_t, \quad z_t \rightarrow$

$\leftarrow x_u, \quad y_u, \quad z_u \rightarrow$

a) $u \phi t \phi s \phi r \phi q$

b) $t \phi^2 r$, $u \phi^3 r$, $u \phi^4 q$, etc.

c) $x_s = (x_r, y_r)$, each $s \phi r$

d) $z_r = (x_s, y_s)$, $s \phi r$

Footnotes

* Earlier versions of this paper were read in Johns Hopkins, Stanford, and the University of California at Berkeley in the Spring of 1980, and a later version at the London School of Economics in 1981. I am grateful to the auditors for their comments, particularly to Peter Hammond for his suggestion that assumption A2 should be called "consistency over time." I had discussed the apparent coincidence that the same normalization should do for time and uncertainty with Kenneth Arrow, Michael Boskin, Frank Hahn, Hugh Rose, Amartya Sen, and John Wise at various times, and the possible relationship between the structure of a criterion function and an information tree such as that depicted in Figure 3 with Bernt Stigum in 1967. I thank them all, especially Bernt Stigum, and Michael Boskin who pointed out the importance of A3 at the London School of Economics.

1/ ↑: monotonically increasing: $f(y) \geq f(x)$ if $y \geq x$,

↟: strictly increasing: $f(y) > f(x)$ if $y > x$,

⤉: very strictly increasing: $f(y) > f(x)$ if $y \geq x$.

Here x is a vector. If a scalar, ↟ and ⤉ coincide. I will sometimes write "cns" for "continuous."

2/ We need to know the potential candidates, too, ahead of time. The path we follow from November depends on what happens at that election.

3/ Often it is the ablest students who believe they did badly in an examination, because they realize what they failed to mention.

4/ Note quite true: Samuelson introduces probabilities; I cover time as well as uncertainty and have annoying end effects.

5/ "Topologically separable": a completely different concept from "separable" as it is used in this paper. It means that the space contains a countably dense subset - such as the rationals among the reals - which can be used in arguments from continuity.

6/ When I read an earlier version of this paper in Stanford in May 1980, Peter Hammond suggested that A2 should be called the consistency axiom and a case can certainly be made for that. It certainly implies (3.27) which I had listed as a separate assumption.

7/ Actually, the results can be strengthened for $\theta(r) < 2$.

References

Debreu, G. [1960], "Topological Methods in Cardinal Utility Theory," in Mathematical Methods in the Social Sciences, 1959: Proceedings of the First Stanford Symposium, Stanford, Califonia: Stanford University Press.

Gorman, W.M. [1968], "The Structure of Utility Functions," Review of Economic Studies, October.

Leontief, W.W. [1947], "A Note on the Interrelations of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives," Bulletin of the American Mathematical Society.

Samuelson, P.A. [1952], "Probability, Utility, and the Independence Axiom," Econometrica, October.

REPORTS IN THIS SERIES

160. "The Structure and Stability of Competitive Dynamical Systems," by David Cass and Karl Shell.
161. "Monopolistic Competition and the Capital Market," by J. E. Stiglitz.
162. "The Corporation Tax," by J. E. Stiglitz.
163. "Measuring Returns to Scale in the Aggregate and the Scale Effect of Public Goods," by David A. Starrett.
164. "Monopoly, Quality, and Regulation," by Michael Spence.
165. "A Note on the Budget Constraint in a Model of Borrowing," by Duncan K. Foley and Martin F. Hellwig.
166. "Incentives, Risk, and Information: Notes Towards a Theory of Hierarchy," by Joseph E. Stiglitz.
167. "Asymptotic Expansions of the Distributions of Estimates in Simultaneous Equations for Alternative Parameter Sequences," by T. W. Anderson.
168. "Estimation of Linear Functional Relationships: Approximate Distributions and Connections with Simultaneous Equations in Econometrics," by T. W. Anderson.
169. "Monopoly and the Rate of Extraction of Exhaustible Resources," by Joseph E. Stiglitz.
170. "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," by Michael Rothschild and Joseph Stiglitz.
171. "Strong Consistency of Least Squares Estimates in Normal Linear Regression," by T. W. Anderson and John B. Taylor.
172. "Incentive Schemes under Differential Information Structures: An Application to Trade Policy," by Partha Dasgupta and Joseph Stiglitz.
173. "The Incidence and Efficiency Effects of Taxes on Income from Capital," by John B. Shoven.
174. "Distribution of a Maximum Likelihood Estimate of a Slope Coefficient: The LIML Estimate for Known Covariance Matrix," by T. W. Anderson and Takamitsu Sawa.
175. "A Comment on the Test of Overidentifying Restrictions," by Joseph B. Kadane and T. W. Anderson.
176. "An Asymptotic Expansion of the Distribution of the Maximum Likelihood Estimate of the Slope Coefficient in a Linear Functional Relationship," by T. W. Anderson.
177. "Some Experimental Results on the Statistical Properties of Least Squares Estimates in Control Problems," by T. W. Anderson and John B. Taylor.
178. "A Note on "Fulfilled Expectations" Equilibria," by David M. Kreps.
179. "Uncertainty and the Rate of Extraction under Alternative Institutional Arrangements," by Partha Dasgupta and Joseph E. Stiglitz.
180. "Budget Displacement Effects of Inflationary Finance," by Jerry Green and E. Sheshinski.
181. "Towards a Marxist Theory of Money," by Duncan K. Foley.
182. "The Existence of Futures Markets, Noisy Rational Expectations and Informational Externalities," by Sanford Grossman.
183. "On the Efficiency of Competitive Stock Markets where Traders have Diverse Information," by Sanford Grossman.
184. "A Bidding Model of Perfect Competition," by Robert Wilson.
185. "A Bayesian Approach to the Production of Information and Learning by Doing," by Sanford J. Grossman, Richard E. Kihlstrom and Leonard J. Mirman.
186. "Disequilibrium Allocations and Recontracting," by Jean-Michel Grandmont, Guy Laroque and Yves Younes.
187. "Agreeing to Disagree," by Robert J. Aumann.
188. "The Maximum Likelihood and the Nonlinear Three Stage Least Squares Estimator in the General Nonlinear Simultaneous Equation Model," by Takeshi Amemiya.
189. "The Modified Second Round Estimator in the General Qualitative Response Model," by Takeshi Amemiya.
190. "Some Theorems in the Linear Probability Model," by Takeshi Amemiya.
191. "The Bilinear Complementarity Problem and Competitive Equilibria of Linear Economic Models," by Robert Wilson.
192. "Noncooperative Equilibrium Concepts for Oligopoly Theory," by L. A. Gerard-Varet.
193. "Inflation and Costs of Price Adjustment," by Eytan Sheshinski and Yoram Weiss.
194. "Power and Taxes in a Multi-Commodity Economy," by R. J. Aumann and M. Kurz.
195. "Distortion of Preferences, Income Distribution and the Case for a Linear Income Tax," by Mordecai Kurz.
196. "Search Strategies for Nonrenewable Resource Deposits," by Richard J. Gilbert.
197. "Demand for Fixed Factors, Inflation and Adjustment Costs," by Eytan Sheshinski and Yoram Weiss.
198. "Bargains and Ripoffs: A Model of Monopolistically Competitive Price Dispersions," by Steve Salop and Joseph Stiglitz.
199. "The Design of Tax Structure: Direct Versus Indirect Taxation by A. B. Atkinson and J. E. Stiglitz.
200. "Market Allocations of Location Choice in a Model with Free Mobility," by David Starrett.
201. "Efficiency in the Optimum Supply of Public Goods," by Lawrence J. Lau, Eytan Sheshinski and Joseph E. Stiglitz.
202. "Risk Sharing, Sharecropping and Uncertain Labor Markets," by David M. G. Newberry.
203. "On Non-Walrasian Equilibria," by Frank Hahn.
204. "A Note on Elasticity of Substitution Functions," by Lawrence J. Lau.
205. "Quantity Constraints as Substitutes for Inoperative Markets: The Case of the Credit Markets," by Mordecai Kurz.
206. "Incremental Consumer's Surplus and Hedonic Price Adjustment," by Robert D. Willig.
207. "Optimal Depletion of an Uncertain Stock," by Richard Gilbert.
208. "Some Minimum Chi-Square Estimators and Comparisons of Normal and LogisticaModels in Qualitative Response Analysis," by Kimio Morimune.
209. "A Characterization of the Optimality of Equilibrium in Incomplete Markets," by Sanford J. Grossman.
210. "Inflation and Taxes in a Growing Economy with Debt and Equity Finance," by M. Feldstein, J. Green and E. Sheshinski.
211. "The Specification and Estimation of a Multivariate Logit Model," by Takeshi Amemiya.
212. Prices and Queues as Screening Devices in Competitive Markets," by Joseph E. Stiglitz.
213. "Conditions for Strong Consistency of Least Squares Estimates in Linear Models," by T. W. Anderson and John B. Taylor.
214. "Utilitarianism and Horizontal Equity: The Case for Random Taxation," by Joseph E. Stiglitz.
215. "Simple Formulae for Optimal Income Taxation and the Measurement of Inequality," by Joseph E. Stiglitz.
216. "Temporal Resolution of Uncertainty and Dynamic Choice Behavior," by David M. Kreps and Evan L. Porteus.
217. "The Estimation of Nonlinear Labor Supply Functions with Taxes from a Truncated Sample," by Michael Hurd.
218. "The Welfare Implications of the Unemployment Rate," by Michael Hurd.
219. "Keynesian Economics and General Equilibrium Theory: Reflections on Some Current Debates," by Frank Hahn.
220. "The Core of an Exchange Economy with Differential Information," by Robert Wilson.
221. "A Competitive Model of Exchange," by Robert Wilson.
222. "Intermediate Preferences and the Majority Rule," by Jean-Michel Grandmont.
223. "The Fixed Price Equilibria: Some Results in Local Comparative Statics," by Guy Laroque.
224. "On Stockholder Unanimity in Making Production and Financial Decisions," by Sanford J. Grossman and Joseph E. Stiglitz.
225. "Selection of Regressors," by Takeshi Amemiya.
226. "A Note on A Random Coefficients Model," by Takeshi Amemiya.
227. "A Note on a Heteroscedastic Model," by Takeshi Amemiya.
228. "Welfare Measurement for Local Public Finance," by David Starrett.
229. "Unemployment Equilibrium with Rational Expectations," by W. P. Heller and R. M. Starr.
230. "A Theory of Competitive Equilibrium in Stock Market Economies," by Sanford J. Grossman and Oliver D. Hart.
231. "An Application of Stein's Methods to the Problem of Single Period Control of Regression Models," by Asad Zaman.
232. "Second Best Welfare Economics in the Mixed Economy," by David Starrett.
233. "The Logic of the Fix-Price Method," by Jean-Michel Grandmont.
234. "Tables of the Distribution of the Maximum Likelihood Estimate of the Slope Coefficient and Approximations," by T. W. Anderson and Takamitsu Sawa.
235. "Further Results on the Informational Efficiency of Competitive Stock Markets," by Sanford Grossman.
236. "The Estimation of a Simultaneous-Equation Tobit Model," by Takeshi Amemiya.
237. "The Estimation of a Simultaneous-Equation Generalized Probit Model," by Takeshi Amemiya.
238. "The Consistency of the Maximum Likelihood Estimator in a Disequilibrium Model," by T. Amemiya and G. Sen.
239. "Numerical Evaluation of the Exact and Approximate Distribution Functions of the Two-Stage Least Squares Estimate," by T. W. *Anderson and Takamitsu Sawa.*
240. "Risk Measurement of Public Projects," by Robert Wilson.
241. "On the Capitalization Hypothesis in Closed Communities," by David Starrett.
242. "A Note on the Uniqueness of the Representation of Commodity-Augmenting Technical Change," by Lawrence J. Lau.
243. "The Property Rights Doctrine and Demand Revelation under Incomplete Information," by Kenneth J. Arrow.
244. "Optimal Capital Gains Taxation Under Limited Information," by Jerry R. Green and Eytan Sheshinski.
245. "Straightforward Individual Incentive Compatibility in Large Economies," by Peter J. Hammond.
246. "On the Rate of Convergence of the Core," by Robert J. Aumann.
247. "Unsatisfactory Equilibria," by Frank Hahn.
248. "Existence Conditions for Aggregate Demand Functions: The Case of a Single Index," by Lawrence J. Lau.
249. "Existence Conditions for Aggregate Demand Functions: The Case of Multiple Indexes," by Lawrence J. Lau.
250. "A Note on Exact Index Numbers," by Lawrence J. Lau.
251. "Linear Regression Using Both Temporally Aggregated and Temporally Disaggregated Data," by Cheng Hsiao.
252. "The Existence of Economic Equilibria: Continuity and Mixed Strategies," by Partha Dasgupta and Eric Maskin.
253. "A Complete Class Theorem for the Control Problem and Further Results on Admissibility and Inadmissibility," by Asad Zaman.
254. "Measure-Based Values of Market Games," by Sergiu Hart.
255. "Altruism as an Outcome of Social Interaction," by Mordecai Kurz.
256. "A Representation Theorem for 'Preference for Flexibility'," by David M. Kreps.
257. "The Existence of Efficient and Incentive Compatible Equilibria with Public Goods," by Theodore Groves and John O. Ledyard.
258. "Efficient Collective Choice with Compensation," by Theodore Groves.
259. "On the Impossibility of Informationally Efficient Markets," by Sanford J. Grossman and Joseph E. Stiglitz.

REPORTS IN THIS SERIES

260. "Values for Games Without Sidepayments: Some Difficulties With Current Concepts," by Alvin E. Roth.
261. "Martingales and the Valuation of Redundant Assets," by J. Michael Harrison and David M. Kreps.
262. "Autoregressive Modelling and Money: Income Causality Detection," by Cheng Hsiao.
263. "Measurement Error in a Dynamic Simultaneous Equations Model without Stationary Disturbances," by Cheng Hsiao.
264. "The Measurement of Deadweight Loss Revisited," by W. E. Diewert.
265. "The Elasticity of Derived Net Supply and a Generalized Le Chatelier Principle," by W. E. Diewert.
266. "Income Distribution and Distortion of Preferences: the ℓ Commodity Case," by Mordecai Kurz.
267. "The n^2 Order Mean Squared Errors of the Maximum Likelihood and the Minimum Logit Chi-Square Estimator," by Takeshi Amemiya.
268. "Temporal Von Neumann-Morgenstern and Induced Preferences," by David M. Kreps and Evan L. Porteus.
269. "Take Over Bids and the Theory of the Corporation," by Stanford Grossman and Oliver D. Hart.
270. "The Numerical Values of Some Key Parameters in Econometric Models," by T. W. Anderson, Kimio Morimune and Takamitsu Sawa.
271. "Two Representations of Information Structures and their Comparisons," by Jerry Green and Nancy Stokey.
272. "Asymptotic Expansions of the Distributions of Estimators in a Linear Functional Relationship when the Sample Size is Large," by Naoto Kunitomo.
273. "Public Goods and Power," by R. J. Aumann, M. Kurz and A. Neyman.
274. "An Axiomatic Approach to the Efficiency of Non-Cooperative Equilibrium in Economies with a Continuum of Traders," by A. Mas-Colell.
275. "Tables of the Exact Distribution Function of the Limited Information Maximum Likelihood Estimator when the Covariance Matrix in Known," by T. W. Anderson and Takamitsu Sawa.
276. "Autoregressive Modeling of Canadian Money and Income Data," by Cheng Hsiao.
277. "We Can't Disagree Forever," by John D. Geanakoplos and Heraklis Polemarchakis.
278. "Constrained Excess Demand Functions," by Herklis M. Polemarchakis.
279. "On the Bayesian Selection of Nash Equilibrium," by Akira Tomioka.
280. "Disequilibrium Econometrics in Simultaneous Equations Systems," by C. Gourieroux, J. J. Laffont and A. Monfort.
281. "Duality Approaches to Microeconomic Theory," by W. E. Diewert.
282. "A Time Series Analysis of the Impact of Canadian Wage and Price Controls," by Cheng Hsiao and Oluwatayo Fakiyesi.
283. "A Strategic Theory of Inflation," by Mordecai Kurz.
284. "A Characterization of Vector Measure Games in pNA," by *Yair Tauman*.
285. "On the Method of Taxation and the Provision of Local Public Goods," by David A. Starrett.
286. "An Optimization Problem Arising in Economics: Approximate Solutions, Linearity, and a Law of Large Numbers," by Sergiu Hart.
287. "Asymptotic Expansions of the Distributions of the Estimates of Coefficients in a Simultaneous Equation System," by Yasunori Fujikoshi, Kimio Morimune, Naoto Kunitomo and Masanobu Taniguchi.
288. "Optimal & Voluntary Income Distribution," by K. J. Arrow.
289. "Asymptotic Values of Mixed Games," by Abraham Neyman.
290. "Time Series Modelling and Causal Ordering of Canadian Money, Income and Interest Rate," by Cheng Hsiao.
291. "An Analysis of Power in Exchange Economies," by Martin J. Osborne.
292. "*Estimation of the Reciprocal of a Normal Mean*," by *Asad Zaman*.
293. "Improving the Maximum Likelihood Estimate in Linear Functional Relationships for Alternative Parameter Sequences," by Kimio Morimune and Naoto Kunitomo.
294. "Calculation of Bivariate Normal Integrals by the Use of Incomplete Negative-Order Moments," by Kei Takeuchi and Akimichi Takemura.
295. "On Partitioning of a Sample with Binary-Type Questions in Lieu of Collecting Observations," by Kenneth J. Arrow, Leon Pesotchinsky and Milton Sobel.
296.
297. "The Two Stage Least Absolute Deviations Estimators," by Takeshi Amemiya.
298. "Three Essays on Capital Markets," by David M. Kreps.
299. "Infinite Horizon Programs," by Michael J. P. Magill.
300. "Electoral Outcomes and Social Log-Likelihood Maxima," by Peter Coughlin and Shmuel Nitzan.
301. "Notes on Social Choice and Voting," by Peter Coughlin.
302. "Overlapping Expectations and Hart's Conditions for Equilibrium in a Securities Model," by Peter J. Hammond.
303. "Directional and Local Electorate Competitions with Probabilistic Voting," by Peter Coughlin and Shmuel Nitzan.
304. "Asymptotic Expansions of the Distributions of the *Test Statistics for Overidentifying Restrictions in a System of Simultaneous Equations*," by Kunitomo, Morimune, and Tsukuda.
305. "Incomplete Markets and the Observability of Risk Preference Properties," by H. H. Polemarchakis and L. Selden.
306. "Multiperiod Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model," by David M. Kreps.
307. "Asymptotic Expansions of the *Distributions of k-Class Estimators when the Disturbances are Small*," by Naoto Kunitomo, Kimio Morimune, and Yoshihiko Tsukuda.
308. "Arbitrage and Equilibrium in Economies with Infinitely Many Commodities," by David M. Kreps.
309. "Unemployment Equilibrium in an Economy with Linked Prices," by Mordecai Kurz.
310. "Pareto Optimal Nash Equilibria are Competitive in a Repeated Economy," by Mordecai Kurz and Sergiu Hart.
311. "Identification," by Cheng Hsiao.
312. "An Introduction to Two-Person Zero Sum Repeated Games with Incomplete Information," by Sylvain Sorin.
313. "Estimation of Dynamic Models With Error Components," by T. W. Anderson and Cheng Hsiao.
314. "On Robust Estimation in Certainty Equivalence Control," by Anders H. Westlund and Hans Stenlund.
315. "On Industry Equilibrium Under Uncertainty," by J. Drèze and E. Sheshinski.
316. "Cost Benefit Analysis and Project Evaluation From the Viewpoint of Productive Efficiency" by W. E. Diewert.
317. "On the Chain-Store Paradox and Predation: Reputation for *Toughness*," by *David M. Kreps and Robert Wilson*.
318. "On the Number of Commodities Required to Represent a Market Games," Sergiu Hart.
319. "Evaluation of the Distribution Function of the Limited Information Maximum Likelihood Estimator," by T. W. Anderson, Naoto Kunitomo, and Takamitsu Sawa.
320. "A Comparison of the Logit Model and Normal Discriminant *Analysis When the Independent Variables Are Binary*," by Takeshi Amemiya and James L. Powell.
321. "Efficiency of Resource Allocation by Uninformed Demand," by Theodore Groves and Sergiu Hart.
322. "A Comparison of the Box-Cox Maximum Likelihood Estimator and the Nonlinear Two Stage Least Squares Estimator," by Takeshi Amemiya and James L. Powell.
323. "*Comparison of the Densities of the TSLS and LIMLK Estimators* for Simultaneous Equations," by T. W. Anderson, Naoto Kunitomo, and Takamitsu Sawa.
324. "Admissibility of the Bayes Procedure Corresponding to the Uniform Prior Distribution for the Control Problem in Four Dimensions but Not in Five," by Charles Stein and Asad Zaman.
325. "Some Recent Developments on the Distributions of Single-Equation Estimators," by T. W. Anderson.
326. "On Inflation", by Frank Hahn
327. Two Papers on Majority Rule: "Continuity Properties of Majority Rule with Intermediate Preferences," by Peter Coughlin and Kuan-Pin Lin; and, "Electoral Outcomes with Probabilistic Voting and Nash Social Welfare Maxima," by Peter Coughlin and Shmuel Nitzan.
328. "On the Endogenous Formation of Coalitions," by Sergiu Hart and Mordecai Kurz.
329. "Controliability, Pecuniary Externalities and Optimal Taxation," by David Starrett.
330. "Nonlinear Regression Models," by Takeshi Amemiya.
331. "Paradoxical Results From Inada's Conditions for Majority Rule," by Herve Raynaud.
332. "On Welfare Economics with Incomplete Information and the Social Value of Public Information," by Peter J. Hammond.
333. "Equilibrium Policy Proposals With Abstentions," by Peter J. Coughlin.
334. "Infinite Excessive and Invariant Measures," by Michael I. Taksar.
335. "The Life-Cycle Hypothesis and the Effects of Social Security and Private Pensions on Family Savings," by Mordecai Kurz.
336. "Optimal Retirement Age," by Mordecai Kurz.
337. "Bayesian Incentive Compatible Beliefs," by Claude d'Aspremont and Louis-André Gerard-Varet.
338. "Qualitative Response Models: A Survey," by Takeshi Amemiya.
339. "The Social Costs of Monopoly and Regulation: A Game Theoretic Analysis," by William P. Rogerson.

Reports in this Series

340. "Sequential Equilibria," by David M. Kreps and Robert Wilson.

341. "Enhancing of Semigroups," by Michael I. Taksar.

342. "Formulation and Estimation of Dynamic Models Using Panel Data," by T.W. Anderson and Cheng Hsiao.

343. "Ex-Post Optimality as a Dynamically Consistent Objective for Collective Choice Under Uncertainty," by Peter Hammond.

344. "Three Lectures In Monetary Theory," by Frank H. Hahn.

345. "Socially Optimal Investment Rules in the Presence of Incomplete Markets and Other Second Best Distortions," by Frank Milne and David A. Starrett.

346. "Approximate Purification of Mixed Strategies," by Robert Aumann, Yitzhak Katznelson, Roy Radner, Robert W. Rosenthal, and Benjamin Weiss.

347. "Conditions for Transitivity of Majority Rule with Algorithmic Interpretations," by Herve Raynaud.

348. "How Restrictive Actually are the Value Restriction Conditions," by Herve Raynaud.

349. "Cournot Duopoly in the Style of Fulfilled Expectations Equilibrium," by William Novshek and Hugo Sonnenschein.

350. "Law of Large Numbers for Random Sets and Allocation Processes," by Zvi Artstein and Sergiu Hart.

351. "Risk Perception in Psychology and Economics," by Kenneth J. Arrow.

352. "Shrunken Predictors for Autoregressive Models," by Taku Yamamoto

353. "Predation, Reputation, and Entry Deterrence," by Paul Milgrom and John Roberts.

354. "Social and Private Production Objectives in the Sequence Economy" by David Starrett

355. "Recursive Rational Choice" by Alain Lewis

356. "Least Absolute Deviations Estimation for Censored and Truncated Regression Models" by James Powell

357. "Relatively Recursive Rational Choice" by Alain Lewis

358. "A Theory of Auctions and Competitive Bidding" by Paul Milgrom and Robert Weber.

359. "Facing an Uncertain Future" by W. M. Gorman

END

FILMED

2-83

DTIC