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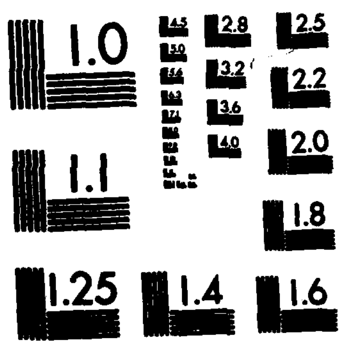

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N-WIDTH AND ENTROPY OF H_p-CLASSES IN $L_q(-1,1)$

H. G. Burchard and K. Höllig

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ABSTRACT

The n-width d_n, approximation numbers δ_n and entropy ε_n of the Hardy spaces H_p in $L_q(-1,1)$ are estimated. More precisely, denote by F^r the space of continuous functions which satisfy a Lipschitz condition of order r at ± 1. It is shown that

$$\exp(-2\alpha n^{1/2}) << \delta_n(H_p \cap F^r, L_\infty), d_n(H_p \cap F^r, L_\infty) << \exp(-\alpha n^{1/2})$$

$$\exp(-2\beta n^{1/2}) << \delta_n(H_p, L_q), d_n(H_p, L_q) << \exp(-\beta n^{1/2}), \quad \text{for } p > q$$

$$\exp(-2\gamma n^{1/3}) << \varepsilon_n(H_p \cap F^r, L_\infty) << \exp(-\gamma n^{1/3})$$

where "<<" indicates that the inequalities hold except for polynomial factors in n. The constants α, β, γ depend on p, q and r. For $p = \infty$, the factor 2 in the lower bound of the first inequality can be omitted.

AMS (MOS) Subject Classifications: 41A46, 30D55

Key Words: Hardy spaces, n-width, entropy, upper and lower bounds, Wittaker series

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SIGNIFICANCE AND EXPLANATION

For analytic functions many of the standard approximation processes converge at an exponential rate. Using more sophisticated methods, it is still possible to obtain exponential convergence even in the presence of singularities at the boundary. F. Stenger and A. A. Goncar, e.g., constructed rank n approximation methods P_n such that

$$\|f - P_n f\|_{\infty,[-1,1]} < C \exp(-rn^{1/2})$$

for f an analytic function, bounded in the unit disc, which satisfies a Lipschitz condition of order r at ± 1.

In this report it is shown that estimates of this type are optimal in the sense of n-width, i.e. the above rate of convergence is best possible for approximation by rank n methods.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

N-WIDTH AND ENTROPY OF H_p-CLASSES IN $L_q(-1,1)$

H. G. Burchard and K. Höllig

1. INTRODUCTION.

For analytic functions many of the standard approximation processes converge at an exponential rate. Using more sophisticated methods, it is still possible to obtain exponential convergence, even in the presence of singularities at the endpoints of an interval of approximation.

In this paper we obtain precise upper and lower bounds for optimal convergence rates of approximation processes for the natural imbeddings of Hardy spaces into sub Q $L_q(-1,1)$ in the sense of n-width, approximation numbers (linear n-width) and also entropy. This makes it possible to assess the optimality of bounds previously obtained for special approximation operators.

As a model example, consider the class H_∞ of analytic functions f bounded in the unit disc. To obtain convergence in $L_\infty(-1,1)$ of approximation methods, some mild additional assumptions must be imposed about the behaviour of f at ± 1. For this, let F^r denote the class of functions in $L_\infty(-1,1)$ which satisfy a Lipschitz condition of order $r > 0$ at ± 1 (c.f. (2.4)).

In [6] A. Goncar has constructed piecewise polynomial approximation operators P_n of rank n such that

$$(1.1) \qquad \|f - P_n f\|_{L_\infty(-1,1)} \ll \exp(-\alpha n^{1/2})(\|f\|_{H_\infty} + \|f\|_{F^r})$$

where $\alpha = \log(1 + \sqrt{2})r^{1/2}$. Here, "$\ll$" indicates that the inequality holds except for a polynomial factor in n. R. De Vore and K. Scherer [4] showed that $\exp(-\sqrt{2}\,\alpha n^{1/2})$ is a lower bound for approximation by piecewise polynomial operators. In [9, 13-16] F. Stenger developed a theory for approximating analytic functions using Whittaker's cardinal series. In particular he obtained (1.1) with an improved value of the constant,

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$\alpha = \frac{\pi}{2} r^{1/2}$. For the approximation of functions f in $H_\infty \cap F^r$ we obtain the sharp lower bound

$$(1.2) \qquad \|f - P_n f\|_{L_\infty(-1,1)} >> \exp(-\frac{\pi}{2}(rn)^{1/2})(\|f\|_{H_\infty} + \|f\|_{F^r}) ,$$

valid for an arbitrary rank n operator P_n (c.f. Theorem 1). This establishes that approximation by Whittaker's cardinal series is optimal and $\exp(-\frac{\pi}{2}(rn)^{1/2})$ is the precise asymptotic order of the n-width of $H_\infty \cap F^r$ in $L_\infty(-1,1)$ up to a polynomial factor in n. We obtain results analagous to (1.1) and (1.2) for the n-width and approximation numbers of $H_p \cap F^r$ in $L_\infty(-1,1)$ (c.f. Theorem 2) and of H_p in $L_q(-1,1)$, $p > q$, (c.f. Theorem 3).

For entropy, however, the asymptotic behavior is different. For our model example we obtain (c.f. Theorem 4)

$$\exp(-2\gamma n^{1/3}) << \varepsilon_n(H_\infty \cap F^r, L_\infty) << \exp(-\gamma n^{1/3})$$

where $\gamma = (\frac{\pi^2}{2} \log 2\ r)^{1/3}$. Thus, our results show that n-width tends to zero more rapidly than entropy. These estimates are in remarkable contrast to the results for Sobolev spaces where $\varepsilon_n < d_n$ [7]. The slower decay of entropy seems to be typical for classes of analytic functions. E.g. the best known example appears to be the following. Set $\Delta = \{w \in \mathbb{C} : |w| < 1\}$. Then we have for $\rho < 1$

$$d_n(H_\infty, L_\infty(\rho\Delta)) >< \exp(-|\log \rho| n)$$

$$\varepsilon_n(H_\infty, L_\infty(\rho\Delta)) >< \exp(-(\log 2\ |\log \rho|\ n^{1/2}) .$$

After stating our main results in section 2 we prove in section 3 auxiliary results regarding n-width and entropy. In section 4 we introduce equivalent approximation problems on the real line and obtain basic approximation properties of weighted cardinal series. The proofs of Theorems 1-4 are given in section 5.

2. MAIN RESULTS

Let $T : X \to Y$ be a bounded linear operator between Banach spaces X and Y. The <u>n-width</u> d_n, the <u>approximation numbers</u> (linear n-width) δ_n and the <u>entropy</u> ε_n of T are defined by

$$d_n(T) = \inf_{\substack{V \subset Y \\ \dim V < n}} \sup_{\|x\|_X \le 1} \operatorname{dist}_Y(Tx, V) \tag{2.1}$$

$$\delta_n(T) = \inf_{\substack{P \in L(X,Y) \\ \operatorname{rank} P < n}} \|T - P : X \to Y\| \tag{2.2}$$

$$\varepsilon_n(T) = \inf\{\varepsilon : \exists\, y_1, \ldots, y_{2^n} \in Y \text{ such that } TB(X) \subset \bigcup_{\nu=1}^{2^n} (y_\nu + \varepsilon B(Y))\} \tag{2.3}$$

where $B(X)$ denotes the closed unit ball of the B-space X. If $T : X \to Y$ is a continuous embedding we write $a_n(X,Y)$ in place of $a_n(T)$. Here and in the sequel a_n stands for either one of the numbers d_n, δ_n or ε_n.

Let H_p, $1 \le p \le \infty$, denote the Hardy space [5], i.e. H_p is the class of analytic functions in the unit disc Δ for which

$$\|f\|_{H_p} = \sup_{0 \le s < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(se^{i\theta})|^p d\theta\right)^{1/p}, \qquad 1 \le p < \infty,$$

$$\|f\|_{H_\infty} = \sup_{|z| < 1} |f(z)|$$

is finite. Given a conformal homeomorphism h of Δ onto a simply connected region $\Omega \subset \mathbb{C}$ one can define [6]

$$H_p(\Omega) = \{f : \Omega \to \mathbb{C} : f \circ h \in H_p\} .$$

Different conformal maps result in equivalent norms.

We denote by F^r the class of functions in $L_\infty(-1,1)$ which satisfy a Lipschitz condition of order $r > 0$ at ± 1. Let $\lfloor r \rfloor$ ($\lceil r \rceil$) denote the least (largest) integer not less (greater) than r. F^r is the direct sum of $P_{2\lfloor r \rfloor - 1}$, the space of polynomials of degree at most $2\lfloor r \rfloor - 1$, and the space $\psi^r L_\infty(-1,1)$ of functions with zeros of order r

at ± 1, $\psi(w) = 1 - w^2$. Let p_r be the projection of F^r onto $P_{2\lfloor r\rfloor -1}$ defined by the conditions

$$(f - p_r f)\psi^{-r} \in L_\infty(-1,1) \ .$$

Then the norm on F^r can be defined by

$$\|f\|_{F^r} = \|p_r f\|_{L_\infty(-1,1)} + \|\psi^{-r}(f - p_r f)\|_{L_\infty(-1,1)} \ . \tag{2.4}$$

We study approximation of functions in $H_p \cap F^r$. To state our results we use the following notions of asymptotic equivalence. Let a_n, b_n, $n \in \mathbb{N}$, be two sequences of positive numbers. We write $a_n \lesssim b_n$ if there exists a positive constant C such that $a_n \le C\, b_n$ and $a_n \ll b_n$ if there exists a positive constant j such that $a_n \lesssim n^j b_n$. The symbols $\gtrsim$, $\gg$ and $\asymp$, $\gg\ll$ are defined similarly.

THEOREM 1. *For* $r > 0$ *we have*

$$\delta_n(H_\infty \cap F^r, L_\infty(-1,1)),\ d_n(H_\infty \cap F^r, L_\infty(-1,1)) \gg\ll \exp\left(-\frac{\pi}{2}(rn)^{1/2}\right) \ .$$

This result has already been mentioned in the introduction (c.f. (1.1), (1.2)). The upper estimate is due to F. Stenger [13] and our lower bound shows the optimality of the order $\exp\left(-\frac{\pi}{2}(rn)^{1/2}\right)$.

THEOREM 2. *For* $r > 0$ *and* $1 \le p \le \infty$ *we have*

$$\exp(-2\alpha n^{1/2}) \ll \delta_n(H_p \cap F^r, L_\infty(-1,1)),\ d_n(H_p \cap F^r, L_\infty(-1,1)) \ll \exp(-\alpha n^{1/2})$$

where $\alpha = \dfrac{\pi r}{2(r + 1/p)^{1/2}}$.

It is interesting to compare these rates with the estimates of F. Stenger [15], who considered the classes $H_p^* = \psi H_p$, with $\psi(w) = 1 - w^2$, and obtained

$$\exp(-(\sqrt{5}\,\pi + \varepsilon)n^{1/2}) \le \delta_n^*(H_p^*, L_\infty(-1,1))$$
$$\le \exp\left(-\left(\frac{\pi}{2(p')^{1/2}} - \varepsilon\right)n^{1/2}\right), \qquad n \ge N(\varepsilon) \ .$$

Here δ_n^* is defined analoguous to δ_n but restricting the class of rank n approximations P to methods based on point evaluation, i.e.

$$Pf = \sum_{j=1}^{n} f(x_j)a_j .$$

As we shall see in section 5, H_p^* is similar to the (smaller) class $H_\infty \cap F^r$, $r = 1 - 1/p = 1/p'$. We obtain the bounds (valid also for δ_n^*, d_n)

(2.5) $$\exp(-\pi \frac{1}{\sqrt{p'}} n^{1/2}) \ll \delta_n(H_p^*, L_\infty(-1,1)) \ll \exp(-\frac{\pi}{2} \frac{1}{\sqrt{p'}} n^{1/2}) .$$

In view of Theorem 1 we conjecture that the factor 2 in the lower bound of Theorem 2 can be omitted and $\exp(-cn^{1/2})$ is the precise asymptotic rate of the n-width.

THEOREM 3. <u>For</u> $1 \le q < p \le \infty$ <u>we have</u>

$$\exp(-2\beta n^{1/2}) \ll \delta_n(H_p, L_q(-1,1)),\ d_n(H_p, L_q(-1,1)) \ll \exp(-\beta n^{1/2})$$

<u>where</u> $\beta = \frac{\pi}{2} (\frac{1}{q} - \frac{1}{p})^{1/2}$.

REMARK. The proofs of Theorems 1-3 will show that the results are true for any s-number in the sense of Pietsch [11].

As mentioned in the introduction, Vitushkin's [19] estimates for entropy of classes of analytic functions show exponential decay of $\varepsilon_n(H_\infty, L_\infty(\rho\Delta))$ as $\exp(-cn^{1/2})$ for $0 < \rho < 1$. Notice that in this case the functions approximated are analytic in a neighborhood of the domain $\rho\Delta$ of approximation. In this paper we obtain estimates for entropy of imbeddings of analytic functions with singularities on the boundary $\{-1,1\}$ of the interval of approximation, the rate being $\exp(-cn^{1/3})$. We attribute the curious exponent 1/3 to the fact that singularities are allowed here. A typical result is as follows.

THEOREM 4. *For* $r > 0$ *and* $1 \leq p \leq \infty$ *we have*

$$\exp(-2\gamma n^{1/3}) \ll \varepsilon_n(H_p \cap F^r, L_\infty(-1,1)) \ll \exp(-\gamma n^{1/3})$$

where $\gamma = \left(\frac{\pi^2 r^2 \log 2}{2(r + 1/p)}\right)^{1/3}$.

3. GENERAL PROPERTIES OF d_n, δ_n AND ε_n

A. Pietsch has developed in [11] a general theory of "s-numbers" which includes n-width and approximation numbers as special cases. We list below some basic properties of d_n and δ_n which hold for entropy as well. Let a_n denote either one of the numbers d_n, δ_n or ε_n and let $T : X \to Y$ be a bounded linear operator, then we have

The numbers a_n *form a monotone decreasing sequence, i.e.*

$$\|T\| = a_0(T) \geq a_1(T) \geq \ldots \tag{3.1}$$

If T *admits the factorization* $T : Y \xrightarrow{R} Y' \xrightarrow{T'} X' \xrightarrow{S} X$ we have

$$a_n(T) \leq \|R\| a_n(T') \|S\| . \tag{3.2}$$

The numbers a_n *are additive, i.e.*

$$a_{n_0+n_1}(T_0 + T_1) \leq a_{n_0}(T_0) + a_{n_1}(T_1) . \tag{3.3}$$

Properties (3.1)-(3.3) are direct consequences of the definitions (c.f. [11]).

The following result is useful for obtaining lower bounds for n-width and approximation numbers.

LEMMA 1 [8]. *Let* V *be an* $n+1$ *dimensional subspace of* X *and let* $i : V \to X$ *be the canonical injection. Then we have*

$$d_n(i) = \delta_n(i) = 1 .$$

We shall need some estimates for a_n in sequence spaces. By ℓ^m_∞, ℓ_∞ we denote $\mathbb{R}^m$, $\mathbb{R}^{\mathbb{Z}}$ with supremum norm. In addition, we define the weighted spaces $\ell_{\infty,\rho}$ by

$$\ell_{\infty,\rho} = \{f \in \ell_\infty : \|f\|_{\ell_{\infty,\rho}} = \sup_{\nu\in\mathbb{Z}} \exp(\rho|\nu|)|f_\nu| < \infty\} . \tag{3.4}$$

LEMMA 2. *For* $m > n$ *we have*

$$\delta_n(\ell^m_\infty, \ell^m_\infty) = d_n(\ell^m_\infty, \ell^m_\infty) = 1$$

and for $N = 2n-1, 2n$

$$d_N(\ell_{\infty,\rho}, \ell_\infty) = \delta_N(\ell_{\infty,\rho}, \ell_\infty) = \exp(-\rho n) .$$

Proof. The first part of the Lemma is a consequence of Lemma 1.

Let P_N be the projection of ℓ_∞ onto the span of the first N basis vectors $(\delta_{\nu,\mu})_{\mu\in\mathbb{Z}}, |\nu| < N/2$. Then

$$\|i - P_N : \ell_{\infty,\rho} \to \ell_\infty\| = \exp(-\rho\lfloor N/2\rfloor)$$

is an upper bound for $\delta_N(i)$, $i : \ell_{\infty,\rho} \to \ell_\infty$ being the natural injection.

For the lower estimate consider the factorization of the identity

$$\ell_\infty^{N+1} \xrightarrow{I} \ell_{\infty,\rho} \hookrightarrow \ell_\infty \xrightarrow{P_{N+1}} \ell_\infty^{N+1}$$

where I is the canonical injection. Using (3.2) and Lemma 1 this yields

$$1 = d_N(\ell_\infty^{N+1},\ell_\infty^{N+1}) < \|I\| d_N(\ell_{\infty,\rho},\ell_\infty)\|P_{N+1}\| < \exp(\rho\lceil(N + 1)/2\rceil)d_N(\ell_{\infty,\rho},\ell_\infty) .$$

LEMMA 3. For $\rho > 0$ we have

$$e^{-2\rho}\exp(-(\log 2\ \rho n)^{1/2}) \lesssim \varepsilon_n(\ell_{\infty,\rho},\ell_\infty) \lesssim e^{\rho}\exp(-(\log 2\ \rho n)^{1/2}) .$$

Proof. For $\varepsilon > 0$ the unit ball B of $\ell_{\infty,\rho}$ contains the finite subset

$$A(\varepsilon) = \{a \in \ell_\infty : a_\nu \in \varepsilon\mathbb{Z},\ |a_\nu| < \exp(-|\nu|\rho)\} .$$

For the upper bound note that $A(2\varepsilon)$ is an ℓ_∞ ε-net for B. It suffices therefore to show $\operatorname{card} A(2\varepsilon) < 2^n$ when $\varepsilon = 2e^{-\delta}$, $\delta = (\log 2\ \rho n)^{1/2} - \rho$. We estimate

$$\operatorname{card} A(2\varepsilon) = \prod_{\nu\in\mathbb{Z}} (2\lceil\exp(-|\nu|\rho + \delta)/4\rceil + 1) < \prod_{|\nu|<\delta/\rho} \exp(-|\nu|\rho + \delta)$$

$$< \exp\left(-\rho\left(\frac{\delta}{\rho} - 1\right)\frac{\delta}{\rho} + \delta\left(2\frac{\delta}{\rho} + 1\right)\right) = \exp\left(\frac{\delta^2}{\rho} + 2\delta\right) < 2^n$$

as claimed.

For the lower bound fix $\tilde{\varepsilon} = \frac{1}{2}e^{-\delta}$, $\delta = (\log 2\ \rho n)^{1/2} + 2\rho$. Then

$$\operatorname{card} A(2\tilde{\varepsilon}) = \prod_{\nu\in\mathbb{Z}} (2\lceil\exp(-|\nu|\rho + \delta)\rceil + 1) > \prod_{|\nu|<\delta/\rho} \exp(-|\nu|\rho + \delta)$$

$$> \exp\left(-\rho\left(\frac{\delta}{\rho} + 1\right)\frac{\delta}{\rho} + \delta\left(2\frac{\delta}{\rho} - 1\right)\right) = \exp\left(\frac{\delta^2}{\rho} - 2\delta\right) > 2^n .$$

Given any ℓ_∞ ε-net N for B with cardinality 2^n, since card $A(2\tilde{\varepsilon}) > 2^n$, at least one of the ℓ_∞-balls of radius ε with center in N must contain two distinct points of $A(2\tilde{\varepsilon})$, and this implies $\tilde{\varepsilon} < \varepsilon$. This establishes the lower bound.

4. APPROXIMATION PROCESSES ON THE LINE

For the proofs of our theorems in §5 it will be convenient to consider equivalent approximation problems on the line. The conformal mapping

$$z = \log \frac{1 + w}{1 - w}$$

transforms the unit disk Δ one-to-one onto the parallel strip $\Omega = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \frac{\pi}{2}\}$ and at the same time maps the interval $(-1,1)$ onto $\mathbb{R}$. This substitution induces isometries from H_p onto $H_p(\Omega)$ and from $L_q(-1,1)$ onto the weighted space $\phi^{1/q}L_q(\mathbb{R})$, where

$$\phi(z) = \frac{dz}{dw} = 1 + \cosh z \ .$$

The norm on $\phi^{1/q}L_q(\mathbb{R})$ is given by

$$\|f\|_{\phi^{1/q}L_q(R)} = \|\phi^{-1/q}f\|_{L_q(R)} \ .$$

We also need other weighted function spaces. Let

$$\Omega_d = \frac{2d}{\pi}\,\Omega, \qquad 0 < d < \pi \ ,$$

i.e., $z \in \Omega_d$ iff $|\mathrm{Im} z| < d$. Then, for real λ, $f \in \phi^{\lambda}H_\infty(\Omega_d)$ iff $\phi^{-\lambda}f \in H_\infty(\Omega_d)$ and

$$\|f\|_{\phi^{\lambda}H_\infty(\Omega d)} = \|\phi^{-\lambda}f\|_{H_\infty(\Omega d)} \ .$$

Notice that ϕ maps Ω_π onto $\mathbb{C}\backslash\{x \in \mathbb{R} : x \le 0\}$ and so ϕ^{λ} is holomorphic on Ω_π.

We note some simple properties of $\phi(z)$. If $z = x + iy \in \Omega_\pi$, then

$$c_y\, e^{|x|} \le |\phi(z)| = \cosh x + \cos y \le 2e^{|x|} \ , \tag{4.1}$$

where $c_y = \frac{1}{2}$ for $|y| \le \frac{\pi}{2}$ and $c_y = (1 + \cos y)/2$ for $\frac{\pi}{2} < |y| < \pi$. If $w \in \Delta$, then $z = \log \frac{1 + w}{1 - w} \in \Omega_{\pi/2}$ and

$$1 - |w|^2 = 2 \cos y/|\phi(z)| \ . \tag{4.2}$$

We now establish equivalent approximation problems on the line. To do this, we first replace F^r by the simpler space $\psi^r L_\infty(-1,1)$.

LEMMA 4. *Let* $r > 0$, $k = 2\lfloor r\rfloor$, $1 < p,q < \infty$. *Then for* $a_n = \delta_n, d_n$

$$a_{n+k}(H_p \cap F^r, L_q(-1,1)) \lesssim a_n(H_p \cap \psi^r L_\infty(-1,1), L_q(-1,1))$$

$$< a_n(H_p \cap F^r, L_q(-1,1)) .$$

For $a_n = \varepsilon_n$, *the second inequality is still valid but the left-hand inequality is replaced by*

$$\varepsilon_{m+n}(H_p \cap F^r, L_q(-1,1)) - 2^{-m/k} \lesssim \varepsilon_n(H_p \cap \psi^r L_\infty(-1,1), L_q(-1,1)) .$$

Proof. Write the natural injection $i : H_p \cap F^r \to L_q(-1,1)$ in the form $i = (i - p_r) + p_r$, where p_r is the projection of F^r onto P_{k-1}, cf. (2.4). When $a_n = \delta_n, d_n$ it follows from (3.3) and rank $p_r = k$ that

(4.3) $$a_{n+k}(i) < a_n(i - p_r) .$$

This is the only step in the proof where entropy requires a different treatment, and we have

(4.4) $$\varepsilon_{m+n}(i) < \varepsilon_n(i - p_r) + \varepsilon_m(p_r) \lesssim \varepsilon_n(i - p_r) + 2^{-m/k} .$$

The second inequality comes from the factorization

$$p_r : H_p \cap F^r \to \ell_\infty^k \xrightarrow{j} \ell_\infty^k \to L_q(-1,1) ,$$

where j is the identify on ℓ_∞^k and $\varepsilon_m(j) \asymp 2^{-m/k}$. We now proceed with $a_n = \delta_n, d_n$ or ε_n. We may factor $i - p_r$ as

$$i - p_r : H_p \cap F^r \xrightarrow{T} H_p \cap \psi^r L_\infty(-1,1) \to L_q(-1,1) ,$$

with $Tf = (i - p_r)f$, and $\|T\| = 1$ in view of (2.4) and the inequalities

$$\|f - p_r f\|_{H_p} < \|f\|_{H_p} + \|p_r f\|_{H_p} \lesssim \|f\|_{H_p} + \|p_r f\|_{L_\infty(-1,1)} = \|f\|_{H_p \cap F^r} .$$

The left-hand inequalities of the Lemma now follow from (4.3), (4.4) and (3.2). The right-hand inequality is obvious from (2.4).

The substitution $z = \log \frac{1+w}{1-w}$ induces isometries $H_p \to H_p(\Omega)$, $L_q(-1,1) \to \phi^{1/q} L_q(\mathbb{R})$ and $\psi^r L_\infty(-1,1) \to 2^r \phi^r L_\infty(\mathbb{R})$, noting that $\phi(z) = \frac{2}{\psi(w)}$. As a consequence we obtain the following equivalence.

LEMMA 5.

$$a_n(H_p \cap \psi^r L_\infty(-1,1), L_q(-1,1)) \asymp a_n(H_p(\Omega) \cap \phi^{-r} L_\infty(\mathbb{R}), \phi^{1/q} L_q(\mathbb{R})) .$$

As one might expect when approximating functions on the line, the precise behavior of the functions near the boundary of Ω is not important for the rate of a_n. In fact, it turns out that what matters is merely the approximate rate of growth of $|f(z)|$ for $f \in H_p(\Omega)$. The next lemma is what we need for the reduction from $H_p(\Omega)$ to $\phi^\lambda H_\infty(\Omega_d)$.

LEMMA 6. <u>For</u> $\varepsilon > 0$ <u>one has</u>

$$\varepsilon^{1/p} \|f\|_{\phi^{1/p} H_\infty(\Omega_{\pi/2-\varepsilon})} \lesssim \|f\|_{H_p(\Omega)} \lesssim \varepsilon^{-1/p} \|f\|_{\phi^{1/p-\varepsilon} H_\infty(\Omega)} .$$

<u>Proof.</u> The lower bound follows from an inequality of Hardy and Littlewood [5]. For $g \in H_p = H_p(\Delta)$

$$|g(w)| \le 2^{1/p}(1 - |w|^2)^{-1/p} \|g\|_{H_p} .$$

For $f \in H_p(\Omega)$ let $g(w) = f\left(\log \frac{1+w}{1-w}\right)$. Then $\|f\|_{H_p(\Omega)} = \|g\|_{H_p}$ and hence by (4.2)

$$|f(z)| \le \cos^{-1/p} y |\phi(z)|^{1/p} \|f\|_{H_p(\Omega)} .$$

For the upper bound we observe that

$$\|\psi^{-1/p+\varepsilon} g\|_{H_p} \le \|\psi^{-1/p+\varepsilon}\|_{H_p} \|g\|_{H_\infty}$$

and $\|\psi^{-1/p+\varepsilon}\|_{H_p} \lesssim \left(\int_0^\pi \sin^{\varepsilon p-1}\theta d\theta\right)^{1/p} \simeq \varepsilon^{-1/p}$.

<u>Remark.</u> In analogy to similar characterizations of Hardy spaces on the upper half-plane [5], one can show that $H_p(\Omega) = \phi^{1/p} \mathcal{H}_p \neq \mathcal{H}_p$, where $f \in \mathcal{H}_p$ iff f is analytic in Ω and

$$\|f\|_{\mathcal{H}_p} = \sup_{|y|<\pi/2} \left\{ \int_{-\infty}^{\infty} |f(x+iy)|^p dx \right\}^{1/p} < \infty .$$

The proof of this non-elementary result makes use of the factorization theorem for the Nevanlinna class N^+ and the fact that $(1 - w^2)^{-1/p}$ is an outer function.

LEMMA 7. *Let* $\varepsilon > 0$. *Then* $\|f\|_{L_q(\mathbb{R})} \lesssim \varepsilon^{-1/q} \|\phi^{\varepsilon} f\|_{L_\infty(\mathbb{R})}$.

This inequality is useful when proving upper bounds, as it implies $a_n(X, \phi^{1/q}(\mathbb{R})) \lesssim \varepsilon^{-1/q} a_n(X, \phi^{1/q-\varepsilon} L_\infty(\mathbb{R}))$. The lower bounds require a different technique employing a regularization mapping $\phi^{1/q} L_q(\mathbb{R})$ into a weighted L_∞-space, cf. Lemma 9 below.

Next, we describe the approximation processes on the line to be used in the proofs of our main theorems in §5. Let $\rho > 0$, $t > 0$, $\nu \in \mathbb{Z}$ and define the functions

$$s_\nu(z) = s_{\nu t\rho}(z) = \phi^{-\rho}(z - \nu/t)\,\frac{\sin\pi(tz - \nu)}{\pi(tz - \nu)}\,.$$

Notice that s_ν is holomorphic in Ω_π and $s_\nu(\mu/t) = \delta_{\mu\nu}$. Let $S_{nt\rho} = \operatorname{span}\{s_{\nu t\rho} : |\nu| < n\}$ and define the interpolatory projections

$$P_{nt\rho} : C(\mathbb{R}) \to S_{nt\rho}\,,$$

$$P_{nt\rho} f = \sum_{|\nu| < n} f(\nu/t)\, s_{\nu t\rho}$$

In addition to these finite-rank approximations we also need the series

$$P_{t\rho} f = \sum_{\nu\in\mathbb{Z}} f(\nu/t)\, s_{\nu t\rho}\,.$$

For $\rho = 0$ this is the Whittaker cardinal series [18]. As mentioned in the introduction, the cardinal series was employed by Stenger [13] in obtaining his upper bounds. Lundin and Stenger [9] and Stenger [15] also used weighted cardinal series similar to ours. The next lemma implies that the series $P_{t,\rho} f$ converges uniformly on $\mathbb{R}$ if $(f(\nu/t))_{\nu\in\mathbb{Z}}$ is a bounded sequence.

We now establish bounds on the condition number of the basis $\{s_\nu\}$. It is perhaps surprising that these crude estimates, where the coefficients grow as powers of n (the parameter t will turn out to be proportional to $n^{1/2}$), suffice for determining the order of a_n.

LEMMA 8. For $(a_\nu)_{\nu\in\mathbb{Z}}$ in $\ell_\infty(\mathbb{Z})$, $t > 1$

$$\|(a_\nu)\|_{\ell_\infty} \le \|\sum_{\nu\in\mathbb{Z}} a_\nu s_{\nu t\rho}\|_{L_\infty(\mathbb{R})} \le t\,\frac{e^{\rho/t}}{\rho}\,\|(a_\nu)\|_{\ell_\infty} \tag{4.5}$$

Proof. For the upper bound we must estimate the Lebesque function $L(x) = \sum_{\nu\in\mathbb{Z}} |s_{\nu t\rho}(x)|$. By (4.1)

$$\|L\|_{L_\infty(\mathbb{R})} \le \sup_{x\in\mathbb{R}} \sum_{\nu\in\mathbb{Z}} e^{-\rho|x-\nu/t|} .$$

The value of the last sum evidently depends only on the residue of xt mod 1, hence let $0 \le xt \le 1$. Then $|x - \nu/t| \ge (|\nu| - 1)/t$ and for $t \ge 1$

$$\|L\|_{L_\infty(\mathbb{R})} \lesssim e^{\rho/t} \sum_{\nu\in\mathbb{Z}} e^{-\rho\nu/t} \lesssim te^{\rho/t}/\rho .$$

A similar estimate can also be proved for $L_\infty(\mathbb{R})$ replaced by $L_q(\mathbb{R})$.

LEMMA 9. For a positive integer n, $\rho > 0$ and $1 \le q \le \infty$

$$(\rho + t)^{-1/q}\, n^{-1/q}\, \|(a_\nu)_{|\nu|\le n}\|_{\ell_\infty^{2n+1}} \le \|\sum_{|\nu|\le n} a_\nu s_{\nu t\rho}\|_{L_q(\mathbb{R})} \tag{4.6}$$

$$\le \frac{n}{(\rho q)^{1/q}}\, \|(a_\nu)_{|\nu|\le n}\|_{\ell_\infty^{2n+1}} .$$

Proof. The right-hand inequality follows from (4.1). To show the lower bound, for each $|\nu| \le n$ extend to $L_q(\mathbb{R})$ the linear functional on $S_{nt\rho}$ given by $\sum_{|\kappa|\le n} a_\kappa s_{\kappa t\rho} \to a_\nu$. A convenient extension is L_ν, where

$$L_\nu s = \sum_{|\mu|\le n} b_{\nu\mu} \frac{1}{2\varepsilon} \int_{\mu/t-\varepsilon}^{\mu/t+\varepsilon} s(x)dx, \quad s \in L_q(\mathbb{R}) ,$$

and $B = (b_{\nu\mu})$ is the inverse of $A = (a_{\mu\kappa})$,

$$a_{\mu\kappa} = \frac{1}{2\varepsilon} \int_{\mu/t-\varepsilon}^{\mu/t+\varepsilon} s_{\kappa t\rho}(x)dx, \quad \varepsilon > 0; \; |\mu|,|\kappa| \le n .$$

As $s_{\kappa t\rho}(\mu/t) = \delta_{\kappa\mu}$, we have the bound

$$|a_{\mu\kappa} - \delta_{\mu\kappa}| \le \frac{1}{2\varepsilon} \int_{\mu/t-\varepsilon}^{\mu/t+\varepsilon} |s_{\kappa t\rho}(x) - s_{\kappa t\rho}(\mu/t)|dx$$

$$\le \varepsilon \sup_{x\in\mathbb{R}} |\frac{d}{dx} s_{\kappa t\rho}(x)| \le M\varepsilon(\rho + t)$$

where M is a constant. We choose $\varepsilon = \frac{1}{8M} (\rho + t)^{-1} n^{-1}$ and obtain

$$\|A - I\|_\infty \le (2n + 1)M\varepsilon(\rho + t) \le \frac{1}{2} .$$

Therefore

$$\|B\|_\infty \le \frac{1}{1 - \|A - I\|_\infty} \le 2 .$$

This gives for $\frac{1}{q'} + \frac{1}{q} = 1$

$$\|L_\nu\|_{L_{q'}(\mathbb{R})} \lesssim \frac{1}{\varepsilon} \varepsilon^{1/q'} = \varepsilon^{-1/q} \asymp (\rho + t)^{1/q} n^{1/q} .$$

Finally, in this section, we state a formula for the error of approximation $f - P_{nt\rho}f$ which follows from the calculus of residues and was extensively used by F. Stenger [13].

LEMMA 10. *If* $f \in H_\infty(\Omega_d)$, $0 < d < \pi$ *and* $x \in \Omega_d$, *then*

$$(4.7) \qquad (f - P_{t,\rho}f)(x) = \frac{\sin\pi tx}{2\pi i} \int_{\partial\Omega_d} \frac{\phi^{-\rho}(x - z)f(z)}{(z - x)\sin(\pi tz)} dz .$$

Proof. Denote by R_n the rectangle

$$\{x + iy \in \Omega_{d-\varepsilon} : |x| < \frac{n + 1/2}{t}\} .$$

If we replace $P_{t,\rho}$ by $P_{n,t,\rho}$ and $\partial\Omega_d$ by ∂R_n, then (4.7) follows from the residue theorem when $x \in R_n$. We now let $\varepsilon \to 0$ and $n \to \infty$ and apply the dominated convergence theorem. Here we are using the existence of nontangential limits in $L_\infty(\partial\Delta)$ for $f \in H_\infty = H_\infty(\Delta)$, as the lines $|\mathrm{Re}\ z| = \text{const}$, $z \in \Omega_d$, transform conformally to nontangential curves in Δ under the substitution $z = \log \frac{1 + w}{1 - w}$. We also make use of (4.1) and of

$$(4.8) \qquad |\sin(x + iy)| = \frac{1}{2}\,|2 \cosh 2y - 2 \cos 2x|^{1/2} \geq \frac{1}{2}\,(e^{|y|} - 1) .$$

LEMMA 11. *Under the hypotheses of Lemma 10, when* $d = \frac{\pi}{2}$ *and* $t \geq 1$

$$(4.9) \qquad \|f - P_{t,\rho}f\|_{L_\infty(\mathbb{R})} \lesssim \exp\left(-\frac{\pi^2}{2}\,t\right)\|f\|_{H_\infty(\Omega_d)} .$$

Proof. From (4.7) and (4.8)

$$\|f - P_{t,\rho}f\|_{L_\infty(\mathbb{R})} \lesssim \left(\exp\left(\frac{\pi^2}{2}\,t\right) - 1\right)^{-1}\|f\|_{H_\infty(\Omega_d)} \lesssim \exp\left(-\frac{\pi^2}{2}\,t\right)\|f\|_{H_\infty(\Omega_d)} .$$

For reference below we note

$$(4.10) \qquad \sup_{z\in\Omega_d}\left|\frac{\sin\pi(tz - \nu)}{\pi(tz - \nu)}\right| \lesssim \exp(\pi dt) ,$$

a consequence of the maximum modulus theorem and (4.8).

We note that approximation in $L_\infty(\mathbb{R})$ can be replaced by approximation in $C(\mathbb{R})$, the subspace of continuous functions in $L_\infty(\mathbb{R})$. More precisely, for a compact linear operator $T : X \to C(\mathbb{R})$ we have

$$(3.5) \qquad a_n(T) = a_n(jT)$$

where $j : C \to L_\infty$ is the injection. It is clear from (3.2) that $a_n(jT) \leq a_n(T)$ as

$\|j\| = 1$. To prove the reverse inequality note that the compactness of T implies that there exists a modulus of continuity $w(\delta,t)$ such that

$$w(\delta,t) \to 0, \quad \delta \to 0 ,$$

$$w(\delta,t) \le w(s,t'), \quad t \le t' ,$$

and for all $f \in T(B(x))$

$$\sup_{\substack{|s|,|s'|\le t \\ |s-s'|\le\delta}} |f(s') - f(s)| \le w(\delta,t) .$$

For $\varepsilon > 0$ we choose a function $h > 0$ such that $w(h(t),t) \le \varepsilon$ and define a smoothing operator $R_\varepsilon : L_\infty \to C$ by

$$(R_\varepsilon f)(t) = \frac{1}{2h(t)} \int_{t-h(|t|)}^{t+h(|t|)} f(s)ds .$$

Then $\|R_\varepsilon\| = 1$ and $a_n(jT) \ge a_n(R_\varepsilon jT) = a_n(R_\varepsilon T) \ge a_n(T) - \|T - R_\varepsilon T\|$ by (3.1), (3.2) and (3.3). From the definition of h we see that $\|T - R_\varepsilon T\| \le \varepsilon$ which, since ε is arbitrary, finishes the proof.

5. PROOFS

To simplify notation we shall use the abbreviations $X_{rd} = \phi^r H_\infty(\Omega_d)$, $X_r = X_{r,\pi/2}$, $X = X_0$, $Y_r = \phi^r C(R)$, $Y = Y_0$.

Proof of Theorem 1. By Lemmas 4 and 5 it suffices to estimate $d_n(i)$, $\delta_n(i)$ with $i : X \cap Y_{-r} \to Y$. In view of the obvious inequality $d_n \le \delta_n$ we shall bound δ_n from above and d_n from below. To prove the upper estimate we write $i = (i - P_{tr}) + P_{tr}$ and by (3.1), (3.3) we have

$$\delta_n(i) \le \|i - P_{tr}\| + \delta_n(P_{tr}) .$$

By Lemma 11 we have

$$\|i - P_{tr}\| \lesssim \exp\left(-\frac{\pi^2}{2} t\right) . \tag{5.1}$$

To estimate the second term we factor P_{tr} as

$$P_{tr} : Y_{-r} \xrightarrow{I} \ell_{\infty,r} \longrightarrow \ell_\infty \xrightarrow{J} Y \tag{5.2}$$

where $(If)_\nu = f(\nu/t)$ and $Ja = \sum a_\nu s_{\nu tr}$. Clearly $\|I\| = 1$ and since by Lemma 8 $\|J\| \lesssim t$ we obtain using (3.2) and Lemma 2

$$\delta_n(P_{tr}) \lesssim t \exp\left(-\frac{r}{2t} n\right) . \tag{5.3}$$

Combining the estimate (5.1) with (5.2) and choosing $t = \frac{1}{\pi}(rn)^{1/2}$ gives the upper bound.

To prove the lower estimate we consider the following factorization of the identity on ℓ_∞^{2m+1},

$$\ell_\infty^{2m+1} \xrightarrow{J_m} X \cap Y_{-r} \xrightarrow{i} Y \xrightarrow{I_m} \ell_\infty^{2m+1} ,$$

where I_m and J_m are defined analogous to I and J. Using the estimates

$$\|s_\nu\|_X \lesssim \exp\left(\frac{\pi^2}{2} t\right)$$

$$\|s_\nu\|_{Y_{-r}} \lesssim \exp\left(\frac{r}{t} m\right), \quad |\nu| \le m$$

for the norms of the basis functions s_ν we obtain, choosing $t = \frac{1}{\pi}(2rm)^{1/2}$, $m = \lfloor n/2 \rfloor$

$$\|J_m\| \lesssim m\left(\exp\left(\frac{\pi}{t}\, m\right) + \exp\left(\frac{\pi^2}{2}\, t\right)\right) \lesssim n \exp\left(\frac{\pi}{\sqrt{2}}\, (rn)^{1/2}\right) .$$

The lower bound follows now from (3.2) and Lemma 1.

We now formulate a general result which allows a unified treatment of the proofs of Theorems 2-4 and is of independent interest.

THEOREM 5. Let a_n denote either one of the numbers d_n, δ_n or ε_n. For $\lambda > 0$; $\rho > 0$ and $t > 0$ we have

$$(5.4)\qquad \begin{aligned} a_n(\ell_{\infty\rho/t}, \ell_\infty) \times \exp\left(-\frac{\pi^2}{2}\frac{\rho}{\lambda+\rho}\, t\right) &<< a_n(X_\lambda \cap Y_{-\rho}, Y) << \\ & a_n(\ell_{\infty\rho/t}, \ell_\infty) + \exp\left(-\frac{\pi^2}{2}\frac{\rho}{\lambda+\rho}\, t\right) . \end{aligned}$$

As a consequence we obtain

$$(5.5)\qquad \begin{aligned} \exp\left(-\pi \left(\frac{\rho^2}{\lambda+\rho}\, n\right)^{1/2}\right) &<< d_n(X_\lambda \cap Y_{-\rho}, Y), \delta_n(X_\lambda \cap Y_{-\rho}, Y) \\ &<< \exp\left(-\frac{\pi}{2} \left(\frac{\rho^2}{\lambda+\rho}\, n\right)^{1/2}\right) \end{aligned}$$

and

$$(5.6)\qquad \begin{aligned} \exp\left(-(4\pi^2\log 2)^{1/3}\left(\frac{\rho^2}{\lambda+\rho}\, n\right)^{1/3}\right) &<< \varepsilon_n(X_\lambda \cap Y_{-\rho}, Y) \\ &<< \exp\left(-\left(\frac{\pi^2}{2}\log 2\right)^{1/3}\left(\frac{\rho^2}{\lambda+\rho}\, n\right)^{1/3}\right) . \end{aligned}$$

Proof. To prove the upper estimate in (5.4) we write the embedding $i : X_\lambda \cap Y_{-\rho} \to Y$ in the form $i = (i - P_{t\sigma}) + P_{t,\sigma}$ where we choose $\sigma > \lambda, \rho$. As in the proof of Theorem 1 we estimate

$$a_n(i) \leq \|i - P_{t\sigma}\| + a_n(P_{t,\sigma})$$

and obtain for the second term (c.f. (5.2))

$$a_n(P_{t\sigma}) \leq t\, a_n(\ell_{\infty\rho/t}, \ell_\infty) \;.$$

Therefore we have to show

$$\|i - P_{t\sigma}\| \ll \exp\Big(-\frac{\pi^2}{2}\,\frac{\rho}{\lambda+\rho}\,t\Big) \;.$$

To estimate $\sup_{x\in\mathbb{R}} |f(x) - P_{t\sigma}f(x)|$ we set $A = \frac{\pi^2}{2}\,\frac{1}{\lambda+\rho}\,t$ and consider two cases.

(i) $|x| \leq A$. Lemma 10 and the estimates (4.1), (4.8) imply that

$$|f(x) - P_{t\sigma}f(x)| \lesssim \exp\Big(-\frac{\pi^2}{2}t\Big)\int_{\mathbb{R}} \exp(-\sigma|x - z| + \lambda|z|)\,dz\; \|f\|_{X_\lambda} \lesssim$$

$$\exp\Big(-\frac{\pi^2}{2}t + \lambda A\Big)\,\|f\|_{X_\lambda} \;.$$

(ii) $|x| > A$. Since $|f(x)| \lesssim \exp(-\rho|x|)\,\|f\|_{Y_{-\rho}}$ it follows that

$$|P_{t\sigma}f(x)| \lesssim \sum_\nu \exp(-\rho|\nu/t| - \sigma|x - \nu/t|)\,\|f\|_{Y_{-\rho}} \lesssim t\,\exp(-\rho|x|)\,\|f\|_{Y_{-\rho}} \;.$$

This implies that for $|x| > A$

$$|f(x) - P_{t\sigma}f(x)| \lesssim \exp(-\rho A)\,\|f\|_{Y_{-\rho}}$$

Combining the estimates (i) and (ii) completes the proof by our choice of A.

To prove the lower bound in (5.4) we consider for $\varepsilon > 0$ and $A = \Big\lceil \frac{\pi^2}{2}\,\frac{1}{\lambda+\rho}\,t^2 \Big\rceil$ the following factorization of the embedding $j : \ell_{\infty,\rho/t+\varepsilon} \to \ell_\infty$

$$\ell_{\infty,\rho/t+\varepsilon} \xrightarrow{J_A} X_\lambda \cap Y_{-\rho} \xrightarrow{i} Y \xrightarrow{I_A} \ell_\infty$$

where $(I_A f)_\nu = f((\nu + b(\nu))/t)$ and $J_A a = \sum_{\nu\in\mathbb{Z}} a_\nu s_{\nu+b(\nu),t,\sigma}$. Here $b(\nu)$ is defined as $b(\nu) = \nu + A \operatorname{sgn} \nu$ and σ is chosen larger than λ and ρ. Using the inequalities (4.1), (4.10) we obtain by a simple calculation, keeping in mind the choice of σ,

$$\|s_{\nu+b(\nu)}\|_{X_\lambda} \lesssim \exp\left(\frac{\pi^2}{2} t - \lambda\Lambda/t\right)$$

$$\|s_{\nu+b(\nu)}\|_{Y_{-\rho}} \lesssim \exp(\rho(\Lambda + |\nu|)/t) .$$

Therefore

$$\|J_\Lambda a\|_{X_\lambda \cap Y_{-\rho}} \leq \sum |a_\nu| \|s_{\nu+b(\nu)}\|_{X_\lambda \cap Y_{-\rho}}$$

$$\lesssim \exp\left(\frac{\pi^2}{2} t - \lambda\Lambda/t\right) \sum |a_\nu| + \exp(\rho\Lambda/t) \sum \exp(\rho|\nu|/t)|a_\nu|$$

$$\lesssim \left(\exp\left(\frac{\pi^2}{2} t - \lambda\Lambda/t\right) + \exp(\rho\Lambda/t)\right) \sum \exp(-\varepsilon|\nu|) \|a\|_{\ell_{\infty,\rho/t+\varepsilon}}$$

and by our choice of Λ we get

$$\|J_\Lambda\| \lesssim \varepsilon^{-1} \exp\left(\frac{\pi^2}{2} \frac{\rho}{\lambda + \rho} t\right) .$$

Since $\|I_\Lambda\| = 1$ this implies

$$a_n(j) \lesssim \varepsilon^{-1} \exp\left(\frac{\pi^2}{2} \frac{\rho}{\lambda + \rho} t\right) a_n(i) .$$

Taking into account the asymptotic behaviour of $a_n(j) = a_n(\ell_{\infty,\rho/t+\varepsilon}, \ell_\infty)$ the lower estimate follows by the appropriate choice of ε.

The inequalities (5.5) and (5.6) follow from (5.4) by substituting the bounds for $a_n(\ell_{\infty,\rho/t}, \ell_\infty)$ obtained in Lemmas 2 and 3 and choosing t appropriately. More precisely for $a_n = \delta_n, d_n$ we choose $t = \left(\frac{2}{\pi^2}(\lambda + \rho)n\right)^{1/2}$ and for $a_n = \varepsilon_n$ let $t = \left(\frac{4 \log 2\ (\lambda + \rho)^2 n}{\pi^4 \rho}\right)^{1/3}$.

Using Theorem 5 we can now easily give the proofs of Theorems 2-4.

<u>Proof of Theorem 2.</u> In view of Lemmas 4 and 5 we have to estimate the n-width and approximation numbers of $i : H_p(\Omega) \cap Y_{-r} \to Y$.

For the upper estimate consider for $\varepsilon > 0$ the following factorization of i

$$H_p(\Omega) \cap Y_{-r} \xrightarrow{j_1} X_{1/p,\pi/2-\varepsilon} \cap Y_{-r} \xrightarrow{T} X_{\lambda/p} \cap Y_{-\lambda r} \xrightarrow{j_2} Y \xrightarrow{T^{-1}} Y$$

where $\lambda = \frac{\pi/2 - \varepsilon}{\pi/2}$ and T is defined by

$$(Tg)(z) = g(\lambda z) .$$

Since by (4.1)

$$|\phi^{\lambda\rho}(z)| = |\phi^{\rho}(\lambda z)|, \quad \text{for} \quad z, \lambda z \in \Omega_d, \quad d < \pi ,$$

we have that

$$\|T : X_{\rho,d} \to X_{\lambda\rho,d/\lambda}\| \lesssim 1, \quad d, d/\lambda < \pi$$

$$\|T : Y_{-\rho} \to Y_{-\lambda\rho}\| \lesssim 1 .$$

Using (3.2), Lemma 6 and (5.5) we obtain from the above factorization

$$\delta_n(i) < \|j_1\|\|T\|\delta_n(j_2)\|T^{-1}\| << \varepsilon^{-1/p}\exp(-f(\varepsilon)n^{1/2})$$

where

$$f(\varepsilon) = \frac{\pi}{2}\left(\frac{(\lambda r)^2}{\lambda/p + \lambda r}\right)^{1/2}, \quad \lambda = \frac{\pi/2 - \varepsilon}{\pi/2} .$$

Since f is a smooth function of ε we get by setting $\varepsilon = n^{-1/2}$

$$\delta_n(i) << \exp(-f(0)n^{1/2} - f'(\xi)), \quad \xi \in [0, n^{-1/2}] .$$

This proves the upper bound in view of $f(0) = \frac{\pi}{2}\left(\frac{r^2}{r + 1/p}\right)^{1/2}$.

To prove the lower bound we consider for $\varepsilon > 0$ the embedding

$$j_1 : X_{1/p-\varepsilon} \cap Y_{-r} \longrightarrow Y ,$$

which may be factored as

$$X_{1/p-\varepsilon} \cap Y_{-r} \xrightarrow{j_2} H_p(\Omega) \cap Y_{-r} \xrightarrow{i} Y .$$

By Lemma 6 we have $\|j_2\| \lesssim \varepsilon^{-1/p}$. Applying (3.2) and substituting the estimate (5.5) for $d_n(j_1)$ we get

$$\exp\left(-\pi\left(\frac{r^2}{1/p - \varepsilon + r}\right)^{1/2} n^{1/2}\right) << d_n(j_1) \lesssim \varepsilon^{-1/p} d_n(i) .$$

As in the proof of the upper bound we write this inequality in the form

$$\varepsilon^{1/p}\exp(-g(\varepsilon)n^{1/2}) << d_n(i) .$$

Writing $g(\varepsilon) = g(0) + \varepsilon g'(\xi)$ and setting $\varepsilon = n^{-1/2}$ finishes the proof.

The proof of Theorem 4 is completely analogous and therefore omitted. We simply use the estimate (5.6) for ε_n instead of (5.5).

Proof of Theorem 3. Recall that by Lemmas 4 and 5 we may estimate $d_n(i), \delta_n(i)$, where $i : H_p(\Omega) \to \phi^{1/q}L_q(\mathbb{R})$.

To prove the upper estimate we consider for $\varepsilon > 0$, $\lambda = \frac{\pi/2 - \varepsilon}{\pi/2}$ and $\rho = 1/q - \frac{\lambda}{2}(1/q + 1/p)$ the following factorization of i

$$i : H_p(\Omega) \xrightarrow{j_1} X_{1/p,\pi/2-\varepsilon} \xrightarrow{T} X_{\lambda/p} \xrightarrow{j_2} Y_{1/q-\rho} \xrightarrow{T^{-1}} Y_{(1/q-\rho)/\lambda} \xrightarrow{j_3} \phi^{1/q}L_q(R)$$

where T is defined by $(Tg)(z) = g(\lambda z)$. As we already pointed out in the proof of Theorem 2, $\|T\|, \|T^{-1}\| \lesssim 1$. By our choice of λ, ρ and since $p > q$ we have $\lambda/p < 1/q - \rho$ and $(1/q - \rho)/\lambda < 1/q$. Therefore the embeddings j_1, j_2, j_3, are well defined. By Lemmas 6, 7 $\|j_1\| \lesssim \varepsilon^{-1/p}$, $\|j_3\| \lesssim 1$. Using (3.2) we obtain

$$\delta_n(i) \lesssim \varepsilon^{-1/p}\delta_n(j_2) .$$

Since $a_n(X_\sigma, Y_\tau) = a_n(X_{\sigma-\tau}, Y) \lesssim a_n(X \cap Y_{\sigma-\tau}, Y)$ this and (5.5) imply

$$\delta_n(i) \lesssim \varepsilon^{-1/p}\delta_n(X \cap Y_{\lambda/p-1/q+\rho}, Y) \lesssim \varepsilon^{-1/p}\exp(-h(\varepsilon)n^{1/2}) ,$$

where $h(\varepsilon) = \frac{\pi}{2}(\lambda/p - 1/q + \rho)$ with $\lambda = \lambda(\varepsilon)$, $\rho = \rho(\varepsilon)$ as defined above. Since $\lambda(0) = 1$, $\rho(0) = 0$ we may set $\varepsilon = n^{-1/2}$ and complete the proof as in the previous cases.

The lower estimate, however, cannot be obtained by this technique since there is no embedding $L_q(\mathbb{R}) \to Y_\lambda$. But we may work with L_q directly and proceed similarly as in the proof of Theorem 1. By Lemma 6 and isometries we obtain

$$d_n(H_p(\Omega), \phi^{1/q}L_q(\mathbb{R})) \gtrsim \varepsilon^{1/p}d_n(X_{1/p-\varepsilon}, \phi^{1/q}L_q(R)) = \varepsilon^{1/p}d_n(X_{1/p-1/q-\varepsilon}, L_q(R))$$

Let $\rho = 1/q - 1/p + \varepsilon$. Then $\rho > \varepsilon$ and we can factor the identity $I : \ell_\infty^{2n+1} \xrightarrow{I_2} X_{-\rho} \xrightarrow{i} L_q(R) \xrightarrow{I_1} \ell_\infty^{2n+1}$. Here, $I_2((a_\nu)) = \sum_{|\nu|\le n} a_\nu s_{\nu t\rho}$. From Lemma 9 and the Hahn-Banach Theorem there exist linear functionals L_ν on $L_q(\mathbb{R})$, $|\nu| \le n$, such that

$$L_\nu\Big(\sum_{|\mu|<n} a_\mu s_{\mu t\rho}\Big) = a_\nu, \quad \|L_\nu\| \lesssim (\rho + t)^{1/q} n^{1/q}$$

(actually, an explicit construction of L_ν is provided in the proof of the lemma). Let I_1 be defined by $I_1 f = (L_\nu f)_{|\nu|<n}$. Then

$$\|I_1\| < (\rho + t)^{1/q}\, n^{1/q} .$$

To estimate $\|I_2\|$ we have from (4.1), (4.9)

$$\|s_{\nu t\rho}\|_{\phi^{-\rho} H_\infty(\Omega)} \lesssim \sup_{x\in\mathbb{R}} e^{\rho|x|-\rho|x-\nu/t|} \exp\Big(\frac{\pi^2}{2}\, t\Big) = \exp\Big(\rho|\nu|/t + \frac{\pi^2}{2}\, t\Big) .$$

Therefore

$$\|I_2\| \lesssim n \exp\Big(\rho n/t + \frac{\pi^2}{2}\, t\Big) .$$

From Lemma 1 and (3.2) there follows now

$$1 = d_{2n}(I) < \|I_1\| d_{2n}(j) \|I_2\| ,$$

hence

$$d_n(j) \gtrsim (\rho + t)^{-1/q}\, n^{-1-1/q}\, \varepsilon^{1/p} \exp(-\rho n/(2t) - \frac{\pi^2}{2}\, t) .$$

The proof is completed by the choice of $t = \sqrt{\rho n}/\pi$ and then $\varepsilon = n^{-1/2}$.

Remark. Combining the ideas of the preceding proofs, we obtain the estimates (2.5). Indeed we have, cf. Lemma 5,

$$\delta_n(H_p^*, L_\infty(-1,1)) = 2\delta_n(\phi^{-1} H_p(\Omega), L_\infty(\mathbb{R})) .$$

Let $\rho = 1 - 1/p = 1/p'$. From the obvious injection $X_{-\rho} \hookrightarrow X \cap Y_{-\rho}$, by Lemma 6, (3.5), (5.5) and using T, λ as defined above

$$\begin{aligned}\delta_n(\phi^{-1} H_p(\Omega), L_\infty(\mathbb{R})) &\lesssim \varepsilon^{-1/p}\ \delta_n(X_{-\rho,\pi/2-\varepsilon}, Y)\\ &\lesssim \varepsilon^{-1/p}\ \delta_n(X_{-\rho}, Y) \lesssim \varepsilon^{-1/p}\ \delta_n(X \cap Y_{-\rho}, Y) << \exp\Big(-\frac{\pi}{2}\,(\rho n)^{1/2}\Big) ,\end{aligned}$$

$\varepsilon = n^{-1/2}$, as in the proof of Theorem 2.

For the lower bound

$$\delta_n(\phi^{-1}H_p(\Omega),L_\infty(\mathbb{R})) > \varepsilon^{1/p}\delta_n(Y_{-\rho-\varepsilon},Y) >> \exp(-\pi(\rho n)^{1/2}) ,$$

were we procede as in the proof of Theorem 3, but using Lemma 8 instead of Lemma 9.

The following slight extension of our results is now easily obtained. It concerns the question about the dependence of $a_n(H_p(D) \cap F^r,L_q(-1,1))$ on the domain D. A domain much smaller than Δ might be useful to consider when it is a matter of approximating functions with singularities very near the interval of approximation $(-1,1)$. Or, conversely, one might be interested in functions known to have singularities far from $(-1,1)$. Suitable domains generalising Δ are given by

$$\Delta_d = \left\{w \in \mathbb{C} : |\arg \frac{1+w}{1-w}| < d\right\}, \quad 0 < d < \pi , \quad \Delta = \Delta_{\pi/2} .$$

These were considered by Stenger [13-16], who obtained upper bounds. The substitution $z = \log \frac{1+w}{1-w}$ maps Δ_d conformally and 1-to-1 onto Ω_d. Using the isometry $(Tg)(z) = g(\lambda z)$, $\lambda = \frac{2d}{\pi}$, we find we can reduce the problem of bounding

$$a_n(H_p(\Delta_d) \cap F^r,L_q(-1,1))$$

to that of bounding

$$a_n(H_p(\Omega) \cap Y_{-\lambda r},\phi^{\lambda/q}L_q(\mathbb{R}))$$

which we have already solved. We state only two of the resulting estimates for illustration:

$$d_n(H_\infty(\Delta_d) \cap F^r,L_\infty(-1,1)) >< \exp\left(-\frac{\pi}{2}(\lambda rn)^{1/2}\right)$$

where $r > 0$, $\lambda = \frac{2\pi}{d}$, and for $p > q/\lambda$

$$\exp(-2\beta n^{1/2}) << d_n(H_p(\Delta_d),L_q(-1,1)) << \exp(-\beta n^{1/2})$$

when $\beta = \frac{\pi}{2}\left(\frac{\lambda}{q} - \frac{1}{p}\right)^{1/2}$ and similarly in the remaining cases.

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The n-width d_n, approximation numbers δ_n and entropy ϵ_n of the Hardy spaces H_p in $L_q(-1,1)$ are estimated. More precisely, denote by F^r the space of continuous functions which satisfy a Lipschitz condition of order r at ± 1. It is shown that (cont.)

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20. ABSTRACT (cont.)

$$\exp(-2\alpha n^{1/2}) << \delta_n(H_p \cap F^r, L_\infty), d_n(H_p \cap F^r, L_\infty) << \exp(-\alpha n^{1/2})$$

$$\exp(-2\beta n^{1/2}) << \delta_n(H_p, L_q), d_n(H_p, L_q) << \exp(-\beta n^{1/2}), \quad \text{for } p > q$$

$$\exp(-2\gamma n^{1/3}) << \varepsilon_n(H_p \cap F^r, L_\infty) << \exp(-\gamma n^{1/3})$$

where "<<" indicates that the inequalities hold except for polynomial factors in n. The constants α, β, γ depend on p, q and r. For $p = \infty$, the factor 2 in the lower bound of the first inequality can be omitted.