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THESIS

SOME COMPUTER ALGORITHMS TO IMPLEMENT  
A RELIABILITY SHORTHAND

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October 1982

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→ variables.

↳ This paper presents an algorithm for the numerical convolution of exponentially distributed random variables. After reducing the system scenario to its shorthand format, one can use the programs that are given in the appendix to obtain numerical values for the reliability of the system. ↗



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Some Computer Algorithms to Implement  
a Reliability Shorthand

by

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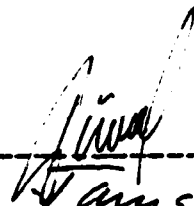
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requirements for the degree of

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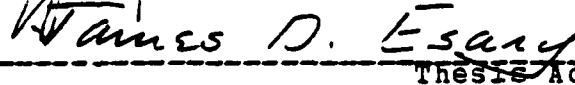
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## ABSTRACT

Under the assumption of constant failure rates it is possible to build a "reliability shorthand" which gives a simple, unified approach to reliability computations for systems in the presence of complications like support by shared spares or changes in the failure rates of surviving components when other components fail. The computational implementation of the shorthand depends upon the convolution of strings of exponentially distributed random variables.

This paper presents an algorithm for the numerical convolution of exponentially distributed random variables. After reducing the system scenario to its shorthand format, one can use the programs that are given in the appendix to obtain numerical values for the reliability of the system.

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## I. INTRODUCTION

The reliability shorthand considered in this paper has been developed in conjunction with the course OA 4302 "Reliability And Weapon Systems Effectiveness Measurement" at the Naval Postgraduate School. A tutorial introduction to the reliability shorthand was given by Repicky[Ref.2]. This paper is devoted to a complementary part of the idea.

Any study on system reliability always requires two steps; one is the description of the system's life and the other is the derivation of its survival function.

Under the assumption of constant component failure rates this paper presents a way of obtaining the system's reliability which requires little beyond the description of the system's life.

Section II contains an approach to the convolution of exponentially distributed random variables. Also there is a presentation of a computational algorithm for the convolution of exponentially distributed random variables.

Appendix A gives survival functions corresponding to several reliability shorthand notations and a program written in Fortran for computations from shorthand notations.

Section III deals with the reliability of redundant systems under the assumption of constant component failure rates.

Appendix B consists of a program written in Fortran. The program supports the approach of Section III. There is a crude Monte Carlo simulation program in Appendix D which is a simulation program parallel to the program in Appendix B.

There is another program in Appendix C written in Fortran. This program uses the network approach to systems described in Section III.

Appendix E summarizes the definitions used in this paper.

## II. AN APPROACH TO COMPUTING CONVOLUTIONS OF EXPONENTIAL RANDOM VARIABLES

This section introduces a general algorithm for computing the survival function of any convolution of exponential random variables.

In reliability, the term convolution refers to the summation of independent random lives. In order to have simplicity in specifying convolutions, the reliability shorthand introduces a special notation. In the following sections we will use this notation.

### A. THE SURVIVAL FUNCTION FOR A CONVOLUTION OF RANDOM VARIABLES

Let  $\bar{F}_1(t)$  and  $\bar{F}_2(t)$  be the survival functions for the random variables  $T_1$  and  $T_2$  respectively. Let  $f_1(t)$  and  $f_2(t)$  be the corresponding densities. Let  $\bar{F}(t)$  be the survival function for the random variable  $T$ , where  $T=T_1+T_2$ .

Then the likelihood expression for  $\bar{F}(t)$  is

$$\bar{F}(t) = \bar{F}_1(t) + \int_0^t \bar{F}_2(t-s) f_1(s) ds$$

In the right hand side of equation,  $\bar{F}_1(t)$  is the probability that component one completes the mission,  $\int_0^t \bar{F}_2(t-s)f_1(s)ds$  is the probability that at some time  $s$  ( $0 \leq s \leq t$ ) component one fails, component two takes its place and carries the system to the end of the mission duration  $t$ .

In order to illustrate consider some applications.

1. Example

Reliability Shorthand Notation : EXP $\{\lambda_1\}$ +EXP $\{\lambda_2\}$

SYSTEM: One component having one spare with a dissimilar failure rate. If the active component fails, the spare will replace it immediately.

Here the life for the system is  $T = T_1 + T_2$ . The reliability shorthand notation indicates that this system has an exponential life with failure rate  $\lambda_1$  followed by an exponential life with failure rate  $\lambda_2$ .

The survival function for the active component is

$$\bar{F}_1(t) = e^{-\lambda_1 t}, \quad t \geq 0.$$

The survival function for the spare is

$$\bar{F}_2(t) = e^{-\lambda_2 t}, \quad t \geq 0.$$

The survival function for the system is

$$\bar{F}(t) = \bar{F}_1(t) + \int_0^t \bar{F}_2(t-s) f_1(s) ds$$

$$\bar{F}(t) = e^{-\lambda_1 t} + \int_0^t e^{-\lambda_2(t-s)} \lambda_1 e^{-\lambda_1 s} ds, \quad t \geq 0.$$

If we complete the integration, the result is

$$\bar{F}(t) = \lambda_2 / (\lambda_2 - \lambda_1) e^{-\lambda_1 t} + \lambda_1 / (\lambda_1 - \lambda_2) e^{-\lambda_2 t}, \quad t \geq 0.$$

Which is the well known result.

Another way to establish this formula is the use of the moment generating function. (Freund and Walpole[Ref.3])

## 2. Example

Reliability Shorthand Notation : EXP{λ}+EXP{λ}

SYSTEM: One component having one identical spare. If the active component fails, the spare will replace it immediately.

The formula that we derived above becomes meaningless, because the denominators become zero. If we proceed as before

$$\bar{F}(t) = \bar{F}_1(t) + \int_0^t \bar{F}_2(t-s) f_1(s) ds$$

$$\bar{F}(t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \lambda e^{-\lambda s} ds$$

$$\bar{F}(t) = (1 + \lambda t) e^{-\lambda t}, \quad t \geq 0.$$

The result comes out as expected to be the Erlang  $\{2, \lambda\}$  survival function.

#### B. RELIABILITY SHORTHAND NOTATION

$\text{EXP}\{\lambda\} + \text{EXP}\{\lambda\} + \dots + \text{EXP}\{\lambda\}$  (n identical  
exponential lives)

This leads to the Erlang  $\{n, \lambda\}$  survival function

$$\bar{F}(t) = \sum_{i=1}^n (\lambda t)^{i-1} / (i-1)! e^{-\lambda t}, \quad t \geq 0.$$

The use of moment generating function gives the result immediately.

#### C. THE RELIABILITY SHORTHAND NOTATION

$\text{EXP}\{\lambda_1\} + \text{EXP}\{\lambda_2\} + \dots + \text{EXP}\{\lambda_n\}$

This is the expression for the convolution of n random variables where each has a distinct failure rate.

By the approach used in Section 2.1, adding one exponential life at once, one can obtain the formula for the survival function

$$\bar{F}(t) = \sum_{i=1}^n \prod_{j \neq i} \lambda_j / \prod_{j \neq i} (\lambda_1 - \lambda_j) e^{-\lambda_i t}, \quad t \geq 0.$$



(a).

Let  $T_1, T_2, T_3$  be random variables distributed as  $\text{EXP}\{\lambda_1\}$ ,  $\text{EXP}\{\lambda_1\}$ ,  $\text{EXP}\{\lambda_2\}$  respectively.

Let  $T_1' = T_1 + T_2$ . This random variable has the Erlang  $\{2, \lambda\}$  distribution.

The convolution formula for  $T = T_1' + T_3$  is;

$$\bar{F}_T(t) = \bar{F}_{T_1'}(t) + \int_0^t \bar{F}_{T_3}(t-s) f_{T_1}(s) ds, \quad t \geq 0.$$

$$\bar{F}_T(t) = (1 + \lambda_1 t) e^{-\lambda_1 t} + \int_0^t e^{-\lambda_2(t-s)} \lambda_1^2 s e^{-\lambda_1 s} ds$$

$$\bar{F}_T(t) = e^{-\lambda_2 t} \left\{ (\lambda_2^2 - 2\lambda_1\lambda_2) / (\lambda_2 - \lambda_1)^2 + \lambda_1\lambda_2 / (\lambda_2 - \lambda_1) t \right\} + e^{-\lambda_1 t} \left\{ \lambda_1^2 / (\lambda_1 - \lambda_2)^2 \right\}, \quad t \geq 0.$$

(b).

Let  $T_1, T_2, T_3$  be random variables distributed as  $\text{EXP}\{\lambda_1\}$ ,  $\text{EXP}\{\lambda_1\}$ ,  $\text{EXP}\{\lambda_2\}$  respectively.

Let  $T_2' = T_2 + T_3$ . This random variable has the survival function

$$\bar{F}_{T_2'}(t) = \lambda_2 / (\lambda_2 - \lambda_1) e^{-\lambda_1 t} + \lambda_1 / (\lambda_1 - \lambda_2) e^{-\lambda_2 t}, \quad t \geq 0.$$

The convolution formula for  $T = T_1 + T_2'$  is

$$\bar{F}_T(t) = \bar{F}_{T_2'}(t) + \int_0^t \bar{F}_{T_1}(t-s) f_{T_2'}(s) ds, \quad t \geq 0.$$

$$\bar{F}_T(t) = \lambda_2 / (\lambda_2 - \lambda_1) e^{-\lambda_1 t} + \lambda_1 / (\lambda_1 - \lambda_2) e^{-\lambda_2 t} + \int_0^t \frac{e^{-\lambda_1(t-s)}}{\lambda_1 \lambda_2} (\lambda_2 - \lambda_1) (e^{-\lambda_1 s} - e^{-\lambda_2 s}) ds$$

$$\bar{F}_T(t) = e^{-\lambda_1 t} \left\{ (\lambda_2^2 - 2\lambda_1\lambda_2) / (\lambda_2 - \lambda_1)^2 + \lambda_1 / (\lambda_2 - \lambda_1) t \right\} + e^{-\lambda_2 t} \left\{ \lambda_1^2 / (\lambda_1 - \lambda_2)^2 \right\}, t \geq 0.$$

Subsections (a) and (b) illustrate that the convolution formula gives a unique result, regardless of the way of choosing the prior random life.

## 2. Reliability Shorthand Notation

$$\text{EXP} \{ \lambda_1 \} + \text{EXP} \{ \lambda_1 \} + \text{EXP} \{ \lambda_2 \} + \text{EXP} \{ \lambda_2 \}$$

Let  $T_1, T_2, T_3, T_4$  be random variables distributed as  $\text{EXP} \{ \lambda_1 \}, \text{EXP} \{ \lambda_1 \}, \text{EXP} \{ \lambda_2 \}, \text{EXP} \{ \lambda_2 \}$  respectively.

Then from the derivation in Section 2,  $T_1' = T_1 + T_2 + T_3$  is a random variable having the survival function

$$\bar{F}_{T_1'}(t) = (a_{11} + a_{12}t) e^{-\lambda_1 t} + a_{21} e^{-\lambda_2 t}, t \geq 0.$$

where  $a_{11} = (\lambda_2^2 - 2\lambda_1\lambda_2) / (\lambda_2 - \lambda_1)^2$ ,  $a_{12} = \lambda_1\lambda_2 / (\lambda_2 - \lambda_1)$ ,

$$a_{21} = \lambda_1^2 / (\lambda_1 - \lambda_2)^2.$$

The convolution formula for  $T = T_1' + T_4$  is

$$\bar{F}_T(t) = \bar{F}_{T_1'}(t) + \int_0^t \bar{F}_{T_1'}(t-s) f_{T_4}(s) ds, t \geq 0.$$

$$\bar{F}_T(t) = (a_{11} + a_{12}t) e^{-\lambda_1 t} + a_{21} e^{-\lambda_2 t} + \int_0^t \frac{e^{-\lambda_2(t-s)}}{\lambda_1 \lambda_2} ((a_{11}\lambda_1 - a_{12} + \lambda_1 a_{12}s) e^{-\lambda_1 s} + \lambda_2 a_{21} e^{-\lambda_2 s}) ds$$



SURVIVAL FUNCTION :

$$\bar{F}(t) = A_1(t) e^{-\lambda_1 t} + A_2(t) e^{-\lambda_2 t} + \dots + A_k(t) e^{-\lambda_k t}, \quad t \geq 0$$

where  $A_1(t) = a_{11} + a_{12}t + a_{13}t^2 + \dots + a_{1n_1}t^{n_1-1}$ ,

$$A_2(t) = a_{21} + a_{22}t + a_{23}t^2 + \dots + a_{2n_2}t^{n_2-1},$$

.....

$$A_k(t) = a_{k1} + a_{k2}t + a_{k3}t^2 + \dots + a_{kn_k}t^{n_k-1}.$$

a. Example:

SHORTHAND NOTATION :

$$\text{EXP} \{ \lambda_1 \} + \text{EXP} \{ \lambda_2 \}$$

$$+ \text{EXP} \{ \lambda_2 \}$$

$$+ \text{EXP} \{ \lambda_3 \} + \text{EXP} \{ \lambda_3 \} + \text{EXP} \{ \lambda_3 \}$$

SURVIVAL FUNCTION :

$$\bar{F}(t) = (a_{11} + a_{12}t) e^{-\lambda_1 t} + a_{21} e^{-\lambda_2 t} + (a_{31} + a_{32}t + a_{33}t^2) e^{-\lambda_3 t}, \quad t \geq 0.$$

b. An Algorithm to Compute the Coefficients

The algorithm represented below develops the survival function by adding one random variable in each run. As an example, in order to compute the survival function for the convolution of ten exponentially distributed random variables, the algorithm is supposed to run ten times.

The notation used in the algorithm is:

$K$         number of dissimilar failure rates  
 $\lambda_i$       ith type failure rate  
 $\lambda_{ie}$      failure rate for the currently entering life  
 $a_{jk}$      kth coefficient on the jth polynomial  
 $n_i$       current number of identical lives having  
          the ith failure rate  
 $nn_i$      number of random variables having ith  
          failure rate.

Initial:     $a_{jk} = 0$ ,  $\forall j, k$  where  $j=1, 2, \dots, K$ ,  $k=1, 2, \dots, n_j$   
 $n_i = 0$ ,  $\forall i$  where  $i=1, 2, \dots, K$

Input :     $\lambda_i$ ,  $\forall i$  where  $i=1, 2, \dots, K$   
 $nn_i$ ,  $\forall i$  where  $i=1, 2, \dots, K$

The first run is :     $n_1 = 1$ ,     $a_{11} = 1$

Algorithm :

$$n_{ie} = n_{ie} + 1, \text{ until } n_{ie} = n n_{ie}.$$

1. Update the coefficients :  $a_{ie}^k$  for  $k=2,3,\dots,n_{ie}$

$$a_{ie}^k = \lambda_i a_{ie}^{m-1} / (m-1), \text{ where } m = n_{ie} - j \text{ for } j=0,1,\dots,n_{ie}-1$$

2. Update the coefficient :  $a_{ie}^{n_{ie}}$

$$a_{ie}^{n_{ie}} = a_{ie}^{n_{ie}-1} + \sum_{i \neq i_e} \{ a_{ie}^{n_{ie}-1} \lambda_i / (\lambda_i - \lambda_{i_e}) + \sum_{j=2}^{n_{ie}} (j-1)! \lambda_i a_{ie}^{n_{ie}-j} / (\lambda_i - \lambda_{i_e})^j \}$$

$n_{ie} > 0 \qquad n_{ie} > 1$

3. Update the other coefficients:  $a_{ik} \forall i, k$

where  $i \neq i_e, n_i \neq 0$  for  $i=1,2,\dots,K, k=1,2,\dots,n_i$

$$a_{in_i} = a_{in_{i_e}} \lambda_i / (\lambda_i - \lambda_{i_e}), \forall i \text{ where } i \neq i_e.$$

$$a_{in} = (\lambda_i a_{in} - m a_{imH}) / (\lambda_i - \lambda_{i_e}), \forall i \text{ where } i \neq i_e \text{ and } n_i \geq 1$$

for  $m = n_i - j, j=1,2,\dots,n_i-1$

c. Example:

Reliability Shorthand Notation :

$$\text{EXP} \{ \lambda_1 \} + \text{EXP} \{ \lambda_1 \} + \text{EXP} \{ \lambda_2 \} + \text{EXP} \{ \lambda_2 \}$$

Let  $T_1, T_2, T_3, T_4$  be random variables distributed as  $\text{EXP} \{ \lambda_1 \}, \text{EXP} \{ \lambda_1 \}, \text{EXP} \{ \lambda_2 \}, \text{EXP} \{ \lambda_2 \}$  respectively.

We would like to derive a formula for the survival function of the random variable  $T=T_1+T_2+T_3+T_4$ .

The use of algorithm:

1<sup>st</sup> RUN;

$$n_1 = 1, a_{11} = 1$$

At the end of this run we have only one random variable, which is distributed  $EXP\{\lambda_1\}$ .

2<sup>nd</sup> RUN;

$$i_e = 1, n_1 = 2$$

$$a_{12} = \lambda_1 a_{11} \quad \text{then} \quad a_{12} = \lambda_1$$

$$a_{11} = a_{11} + 0 \quad \text{then} \quad a_{11} = 1$$

At the end of the 2. run we have the survival function for the  $T'=T_1+T_2$ , where  $T_1$  and  $T_2$  are identically distributed as  $EXP\{\lambda_1\}$ . The survival function is

$$\bar{F}(t) = (a_{11} + a_{12}t) e^{-\lambda_1 t}$$

$$\bar{F}(t) = (1 + \lambda_1 t) e^{-\lambda_1 t}, \quad t \geq 0$$

Which is  $ERLANG\{2, \lambda_1\}$ .

3<sup>rd</sup> RUN;

$$i_e = 2, n_1 = 2, n_2 = 1$$

$$a_{21} = 0 + a_{11}\lambda_1 / (\lambda_1 - \lambda_2) + a_{12}\lambda_2 / (\lambda_1 - \lambda_2)^2$$

$$= \lambda_1 / (\lambda_1 - \lambda_2) + \lambda_1\lambda_2 / (\lambda_1 - \lambda_2)^2$$

$$= \lambda_1^2 / (\lambda_1 - \lambda_2)^2$$

$$a_{12} = a_{12}\lambda_2 / (\lambda_2 - \lambda_1) \quad \text{then} \quad a_{12} = \lambda_1\lambda_2 / (\lambda_2 - \lambda_1)$$

$$a_{11} = (\lambda_2 a_{11} - a_{12}) / (\lambda_2 - \lambda_1) \quad \text{then} \quad a_{11} = (\lambda_2^3 - 2\lambda_1\lambda_2) / (\lambda_2 - \lambda_1)^2$$

Note that, here the coefficient  $a_{12}$  is the updated one.

At the end of the 3. run we have the survival function for the random variable  $T'' = T_1 + T_2 + T_3$ , where  $T_1, T_2, T_3$  are as defined before.

$$\bar{F}_{T''}(t) = (a_{11} + a_{12}t) e^{-\lambda_1 t} + a_{21} e^{-\lambda_2 t}, \quad t \geq 0$$

where  $a_{11} = (\lambda_2^3 - 2\lambda_1\lambda_2) / (\lambda_2 - \lambda_1)^2$ ,  $a_{12} = \lambda_1\lambda_2 / (\lambda_2 - \lambda_1)$ ,

$$a_{21} = \lambda_1^2 / (\lambda_1 - \lambda_2)^2.$$

Note that the coefficients are identical to the result of Subsection (b).

4<sup>th</sup> RUN;

$$i_e = 2, \quad n_1 = 2, \quad n_2 = 2$$

$$a_{22} = \lambda_2 a_{21} \quad \text{then} \quad a_{22} = \lambda_1^2 \lambda_2 / (\lambda_1 - \lambda_2)^2$$

$$a_{21} = a_{21} + a_{11}\lambda_1 / (\lambda_1 - \lambda_2) + a_{12}\lambda_2 / (\lambda_1 - \lambda_2)^2$$

$$\text{then} \quad a_{21} = (\lambda_1^3 - 3\lambda_2^2\lambda_2) / (\lambda_1 - \lambda_2)^3$$

$$a_{12} = a_{12}\lambda_2 / (\lambda_2 - \lambda_1) \quad \text{then} \quad a_{12} = \lambda_1\lambda_2^2 / (\lambda_2 - \lambda_1)^2$$

$$a_{11} = (\lambda_2 a_{11} - a_{12}) / (\lambda_2 - \lambda_1) \quad \text{then} \quad a_{11} = (\lambda_2^3 - 3\lambda_1\lambda_2^2) / (\lambda_2 - \lambda_1)^3$$

At the end of the 4. run, we have the survival function for the variable  $T = T_1 + T_2 + T_3 + T_4$ .

$$\bar{F}_T(t) = (a_{11} + a_{12}t) e^{-\lambda_1 t} + (a_{21} + a_{22}t) e^{-\lambda_2 t}, \quad t \geq 0$$

$$\text{where } a_{11} = (\lambda_2^3 - 3\lambda_1\lambda_2^2) / (\lambda_2 - \lambda_1)^3, \quad a_{12} = \lambda_1\lambda_2^2 / (\lambda_2 - \lambda_1)^2,$$

$$a_{21} = (\lambda_1^3 - 3\lambda_1^2\lambda_2) / (\lambda_1 - \lambda_2)^3, \quad a_{22} = \lambda_1^2\lambda_2 / (\lambda_1 - \lambda_2)^2.$$

Note that the coefficients are identical to the result of Subsection 2.

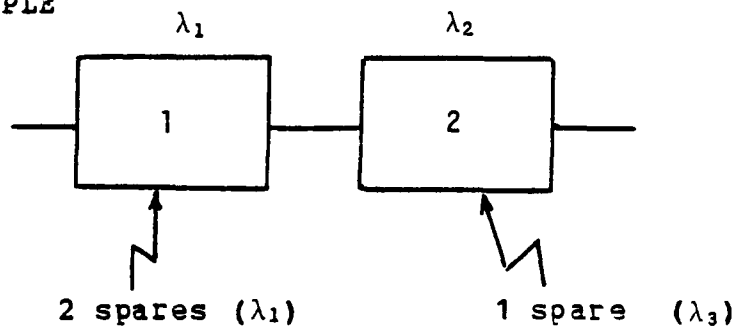
### III. RELIABILITY SHORTHAND APPROACH TO SYSTEM RELIABILITY

This section deals with a system whose components have constant failure rates.

Having the reliability network for a system in which each component has an exponential life and knowing the probabilities for failures (discussed in Appendix E2) makes it easy to describe the system's life.

In order to make the idea clear, we will go through some examples.

#### A. EXAMPLE

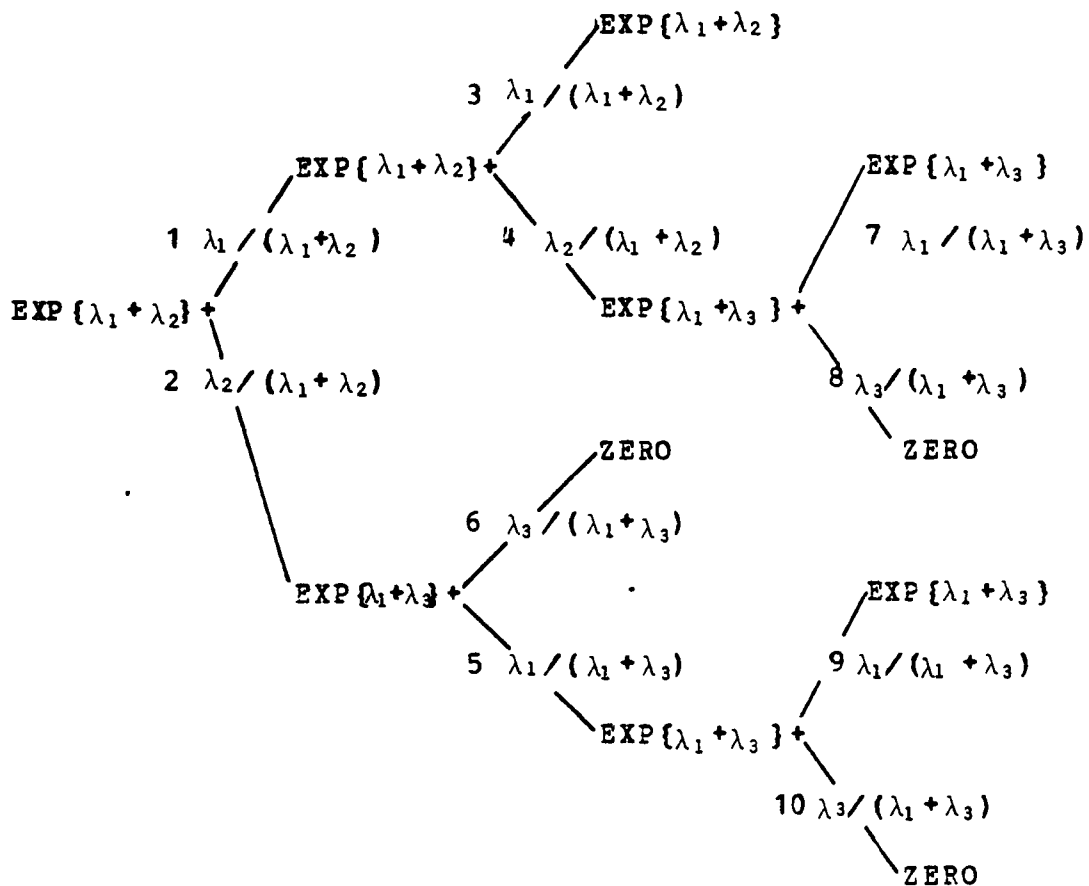


SYSTEM : 2 components in series with spares.

DESCRIPTION : System has 2 components in series. Component 1 has a life distributed as  $EXP\{\lambda_1\}$  and there are 2 identical spares. Component 2 has a life distributed as

EXP $\{\lambda_2\}$  and there is one non-identical spare, whose life is distributed as EXP $\{\lambda_3\}$ .

LIFE :



Explanation for the derivation of system's life:

At the beginning the system has an EXP $\{\lambda_1 + \lambda_2\}$  life. The failure of the system is by the failure of component 1 with a probability of  $\lambda_1 / (\lambda_1 + \lambda_2)$  or by the failure of component 2 with probability of  $\lambda_2 / (\lambda_1 + \lambda_2)$ .

In the life figure, number 1 denotes the event "failure of component 1" and number 2 denotes the event "failure of component 2".

If event 1 occurs, component 1 is replaced by one of the spares and system again functions with a life distributed as  $EXP\{\lambda_1 + \lambda_2\}$ , since the exponential distribution has the memoryless property and component 2 still has the same failure rate.

If the event 2 occurs, component 2 is replaced by its spare and the system has a life distributed as  $EXP\{\lambda_1 + \lambda_3\}$ .

The numbers on the life figure correspond to the transitions that can occur. The probability on each arc shows the conditional probability of the transition. As an example event 3 can occur with probability of  $\lambda_1 / (\lambda_1 + \lambda_2)$ , given that event 1 has occurred before.

The distribution ZERO defined by Esary [Ref.1] and Repicky [Ref.2] (also defined in Appendix E3) enters when life is exhausted.

For convenience of description, it is helpful to define the concept of path used in this paper. Path denotes the

sequence of events in the system's life from the starting point to the point where the system is not functioning.

Examples;

Events 1 and 3 are a path, which denotes a sequence of lives for the system. In this case, the system has 3 exponentially distributed lives  $EXP\{\lambda_1 + \lambda_2\} + EXP\{\lambda_1 + \lambda_2\} + EXP\{\lambda_1 + \lambda_2\}$  with the probability of  $\lambda_1 / (\lambda_1 + \lambda_2) \lambda_1 / (\lambda_1 + \lambda_2)$ .

Events 1, 4 and 8 form another path, which describes a sequence of lives for the system. In this path, the system life is  $EXP\{\lambda_1 + \lambda_2\} + EXP\{\lambda_1 + \lambda_2\} + EXP\{\lambda_1 + \lambda_3\} + ZERO$ . The probability of this path is

$$\left[ \lambda_1 / (\lambda_1 + \lambda_2) \right] \left[ \lambda_2 / (\lambda_1 + \lambda_2) \right] \left[ \lambda_3 / (\lambda_1 + \lambda_3) \right].$$

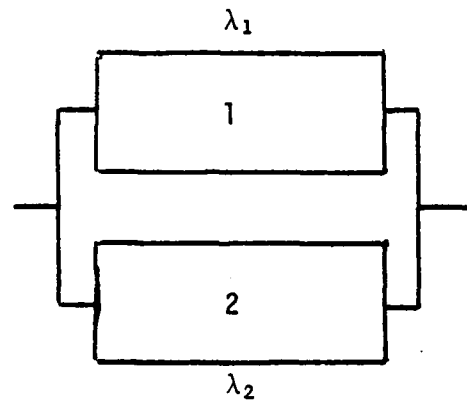
The ZERO distribution contributes zero additional life to the system, so we can omit it. Nevertheless, we can not omit its probability in the calculation of path probability, since the event numbered 8 has a probability of occurring.

## B. THE SURVIVAL FUNCTION FOR THE SYSTEM

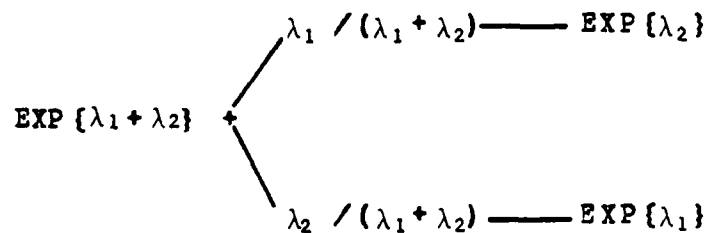
In this section we will deal with an approach to obtaining the system's survival function.

### 1. EXAMPLES

SYSTEM: Two components in parallel.



LIFE :



DESCRIPTION: Component 1 has  $\text{EXP}\{\lambda_1\}$  life and component 2 has  $\text{EXP}\{\lambda_2\}$  life.

There are two failure events that can be allowed. The first is that component 1 fails at some time  $t$  and component 2 carries the system for the rest of the time. The other one

is that component 2 fails at some time during the mission and component 1 carries the system for the rest of the time. There are more events such as no failures during the mission duration, which are taken care of by the representation.

Now we have two paths

<u>No. of Path</u>	<u>Weight</u>	<u>Life</u>
1	$P_1 = \lambda_1 / (\lambda_1 + \lambda_2)$	$\text{EXP} \{ \lambda_1 + \lambda_2 \} + \text{EXP} \{ \lambda_2 \}$
2	$P_2 = \lambda_2 / (\lambda_1 + \lambda_2)$	$\text{EXP} \{ \lambda_1 + \lambda_2 \} + \text{EXP} \{ \lambda_1 \}$

Let  $T$  be the system's time to failure and let  $T_1, T_2, T_3$  be random variables exponentially distributed with the failure rates  $\lambda_1 + \lambda_2, \lambda_2, \lambda_1$  respectively.

Then

$$T = \begin{cases} T_1 + T_2 & \text{with probability } p_1 \\ T_1 + T_3 & \text{with probability } p_2 \end{cases}$$

The survival function can be written as

$$\bar{F}(t) = p_1 \bar{F}_1(t) + p_2 \bar{F}_2(t) \quad , \quad t \geq 0$$

where  $p_1 = \lambda_1 / (\lambda_1 + \lambda_2)$ ,  $p_2 = \lambda_2 / (\lambda_1 + \lambda_2)$  and  $\bar{F}_1(t)$ ,  $\bar{F}_2(t)$  denote the survival functions for the shorthand notations  $\text{EXP} \{ \lambda_1 + \lambda_2 \} + \text{EXP} \{ \lambda_2 \}$ ,  $\text{EXP} \{ \lambda_1 + \lambda_2 \} + \text{EXP} \{ \lambda_1 \}$  respectively.

The survival function for the convolution of two exponentially distributed random variables with dissimilar failure rates is

$$\bar{F}(t) = \lambda_2 / (\lambda_2 - \lambda_1) e^{-\lambda_1 t} + \lambda_1 / (\lambda_1 - \lambda_2) e^{-\lambda_2 t}, t \geq 0.$$

If we do the substitutions for  $\bar{F}_1(t)$  and  $\bar{F}_2(t)$  as  $\lambda_1 = \lambda_1 + \lambda_2$ ,  $\lambda_2 = \lambda_2$  and  $\lambda_1 = \lambda_1 + \lambda_2$ ,  $\lambda_2 = \lambda_1$  respectively,  $\bar{F}_1(t)$  becomes

$$\bar{F}_1(t) = \lambda_2 / (-\lambda_1) e^{-(\lambda_1 + \lambda_2)t} + (\lambda_1 + \lambda_2) / \lambda_1 e^{-\lambda_2 t}, t \geq 0,$$

and  $\bar{F}_2(t)$  becomes

$$\bar{F}_2(t) = \lambda_1 / (-\lambda_2) e^{-(\lambda_1 + \lambda_2)t} + (\lambda_1 + \lambda_2) / \lambda_2 e^{-\lambda_2 t}, t \geq 0.$$

Then

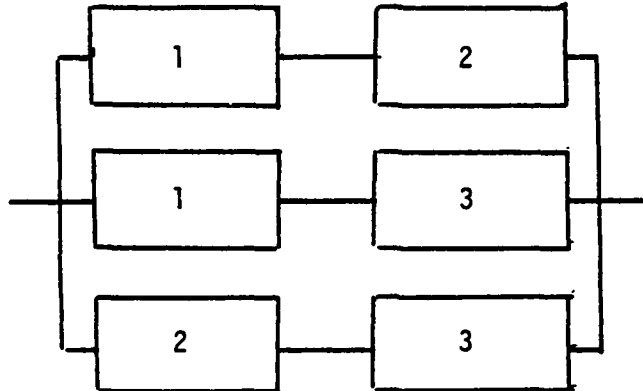
$$\bar{F}(t) = \lambda_1 / (\lambda_1 + \lambda_2) \bar{F}_1(t) + \lambda_2 / (\lambda_1 + \lambda_2) \bar{F}_2(t)$$

$$\bar{F}(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}, t \geq 0.$$

The result gives the survival function that is well known for this system.

Another example is the 2 out of 3 system.

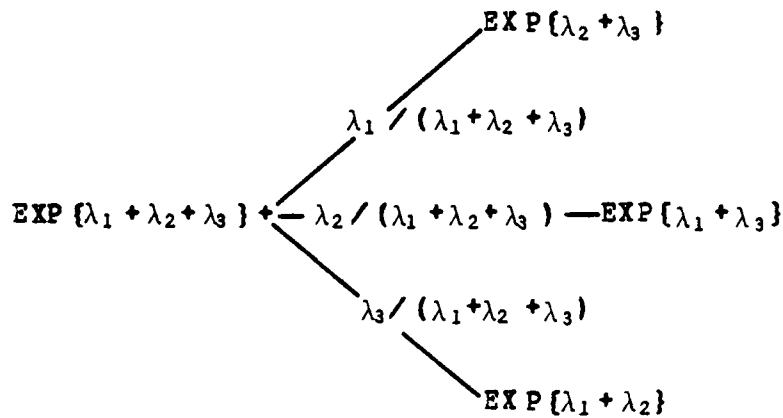
RELIABILITY NETWORK:



The known survival function is

$$\bar{F}(t) = e^{-(\lambda_1 + \lambda_2)t} + e^{-(\lambda_1 + \lambda_3)t} + e^{-(\lambda_2 + \lambda_3)t} - 2e^{-(\lambda_1 + \lambda_2 + \lambda_3)t}, \quad t \geq 0.$$

LIFE:



$$T = \begin{cases} \text{EXP}\{\lambda_1 + \lambda_2 + \lambda_2\} + \text{EXP}\{\lambda_2 + \lambda_3\} & \text{with probability } \lambda_1 / (\lambda_1 + \lambda_2 + \lambda_3) \\ \text{EXP}\{\lambda_1 + \lambda_2 + \lambda_3\} + \text{EXP}\{\lambda_1 + \lambda_3\} & \text{with probability } \lambda_2 / (\lambda_1 + \lambda_2 + \lambda_3) \\ \text{EXP}\{\lambda_1 + \lambda_2 + \lambda_3\} + \text{EXP}\{\lambda_1 + \lambda_2\} & \text{with probability } \lambda_3 / (\lambda_1 + \lambda_2 + \lambda_3) \end{cases}$$

The survival function is

$$\bar{F}(t) = p_1 \bar{F}_1(t) + p_2 \bar{F}_2(t) + p_3 \bar{F}_3(t), \quad t \geq 0.$$

where

$$\bar{F}_1(t) = (\lambda_2 + \lambda_3) / (-\lambda_1) e^{-(\lambda_1 + \lambda_2 + \lambda_3)t} + (\lambda_1 + \lambda_2 + \lambda_3) / \lambda_1 e^{-(\lambda_2 + \lambda_3)t}$$

$$\bar{F}_2(t) = (\lambda_1 + \lambda_3) / (-\lambda_2) e^{-(\lambda_1 + \lambda_2 + \lambda_3)t} + (\lambda_1 + \lambda_2 + \lambda_3) / \lambda_2 e^{-(\lambda_1 + \lambda_3)t}$$

$$\bar{F}_3(t) = (\lambda_1 + \lambda_2) / (-\lambda_3) e^{-(\lambda_1 + \lambda_2 + \lambda_3)t} + (\lambda_1 + \lambda_2 + \lambda_3) / \lambda_3 e^{-(\lambda_1 + \lambda_2)t}$$

If we do the necessary cancellations, we can get,

$$\bar{F}(t) = e^{-(\lambda_2 + \lambda_3)t} + e^{-(\lambda_1 + \lambda_3)t} + e^{-(\lambda_1 + \lambda_2)t} - 2e^{-(\lambda_1 + \lambda_2 + \lambda_3)t}, \quad t \geq 0.$$

as desired.

## 2. General Procedure

Having the algorithm presented in Section II, we can treat more complicated systems under similar assumptions.

The procedure is

- i. Set up the reliability network for the system.
- ii. According to this network, set up the system life.
- iii. Using the proper reliability shorthand formulas for the related convolutions of exponential lives, set up the survival function.

#### IV. SUMMARY

The reliability shorthand is an easy way to describe a system's life, but it is difficult to implement computationally since there is considerable complexity in handling convolutions.

The algorithm presented in this paper gives some relief from this difficulty. However, the accuracy in obtained from this algorithm is very much related to the differences in the failure rates.

Another aspect in the algorithm is that distributions are convolved one at a time and this requires very accurate running conditions in the case of a complicated system.

It is believed that it is possible to derive another algorithm which is more powerful than the one introduced here. Instead of adding one distribution at a time, one can try to add several at a time.

## APPENDIX A

A-1

This section contains the survival functions for several shorthand notations which were derived by the use of the approach described in Section II.

A. 1. 1 Shorthand Notation : EXP{  $\lambda$  }.

Survival Function:  $\bar{F}(t) = e^{-\lambda t}$  ,  $t \geq 0$ .

A. 1. 2. 1 Shorthand Notation: EXP{  $\lambda_1$  } + EXP{  $\lambda_2$  }

Survival Function:

$$\bar{F}(t) = \lambda_2 / (\lambda_2 - \lambda_1) e^{-\lambda_1 t} + \lambda_1 / (\lambda_1 - \lambda_2) e^{-\lambda_2 t} , t \geq 0.$$

A. 1. 2. 2 Shorthand Notation : EXP{  $\lambda$  } + EXP{  $\lambda$  }.

Survival Function:

$$\bar{F}(t) = (1 + \lambda t) e^{-\lambda t} , t \geq 0.$$

A. 1. 3. 1 Shorthand Notation : EXP{  $\lambda_1$  } + EXP{  $\lambda_2$  } + EXP{  $\lambda_3$  }

Survival Function:

$$\bar{F}(t) = a_{11} e^{-\lambda_1 t} + a_{21} e^{-\lambda_2 t} + a_{31} e^{-\lambda_3 t} , t \geq 0.$$

where  $a_{11} = \lambda_2 \lambda_3 / (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_1)$  ,  $a_{21} = \lambda_1 \lambda_3 / (\lambda_1 - \lambda_2) (\lambda_3 - \lambda_2)$  ,

$$a_{31} = \lambda_1 \lambda_2 / (\lambda_1 - \lambda_3) (\lambda_2 - \lambda_3) .$$

A. 1.3.2 EXP{λ} + EXP{λ} + EXP{λ}

Survival Function:

$$\bar{F}(t) = (1 + \lambda t + 1/2 \lambda^2 t^2) e^{-\lambda t} , t \geq 0 .$$

A. 1.3.3 EXP{λ<sub>1</sub>} + EXP{λ<sub>2</sub>} + EXP{λ<sub>2</sub>}

Survival Function:

$$\bar{F}(t) = a_{11} e^{-\lambda_1 t} + (a_{21} + a_{22} t) e^{-\lambda_2 t} , t \geq 0$$

where  $a_{11} = \lambda_2^2 / (\lambda_2 - \lambda_1)^2$  ,  $a_{21} = (\lambda_1^2 - 2\lambda_1 \lambda_2) / (\lambda_1 - \lambda_2)^2$

$$a_{22} = \lambda_1 \lambda_2 / (\lambda_1 - \lambda_2) .$$

A. 1.4.1 EXP{λ<sub>1</sub>} + EXP{λ<sub>2</sub>} + EXP{λ<sub>3</sub>} + EXP{λ<sub>4</sub>}

Survival Function:

$$\bar{F}(t) = a_{11} e^{-\lambda_1 t} + a_{21} e^{-\lambda_2 t} + a_{31} e^{-\lambda_3 t} + a_{41} e^{-\lambda_4 t} , t \geq 0 .$$

where  $a_{i1} = \prod_{j \neq i} \lambda_j / \prod_{j \neq i} (\lambda_j - \lambda_i) \quad \forall i=1,2,3,4$

A. 1.4.2 EXP{λ} + EXP{λ} + EXP{λ} + EXP{λ}

Survival Function:

$$\bar{F}(t) = (1 + \lambda t + 1/2 \lambda^2 t^2 + 1/6 \lambda^3 t^3) e^{-\lambda t} , t \geq 0 .$$

A. 1.4.3 EXP{λ<sub>1</sub>} + EXP{λ<sub>2</sub>} + EXP{λ<sub>2</sub>} + EXP{λ<sub>2</sub>}

Survival Function:

$$\bar{F}(t) = a_{11} e^{-\lambda_1 t} + (a_{21} + a_{22} t + a_{23} t^2) e^{-\lambda_2 t}, \quad t \geq 0.$$

where  $a_{11} = \lambda_2^3 / (\lambda_2 - \lambda_1)^3$ ,  $a_{21} = 1 - a_{11}$ ,

$$a_{22} = \lambda_1 \lambda_2 (\lambda_1 - 2\lambda_2) / (\lambda_1 - \lambda_2)^2, \quad a_{23} = \lambda_1 \lambda_2^2 / 2 (\lambda_1 - \lambda_2).$$

#### A. 1.4.4 EXP( $\lambda_1$ ) + EXP( $\lambda_1$ ) + EXP( $\lambda_2$ ) + EXP( $\lambda_2$ )

Survival Function:

$$\bar{F}(t) = (a_{11} + a_{12} t) e^{-\lambda_1 t} + (a_{21} + a_{22} t) e^{-\lambda_2 t}, \quad t \geq 0.$$

where  $a_{11} = (\lambda_2^3 - 3\lambda_2^2 \lambda_1) / (\lambda_2 - \lambda_1)^3$ ,  $a_{12} = \lambda_1 \lambda_2^2 / (\lambda_2 - \lambda_1)^2$ ,

$$a_{21} = (\lambda_1^3 - 3\lambda_1^2 \lambda_2) / (\lambda_1 - \lambda_2)^3, \quad a_{22} = \lambda_2 \lambda_1^2 / (\lambda_1 - \lambda_2)^2.$$

#### A. 1.4.5 EXP( $\lambda_1$ ) + EXP( $\lambda_2$ ) + EXP( $\lambda_3$ ) + EXP( $\lambda_3$ )

Survival Function:

$$\bar{F}(t) = a_{11} e^{-\lambda_1 t} + a_{21} e^{-\lambda_2 t} + (a_{31} + a_{32} t) e^{-\lambda_3 t}, \quad t \geq 0.$$

where  $a_{11} = \lambda_2 \lambda_3^2 / (\lambda_2 - \lambda_1) (\lambda_3 - \lambda_1)^2$ ,  $a_{21} = \lambda_1 \lambda_2^2 / (\lambda_1 - \lambda_2) (\lambda_3 - \lambda_2)^2$ ,

$$a_{31} = \lambda_1 \lambda_2 / (\lambda_1 - \lambda_3) (\lambda_2 - \lambda_3) + \lambda_1 \lambda_2 \lambda_3 [ 1 / (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3)^2 - 1 / (\lambda_1 - \lambda_2) (\lambda_2 - \lambda_3)^2 ]$$

$$a_{32} = \lambda_1 \lambda_2 \lambda_3 / (\lambda_1 - \lambda_3) (\lambda_2 - \lambda_3).$$

This section introduces a Fortran program using the algorithm described in Section II.

A PROGRAM FOR THE ALGORITHM ONE AT A TIME:

```

CCCCCCCCCCCCCCCC
THIS IS A PROGRAM TO COMPUTE THE RELIABILITY OF A SYSTEM
      WHICH HAS THE RELIABILITY SHORTHAND NOTATION
      EXP (L1)+...+EXP (LN)
WHERE THERE IS NO RESTRICTION FOR THE FAILURE RATES.
      VARIABLES:
A (I,J)      : I.TH TYPE FAILURE RATE, J.TH COEFFICIENT
              ON THE POLINOM.
NI (I)       : AMOUNT OF LIVES HAVING THE I.TH TYPE
              FAILURE RATE.
NINIT (I)    : AUXILAURY ARRAY FOR THE NI (I). INPUT
              FOR THE PROGRAM.
L (I)        : THE ARRAY FOR THE FAILURE RATES

      REAL L (20), A (20,10)
      INTEGER NINIT (20), NI (20)

              GET INPUT

      READ (5,499) T
      WRITE (6,498) T
      READ (5,500) K
      WRITE (6,1500) K
      READ (5,501) (NINIT (I), L (I), I=1, K)
      WRITE (6,1501) (NINIT (I), L (I), I=1, K)

              COMPUTE COEFFICIENTS ONE ALL
              DISSIMILAR

      KK=1
2     A (KK, 1) = 1.
      NI (KK) = 1
          JJ=1
4     IF (JJ.EQ.KK) GO TO 6
      A (KK, 1) = A (KK, 1) * L (JJ) / (L (JJ) - L (KK))
6     IF (JJ.EQ.K) GO TO 9
          JJ=JJ+1
          GO TO 4
9     IF (KK.EQ.K) GO TO 8
      KK=KK+1
      GO TO 2
8     CONTINUE

              BEGIN TO ADD ONE AT A TIME

      IE=1
32    IF (NINIT (IE).EQ.NI (IE)) GO TO 99
      NI (IE) = NI (IE) + 1

              UPDATE IE

```

```

C
12   J=0
    NNNN=NI(IE)-J
    A(IE,NNNN)=L(IE)*A(IE,NNNN-1)/FLOAT(NNNN-1)
    IF(NNNN.EQ.2) GO TO 20
    J=J+1
    GO TO 12
20   SUM=0.
    DO 101 I=1,K
    IF(I.EQ.IE) GO TO 101
    SUM=SUM+A(I,1)*L(I)/(L(I)-L(IE))
    IF(NI(I).LT.2) GO TO 101
    FACT=1.
    NKK=NI(I)
    DO 102 II=2,NKK
    FACT=FACT*(II-1)
    SUM=SUM+A(I,II)*FACT*L(IE)/(L(I)-L(IE)**II
102  CONTINUE
101  CONTINUE
    A(IE,1)=SUM+A(IE,1)

                                UPDATE I.NE.IE

C
C
C
21   I=1
    IF(I.EQ.IE) GO TO 26
    A(I,NI(I))=A(I,NI(I))*L(IE)/(L(IE)-L(I))
    IF(NI(I).EQ.1) GO TO 26
    J=1
24   NNI=NI(I)-J
    A(I,NNI)=(L(IE)*A(I,NNI)-FLOAT(NNI)*A(I,NNI+1))
    * / (L(IE)-L(I))
    IF(J.EQ.(NI(I)-1)) GO TO 26
    J=J+1
    GO TO 24
26   IF(I.EQ.K) GO TO 32
    I=I+1
    GO TO 21
99   CONTINUE
    IF(IE.EQ.K) GO TO 35
    IE=IE+1
    GO TO 32

                                CALCULATE THE PROBABILITY

C
C
C
35   CONTINUE
    PRO=0.
    DO 40 I=1,K
    SUM=0.
    NKK=NI(I)
    DO 41 J=1,NKK
    SUM=SUM+A(I,J)*T*(J-1)
41   CONTINUE
    PRO=PRO+SUM*EXP(-L(I)*T)
40   CONTINUE

                                PRINT OUT THE RESULTS

C
C
C
    WRITE(6,600)
    DO 17 I=1,K
    NKK=NI(I)
    WRITE(6,600) L(I),NI(I)
17   WRITE(6,603) (A(I,J),J=1,NKK)
    WRITE(6,601) T,PRO
    STOP
498  FORMAT('1',10X,'THE INPUT IS:',//15X,'TIME=',F10.4)
499  FORMAT(F10.4)
500  FORMAT(3X,I3)
501  FORMAT(3X,I3,F10.5)

```





```

WRITE(6,1499) T
READ(5,500) IK
WRITE(6,1500) IK
READ(5,3) (P(I),I=1,IK)
RETURN
3  FORMAT(7F10.7)
499  FORMAT(5X,F10.4)
500  FORMAT(5X,I5)
1499  FORMAT('1',10X,'TIME=',F10.4)
1500  FORMAT('0',10X,'IK  ='5X,I3)
END

```

C  
C  
C

```

SUBROUTINE ONECON(PRO,T)
REAL A(20,10),L(20)
INTEGER NI(20),NINIT(20)
CALL READ(K,L,NINIT)
CALL ONEDIS(K,A,NI,L)
CALL ONEATA(K,A,NI,L,NINIT)
CALL CALPRO(K,A,NI,L,PRO,T)
RETURN
END

```

C  
C  
C

```

SUBROUTINE READ(K,L,NINIT)
REAL L(20)
INTEGER NINIT(20)
READ(5,1) K
WRITE(6,1501) K
READ(5,52) (L(I),NINIT(I),I=1,K)
RETURN
1  FORMAT(5X,I5)
52  FORMAT(5X,F10.5,I5)
1501  FORMAT(10X,'K  ='5X,I2,5X,' (# OF DISSIMILAR FAILURE
* RATES.) ')
END

```

C

```

SUBROUTINE ONEDIS(K,A,NI,L)
REAL A(20,10),L(20)
INTEGER NI(20)
KK=1
2  A(KK,1)=1.
   NI(KK)=1
   JJ=1
4  IF(JJ.EQ.KK) GO TO 6
   A(KK,1)=A(KK,1)*L(JJ)/(L(JJ)-L(KK))
6  IF(JJ.EQ.K) GO TO 9
   JJ=JJ+1
   GO TO 4
9  IF(KK.EQ.K) GO TO 8
   KK=KK+1
   GO TO 2
8  RETURN
END

```

C  
C  
C

BEGIN TO ADD ONE AT A TIME

```

SUBROUTINE ONEATA(K,A,NI,L,NINIT)
REAL L(20),A(20,10)
INTEGER NINIT(20),NI(20)
IE=1
32  IF(NINIT(IE).EQ.NI(IE)) GO TO 99
   NI(IE)=NI(IE)+1

```

C  
C  
C

UPDATE IE

```

12      J=0
      NN=NI(IE)-J
      A(IE,NN)=L(IE)*A(IE,NN-1)/FLOAT(NN-1)
      IF(NN.EQ.2) GO TO 20
      J=J+1
      GO TO 12
20 CONTINUE
      DO 101 I=1,K
      IF(I.EQ.IE) GO TO 101
      A(IE,I)=A(IE,I)+A(I,I)*L(I)/(L(I)-L(IE))
      IF(NI(I).LT.2) GO TO 101
      FACT=1.
      NKK=NI(I)
      DO 102 II=2,NKK
      FACT=FACT*(II-1)
      A(IE,I)=A(IE,I)+A(I,II)*FACT*L(IE)/(L(I)-L(IE))*II
102 CONTINUE
101 CONTINUE

```

UPDATE I.NE.IE

```

I=1
21 IF(I.EQ.IE) GO TO 26
   A(I,NI(I))=A(I,NI(I))*L(IE)/(L(IE)-L(I))
   IF(NI(I).EQ.1) GO TO 26
   J=1
24   NNI=NI(I)-J
   * A(I,NNI)=(L(IE)*A(I,NNI)-FLOAT(NNI)*A(I,NNI+1))
     / (L(IE)-L(I))
   IF(J.EQ.(NI(I)-1)) GO TO 26
   J=J+1
   GO TO 24
26 IF(I.EQ.K) GO TO 32
   I=I+1
   GO TO 21
99 CONTINUE
   IF(IE.EQ.K) GO TO 35
   IE=IE+1
   GO TO 32
35 RETURN
   END

```

```

SUBROUTINE CALPRO(K,A,NI,L,PRO,T)
REAL L(20),A(20,10)
INTEGER NI(20)
PRO=0.
DO 40 I=1,K
SUM=0.
NKK=NI(I)
DO 41 J=1,NKK
SUM=SUM+A(I,J)*T*(J-1)
41 CONTINUE
TTT=-L(I)*T
PRO=PRO+SUM*EXP(TTT)
40 CONTINUE
WRITE(6,1010)
DO 1000 I=1,K
NNN=NI(I)
WRITE(6,1001) L(I),NI(I),(A(I,J),J=1,NNN)
1000 CONTINUE
RETURN
1001 FORMAT(10X,F10.5,5X,I3,5X,2(5(E12.5,5X),/30X))
1010 FORMAT(15X,'LAMBDA ',6X,'NI',5X,'COEFFICIENTS',/14X,
135(' - '))
END

```

INPUT FOR THE PROGRAM :

TIME	15.				
#ROWS	5				
1. ST	3	.2	.2	.2	.2
	.1		3		
	.8		2		
	4		1		
2. ND	3		2		
	.2		4		
	.5		1		
	6		3		
3. RD	3		4		
	.1		2		
	.5		3		
	3		2		
4. TH	2		3		
	.1		3		
	.2		3		
5. TH	3		2		
	.1		4		
	.2		3		
	.4				

OUTPUT FROM THE PROGRAM :

```

TIME= 15.0000
IK = 5
1. P(1) = 0.2
   K = 3 (# OF DISSIMILAR FAILURE RATES.)
   LAMDA NI COEFFICIENTS
-----
0.10 3 0.11294E+01 0.66342E-01 0.87075E-02
0.80 2 0.18742E-01 0.23324E-02
0.40 1 -0.14815E+00
2. P(2) = 0.2
   K = 3 (# OF DISSIMILAR FAILURE RATES.)
   LAMDA NI COEFFICIENTS
-----
0.20 2 -0.25077E+02 0.23148E+01
0.50 4 -0.13017E+03 0.21333E+02 -0.22222E-01 0.55556E-01
0.60 1 0.15625E+03
3. P(3) = 0.2
   K = 3 (# OF DISSIMILAR FAILURE RATES.)
   LAMDA NI COEFFICIENTS
-----
0.10 3 0.75531E+01 -0.54932E+00 0.27466E-01
0.50 4 -0.89945E+01 -0.99536E+00 -0.42847E-01 -0.73242E-03
0.30 2 0.24414E+01 -0.14648E+01
4. P(4) = 0.2
   K = 2 (# OF DISSIMILAR FAILURE RATES.)
   LAMDA NI COEFFICIENTS
-----
0.10 3 0.32000E+02 -0.16000E+01 0.40000E-01
0.20 3 -0.31000E+02 -0.14000E+01 -0.20000E-01
5. P(5) = 0.2
   K = 3 (# OF DISSIMILAR FAILURE RATES.)
   LAMDA NI COEFFICIENTS
-----
0.10 2 -0.15170E+03 0.37926E+01
0.20 4 0.14400E+03 0.12800E+02 0.32000E+00 0.10667E-01
0.40 3 0.87037E+01 0.51851E+00 0.88889E-02
PROBABILITY ( TIME > 15.0 ) = 0.911188

```

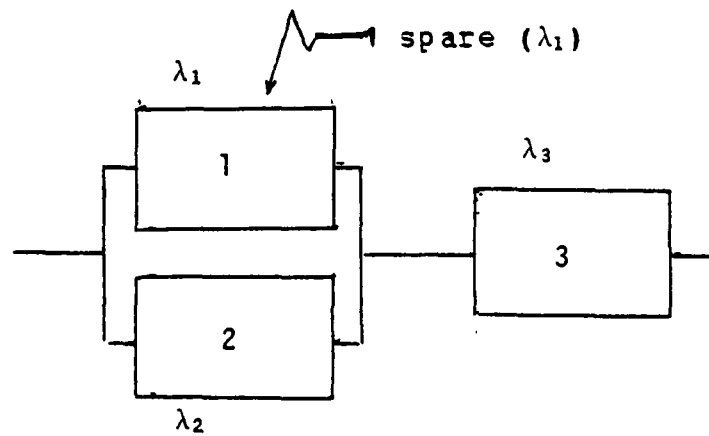
## APPENDIX C

### INTRODUCTION

This section consists of a computer program to compute system reliabilities as described in Section III.

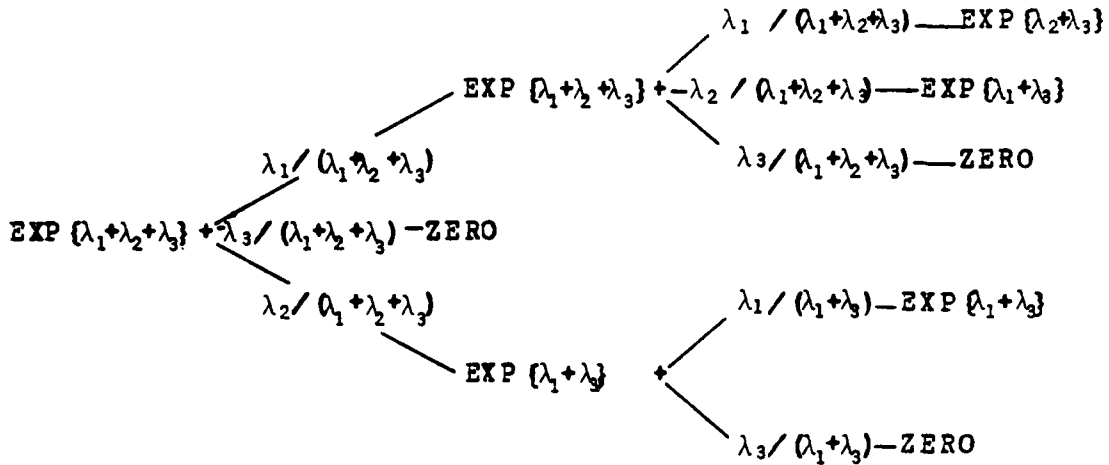
Again for simplicity, we will go through an example.

The structure of the system is



Components 1, 2 and 3 have lives exponentially distributed with failure rates  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  respectively. Also we have a spare for component 1 which is identical to component 1.

Using the shorthand approach, the system life would be determined as follows

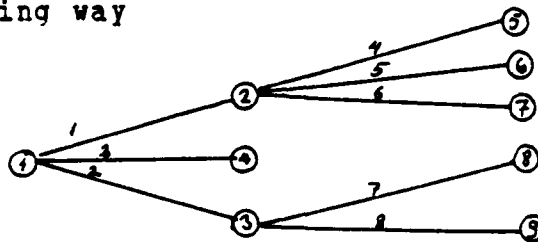


Some definitions are necessary before describing the program

**NODE:** Each node represents an exponentially distributed random variable with a certain failure rate. The number for any node can be chosen arbitrarily but can not be used more than once.

**ARC :** Each arc originates at a node and leads to another node. Only one arc can terminate at a given node. Arc numbers can be chosen arbitrarily. If  $n$  is the total number of nodes in a system life then there will be  $n-1$  arcs in this system.

With these definitions, we can represent a system life in the following way



BACK POINTER LIST (IPB): Each node has a back pointer. A back pointer is an arc number which shows which arc connects the node to the tree. If the pointer is zero, then the related node is the root of the tree.

<u>NODE NO</u>	<u>BACK POINTER IPB(I)</u>
1	0
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8

(Here node 1 is the root of the tree.)

NODE CODE LIST: As we mentioned, each node represents an exponential lifetime, for simplicity we can use some integer code numbers for each failure rate.

<u>Code No.</u>	<u>Related Failure Rate-L(I)</u>
0	(distribution ZERO)
1	$\lambda_1 + \lambda_2 + \lambda_3$
2	$\lambda_1 + \lambda_3$
3	$\lambda_1 + \lambda_2$

Here only the code number 0 is not arbitrary and 0 can be used for the ZERO distribution. The others can be picked out arbitrarily.

ARC CODE LIST: This list is similar to the node code list. Arc code numbers represent probabilities.

<u>Code No.</u>	<u>Probability-PA(I)</u>
1	$\lambda_1 / (\lambda_1 + \lambda_2 + \lambda_3)$
2	$\lambda_2 / (\lambda_1 + \lambda_2 + \lambda_3)$
3	$\lambda_3 / (\lambda_1 + \lambda_2 + \lambda_3)$
4	$\lambda_1 / (\lambda_1 + \lambda_3)$
5	$\lambda_3 / (\lambda_1 + \lambda_3)$

ARC ORIGIN LIST : Each arc has an origin node and a terminal node. In the program we need to use only the origin list.

<u>Arc No.</u>	<u>Origin Node IO(I)</u>
1	1
2	1
3	1
4	2
5	2
6	2
7	3
8	3

LAST POINT NODE LIST (LP): This list indicates the nodes on the end of each path. There is no necessary order in the list.

LAST POINT DEPTH LIST (IPD): This list indicates the number of arcs from last node to the root of the tree.

<u>I</u>	<u>L. P. Node List LP(I)</u>	<u>L. P. Depth List IPD(I)</u>
1	5	2
2	6	2
3	7	2
4	8	2
5	9	2
6	4	1





```

1 P=P*PA(IPA(MPATH(I)))
  IPP=IP+1
  II=1
  IF(ICL(NPATH(1)).EQ.0) II=2
2  DO 2 I=II,IPP
  LCODE(I)=ICL(NPATH(I))
  IK=0
  DO 3 I=II,IPP
  IF(LCODE(I).EQ.0) GO TO 3
  IK=IK+1
  LL(IK)=L(LCODE(I))
  LLL=1
  JJ=I+1
  IF(JJ.GT.IPP) GO TO 13
  DO 4 J=JJ,IPP
  IF(LCODE(I).NE.LCODE(J)) GO TO 4
  LLL=LLL+1
  LCODE(J)=0
  CONTINUE
4  NI(IK)=LLL
13 LCODE(I)=0
3  CONTINUE
  RETURN
  END C

```

C  
C  
C THIS SUBROUTINE CONTROLS THE  
COMPUTING OF THE SURVIVAL FUNCTION OF THE CURRENT PATH

```

SUBROUTINE ONECON(PRO,T,L,NINIT,K)
REAL A(11,10),L(11)
INTEGER NI(11),NINIT(11)
CALL ONEDIS(K,A,NI,L)
CALL ONEATA(K,A,NI,L,NINIT)
CALL CALPRO(K,A,NI,L,PRO,T)
RETURN
END

```

C  
C  
C THIS ROUTINE COMPUTES THE COEFFICIENTS  
AS EACH IS DISSIMILAR

```

SUBROUTINE ONEDIS(K,A,NI,L)
REAL A(11,10),L(11)
INTEGER NI(20)
KK=1
2 A(KK,1)=1.
  NI(KK)=1
  JJ=1
4 IF(JJ.EQ.KK) GO TO 6
  A(KK,1)=A(KK,1)*L(JJ)/(L(JJ)-L(KK))
6 IF(JJ.EQ.K) GO TO 9
  JJ=JJ+1
  GO TO 4
9 IF(KK.EQ.K) GO TO 3
  KK=KK+1
  GO TO 2
8 RETURN
END

```

C  
C  
C BEGIN TO ADD ONE AT A TIME

```

SUBROUTINE ONEATA(K,A,NI,L,NINIT)
REAL L(11),A(11,10)
INTEGER NINIT(11),NI(11)
IE=1
32 IF(NINIT(IE).EQ.NI(IE)) GO TO 99
  NI(IE)=NI(IE)+1

```

UPDATE IE

```

12      J=0
      NN=NI(IE)-J
      A(IE,NN)=L(IE)*A(IE,NN-1)/FLOAT(NN-1)
      IF(NN.EQ.2) GO TO 20
      J=J+1
      GO TO 12
20     CONTINUE
      DO 101 I=1,K
      IF(I.EQ.IE) GO TO 101
      A(IE,I)=A(IE,I)+A(I,I)*L(I)/(L(I)-L(IE))
      IF(NI(I).LT.2) GO TO 101
      FACT=1.
      NKK=NI(I)
      DO 102 II=2,NKK
      FACT=FACT*(II-1)
102    A(IE,I)=A(IE,I)+A(I,II)*FACT*L(IE)/(L(I)-L(IE))**II
101    CONTINUE
      UPDATE I,NE,IE
      I=1
21    IF(I.EQ.IE) GO TO 26
      A(I,NI(I))=A(I,NI(I))*L(IE)/(L(IE)-L(I))
      IF(NI(I).EQ.1) GO TO 26
      J=1
24    NNI=NI(I)-J
      A(I,NNI)=(L(IE)*A(I,NNI)-FLOAT(NNI)*A(I,NNI+1))
      / (L(IE)-L(I))
      *
      IF(J.EQ.(NI(I)-1)) GO TO 26
      J=J+1
      GO TO 24
26    IF(I.EQ.K) GO TO 32
      I=I+1
      GO TO 21
99    CONTINUE
      IF(IE.EQ.K) GO TO 35
      IE=IE+1
      GO TO 32
35    RETURN
      END

```

C  
C  
C

C  
C

CALCULATES THE PROBABILITY FOR THE CURRENT PATH

```

SUBROUTINE CALPRO(K,A,NI,L,PRO,T)
REAL L(11),A(11,10)
INTEGER NI(11)
PRO=0.
DO 40 I=1,K
SUM=0.
NKK=NI(I)
DO 41 J=1,NKK
SUM=SUM+A(I,J)*T*(J-1)
41  CONTINUE
TTT=-L(I)*T
PRO=PRO+SUM*EXP(TTT)
40  CONTINUE
WRITE(6,1011)
DO 1000 I=1,K
NNN=NI(I)
WRITE(6,1010) L(I),NI(I),(A(I,J),J=1,NNN)
1000 CONTINUE
RETURN
1010 FORMAT('0',10X,F10.7,5X,I2,5X,2(5(E12.5,2X),/25X))
1011 FORMAT(15X,'LAMBDA',7X,'NI',5X,'COEFFICIENTS',/10X,
135(' ',1))
END

```

INPUT DATA FOR THE PROGRAM:

T.N	2	1	13							
NO. 1	0	1	1	1	2	1	3	3	4	1
NO. 6	5	1	6	2	7	3	8	3	9	2
NO. 11	10	3	11	2	12	2				
AR. 1	1	1	1	2	1	2	2	2	3	1
AR. 6	3	2	4	3	5	1	5	2	6	1
AR. 11	6	2	7	3						
NL MN	6	3	3							
LPI PD	8	3	9	3	10	3	11	3	12	3
LP6 PD	13	3								
LAMDA	2.0		.5		1.5					
PA'S	.25		.75		1.0					

OUTPUT FROM THE PROGRAM :

BEGIN TO CALCULATION

```

1. P ( 1 ) = 0.0625000
LAMDA NI COEFFICIENTS
-----
1.5 2 -0.80000E+02 0.24000E+02
2.0 2 0.81000E+02 0.18000E+02
2. P ( 2 ) = 0.0468750
LAMDA NI COEFFICIENTS
-----
1.5 2 -0.51200E+03 0.96000E+02
2.0 3 0.51300E+03 0.16200E+03 0.18000E+02
3. P ( 3 ) = 0.1406250
LAMDA NI COEFFICIENTS
-----
0.5 2 -0.63578E-06 0.11852E+01
2.0 3 0.10000E+01 0.81481E+00 0.22222E+00
4. P ( 4 ) = 0.0468750
LAMDA NI COEFFICIENTS
-----
1.5 2 -0.51200E+03 0.96000E+02
2.0 3 0.51300E+03 0.16200E+03 0.18000E+02
5. P ( 5 ) = 0.1406250
LAMDA NI COEFFICIENTS
-----
0.5 2 -0.63578E-06 0.11852E+01
2.0 3 0.10000E+01 0.81481E+00 0.22222E+00
6. P ( 6 ) = 0.5625000
LAMDA NI COEFFICIENTS
-----
0.5 2 0.59259E+00 0.88889E+00
2.0 2 0.40741E+00 0.22222E+00
PROBABILITY ( TIME > 2.0 ) = 0.0173515

```



```

C   GET THE TIME
C   READ (5,505) TIME
C   GET THE NUMBER OF ROWS
C   READ (5,501) N
C   GET ALL P'S
C   READ (5,504) (P(I),I=1,N)
C   GET ALL LANDAS IN THE ORDER OF ROW BY ROW
C   MIDO=1
C   MIUP=0
C   DO 100 J=1,N
C   IF (J.NE.1) MIDO=MIUP+1
C   MT(J)=MIDO
C   READ (5,520) MI(J)
C   MIUP=MIUP+MI(J)
100  READ (5,503) (L(I),I=MIDO,MIUP)
C   CONTINUE
C   RETURN
501  FORMAT (5X,I5)
503  FORMAT (5F10.5)
504  FORMAT (5F10.7)
505  FORMAT (5X,F10.3)
520  FORMAT (5X,I5)
C   END

C   SUBROUTINE CONTRO
C   SUBROUTINE CONTRO(L,N,TIME,MI,MT,P,IX,PRO)
C   REAL L(500),P(50)
C   INTEGER MI(50),MT(50)
C   SUM=0.
C   DO 1 I=1,N
C   NN=1000000*P(I)
C   CALL SIMULA(NN,L,MI(I),MT(I),X,IX,TIME)
1    SUM=SUM+X
C   PRO=SUM/1000000.
C   RETURN
C   END

C   SUBROUTINE FOR SIMULATION
C   SUBROUTINE SIMULA(NN,L,M,MTT,X,IX,TIME)
C   REAL L(500),RN(50)
C   INTEGER M,I,J,IX
C   X=0.
C   MMT=MTT+M-1
C   DO 11 I=1,NN
C   TEST=0.
C   CALL LEXPN(IX,RN,M,16807,0)
C   JJ=0
C   DO 111 J=MMT,MMT
C   JJ=JJ+1
C   TEST=TEST+RN(JJ)/L(J)
111  IF (TEST.GE.TIME) X=1.0+X
11   RETURN
C   END

C   SUBROUTINE OUTPUT
C   SUBROUTINE OUTPUT(L,N,TIME,MI,MT,P,PRO)
C   REAL L(500),P(50)
C   INTEGER MI(50),MT(50)

```

```

C
C
C PRINT OUT ALL THE SYSTEM
WRITE(6,600)
MIUP=MI(1)
WRITE(6,601) P(1), (L(I), I=1, MIUP)
DO 102 I=2, N
MIDO=MT(I)
MIUP=MT(I)+MI(I)-1
WRITE(6,602) P(I), (L(J), J=MIDO, MIUP)
102 CONTINUE
WRITE(6,603) TIME, PRO
RETURN

```

```

C
C
C FORMAT STATEMENTS
600 FORMAT('1', 5X, 'THE SYSTEM IS :')
601 FORMAT('0', 5X, 'F(T) =', F10.7, ' * ', 'EXP L = ', 5
1(F10.4, 3X), 5(//20X, 5(F10.4, 3X))
602 FORMAT('0', 9X, '+', F10.7, ' * ', 'EXP L = ', 5(F1
10.4, 3X), 5(//20X, 5(F10.4, 3X))
603 FORMAT('0', 5X, 'PROBABILITY OF SYSTEM SURVIVAL AT
*T=', F10.5, ' IS ', F10.7)
END

```

INPUT FOR THE PROGRAM :

TIME #ROW	15.				
2	5				
1. ST	6	.2	.2	.2	.2
.1		.8	.4	.1	.1
.8					
2. ND	7				
.2		.2	.5	.5	.5
.5		.5			
3. PD	9	.5	.3	.1	.1
.1		.5	.5	.3	
.5					
4. TH	6	.2	.1	.1	.2
.1					
.2					
5. TH	9		.4	.1	.2
.1		.2	.4	.4	
.2		.2			

OUTPUT FROM THE PROGRAM :

THE SYSTEM IS :

F(T) = 0.2 * EXP	L = 0.1	0.8	0.4	0.1	0.1
+ 0.2 * EXP	L = 0.2	0.2	0.5	0.6	0.5
+ 0.2 * EXP	L = 0.5	0.5	0.3	0.1	0.1
+ 0.2 * EXP	L = 0.1	0.5	0.5	0.3	0.2
+ 0.2 * EXP	L = 0.2	0.2	0.1	0.1	0.2
+ 0.2 * EXP	L = 0.1	0.2	0.4	0.1	0.2
	0.2	0.2	0.4	0.4	

PROBABILITY OF SYSTEM SURVIVAL AT T= 15.00000 IS 0.911241

## APPENDIX E

This section reviews some notions which are found in the references for this paper.

E.1 Redundant systems with exponentially lived components:

In reliability analysis, the term system is used to describe a set of components organized to perform some mission. A system is redundant if, in some fashion, some of the components involved act as back up for other components.

A rough definition might be that a system is not redundant if the failure of any one of its components causes the failure of the system, and is redundant if one or more of its components can fail without causing the system to fail. Thus redundant systems have the property that they can suffer damage through the failure of some of their components and still survive. (ESARY Ref.1)]

E.2 First failure in a set of exponentially lived components:

We have  $n$  components, each independent from the others, we want to compute the probability that the  $j$ .th component fails first.

$$P(\text{j.th component fails first}) = P(T_j < T_i, \forall i \neq j)$$

$$\begin{aligned} &= \int_0^{\infty} P(T_i > T_j, \forall i \neq j | T_j = s) \lambda_j e^{-\lambda_j s} ds \\ &= \int_0^{\infty} \left[ \prod_{i \neq j} \bar{F}_i(s) \right] f_j(s) ds \\ &= \int_0^{\infty} \left[ \prod_{i \neq j} e^{-\lambda_i s} \right] \lambda_j e^{-\lambda_j s} ds \\ &= \lambda_j / \left( \sum_{i=1}^n \lambda_i \right) \int_0^{\infty} \left[ \sum_{i=1}^n \lambda_i \right] e^{-\sum_{i=1}^n \lambda_i s} ds \\ &= \lambda_j / \left( \sum_{i=1}^n \lambda_i \right). \end{aligned}$$

### E.3 Degeneracy at zero (Zero Distribution):

Let ZERO be the name for the distribution of a random variable that is degenerate at zero.

If  $P(T_0 = 0) = 1$ , then we say that  $T_0$  has the distribution ZERO, or

$$T_0 \sim \text{ZERO}.$$

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