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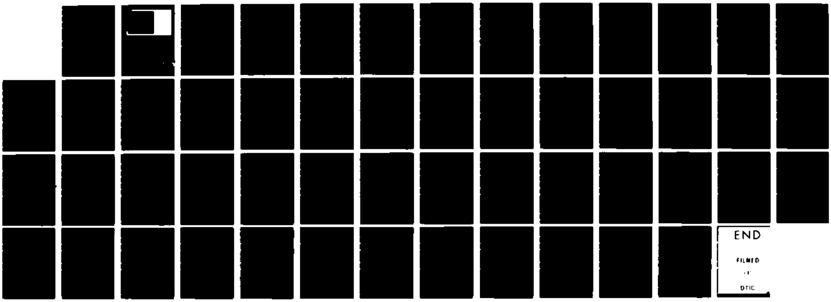

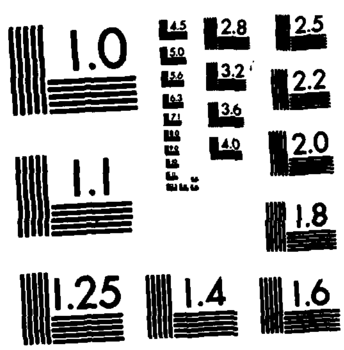
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A NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION OCCURRING IN HEAT FLOW

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ABSTRACT

We study the A nonlinear Volterra integrodifferential equation is studied.

$$\frac{du}{dt} + Bu(t) + a*Au(t) + \frac{d}{dt}(b*u(t)) \ni F(t) \quad \text{a.e. on } \mathbb{R}^+$$

$$u(0) = u_0 \;;$$

A, B are nonlinear operators, a, b, F are functions defined on $[0,\infty)$, $*$ denotes the convolution on $[0,t]$, and u_0 is a given element. Under various assumptions motivated by heat flow in materials with memory, results on existence of solutions are obtained, followed by various results on boundedness and the asymptotic behaviour of solutions ~~as $t \to \infty$~~ with applications to such heat flow problems.

AMS (MOS) Subject Classifications: 45K05, 45D05, 45G10, 45N05, 45M10, 47H05, 47H15, 73D05, 73F99, 80A20

Key Words: nonlinear Volterra equations, maximum monotone operator, subdifferential, global existence, boundedness, asymptotic behaviour, limit equation, heat flow, materials with memory.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Consider nonlinear heat flow in a homogeneous bar of unit length of a material with memory with the ends of the rod maintained at zero temperature and with the history of temperature prescribed for time $t < 0$. For such materials the internal energy and heat flux are functionals (rather than functions) of the temperature and of the gradient of temperature respectively. Under physically reasonable constitutive assumptions for these, generally nonlinear, functionals application of the law of balance of heat leads to a nonlinear Volterra integrodifferential equation, derived in Section 6 (see equation (6.4)), together with appropriate boundary and initial conditions, which model the physical problem. This mathematical model problem, which cannot be solved explicitly and which is difficult to analyse, can be transformed by standard methods to the general nonlinear integrodifferential equation given in the Abstract. The resulting kernels a and b can be expressed in terms of the internal energy and heat flux relaxation functions which are presumed to be known for the physical problem. The operators A and B are nonlinear differential operators which incorporate the boundary conditions, and the forcing term F depends on the given initial temperature distribution, the given external heat supply, and the given history of temperature. In previous studies it was either assumed that the operators A and B are equal or that the kernel $b \equiv 0$, or both. The problem as formulated in this paper appears to model the general physical situation more accurately, although admittedly the experimental evidence for theories of heat flow in materials with memory is rather sparce.

Under physically reasonable assumptions motivated by this physical problem we establish existence of global solutions, followed by a rather complete description of the qualitative behaviour of such solutions, including boundedness and decay as $t \to \infty$; the approach to equilibrium states (other than zero) as $t \to \infty$ is also analysed. These results are obtained for the abstract evolution equation (using techniques of monotone operator theory combined with energy methods and the theory of Volterra operators), and then interpreted and applied to the physical problem. A comparison with other results in the literature is also given.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION
OCCURRING IN HEAT FLOW

S.-O. Londen[(1)] and J. A. Nohel[(2)]

1. Introduction and Discussion of Results. We study the nonlinear Volterra integrodifferential equation

$$\begin{aligned} &\frac{du}{dt} + Bu(t) + a*Au(t) + \frac{d}{dt}(b*u(t)) \ni F(t) \quad \text{a.e. on } \mathbb{R}^+ \\ &u(0) = u_0 \ . \end{aligned} \tag{1.1}$$

In (1.1) A, B are nonlinear operators, a, b and F are given functions defined on $[0,\infty)$, $*$ denotes the convolution $g*h(t) = \int_0^t g(t-\tau)h(\tau)d\tau$, and u_0 is a given element. Under various assumptions, partly motivated by the problem of heat flow in a material with "memory" formulated and discussed in Section 6, existence results are established, followed by L^2, boundedness, and asymptotic results. These are then applied to the physical problem in Section 6. From the abstract viewpoint the present study generalizes the theory developed in [8] for (1.1) with $b \equiv 0$ (see further comments below); the case $b \not\equiv 0$ is the one which arises naturally in the mathematical model for heat flow.

In order to state and discuss the existence results we follow [8] and introduce the hypotheses common to Theorem 1 and 2 under the heading:

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General Assumptions

Let H be a real Hilbert space and W a real reflexive Banach space satisfying

$$W \subset H \subset W' \tag{1.2}$$

where W' is the dual of W. It is assumed that the injections in (1.2) are continuous and dense and $\langle w',w\rangle = (w',w)$ for $w' \in H$, $w \in W$ where $\langle w',w\rangle$ is the value of $w' \in W'$ at $w \in W$ and $(\cdot,\cdot)$ is the inner product of H. We denote the norm in H by $|\cdot|$ and the norm in W by $\|\cdot\|$. Let $\psi : W \to (-\infty,\infty]$ and $\phi : H \to (-\infty,\infty]$ be convex, lower semicontinuous (l.s.c.) and proper functions and define

$$A = \partial\psi, \quad B = \partial\phi \quad , \tag{1.3}$$

where $\partial\psi$, $\partial\phi$ are the subdifferentials of ψ and ϕ respectively (see e.g., [1]). Then A and B are (possibly multivalued) maximal monotone operators from W and H to W' and H respectively. Define $\psi_H : H \to (-\infty,\infty]$ by

$$\psi_H(u) = \lim_{r\downarrow 0} \inf\{\psi(v) : v \in W \text{ and } |v - u| < r\} \quad . \tag{1.4}$$

ψ_H is automatically l.s.c. and ψ_H is convex since ψ is convex. ψ_H is the largest l.s.c. function on H satisfying $\psi_H \leq \psi$ on W. We assume that

$$\psi_H(u) = \psi(u) \quad \text{for} \quad u \in W \quad . \tag{1.5}$$

Let $A_H = \partial\psi_H$; A_H is maximal monotone in H and, in view of (1.5), has the property

$$A_H u \subset Au \quad \text{for} \quad u \in W \quad . \tag{1.6}$$

This follows from the implication: $u \in W$, $v \in H$ and $\psi_H(z) \geq \psi_H(u) + (v,z - u)$ for $z \in H \Longrightarrow \psi(z) \geq \psi(u) + \langle v,z - u\rangle$ for $z \in W$ when (1.5) holds. Note that if $\tilde{\psi} : H \to (-\infty,\infty]$ defined by

$$\tilde{\psi}(u) \begin{cases} = \psi(u), & u \in W \\ = +\infty, & u \in H\backslash W \end{cases}$$

is l.s.c., then $\tilde{\psi} = \psi_H$ and (1.5) holds. Moreover, $\tilde{\psi}$ is l.s.c. if $\lim_{\|u\|\to\infty}\psi(u) = +\infty$.

The Yosida approximations A_λ of A_H are defined for $\lambda > 0$ by

$$A_\lambda = \lambda^{-1}(I - J_\lambda), \quad J_\lambda = (I + \lambda A_H)^{-1} \quad ;$$

see [1] for the properties of A_λ. Relating A_λ and B we assume there exists $\beta \in [0,\infty)$ such that

$$(w,A_\lambda u) \geq -\beta(|w|^2 + |u|^2 + 1) \quad \text{for} \quad u \in W,\ w \in Bu,\ \lambda \in (0,1] \quad . \tag{1.7}$$

We will also require the compactness assumption

$$\text{For every} \quad K > 0,\ \{u \in H : |\phi(u)| + |u| \leq K\} \quad \text{is precompact in} \quad W \quad . \tag{1.8}$$

In particular, we assume $D(\phi) \subseteq W$.

As regards the kernel a, we will require that the following conditions are satisfied.

Conditions (a):

$$a(t) \quad \text{is locally absolutely continuous on} \quad [0,\infty) \quad . \tag{1.9}$$

For every $T > 0$ there is a $K_T > 0$ such that

$$v \in L^2(0,T;H), \quad d_1,d_2 \in [0,\infty)$$

and

$$\int_0^t (a*v(s),v(s))ds \leq d_1 + d_2 \max_{0\leq s\leq t} |\int_0^s v(\tau)d\tau|, \quad 0 \leq t \leq T$$

imply

$$|\int_0^t v(s)ds| \leq K_T(d_1^{1/2} + d_2), \quad 0 \leq t \leq T \quad , \tag{1.10}$$

and

$$|\int_0^t (a*v(s),v(s))ds| \leq K_T(d_1 + d_2^2), \quad 0 \leq t \leq T \quad .$$

Note that if $v \in L^2(0,T_0;H)$ where $T_0 < T$ satisfies the assumptions of (1.10) on $[0,T_0]$, then v extended as 0 on $(T_0,T]$ satisfies the same conditions on $[0,T]$. Thus, without loss of generality, the map $T \to K_T$ can be assumed nondecreasing.

For classes of kernels a satisfying Conditions (a) see Proposition (a) and Theorem (a) of [8]. Finally, regarding the kernel b we assume:

$$b(t) \text{ is locally absolutely continuous on } [0,\infty) \quad . \tag{1.11}$$

This concludes the general assumptions.

The first existence result is:

<u>Theorem 1. Let the general assumptions</u> (1.2) - (1.11) <u>be satisfied. Further assume that</u> $A = \partial\psi$ <u>is single-valued and</u> $D(A) = W$. <u>Then for every</u> $F \in W^{1,1}_{\ell oc}([0,\infty);H)$ <u>and</u> $u_0 \in D(\phi)$ <u>equation</u> (1.1) <u>has a solution</u> u <u>in the sense</u>:

(i) $u \in C([0,\infty);W)$,

(ii) $\frac{du}{dt} \in L^2_{\ell oc}([0,\infty);H)$,

(iii) $F - \frac{d}{dt}(u + b*u) - a*Au \in L^2_{\ell oc}([0,\infty);H)$,

(iv) $F(t) - \frac{d}{dt}(u(t) + b*u(t)) - a*Au(t) \in Bu(t)$ a.e. $t > 0$.

<u>Moreover</u>,

(v) $\int_0^t Au(s)ds \in L^\infty_{\ell oc}([0,\infty);H)$.

In the special case $b \equiv 0$, which is not excluded here, Theorem 1 was proved in [8]. The present result, as well as Theorem 2 below, is a generalization in the spirit of the remarks in [8, p. 717] in which the operator A in (1.1) is replaced by $A + P$ where $P : H \to H$ is a Lipschitz mapping. However, in the present context the perturbation term $\frac{d}{dt}(b*u)$ is different and requires a different treatment in the proof which is sketched in Section 2. Primarily affected is the proof of the analogue of Proposition 2.1 of [8]. A proof of a similar generalization was first given by Mr. M. J. Luo as a part of a research seminar of the second author at the University of Wisconsin during 1979-80.

We remark as in [8, p. 705] that by conclusion (i) of Theorem 1 the map $t \to Au(t)$ is continuous into the weak topology of W' and $a*Au$ is well defined with values in W'. By (v) and $a*Au(t) = a(0) \int_0^t Au(s)ds + a'*(\int_0^t Au(s)ds)$, one also has $a*Au \in L^\infty_{loc}(\mathbb{R}^+, H)$. *However, under the* assumptions of Theorem 1 one cannot obtain estimates on Au in H.

Under suitable additional assumptions estimates on $Au \in L^2_{loc}([0,\infty);H)$ can be obtained. Then, as in [8], existence results can be proved in which neither A nor B is required to be single-valued. We give such a result under the type of compatibility restriction relating the operators A and B which is used in the boundedness and asymptotic analysis for (1.1).

Theorem 2. Let the general assumptions (1.2) - (1.11) be satisfied with $W = H = W'$ (thus $\psi_H = \psi$, $A_H = A$, etc.). In addition, let

$$b(0) > 0 \tag{1.12}$$

and let there exist constants $\gamma > 0$, $\delta > 0$ such that

$$\gamma|u|^2 + (v,w) - \frac{b(0)}{2}|v-u|^2 \geq \delta|v|^2 \tag{1.13}$$

for $v \in Au$, $w \in Bu$. Then for every $F \in W^{1,1}_{loc}([0,\infty);H)$ and $u_0 \in D(\psi) \cap D(\phi)$ equation (1.1) has a solution u satisfying $u(0) = u_0$, u, $u' \in L^2_{loc}([0,\infty);H)$, and there exist $v, w \in L^2_{loc}([0,\infty);H)$ with $v(t) \in Au(t)$, $w \in Bu(t)$ a.e. on $0 < t < \infty$ such that

$$\frac{du}{dt} + w(t) + a*v(t) + \frac{d}{dt}(b*u(t)) = F(t) \quad \text{a.e.} \quad (0 < t < \infty) \ .$$

A sketch of the proofs of Theorems 1 and 2 is given in Section 2. Assumption (1.13) can be replaced by the more general assumption (used in [8, Theorem 2]): for each $r > 0$ there exists a number $k(r)$ such that

$$k(r)\,(1 + |w|) \geq |v| \quad \text{for } v \in Au,\ w \in Bu,\ |u| \leq r \tag{1.14}$$

without affecting the proof of Theorem 2. To verify that (1.13) implies (1.14) take $k(r) = k_0(1 + r^2)$, $k_0 = \max(\delta^{-1}, \gamma\delta^{-1}, 1)$ and consider the cases $|v| > 1$, $|v| \leq 1$ in (1.13). We prefer using (1.13) as it arises naturally in the discussion of L^2, boundedness and asymptotic results for solutions of (1.1) which will be presented next. Concerning Theorems 1 and 2 we note that the question of uniqueness of solutions of (1.1) remains open, even in the case $b \equiv 0$.

We next turn to a discussion of boundedness and asymptotics of solutions. To simplify the exposition we assume in Theorems 3 - 6 that the operators A and B are single-valued, and consequently replace the inclusion by equality in (1.1). In what follows we denote locally absolutely continuous functions by LAC.

<u>Theorem 3</u> <u>Assume that in</u> (1.1)

$$a,\ a' \in L^1(\mathbb{R}^+)\ , \tag{1.15}$$

$$a \ \underline{\text{is strongly positive definite on}}\ \mathbb{R}^+\ , \tag{1.16}$$

$$\begin{aligned} &b \in LAC(\mathbb{R}^+),\ \ \underline{\text{and}}\ \ (-1)^k b^{(k)}(t) \geq 0 \\ &\text{a.e. } \underline{\text{on}}\ \ \mathbb{R}^+\ (k = 0,1)\ , \end{aligned} \tag{1.17}$$

$$F \in L^2(\mathbb{R}^+,H)\ , \tag{1.18}$$

$$A = \partial\psi \ \ \underline{\text{where}}\ \ \psi : H \to (-\infty,\infty]\ \ \underline{\text{is convex, l.s.c. proper}}\ , \tag{1.19}$$

$$\begin{aligned} &B : D(B) \subset H \to H \ \ \underline{\text{with}}\ \ (u,Bu) \geq c|u|^2 \\ &\underline{\text{for some}}\ \ c > 0,\ \ \underline{\text{for every}}\ \ u \in D(B) \end{aligned} \tag{1.20}$$

$$\left.\begin{aligned} &\mu\, c_0^2|u|^2 + (Au,Bu) - \frac{b(0)}{2}\,|Au-u|^2 \geq \delta|Au|^2 \\ &\underline{\text{for some}}\ \ \mu > 0\ \ \underline{\text{and}}\ \ c_0,\ \delta\ \ \underline{\text{satisfying}}\ \ c > c_0,\ \delta > 0\ , \\ &\underline{\text{for every}}\ \ u \in D(A) \cap D(B). \end{aligned}\right\} \tag{1.21}$$

<u>Let</u> u <u>be a solution of</u> (1.1) <u>satisfying</u>

$$u \in LAC(\mathbb{R}^+,D(A) \cap D(B));\ Au,\ Bu \in L^2_{\ell oc}(\mathbb{R}^+,H)\ . \tag{1.22}$$

<u>Then</u>

$$Au \in L^2(\mathbb{R}^+,H),\ u \in L^2(\mathbb{R}^+,H) \cap L^\infty(\mathbb{R}^+,H)\ .$$

By definition the condition (1.16) is the same as the requirement that $a(t) - \alpha \exp(-t)$ is positive definite on $\mathbb{R}^+$ for some $\alpha > 0$. A consequence of (1.15), (1.16) is that (see [19, Lemma 4.2] and [10, Lemma 3.1]) for every $\phi \in L^1_{\ell oc}(\mathbb{R}^+,H)$ and for every $T > 0$

$$\int_0^T |a*\phi(t)|^2 dt \le \mu^{-1} Q(a,\phi,T) \tag{1.23}$$

where $Q(a,\phi,T) = \int_0^T (\phi(t), a*\phi(t))dt$, and where $\mu = \alpha^{-1}\|a\|_{L^1(\mathbb{R}^+)} + 4\alpha^{-1}\|a'\|_{L^1(\mathbb{R}^+)}$. It is important to observe that this constant μ also appears in (1.21). Assumption (1.21) is formally the same as assumption (1.13) in Theorem 2, but the constant γ is now written in the form μc_0^2. It should be noted that the requirement $\inf_{u \in H} \psi(u) > -\infty$ is not imposed in Theorem 3 (compare [8], Theorem 4); thus Theorem 3 is new, even in the special case $b \equiv 0$.

The assumptions (1.15) - (1.21) of Theorem 3 do not imply the existence of solutions of (1.1) satisfying (1.22). However, if one also requires that $a' \in BV_{\ell oc}[0,\infty)$, that $B = \partial\phi$, where $\phi : H \to (-\infty,\infty]$ is a convex, l.s.c., and proper function, that assumptions (1.7), (1.8) are satisfied, and that $F \in W^{1,1}_{\ell oc}(\mathbb{R}^+,H)$, then Theorem 2 yields the existence of solutions u satisfying (1.22). The reader should note that $a' \in BV_{\ell oc}[0,\infty]$, (1.15) and $a(0) > 0$ (which follows from (1.16)), imply that conditions (a) of the general assumptions are satisfied (see Proposition (a) in [8]). Theorem 3 is proved in Section 3.

In order to state a boundedness result for "large" forcing terms F in (1.1) (i.e. F not necessarily in $L^2(\mathbb{R}^+,H)$) we denote by $L^2_\infty(\mathbb{R}^+,H)$ the class of functions $\phi : \mathbb{R}^+ \to H$ such that each ϕ is locally square integrable and such that

$$\sup_{1 \le t < \infty} \int_{t-1}^t |\phi(s)|^2 ds < \infty .$$

Theorem 4. Let the assumptions (1.15) - (1.17), (1.19) - (1.22) *of* Theorem 3 *be satisfied. In addition, assume that*

$$|a(t)| \le Kt^{-\nu}, \quad |b'(t)| \le Kt^{-\nu} \quad \text{a.e. } \underline{on} \quad [1,\infty) \tag{1.24}$$

for some constants K, ν *with* $\nu > 3/2$,

$$F \in L^2_\infty(\mathbb{R}^+,H) \tag{1.25}$$

$$|u| \le \rho|Au|, \quad \underline{\text{for some}} \quad \rho > 0 \quad \underline{\text{and for every}} \quad u \in D(A) . \tag{1.26}$$

Then

$$Au \in L^2_\infty(\mathbb{R}^+,H) \tag{1.27}$$

$$u \in L^\infty(\mathbb{R}^+,H) . \tag{1.28}$$

If, in addition, $B = \partial\phi$ *where* $\phi : H \to (-\infty,\infty]$, *is convex, l.s.c. and proper, then*

$$\frac{du}{dt} \quad \underline{\text{and}} \quad Bu \in L^2_\infty(\mathbb{R}^+,H) . \tag{1.29}$$

Theorem 4 is proved in Section 4.

The common conclusion of Theorems 3 and 4 is $u \in L^\infty(\mathbb{R}^+,H)$. Comparing the two results observe that the assumption (1.18) in Theorem 3 concerning F is weakened to (1.25) in Theorem 4. But in order to establish the conclusions of Theorem 4 the decay rates (1.24) must be added to assumptions (1.15) - (1.17), and assumption (1.26) is needed in addition to (1.21). In the special case $b \equiv 0$ and $B \ne A$ no analogue of Theorem 4 (also of Theorem 5 and 6) was considered previously.

Theorem 4 serves as a basis for the following asymptotic result.

Theorem 5. Let the assumptions of Theorem 4 be satisfied. In addition, assume that assumption (1.25) *is strengthened to*

$$\lim_{t\to\infty} \int_{t-1}^{t} |F(\tau)|^2 d\tau = 0 . \tag{1.30}$$

Then

$$\lim_{t\to\infty} \int_{t-1}^{t} |Au(\tau)|^2 d\tau = 0 , \tag{1.31}$$

$$\lim_{t\to\infty} |u(t)| = 0 \quad . \tag{1.32}$$

Theorem 5 is proved in Section 5.

We next wish to consider the analogues of Theorems 3 and 5 for equation (1.1) when $F(\infty) \neq 0$, of importance for the physical problem discussed in Section 6. To introduce the results proceed formally at first assuming that, e.g., the assumptions of Theorem 3 are satisfied, except that (1.18) is replaced by $\lim_{t\to\infty} F(t) = F(\infty)$ exists. In addition, suppose that $b(\infty) = 0$ and that u is a solution of (1.1) such that $u(\infty) = \lim_{t\to\infty} u(t)$ exists. Then the "limit equation" associated with (1.1) is

$$Bu(\infty) + \left(\int_0^\infty a(s)ds\right)Au(\infty) = F(\infty) \ , \tag{1.33}$$

where $\int_0^\infty a(s)ds > 0$ (by assumption (1.16)). A result of Brézis and Haraux [2] states that equation (1.33) has a unique solution $u(\infty)$ for every value $F(\infty)$ in H (including 0), provided the operators A and B are subdifferentials (of proper, convex l.s.c. functions: $H \to (-\infty,\infty]$), and provided at least one of the operators is onto (this is the case for B satisfying (1.20)).

It is easily seen that if $u(\infty)$ is the solution of the limit equation (1.33) and if $u(t)$ satisfies (1.1) a.e. on R^+, then $u(t) - u(\infty)$ satisfies the equation

$$\begin{gathered} \frac{d}{dt}(u(t) - u(\infty)) + Bu(t) - Bu(\infty) + a*(Au(t) - Au(\infty)) + \\ \frac{d}{dt}[b*(u(t) - u(\infty))] = g(t) \quad \text{a.e. on } R^+ \ , \end{gathered} \tag{1.34}$$

where

$$\begin{gathered} g(t) = f(t) + \left(\int_t^\infty a(s)ds\right)Au(\infty) - b(t)u(\infty) \\ f(t) = F(t) - F(\infty) \quad . \end{gathered} \tag{1.35}$$

The following analogue of Theorem 3 can be proved by examining its proof in Section 3 step by step.

Theorem 6. Let the assumptions (1.15) - (1.17), $b(\infty) = 0$, (1.19) be satisfied. In addition, assume that

$$Bu = \partial\phi(u), \ \phi : H \to (-\infty,\infty] \ \text{is convex, l.s.c. and proper} , \tag{1.36}$$

$$\begin{aligned} &\text{(i) } f(t) = F(t) - F(\infty) \in L^2(\mathbb{R}^+,H), \\ &\text{(ii) } b(t) \in L^2(\mathbb{R}^+), \ \int_t^\infty a(s)ds \in L^2(\mathbb{R}^+) \ . \end{aligned} \tag{1.37}$$

Let u be a solution of (1.1) satisfying (1.22), and let $u(\infty)$ be the solution of the limit equation (1.33) such that assumptions (1.20) and (1.21) hold with u, Au, Bu replaced respectively by $u - u_\infty$, $Au - Au_\infty$, and $Bu - Bu_\infty$. Then

$$(Au - Au(\infty)) \in L^2(\mathbb{R}^+,H), \ (u - u(\infty)) \in L^2(\mathbb{R}^+,H) \cap L^\infty(\mathbb{R}^+,H) \ .$$

It should be observed that if $F(\infty) = 0$, then $u(\infty) = 0$ and Theorem 6 reduces to Theorem 3.

It is also clear that the boundedness result (Theorem 4) does not require any analogue in the present context.

The following analogue of Theorem 5 can be proved by examining its proof in Section 5 step by step.

Theorem 7. Let a, b satisfy (1.15) - (1.17), (1.24) and (1.37(ii)). Assume A satisfies (1.19) and let (1.20), (1.21), (1.26) hold with u, Au, Bu replaced respectively by $u - u(\infty)$, $Au - Au(\infty)$, $Bu - Bu(\infty)$ where $u(\infty)$ is the solution of (1.33). Let u be a solution of (1.1) satisfying (1.22) and suppose $\lim_{t\to\infty} \int_{t-1}^t |F(\tau) - F(\infty)|^2 d\tau = 0$. Then

$$\lim_{t\to\infty} |u(t) - u(\infty)| = 0 \quad , \quad \lim_{t\to\infty} \int_{t-1}^t |Au(s) - Au(\infty)|^2 ds = 0 \ .$$

We conclude the discussion of equation (1.1) with some remarks about the very special case when $B \equiv A$. Define

$$c(t) = 1 + \int_0^t a(\tau)d\tau \quad (t \in \mathbb{R}^+) \quad .$$

Then (1.1) with $B \equiv A$ can be written in the form

$$\frac{d}{dt} [u + c*Au + b*u] \ni F, \; u(0) = u_0 \quad . \tag{1.38}$$

Let

$$G(t) = u_0 + \int_0^t F(\tau)d\tau \quad .$$

Integrating (1.38), equation (1.1) $(B \equiv A)$ is equivalent to the nonlinear Volterra equation

$$u + c*Au + b*u \ni G \quad . \tag{1.39}$$

If $k : [0,\infty) \to \mathbb{R}^+$ is the resolvent kernel of b, uniquely defined (under assumption (1.11)) by

$$k(t) + b*k(t) = -b(t) \quad ,$$

and if

$$d(t) = c(t) + k*c(t), \; g(t) = G(t) + k*G(t) \quad ,$$

then (1.39) is equivalent to the nonlinear Volterra equation

$$u(t) + d*Au(t) \ni g(t) \;\; \text{a.e. on } \mathbb{R}^+ \quad . \tag{1.40}$$

This equation has been studied extensively in the present context. In particular, existence (and also uniqueness) theory has been developed by S.-O. Londen [13], Crandall and Nohel [9], Gripenberg [11], results on boundedness and asymptotic behaviour of solutions of (1.40) have been obtained by R. C. MacCamy [15], S.-O. Londen [13], and particularly analogues of Theorem 3, 5, 7 with applications to a special case of the heat flow problem discussed in Section 6, by Clément, MacCamy, and Nohel [5]. The existence, boundedness, and asymptotic behaviour of positive solutions of (1.40) (when the data are positive) was investigated by Clément and Nohel [3], [4]. The present study can also be regarded as a generalization to (1.1) of some of these results when $B \not\equiv A$.

2. Proof of Theorems 1 and 2. The basic outline of the proof will follow that of Theorems 1 and 2 in [8] which concerns the special case $b \equiv 0$ in (1.1). Several of the technical aspects do however differ; the latter will be emphasized.

Let A_λ be the Yosida approximations of A_H and consider the regularized problem associated with (1.1) (compare [8, (2.11)]):

$$\begin{aligned} &\frac{du_\lambda}{dt} + Bu_\lambda + \varepsilon A_\lambda u_\lambda + a*A_\lambda u_\lambda + \frac{d}{dt}(b*u_\lambda) \ni F, \ \lambda > 0, \ \varepsilon > 0 \\ &u_\lambda(0) = u_0 \ . \end{aligned} \tag{2.1}$$

An easy application of Lemma 2.1 of [8] with

$$G(u) = F - \varepsilon A_\lambda u - a*A_\lambda u - \frac{d}{dt}(b*u)$$

yields the following analogue of Corollary 2.1 of [8]:

Proposition 2.1. Let the general assumptions (1.2) - (1.11) be satisfied. Let $\varepsilon > 0$, $\lambda > 0$ be fixed. Then for every $F \in L^2_{loc}(\mathbb{R}^+;H)$ and $u_0 \in D(\phi)$ the initial value problem (2.1) has a unique solution u_λ on $[0,\infty)$ in the sense

$$u_\lambda \in C([0,\infty);H), \ \frac{du_\lambda}{dt} \in L^2_{loc}(\mathbb{R}^+,H)$$

$$u_\lambda \in D(B) \quad \text{a. e. on} \quad \mathbb{R}^+$$

$$u_\lambda \ \text{satisfies (2.1) a.e. on} \ \mathbb{R}^+ \ .$$

The next step is to obtain various a priori estimates for the solution u_λ of (2.1) which permit first $\lambda \to 0$ for fixed $\varepsilon > 0$, and then $\varepsilon \to 0$ in (2.1). For this purpose we establish the following analogue of Proposition 2.1 of [8]; it is here where the technicalities of the proof differ.

Proposition 2.2. Let $T > 0$, $D = \partial\Phi$, $C = \partial\Psi$ where $\Phi, \Psi : H \to (-\infty,\infty]$ are convex, l.s.c., and proper. Let $\alpha, \beta, c_0 \in [0,\infty)$, $T > 0$, $F \in W^{1,1}(0,T;H)$, $u_0 \in D(\Phi) \cap D(\Psi)$, $a: [0,\infty) \to \mathbb{R}$, $b : [0,\infty) \to \mathbb{R}$ be given such that

$$\left.\begin{array}{ll}\text{(i)} & \Phi(u) \geq -c_0(|u|+1),\ \Psi(u) \geq -c_0(|u|+1) \ \underline{\text{for}}\ u \in H\ ,\\ \text{(ii)} & (v,w) \geq \alpha|v|^2 - \beta(|w|^2 + |u|^2 + 1) \ \text{for}\ u \in H,\ v \in Cu,\ w \in Du\ ,\\ \text{(iii)} & a\ \underline{\text{satisfies conditions}}\ (a),\quad \text{(iv)}\ b\ \underline{\text{satisfies}}\ (1.11)\ ,\end{array}\right\} \quad (2.2)$$

__Then there exists a constant__ C __depending on__ $|u_0|$, a, b, c_0, T, $\Phi(u_0)$, $\Psi(u_0)$, β, $\|F\|_{W^{1,1}}$ (__but not otherwise on__ Φ, Ψ, __and not on__ α), __such that if__

$$\left.\begin{array}{ll}\text{(i)} & u,\ \frac{du}{dt},\ v,\ w \in L^2(0,T;H),\ u(0) = u_0\\ \text{(ii)} & v(t) \in Cu(t),\ w(t) \in Du(t)\ \ \text{a.e.}\ \underline{\text{on}}\ (0,T)\\ \text{(iii)} & \frac{du}{dt} + w(t) + a*v(t) + \frac{d}{dt}[b*u(t)] = F(t)\ \ \text{a.e.}\ \underline{\text{on}}\ (0,T)\ ,\end{array}\right\} \quad (2.3)$$

__then__

$$\max\{\int_0^T |\frac{du}{ds}(s)|^2 ds,\ \int_0^T |w(s)|^2 ds,\ \alpha\int_0^T |v(s)|^2 ds,\ |u(t)|\ ,$$

$$|\Phi(u(t))|,\ |\Psi(u(t))|,\ |\int_0^t v(s)ds|\} \leq C$$

__for__ $0 \leq t \leq T$.

__Sketch of Proof of Proposition 2.2.__ The proof is similar to that of Proposition 2.1 in [8]. In particular, to obtain the analogue of the estimate (2.18) in [8] take the scalar product of (2.3) (iii) with v, integrate over $[0,t]$ and use (2.2) (ii) to obtain (compare with (2.14) in [8, p. 711]):

$$\begin{aligned}&\Psi(u(t)) - \Psi(u_0) + \alpha\int_0^t |v(s)|^2 ds + \int_0^t (a*v(s),v(s))ds \leq\\ &-b(0)\int_0^t (u(s),v(s))ds - \int_0^t (b'*u(s),v(s))ds + \int_0^t (F(s),v(s))ds \qquad (2.4)\\ &+ \beta[\int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds + 1],\ 0 \leq t \leq T\ .\end{aligned}$$

Define as in [8]

$$g_v(t) = \max_{0 \leq s \leq t} |\int_0^s v(s)ds|\ .$$

Using assumption (2.2) (i) and the estimate (see (2.17) (i) in [8])

$$\left|\int_0^t (F(s),v(s))ds\right| \le c_1 g_v(t)$$

in (2.4) yields

$$\alpha \int_0^t |v(s)|^2 ds + \int_0^t (a^*v(s),v(s))ds \le c_0(|u|+1) + \psi(u_0) + c_1 g_v(t)$$

$$+ \beta[1 + \int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds] + |b(0)| \left|\int_0^t (v(s),u(s))ds\right| \qquad (2.5)$$

$$+ \left|\int_0^t (v(s), b'^*u(s))ds\right| \quad 0 \le t \le T \quad .$$

By $c_1, c_2, \ldots$ we denote constants which depend only on $|u_0|$, a, b, c_0, T, $\phi(u_0)$, $\psi(u_0)$, β and $\|F\|_{W^{1,1}(0,T;H)}$.

To estimate the last two terms in (2.5), integrate both by parts and estimate to obtain

$$|b(0)| \left|\int_0^t (v(s),u(s))ds\right| + \left|\int_0^t (v(s),b'^*u(s))ds\right| \le$$

$$g_v(t)[|b(0)|\ |u(t)| + |b(0)| \int_0^t |u'(s)|ds + \|b'\|_{L^1(0,T)} \sup_{0\le s\le t} |u(s)|$$

$$+ |u(0)|\ \|b'\|_{L^1(0,T)} + \|b'\|_{L^1(0,T)} \int_0^t |u'(s)|ds] \quad .$$

Substitution of this estimate into (2.5) yields (compare with (2.18) in [8])

$$\alpha \int_0^t |v(s)|^2 ds + \int_0^t (a^*v(s),v(s))ds \le$$

$$c_2[1 + |u(t)| + \int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds] \qquad (2.6)$$

$$+ c_3[1 + |u(t)| + \int_0^t |u'(s)|ds]g_v(t) \ , \quad 0 \le t \le T \quad .$$

The monotonicity of the maps $t \to \|u\|_{L^\infty(0,t;H)} + \int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds$ and $t \to \|u\|_{L^\infty(0,t;H)} + \int_0^t |u'(s)|ds$ used in conditions (a) and combined with (2.6) yields (compare with (2.19) in [8])

$$g_v(t) \leq c_4(1 + \|u(t)\|_{L^\infty(0,t,H)} + \int_0^t |w(s)|^2 ds + \int_0^t |u(s)|^2 ds)^{1/2}$$

$$+ c_5(1 + \|u(t)\|_{L^\infty(0,t,H)} + \int_0^t |u'(s)| ds), \ 0 \leq t \leq T \ . \tag{2.7}$$

Next, from (2.3) (iii)

$$w(t) = F(t) - u'(t) - a*v(t) - b(0)u(t) - b'*u(t) \quad ,$$

and using the known estimate (see (2.17) (ii) in [8])

$$|a*v(t)| \leq c_6 g_v(t) \quad ,$$

we obtain

$$|w(t)| \leq c_7[1 + |u'(t)| + g_v(t) + \sup_{0 \leq \tau \leq t} |u(\tau)|] \quad .$$

Substitution into (2.7) yields (compare with (2.21) in [8] where the first term under the integral should be $|u'(s)|^2$)

$$g_v(t) \leq c_8[\int_0^t (|u'(s)|^2 + \sup_{0 \leq \tau \leq s} |u(\tau)|^2 + g_v^2(s)) ds]^{1/2}$$

$$+ c_9[\ 1 + \int_0^t |u'(s)| ds], \ 0 \leq t \leq T \quad . \tag{2.8}$$

Squaring (2.8) and using

$$\sup_{0 \leq \tau \leq s} |u(\tau)|^2 \leq (|u(0)| + \int_0^s |u'(\tau) d\tau)^2 \quad ,$$

$$(\int_0^t |u'(s)| ds)^2 \leq t \int_0^t |u'(s)|^2 ds$$

in (2.8) yields (compare with (2.26) in [8])

$$g_v^2(t) \leq c_{10}(1 + \int_0^t |u'(s)|^2 ds + \int_0^t (g_v(s))^2 ds), \ 0 \leq t \leq T \quad , \tag{2.9}$$

The Gronwall inequality, $g_v(0) = 0$, and the monotonicity of the map $t \to \int_0^t |u'(s)|^2 ds$ used in (2.9) imply (compare with (2.28) in [8])

$$g_v^2(t) \leq c_{11}(1 + \int_0^t |u'(s)|^2 ds), \ 0 \leq t \leq T \quad . \tag{2.10}$$

We next estimate $\int_0^t |u'(s)|^2 ds$. Taking the scalar product of (2.3) (iii) with u' and integrating over $[0,t]$ yields

$$\int_0^t |u'(s)|^2 ds + \Phi(u(t)) - \Phi(u_0) + \int_0^t (a*v(s),u'(s))ds$$

$$+ b(0) \int_0^t (u(s),u'(s))ds + \int_0^t (b'*u(s),u'(s))ds \tag{2.11}$$

$$\leq \max_{0\leq s\leq t} |F(s)| \int_0^t |u'(s)|ds, \ 0 \leq t \leq T \ .$$

Using (2.2) (i), the known estimate for $|a*v(t)|$ in terms of $g_v(t)$, and

$$|b(0) \int_0^t (u(s),u'(s))ds + \int_0^t (b'*u(s),u'(s))ds| \leq$$

$$\frac{1}{4} \int_0^t |u'(s)|^2 ds + |b(0)| \int_0^t |u(s)|^2 ds + \frac{1}{4} \int_0^t |u'(s)|^2 ds$$

$$+ \|b'\|^2_{L^1} \int_0^t |u(s)|^2 ds$$

in (2.11) gives (compare with (2.23) in [8])

$$\int_0^t|u'(s)|^2 ds \leq c_{12}(1 + (1+g_v(t)) \int_0^t|u'(s)|ds + |u(t)| + \int_0^t|u(s)|^2 ds) \tag{2.12}$$

The routine estimates $|u(t)| \leq |u(0)| + \int_0^t |u'(s)|ds$, $\int_0^t |u(s)|^2 ds \leq K[1 + (\int_0^t |u'(s)|ds)^2]$ used in (2.12) yield

$$\int_0^t |u'(s)|^2 ds \leq c_{13} + c_{14} g_v(t) \int_0^t |u'(s)|ds + c_{15}(\int_0^t |u'(s)|ds)^2$$

$$\leq c_{13} + c_{14}[\frac{\eta}{2} g_v^2(t) + \frac{1}{2\eta} (\int_0^t |u'(s)|ds)^2] + c_{15}(\int_0^t |u'(s)|ds)^2$$

for any $\eta > 0$. Substitution of (2.10) gives, for $\eta > 0$ sufficiently small, the final estimate

$$\int_0^t |u'(s)|^2 ds \leq c_{16} + c_{17}(\int_0^t |u'(s)|ds)^2, \ 0 \leq t \leq T \ ,$$

which is the same as (2.29) in [8]. The proof of Proposition 2.2 is concluded exactly as in [8], proof of Proposition 2.1.

The proof of Theorems 1 and 2 is completed using Propositions 2.1 and 2.2 following the procedure in [8, p. 714-717]. In particular, Proposition 2.2 applied to solutions of (2.1) yields the estimates (2.31) of [8], with (2.31) (vi) replaced by

$$\int_0^T |F(s) - (u'_\lambda(s) + a*A_\lambda u_\lambda(s) + \frac{d}{ds} (b*u_\lambda(s)))|^2 ds \leq C_T \ .$$

Keeping $\varepsilon > 0$ fixed and letting $\lambda \downarrow 0$ in (2.1), and using the estimates (2.31) in [8] and the compactness assumption (1.8) gives (2.32) of [8] with (iv) replaced by

$$F - (u'_{\lambda_n} + \varepsilon A_{\lambda_n} u_{\lambda_n} + a^* A_{\lambda_n} u_{\lambda_n} + \frac{d}{dt}(b^* u_{\lambda_n})) \to w_\varepsilon$$

weakly in $L^2(0,T;H)$, $T > 0$. Then the limit function u_ε satisfies (compare with (2.33) in [8])

$$u'_\varepsilon + w_\varepsilon + \varepsilon v_\varepsilon + a^* v_\varepsilon + \frac{d}{dt}(b^* u_\varepsilon) = F$$

$$u'_\varepsilon,\ w_\varepsilon,\ v_\varepsilon \in L^2_{\ell oc}(\mathbb{R}^+, H),\ w_\varepsilon(t) \in Bu_\varepsilon(t),\ v_\varepsilon(t) \in A_H u_\varepsilon(t) \quad \text{a.e. on } \mathbb{R}^+ .$$

The remainder of the proof is now exactly as in [8]. In proving Theorem 2 one needs to remark, as was already done in (1.14) Section 1, that the present assumption (1.13) in Theorem 2 is a special case of assumption (1.12) in [8].

3. <u>Proof of Theorem 3</u>. Form the inner product of (1.1) with u and integrate over $[0,t]$ obtaining

$$\begin{aligned} &\frac{|u(t)|^2}{2} - \frac{|u_0|^2}{2} + \int_0^t (u,Bu)d\tau + \int_0^t (u,a^*Au)d\tau \\ &\qquad + Q(u,t;db) = \int_0^t (u,F)d\tau,\ t \in \mathbb{R}^+ , \end{aligned} \tag{3.1}$$

where

$$Q(u,t;db) = \int_0^t (u,u^*db)d\tau,\ u^*db = b(0)u(t) + \int_0^t b'(s)u(t-s)ds .$$

Using (1.20), noting that by (1.17) $Q(u,t;db) \geq 0$ (see the identity (3.7) below with $f_1 = f_2 = u$), and writing

$$\|u\|_t^2 = \int_0^t |u(\tau)|^2 d\tau ,$$

(3.1) implies

$$c\|u\|_t^2 \leq \|u\|_t \|F\|_{L^2(\mathbb{R}^+)} + \|u\|_t \|a^*Au\|_t + 2^{-1}|u_0|^2 . \tag{3.2}$$

By (1.23) $\|a*Au\|_t < \mu^{-1/2}Q^{1/2}(a,Au,t)$, and therefore, from (3.2)

$$\mu^{-1/2}Q^{1/2}(a,Au,t) > c\|u\|_t - \|F\|_{L^2(\mathbb{R})^+} - (2\|u\|_t)^{-1}|u_0|^2 \quad . \tag{3.3}$$

Suppose that

$$\lim_{t\to\infty} \|u\|_t = \infty \quad . \tag{3.4}$$

Recalling (1.18) and $c > c_0$ (see (1.20), (1.21)), (3.3) and (3.4) imply

$$Q(a,Au,t) > c_0^2\mu\|u\|_t^2, \quad \text{for } t \in \mathbb{R}^+ \text{ sufficiently large} . \tag{3.5}$$

To obtain an upper bound for $Q(a,Au,t)$ form the inner product of (1.1) with Au and integrate over $[0,t]$. Using (1.19) one obtains

$$\begin{aligned}\psi(u(t)) - \psi(u_0) + \int_0^t (Au,Bu)d\tau + Q(a,Au,t)\\ + \int_0^t (Au,u*db)d\tau = \int_0^t (Au,F)d\tau,\ t \in \mathbb{R}^+ \quad .\end{aligned} \tag{3.6}$$

To estimate the last term on the left side of (3.6) we use the definition of $u*db$ and the identity (easily checked directly by differentiating both sides)

$$\begin{aligned}&\int_0^t(f_1,f_2*b')d\tau =\\ &-\frac{1}{2}\int_0^t\int_0^\tau|f_1(\tau) - f_2(\tau-s)|^2b'(s)dsd\tau + \frac{1}{2}\int_0^t b(\tau)|f_1(\tau)|^2d\tau\\ &+\frac{1}{2}\int_0^t b(t-\tau)|f_2(\tau)|^2d\tau - \frac{b(0)}{2}\int_0^t(|f_1(\tau)|^2 + |f_2(\tau)|^2)d\tau ,\end{aligned} \tag{3.7}$$

where f_1, $f_2 \in L^2_{\ell oc}(\mathbb{R}^+,H)$, and where we take $f_1 = Au$, $f_2 = u$. Consequently (1.17) and (3.7) imply

$$\int_0^t (Au,u*db)d\tau > -\frac{b(0)}{2}\|Au-u\|_t^2 \quad . \tag{3.8}$$

Using (3.8) in (3.6) yields

$$\begin{aligned}\int_0^t (Au,Bu)d\tau + Q(a,Au,t) - \frac{b(0)}{2}\|Au-u\|_t^2 <\\ \psi(u_0) - \psi(u(t)) + \|Au\|_t\|F\| \quad , \quad t \in \mathbb{R}^+ \quad .\end{aligned} \tag{3.9}$$

To establish the term $-\psi(u(t))$ in (3.9) we argue as follows: Suppose

$$\limsup_{t\to\infty} \frac{\|u\|_t}{\|Au\|_t} = \infty \quad . \tag{3.10}$$

From (3.10) and assumption (3.4) there exist sequences $t_n \to \infty$, $\varepsilon_n \to 0$ such that

$$\left|\int_0^t (u(\tau),a*Au(\tau))d\tau\right| \le \|u\|_{t_n} \|a\|_{L^1(\mathbb{R}^+)} \|Au\|_{t_n} \le \varepsilon_n \|u\|_{t_n}^2 \quad . \tag{3.11}$$

Using (1.20), (3.11), and $Q(u,t;db) \ge 0$ in (3.1) yields

$$\frac{c}{2} \|u\|_{t_n}^2 \le \frac{|u_0|^2}{2} + \|F\|_{t_n} \|u\|_{t_n} \le \frac{|u_0|^2}{2} + \|F\|_{L^2(\mathbb{R}^+,H)} \|u\|_{t_n}$$

which implies $\sup_n \|u\|_{t_n} < \infty$ and $u \in L^2(\mathbb{R}^+,H)$, in violation of (3.4). Thus we may suppose that (3.10) is false, and

$$\limsup_{t\to\infty} \frac{\|u\|_t}{\|Au\|_t} < \infty \quad .$$

Therefore, there exists a constant K, independent of t, such that

$$\|u\|_t \le K\|Au\|_t \quad , \quad t \in \mathbb{R}^+ \quad . \tag{3.12}$$

Suppose next that

$$\limsup_{t\to\infty} \frac{|u(t)|}{\|Au\|_t} = \infty \quad . \tag{3.13}$$

Using (3.12) to estimate the left-hand side of (3.11) yields

$$\left|\int_0^t (u(\tau),a*Au(\tau))d\tau\right| \le K\|a\|_{L^1(\mathbb{R}^+)} \|Au\|_t^2 \quad . \tag{3.14}$$

Using $(u,Bu) \ge 0$, $Q(u,t;db) \ge 0$ and (3.14) in (3.1) gives

$$\frac{|u(t)|^2}{2} \le \frac{|u_0|^2}{2} + K\|a\|_{L^1(\mathbb{R}^+)} \|Au\|_t^2 + \|F\|_{L^2(\mathbb{R}^+,H)} \|Au\|_t, \ t \in \mathbb{R}^+,$$

which violates (3.13). Thus there exists a constant K_1, independent of t, such that

$$|u(t)| \le K_1 + K_1\|Au\|_t \quad , \quad t \in \mathbb{R}^+ \quad . \tag{3.15}$$

Since by hypothesis ψ is bounded below by an affine function there exist constants K_2, K_3 independent of t, such that making use of (3.15) in turn implies that

$$- \psi(u(t)) \leq K_2 + K_3 \|Au\|_t \quad , t \in \mathbb{R}^+ \tag{3.16}$$

which is the desired estimate for $-\psi(u(t))$.

Returning to (3.9) and using (1.18), (1.21), (3.5), (3.16) yields

$$\delta \|Au\|_t^2 \leq K_4 + K_5 \|Au\|_t \quad , \quad t \in \mathbb{R}^+ \quad ,$$

where K_4, K_5 are constants independent of t. Thus

$$\sup_{t\in\mathbb{R}^+} \int_0^t |Au|^2 d\tau < \infty \quad . \tag{3.17}$$

But from (1.15) and (3.17) one has $a^*Au \in L^2(\mathbb{R}^+,H)$; hence (1.1) has the form

$$\frac{du}{dt} + Bu(t) + b(0)u(t) + b'*u(t) = F_1(t), \; t \in \mathbb{R}^+ \tag{3.18}$$

where $F_1 = F - a^*Au \in L^2(\mathbb{R}^+,H)$ by (1.18). Forming the inner product of (3.18) with u, integrating over $[0,t]$, and using (1.20) yields

$$\frac{|u(t)|^2}{2} - \frac{|u_0|^2}{2} + c\|u\|_t^2 + Q(u,t;db) \leq \|F_1\|_{L^2} \|u\|_t, \; t \in \mathbb{R}^+ \quad . \tag{3.19}$$

Since $Q(u,t;db) \geq 0$, $F_1 \in L^2(\mathbb{R}^+,H)$, standard estimates used in (3.19) imply

$$\sup_{t\in\mathbb{R}^+} \int_0^t |u(\tau)|^2 d\tau < \infty \quad . \tag{3.20}$$

Consequently, the assumption (3.4) is false and (3.20) holds.

Using (1.17) - (1.19), (3.20), and $Q(a,Au,t) \geq 0$ (by the positive definitness of a), in (3.6) one has

$$\int_0^t (Au,Bu)d\tau \leq K_6 \|Au\|_t - \psi(u(t)) + K_7, \; t \in \mathbb{R}^+ \tag{3.21}$$

where K_6, K_7 are independent of t. But from (1.15), (1.18), (3.1), (3.20), $(u,Bu) + Q(u,t,db) \geq 0$ follows that (3.15) and hence (3.16) hold even if (3.20) is satisfied. Therefore by (3.21)

$$\int_0^t (Au,Bu)d\tau \leq K_8 \|Au\|_t + K_8, \; t \in \mathbb{R}^+ \tag{3.22}$$

for some constant K_8. From (1.21), (3.20) and $b(0) \geq 0$ follows

$$\int_0^t (Au,Bu) \geq \delta \|Au\|_t^2 - K_9, \ t \in \mathbb{R}^+ ,$$

for some constant K_9 independent of t, and this, together with (3.22) gives

$$\sup_{t\in\mathbb{R}^+} \int_0^t |Au|^2 d\tau < \infty . \tag{3.23}$$

Finally, returning to (3.1) and using $\int_0^t (u,Bu)d\tau + Q(u,t;db) \geq 0$, (1.15), (1.18), (3.20), (3.23) gives that $u \in L^\infty(\mathbb{R}^+,H)$. This completes the proof of Theorem 3.

4. Proof of Theorem 4. We require two technical lemmas for the analysis; their proofs are given at the end of this section.

Lemma 4.1. *Let* $g : [1,\infty) \to \mathbb{R}$ *satisfy*

$$t^\nu g(t) \in L^\infty(1,\infty) \tag{4.1}$$

for some $\nu > 3/2$. *Define*

$$y_g^2(T_0) = \sup \int_{T_0-1}^{T+T_0} \left[\sum_{k=0}^{\infty} \left(\int_{x+kT}^{\infty} g^2(\tau)d\tau\right)^{1/2} \right]^2 dx, \ T_0 > 2 , \tag{4.2}$$

where the sup is taken over $T \in \{T : T_0 < T < \infty\}$. *Then*

$$y_g \in L^\infty(2,\infty), \ y_g(T_0) = O(T_0^{1-\nu}), \ T_0 \to \infty . \tag{4.3}$$

Lemma 4.2. *Let* ε, T_0 *be given positive numbers and let* $f \in L^1_{\ell oc}(\mathbb{R}^+,\mathbb{R}^+)$. *Assume that* $\limsup_{t\to\infty} \int_{t-1}^t f(\tau)d\tau = \infty$. *Then there exists* $T > T_0$ *and a sequence* $t_n \to \infty$ *as* $n \to \infty$ *such that*

$$\int_{t-T}^t f(\tau)d\tau < \int_{t_n-T}^{t_n} f(\tau)d\tau, \ T < t < t_n , \tag{4.4}$$

$$\int_{t_n-T-T_0}^{t_n-T} f(\tau)d\tau \leq \varepsilon \int_{t_n-T}^{t_n} f(\tau)d\tau \quad . \tag{4.5}$$

The proof of Theorem 4 requires the following preliminaries. Fix $T_0 > 2$ such that

$$\max(y_a(T_0),\ \rho y_{b'}(T_0)) < \min(\frac{\delta}{4}\ ,\ \frac{c-c_0}{4}\ \omega^{1/2}) \tag{4.6}$$

where a, b' are the kernels in (1.1), ρ is the constant in (1.26), c is the constant in (1.20), c_0, δ are the constants in (1.21) with $c > c_0$, and where $\omega = \frac{\delta\mu^{-1}}{2c_0^2}$, μ is defined in (1.23); this choice is possible by (1.24) and Lemma 4.1. Next choose $\varepsilon \in (0,1]$ such that

$$4\varepsilon(g+\rho) < \delta \quad , \tag{4.7}$$

$$2\varepsilon(g+\rho^2\omega^{-1/2}) < (c-c_0)\omega^{1/2} \tag{4.8}$$

where we define $g = |a| + \rho|b'|$, $|a| = \int_{R^+} |a(s)|ds$, $|b'| = \int_{R^+} |b'(s)|ds$. Choose $T > 0$, and a sequence $t_n \to \infty$ as $n \to \infty$, such that

$$\int_{t-T}^{t} |Au|^2ds \leq \alpha_n^2,\ T \leq t \leq t_n,\ n = 1,2,\ldots \quad , \tag{4.9}$$

$$a_n \leq \varepsilon\alpha_n,\ n = 1,2,\ldots \tag{4.10}$$

where we define

$$\alpha_n^2 = \int_{t_n-T}^{t_n} |Au|^2ds,\ a_n^2 = \int_{t_n-T-T_0}^{t_n-T} |Au|^2ds \quad ; \tag{4.11}$$

These choices are possible by Lemma 4.2, and because we will assume $\limsup_{t\to\infty} \int_{t-1}^{t} |Au|^2ds = \infty$ (otherwise conclusion (1.27) of Theorem 4 holds).

In the proof of Theorem 4 we will consider the intervals $I_n = [t_n-T-1,\ t_n-T]$. For each n take $\tau_n \in I_n$ such that $|Au(\tau_n)| \leq \varepsilon\alpha_n$. (To see that such τ_n exist, note that if not then $|Au(\tau)| > \varepsilon\alpha_n$ a.e. on I_n, and as $T_0 > 1$

$$\varepsilon^2\alpha_n^2 < \int_{I_n} |Au|^2ds \leq a_n^2 \leq \varepsilon^2\alpha_n^2 \quad ,$$

where the last inequality follows from (4.10).) Define $T_n = t_n - \tau_n$, thus $T \leq T_n \leq T+1$ and

$$|Au(t_n - T_n)| < \varepsilon\alpha_n \quad , \tag{4.12}$$

$$\int_{t_n-T-T_0}^{t_n-T_n} |Au|^2 ds < \varepsilon^2 \alpha_n^2 \quad , \tag{4.13}$$

$$\int_{t_n-T_n}^{t_n} |Au|^2 ds < (1+\varepsilon^2)\alpha_n^2 \quad . \tag{4.14}$$

Define the sequences of numbers β_n, b_n, γ_n, $n = 1,2,\ldots,$ by

$$\beta_n^2 = \int_{t_n-T_n}^{t_n} |u|^2 ds, \quad b_n^2 = \int_{t_n-T-T_0}^{t_n-T_n} |u|^2 ds \quad , \tag{4.15}$$

$$\gamma_n^2 = \sup_{T<t<t_n} \int_{t-T}^{t} |u|^2 ds \quad . \tag{4.16}$$

Then using $|u| < \rho|Au|$ (assumption (1.26)) and (4.9), (4.10), (4.14) as well as $T_n > T$, we have

$$\beta_n < \rho(1+\varepsilon)\alpha_n \quad , \tag{4.17}$$

$$b_n < \varepsilon\rho\alpha_n \quad , \tag{4.18}$$

and

$$\gamma_n < \rho\alpha_n \quad . \tag{4.19}$$

We begin the proof of Theorem 4 by taking the inner product of (1.1) by u and integrating over $[t_n-T_n, t_n]$ obtaining

$$\begin{aligned} &\frac{|u(t_n)|^2}{2} - \frac{|u(t_n-T_n)|^2}{2} + \int_{t_n-T_n}^{t_n} (u,Bu)d\tau + \int_{t_n-T_n}^{t_n} (u,a^*Au)d\tau \\ &+ \int_{t_n-T_n}^{t_n} (u,u^*db)d\tau = \int_{t_n-T_n}^{t_n} (u,F)d\tau \quad . \end{aligned} \tag{4.20}$$

To estimate the terms in (4.20) define $u_n = \chi[t_n-T_n, t_n]u$, $n = 1,2,\ldots,$ where χ is the characteristic function. Then

$$\int_{t_n-T_n}^{t_n} (u,u^*db)d\tau = Q[u_n, t_n; db] + h_n \quad , \tag{4.21}$$

where we define

$$h_n = \int_{t_n - T_n}^{t_n} (u(\tau), \int_0^{t_n - T_n} b'(\tau - s), u(s)ds)d\tau .$$

To estimate h_n we first use (4.15), (4.18) to obtain

$$|\int_{t_n - T_n}^{t_n} (u(\tau), \int_{t_n - T - T_0}^{t_n - T_n} b'(\tau - s)u(s)ds)d\tau| < \beta_n b_n |b'| < \varepsilon\rho |b'| \alpha_u \beta_n . \tag{4.22}$$

Then observe that by (4.2), (4.15), (4.16)

$$|\int_{t_n - T_n}^{t_n} (u(\tau), \int_0^{t_n - T - T_0} b'(\tau - s)u(s)ds)d\tau|$$

$$< \beta_n \Big(\int_{t_n - T_n}^{t_n} [\sum_{k=1}^{\infty} \int_{t_n - (k+1)T - T_0}^{t_n - kT - T_0} |b'(\tau - s)u(s)|ds]^2 d\tau\Big)^{1/2}$$

$$< \beta_n \Big(\int_{t_n - T_n}^{t_n} [\sum_{k=1}^{\infty} (\int_{t_n - (k+1)T - T_0}^{t_n - kT - T_0} |b'(\tau - s)|^2 ds)^{1/2} (\int_{t_n - (k+1)T - T_0}^{t_n - kT - T_0} |u(s)|^2 ds)^{1/2}]^2 d\tau\Big)^{1/2}$$

$$< \beta_n \gamma_n \Big(\int_{t_n - T - 1}^{t_n} [\sum_{k=1}^{\infty} (\int_{t_n - (k+1)T - T_0}^{t_n - kT - T_0} |b'(\tau - s)|^2 ds)^{1/2}]^2 d\tau\Big)^{1/2} \tag{4.23}$$

$$= \beta_n \gamma_n \Big(\int_{T_0 - 1}^{T + T_0} [\sum_{k=0}^{\infty} (\int_{x + kT}^{x + (k+1)T} (b'(v))^2 dv)^{1/2}]^2 dx\Big)^{1/2}$$

$$< \beta_n \gamma_n y_{b'}(T_0) < \rho\alpha_n \beta_n y_{b'}(T_0) ,$$

where the last inequality follows from (4.19). Thus

$$|h_n| < \rho\alpha_n \beta_n (\varepsilon |b'| + y_{b'}(T_0)) . \tag{4.24}$$

In order to bound the term in (4.20) with the kernel a we notice that by (1.23), (4.15)

$$|\int_{t_n - T_n}^{t_n} (u(\tau), Au * a(\tau))d\tau| = |\int_0^{t_n} (u_n(\tau), Au_n * a(\tau))d\tau + g_n|$$

$$< \beta_n \mu^{-1/2} Q^{1/2}(a, Au_n, t_n) + |g_n| , \tag{4.25}$$

where $g_n \overset{\text{def}}{=} \int_{t_n-T_n}^{t_n} (u(\tau), \int_0^{t_n-T_n} a(\tau-s)Au(s)ds)d\tau$. To estimate g_n we proceed as in (4.22) - (4.24). This obviously yields

$$|g_n| \le \alpha_n\beta_n[\varepsilon|a| + y_a(T_0)] \quad . \tag{4.26}$$

To complete the estimation of the terms in (4.20) we finally observe that by (1.25), (4.17),

$$K_1 \overset{\text{def}}{=} \sup_n \alpha_n^{-1} \left|\int_{t_n-T_n}^{t_n} (u,F)d\tau\right| < \infty \quad . \tag{4.27}$$

Now use (1.20), the fact that $Q(u_n, t_n, db) \ge 0$ and (4.21), (4.24) - (4.27) in (4.20) to obtain

$$\begin{aligned} c\beta_n^2 &\le \varepsilon g\alpha_n\beta_n + \beta_n\mu^{-1/2}Q^{1/2}(a,Au_n,t_n) \\ &+ \alpha_n\beta_n[y_a(T_0) + \rho y_{b'}(T_0)] + K_1\alpha_n + 2^{-1}|u(t_n-T_n)|^2 \quad . \end{aligned} \tag{4.28}$$

The relation (4.28) should be viewed as providing a lower bound for $Q(a,Au_n,t_n)$. Our next purpose is consequently to obtain an upper bound for the same quantity.

Form the scalar product of Au and (1.1), then integrate over $[t_n-T_n, t_n]$. This gives

$$\begin{aligned} &\psi(u(t_n)) - \psi(u(t_n-T_n)) + \int_{t_n-T_n}^{t_n} (Au,Bu)d\tau + \int_{t_n-T_n}^{t_n} (Au,Au*a)d\tau \\ &+ \int_{t_n-T_n}^{t_n} (Au,u*db)d\tau = \int_{t_n-T_n}^{t_n} (Au,F)d\tau \quad . \end{aligned} \tag{4.29}$$

Concerning the terms in (4.29) we observe at first that from (4.9), (4.13), (4.14) follows upon estimating as in (4.21) - (4.24)

$$\int_{t_n-T_n}^{t_n}(Au,Au*a)d\tau = Q(a,Au_n,t_n) + \int_{t_n-T_n}^{t_n}(Au(\tau), \int_{t_n-T-T_0}^{t_n-T_n}$$

$$+ \int_0^{t_n-T-T_0} a(\tau-s)Au(s)ds(d\tau \qquad (4.30)$$

$$\geq Q(a,Au_n,t_n) - \alpha_n^2[\varepsilon(1+\varepsilon)|a| + y_a(T_0)] \quad .$$

Then observe that

$$\int_{t_n-T_n}^{t_n}(Au,u*db)d\tau = b(0)\int_0^{t_n}(Au_n,u_n)d\tau + \int_0^{t_n}(Au_n,u_n*b')d\tau$$

$$+ \int_{t_n-T_n}^{t_n}(Au(\tau), \int_{t_n-T-T_0}^{t_n-T_n} + \int_0^{t_n-T-T_0} b'(\tau-s)u(s)ds)d\tau \geq -\frac{b(0)}{2}\int_{t_n-T_n}^{t_n}|u-Au|^2 d\tau \qquad (4.31)$$

$$- \alpha_n[(1+\varepsilon)b_n|b'| + \gamma_n y_{b'}(T_0)] \geq$$

$$-\frac{b(0)}{2}\int_{t_n-T_n}^{t_n}|u-Au|^2 d\tau - \alpha_n^2[\rho\varepsilon(1+\varepsilon)|b'| + \rho y_{b'}(T_0)] \quad ,$$

where the last step uses (4.18), (4.19). Note that the first inequality in (4.31) follows from (1.17) and (3.7). By (1.25) we have

$$K_2 \overset{\text{def}}{=} \sup_n \alpha_n^{-1}\left|\int_{t_n-T_n}^{t_n}(Au,F)d\tau\right| < \infty \quad . \qquad (4.32)$$

Our last problem when estimating the various terms in (4.29) is to bound the difference $\psi(u(t_n)) - \psi(u(t_n-T_n))$. Using (1.26), (4.12), (4.21), (4.32), $(u,Bu) \geq 0$ and the fact that

$$\sup_n \alpha_n^{-2}\int_{t_n-T_n}^{t_n}(u,Au*a)d\tau < \infty$$

in (4.20) gives $\sup_n \alpha_n^{-2}|u(t_n)|^2 < \infty$ and so $\lim_{n\to\infty} \alpha_n^{-2}|u(t_n)| = 0$. But then, for some $\varepsilon_n \to 0$,

$$\psi(u(t_n)) - \psi(u(t_n-T_n)) \geq -K_1 - K_1|u(t_n)| - (Au(t_n-T_n),\, u(t_n-T_n))$$

$$\geq -K_1 - \epsilon_n\alpha_n^2 - \rho|Au(t_n-T_n)|^2 \geq -K_1 - \epsilon_n\alpha_n^2 - \rho\epsilon^2\alpha_n^2$$

and so, for some constant K_1, if n is sufficiently large,

$$\psi(u(t_n)) - \psi(u(t_n-T_n)) \geq -K_1 - 2\rho\epsilon^2\alpha_n^2 \quad . \tag{4.33}$$

Finally, inserting (4.30) - (4.33) into (4.29) and invoking (4.6), (4.7) (also recall that $\alpha_n \to \infty$) one obtains

$$\int_{t_n-T_n}^{t_n} [(Au,Bu) - \frac{b(0)}{2}|u-Au|^2]\,d\tau + Q(a,Au_n,t_n) \leq \frac{\delta}{2}\alpha_n^2$$

$$\leq \frac{\delta}{2}\int_{t_n-T_n}^{t_n} |Au|^2 d\tau \quad . \tag{4.34}$$

We now have both a lower bound (4.28) and an upper bound (4.34) for $Q(a,Au_n,t_n)$. The lower bound does however contain the term $|u(t_n-T_n)|^2$ which must be estimated in terms of $\alpha_n\beta_n$. This we do in what follows. Suppose for a moment that $\beta_n^2 \leq \omega\alpha_n^2$. Then by (4.34), as $Q(a,Au_n,t_n) \geq 0$, and by the definition of ω,

$$\int_{t_n-T_n}^{t_n} \{(Au,Bu) - \frac{b(0)}{2}|u-Au|^2 + c_0^2\mu|u|^2\}d\tau$$

$$\leq \int_{t_n-T_n}^{t_n} \{\frac{\delta}{2}|Au|^2 + c_0^2\mu|u|^2\}d\tau \leq \delta\int_{t_n-T_n}^{t_n} |Au|^2 d\tau \quad ,$$

which violates (1.21). Thus

$$\alpha_n < \omega^{-1/2}\beta_n \tag{4.35}$$

for n sufficiently large. But then

$$|u(t_n-T_n)|^2 \leq \rho^2|Au(t_n-T_n)|^2 \leq \rho^2\epsilon^2\alpha_n^2 \leq \rho^2\epsilon^2\alpha_n\beta_n\omega^{-1/2} \quad . \tag{4.36}$$

The estimate (4.36) is now used in (4.28) to get

$$[\mu^{-1}Q(a,Au_n,t_n)]^{-1/2} \geq c_0\beta_n + \beta_n[(c-c_0) - \omega^{-1/2}$$

$$[\varepsilon c + \rho^2\varepsilon^2\omega^{-1/2} + y_a(T_0) + \rho y_{b'}(T_0)]] \geq c_0\beta_n \tag{4.37}$$

for n sufficiently large where the last inequality follows from (4.6), (4.8). Thus

$$Q(a,Au_n,t_n) \geq \mu c_0^2\beta_n^2 \quad . \tag{4.38}$$

Finally use this lower bound for Q in (4.34). The result violates (1.21) and so (1.27) follows.

By (1.26), (1.27) we have

$$u \in L^2_\infty(\mathbb{R}^+,H) \quad . \tag{4.39}$$

Then observe that as $a,b' \in L^1(\mathbb{R}^+)$ it follows from (1.25), (1.27), (4.39) that

$$F_1 \in L^2_\infty(\mathbb{R}^+,H) \quad , \tag{4.40}$$

where $F_1(t) \overset{\text{def}}{=} F(t) - Au{*}a(t) - u{*}db(t)$. By (1.1)

$$u'(t) + Bu(t) = F_1(t) \quad . \tag{4.41}$$

Form the scalar product of u and (4.41); then integrate over $[t_1,t_2]$; $t_1,t_2 \in \mathbb{R}^+$; $0 < t_2-t_1 \leq 1$. This gives

$$|u(t_2)|^2 - |u(t_1)|^2 + 2\int_{t_1}^{t_2}(u,Bu)d\tau = 2\int_{t_1}^{t_2}(u,F_1)d\tau$$

and so by (4.39), (4.40) and as $(u,Bu) \geq 0$,

$$|u(t_2)|^2 - |u(t_1)|^2 \leq K \tag{4.42}$$

for some $K > 0$ independent of t_1, t_2. But (4.39), (4.42) give (1.28).

Assume next that $B = \partial\phi$, multiply (4.41) by Bu and integrate over $[\tau_n,t_n]$ where $t_n \to \infty$ is such that

$$\int_{t-1}^{t}|Bu|^2d\tau \leq \int_{t_n-1}^{t_n}|Bu|^2d\tau, \quad 1 \leq t \leq t_n \quad , \tag{4.43}$$

(if no such t_n exist then (1.29) follows) and where τ_n satisfies

$$\tau_n \in [t_n-2, t_n-1], \quad u(\tau_n) \in D(B) \quad ,$$
$$|Bu(\tau_n)| \leq 1 + \inf |Bu(\tau)| \quad . \tag{4.44}$$

Here the inf is taken over $\tau \in \{\tau | t_n-2 \leq \tau \leq t_n-1, \ u(\tau) \in D(B)\}$. Then, by (4.40), (4.43)

$$\phi(u(t_n)) - \phi(u(\tau_n)) + 2^{-1} \int_{\tau_n}^{t_n} |Bu|^2 d\tau \leq 0 \quad . \tag{4.45}$$

But by (1.28) and as $B = \partial\phi$

$$\phi(u(t_n)) - \phi(u(\tau_n)) \geq -2\|u\|_{L^\infty(\mathbb{R}^+,H)} |Bu(\tau_n)| \quad . \tag{4.46}$$

From (4.43) - (4.46) follows

$$[4\|u\|_{L^\infty(\mathbb{R}^+,H)}]^{-1} \int_{\tau_n}^{t_n} |Bu|^2 d\tau \leq |Bu(\tau_n)| \leq 1 + \inf |Bu(\tau)|$$
$$\leq 1 + \int_{t_n-2}^{t_n-1} |Bu| d\tau \leq 1 + \left(\int_{t_n-1}^{t_n} |Bu|^2 d\tau\right)^{1/2}$$

from which the second part of (1.29) follows. To obtain the first part one also recalls (4.40), (4.41).

PROOF OF LEMMA 4.1.

By (4.1) and as $x \geq 1$

$$\int_{x+kT}^{\infty} g^2(v) dv \leq K(x + kT)^{1-2\nu}, \quad k = 0,1,2,\ldots\ldots \quad ,$$

for some constant K. Therefore

$$\sum_{k=0}^{\infty} \{\int_{x+kT}^{\infty} g^2(v) dv\}^{1/2} \leq K^{1/2} x^{1/2-\nu} [1 + (1 + \frac{T}{x})^{1/2-\nu} + (1 + \frac{2T}{x})^{1/2-\nu} + \ldots\ldots] \quad .$$

But $x \leq T_0+T \leq 2T$ and so $x^{-1}T \geq 2^{-1}$. This together with $\nu > 3/2$ yields

$$\sum_{k=0}^{\infty} \{\int_{x+kT}^{\infty} g^2(v) dv\}^{1/2} \leq K_1 x^{1/2-\nu} \quad ,$$

for some constant K_1. But then

$$\int_{T_0}^{T+T_0} \left[\sum_{k=0}^{\infty} \{ \int_{x+kT}^{\infty} g^2(v)dv \}^{1/2} \right]^2 dx < K_1^2 \int_{T_0}^{\infty} x^{1-2\nu} dx = O(T_0^{2-2\nu}), \ T_0 \to \infty$$

from which (4.3) follows.

PROOF OF LEMMA 4.2.

Let N be any integer $> \varepsilon^{-1}$ and take T_c such that

$$T_c > NT_0 \quad . \tag{4.47}$$

Let $\tau_n \to \infty$ be a sequence satisfying

$$\int_{t-T_c}^{t} f \, d\tau < \int_{\tau_n - T_c}^{\tau_n} f \, d\tau, \quad T_c < t < \tau_n \quad , \tag{4.48}$$

and suppose the Lemma does not hold. Then in particular

$$\int_{\tau_n - T_c - T_0}^{\tau_n - T_c} f \, d\tau > \varepsilon \int_{\tau_n - T_c}^{\tau_n} f \, d\tau$$

(at least for some subsequence of $\{\tau_n\}$ which without loss of generality we take equal to $\{\tau_n\}$) and so

$$\int_{\tau_n - T_c - T_0}^{\tau_n} f \, d\tau > (1+\varepsilon) \int_{\tau_n - T_c}^{\tau_n} f \, d\tau \quad . \tag{4.49}$$

For each n there exists $t_{1n} \in [0, \tau_n]$ such that

$$\int_{t-T_c-T_0}^{t} f \, d\tau < \int_{t_{1n}-T_c-T_0}^{t_{1n}} f \, d\varepsilon \quad , \quad T_c + T_0 < t < \tau_n \quad . \tag{4.50}$$

Clearly $\lim_{n\to\infty} t_n = \infty$. By (4.49), (4.50)

$$\int_{t_{1n}-T_c-T_0}^{t_{1n}} f \, d\tau > (1+\varepsilon) \int_{\tau_n - T_c}^{\tau_n} f \, d\tau \quad . \tag{4.51}$$

Suppose that

$$\int_{t_{1n}-T_c-2T_0}^{t_{1n}-T_c-T_0} f \, d\tau < \varepsilon \int_{t_{1n}-T_c-T_0}^{t_{1n}} f \, d\tau \quad . \tag{4.52}$$

Then, by (4.50), the choice $T = T_c + T_0$, $t_n = t_{1n}$ would give the Lemma. Therefore (4.52) cannot hold and so, using also (4.51)

$$\int_{t_{1n}-T_c-2T_0}^{t_{1n}} f\,d\tau > (1+\varepsilon) \int_{t_{1n}-T_c-T_0}^{t_{1n}} f\,d\tau > (1+\varepsilon)^2 \int_{\tau_n-T_c}^{\tau_n} f\,d\tau \quad . \tag{4.53}$$

Now repeat the last few arguments. For each n there exists $t_{2n} \in [0,t_{1n}]$ such that

$$\int_{t-T_c-2T_0}^{t} f\,d\tau < \int_{t_{2n}-T_c-2T_0}^{t_{2n}} f\,d\tau,\ T_c + 2T_0 < t < t_{1n} \quad . \tag{4.54}$$

Observe again that $\lim_{n\to\infty} t_{2n} = \infty$ and that $t_{2n} < t_{1n} < \tau_n$. By (4.53), (4.54)

$$\int_{t_{2n}-T_c-2T_0}^{t_{2n}} f\,d\tau > (1+\varepsilon)^2 \int_{\tau_n-T_c}^{\tau_n} f\,d\tau \quad .$$

Analagously to (4.52) now suppose that

$$\int_{t_{2n}-T_c-3T_0}^{t_{2n}-T_c-2T_0} f\,d\tau < \varepsilon \int_{t_{2n}-T_c-2T_0}^{t_{2n}} f\,d\tau \quad .$$

But the choice $T = T_c + 2T_0$, $t_n = t_{2n}$ would now result in the Lemma. Hence

$$\int_{t_{2n}-T_c-3T_0}^{t_{2n}} f\,d\tau > (1+\varepsilon) \int_{t_{2n}-T_c-2T_0}^{t_{2n}} f\,d\tau > (1+\varepsilon)^3 \int_{\tau_n-T_c}^{\tau_n} f\,d\tau \quad .$$

Proceeding in this fashion yields, remembering how N was picked,

$$\int_{t_{N-1,n}-T_c-NT_0}^{t_{N-1,n}} f\,d\tau > (1+\varepsilon)^N \int_{\tau_n-T_c}^{\tau_n} f\,d\tau > 2 \int_{\tau_n-T_c}^{\tau_n} f\,d\tau \tag{4.55}$$

where $t_{N-1,n} < \tau_n$. But by (4.47) and (4.55)

$$\int_{t_{N-1,n}-2T_c}^{t_{N-1,n}} f\,d\tau > 2 \int_{\tau_n-T_c}^{\tau_n} f\,d\tau$$

which by (4.48) cannot possibly hold. This contradiction gives the Lemma.

5. Proof of Theorem 5. Define $p = \limsup_{t\to\infty} \int_{t-1}^{t} |Au(\tau)|^2 d\tau$, and assume that conclusion (1.31) does not hold; then $p > 0$. Recall the conclusions $Au \in L^2_\infty(\mathbb{R}^+,H)$, $u \in L^\infty(\mathbb{R}^+,H)$ of Theorem 4.

Take any $\eta > 0$ such that

$$3(1-\eta) > 2(1+\eta) \quad . \tag{5.1}$$

Choose sequences $\tilde{T}_n \to \infty$, $t_n \to \infty$ as $n \to \infty$ such that

$$\int_{t_n-\tilde{T}_n}^{t_n} |Au|^2 d\tau > (1-\eta) \limsup_{t\to\infty} \int_{t-\tilde{T}_n}^{t} |Au|^2 d\tau, \ \lim_{n\to\infty} \int_{t_n-2\tilde{T}_n}^{t_n} |F|^2 d\tau = 0 \quad . \tag{5.2}$$

Define $g = |a| + \rho|b'|$ (see definitions following (4.8)). Fix $\varepsilon > 0$ such that

$$\varepsilon^{1/4} g < \delta/4 \quad , \tag{5.3}$$

$$\varepsilon^{1/4} g < K^{1/2}(c-c_0), \ K = \frac{\delta}{4\mu c_0^2} \tag{5.4}$$

(where the constants c_0, c, δ appear in assumptions (1.20), (1.21)), and such that there exists a positive integer N satisfying

$$\varepsilon^{-1} > N > 2\varepsilon^{-1/2} \quad . \tag{5.5}$$

We claim that there exist sequences $\{T_n\}$, $\{T_{0n}\}$ such that

$$\tilde{T}_n < T_n < 2\tilde{T}_n \quad , \quad T_{0n} > \frac{\varepsilon}{2} T_n \quad , \tag{5.6}$$

$$\int_{t_n-T_n-T_{0n}}^{t_n-T_n} |Au|^2 d\tau < \varepsilon^{1/2} \int_{t_n-T_n}^{t_n} |Au|^2 d\tau \quad . \tag{5.7}$$

Suppose the claim does not hold. Then in particular

$$\int_{t_n-\tilde{T}_n-\varepsilon\tilde{T}_n}^{t_n-\tilde{T}_n} |Au|^2 d\tau > \varepsilon^{1/2} \int_{t_n-\tilde{T}_n}^{t_n} |Au|^2 d\tau \quad ; \tag{5.8}$$

for if not take $T_n = \tilde{T}_n$, $T_{0n} = \varepsilon\tilde{T}_n$. From (5.8) one has

$$\int_{t_n-\tilde{T}_n-\varepsilon\tilde{T}_n}^{t_n} |Au|^2 d\tau > (1+\varepsilon^{1/2}) \int_{t_n-\tilde{T}_n}^{t_n} |Au|^2 d\tau \quad ;$$

however, the following is also true (otherwise take $T_n = (1+\varepsilon)\tilde{T}_n$, $T_{0n} = \varepsilon\tilde{T}_n$):

$$\int_{t_n-\tilde{T}_n-2\varepsilon\tilde{T}_n}^{t_n-\tilde{T}_n-\varepsilon\tilde{T}_n} |Au|^2 d\tau > \varepsilon^{1/2} \int_{t_n-\tilde{T}_n-\varepsilon\tilde{T}_n}^{t_n} |Au|^2 d\tau \quad ;$$

Consequently one also has

$$\int_{t_n-\tilde{T}_n-2\varepsilon\tilde{T}_n}^{t_n} |Au|^2 d\tau > (1+\varepsilon^{1/2}) \int_{t_n-\tilde{T}_n-\varepsilon\tilde{T}_n}^{t_n} |Au|^2 d\tau > (1+\varepsilon^{1/2})^2 \int_{t_n-\tilde{T}_n}^{t_n} |Au|^2 d\tau \ .$$

Proceeding in this fashion one arrives at

$$\int_{t_n-\tilde{T}_n-N\varepsilon\tilde{T}_n}^{t_n-\tilde{T}_n-(N-1)\varepsilon\tilde{T}_n} |Au|^2 d\tau > \varepsilon^{1/2} \int_{t_n-\tilde{T}_n-(N-1)\varepsilon\tilde{T}_n}^{t_n} |Au|^2 d\tau$$

(note that otherwise take $T_n = \tilde{T}_n + (N-1)\varepsilon\tilde{T}_n$, $T_{0n} = \varepsilon\tilde{T}_n$; since by (5.5) $N\varepsilon < 1$ we then have $T_n \in [\tilde{T}_n, 2\tilde{T}_n]$, $T_{0n} > \frac{\varepsilon}{2} T_n$), and

$$\int_{t_n-\tilde{T}_n-N\varepsilon\tilde{T}_n}^{t_n} |Au|^2 d\tau > (1+\varepsilon^{1/2})^N \int_{t_n-\tilde{T}_n}^{t_n} |Au|^2 d\tau \ . \tag{5.9}$$

But by (5.2), (5.5), (5.9)

$$2(1+\eta)\limsup_{t\to\infty} \int_{t-\tilde{T}_n}^{t} |Au|^2 d\tau > \int_{t_n-2\tilde{T}_n}^{t_n} |Au|^2 d\tau > \int_{t_n-\tilde{T}_n-N\varepsilon\tilde{T}_n}^{t_n} |Au|^2 d\tau$$

$$> (1+\varepsilon^{1/2})^N \int_{t_n-\tilde{T}_n}^{t_n} |Au|^2 d\tau > 3 \int_{t_n-\tilde{T}_n}^{t_n} |Au|^2 d\tau > 3(1-\eta)\limsup_{t\to\infty} \int_{t-\tilde{T}_n}^{t} |Au|^2 d\tau \ ,$$

which cannot hold by (5.1). Thus the claim (5.6), (5.7) is established. It should be noted that by the above arguments and the fact that $Au \in L^2_\infty(\mathbb{R}^+, H)$ one may, without loss of generality, assume

$$\sup_n |Au(t_n - T_n)| < \infty \ . \tag{5.10}$$

Let $\{T_n\}$, $\{T_{0n}\}$ be sequences satisfying (5.6), (5.7), (5.10) and define numbers α_n, a_n, β_n, b_n by

$$\alpha_n^2 = \int_{t_n-T_n}^{t_n} |Au|^2 d\tau, \quad a_n^2 = \int_{t_n-T_n-T_{0n}}^{t_n-T_n} |Au|^2 d\tau \ , \tag{5.11}$$

$$\beta_n^2 = \int_{t_n-T_n}^{t_n} |u|^2 d\tau, \quad b_n^2 = \int_{t_n-T_n-T_{0n}}^{t_n-T_n} |u|^2 d\tau \ . \tag{5.12}$$

Then by (5.7)

$$a_n < \varepsilon^{1/4} \alpha_n \quad . \tag{5.13}$$

Next, take the inner product of (1.1) with u and integrate over $[t_n - T_n, t_n]$ to obtain (4.20). To estimate the convolution terms in (4.20) we first use (5.12) and then (1.26), (5.13) to obtain

$$\left|\int_{t_n-T_n}^{t_n} (u(\tau), \int_0^{t_n-T_n} b'(\tau-s)u(s)ds)d\tau\right| < \beta_n b_n |b'| < \varepsilon^{1/4} \alpha_u \beta_n \rho |b'| \quad . \tag{5.14}$$

But using the fact that $\|u\|_{L^2_\infty(\mathbb{R}^+,H)} = \{\sup_{t\geq 1} \int_{t-1}^{t} |u|^2 d\tau\}^{1/2} < \infty$ we also have the estimate

$$\left.\begin{aligned}
&\left|\int_{t_n-T_n}^{t_n} (u(\tau), \int_0^{t_n-T_n-T_{0n}} b'(\tau-s)u(s)ds)d\tau\right| \\
&< \beta_n\left[\int_{t_n-T_n}^{t_n} \left(\sum_{k=1}^{\infty} \int_{t_n-(k+1)T_n-T_{0n}}^{t_n-kT_n-T_{0n}} |b'(\tau-s)|\ |u(s)|ds\right)^2 d\tau\right]^{1/2} \\
&< \beta_n\left[\int_{t_n-T_n}^{t_n} \left(\sum_{k=1}^{\infty} \left[\int_{t_n-(k+1)T_n-T_{0n}}^{t_n-kT_n-T_{0n}} |b'(\tau-s)|^2 ds\right]^{1/2}\right.\right. \\
&\qquad\qquad \left.\left.\left[\int_{t_n-(k+1)T_n-T_{0n}}^{t_n-kT_n-T_{0n}} |u(s)|^2 ds\right]^{1/2}\right)^2 d\tau\right]^{1/2} \\
&< \beta_n T_n^{1/2} \|u\|_{L^2_\infty(\mathbb{R}^+,H)} \left[\int_{T_{0n}}^{T_{0n}+T_n} \left(\sum_{k=0}^{\infty} \left[\int_{x+kT_n}^{x+(k+1)T_n} |b'^2(\xi)|d\xi\right]^{1/2}\right)^2 dx\right]^{1/2} \\
&= o(\beta_n),\ n \to \infty \quad ,
\end{aligned}\right\} \tag{5.15}$$

where the last inequality follows from(4.3), the second part of (5.6) and from the hypothesis $\nu > 3/2$ in (1.24). To estimate the other convolution term in (4.20) observe that (using (5.11), (5.12), (5.13))

$$|\int_{t_n-T_n}^{t_n}(u(\tau),\ \int_{t_n-T_n-T_{0n}}^{t_n-T_n} a(\tau-s)Au(s)ds)d\tau| \le \varepsilon^{1/4}\alpha_n\beta_n|a| \quad ; \tag{5.16}$$

and that repeating the arguments in (5.15) yields

$$|\int_{t_n-T_n}^{t_n}(u(\tau),\ \int_0^{t_n-T_n-T_{0n}} a(\tau-s)Au(s)ds)d\tau| = o(\beta_n),\ n \to \infty \ . \tag{5.17}$$

From the first part of (5.6), the second part of (5.2), and from (5.12) one has

$$|\int_{t_n-T_n}^{t_n}(u,F)d\tau| \le o(\beta_n),\ n \to \infty \ . \tag{5.18}$$

Returning to (4.20) and using assumption (1.20), as well as (1.23), (5.12), (5.14) - (5.18) and the fact that $Q[u_n,t_n;db] \ge 0$ results in the estimate

$$\mu^{-1/2}Q^{1/2}(a,Au_n,t_n) \ge c\beta_n - (2\beta_n)^{-1}|u(t_n-T_n)|^2 - \varepsilon^{1/4}\alpha_n g - \varepsilon_n \ , \tag{5.19}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Form the inner product of (1.1) by Au and integrate over $[t_n-T_n,\ t_n]$ to obtain (4.29). To estimate the two convolution terms on the left-hand side of (4.29) we argue as in the preceding paragraphs (see also the proof of Theorem 4), and we obtain

$$\begin{aligned}\int_{t_n-T_n}^{t_n}(Au,a*Au + u*db)d\tau &\ge Q(a,Au_n,t_n)\\ &- \frac{b(0)}{2}\int_{t_n-T_n}^{t_n}|u-Au|^2 d\tau - \varepsilon^{1/4}\alpha_n^2 g + o(\alpha_n),\ n \to \infty \ .\end{aligned} \tag{5.20}$$

In addition

$$\int_{t_n-T_n}^{t_n}(F,Au)d\tau = o(\alpha_n),\ n \to \infty \ . \tag{5.21}$$

Also observe that by (1.28) and (5.10)

$$e \overset{\text{def}}{=} \inf_{n\to\infty}\ (\psi(u(t_n)) - \psi(u(t_n-T_n))) > -\infty \ . \tag{5.22}$$

Making use of (5.20) - (5.22) in (4.29) we obtain, after adding $\mu(c_0\beta_n)^2$ to both sides

$$\int_{t_n-T_n}^{t_n} [(Au,Bu) - \frac{b(0)}{2} |u-Au|^2 + \mu c_0^2 |u|^2] d\tau$$
$$+ Q[a,Au_n,t_n] \le -e + \mu(c_0\beta_n)^2 + \varepsilon^{1/4}\alpha_n^2 g + o(\alpha_n) \quad , \tag{5.23}$$

where c_0 is the constant in (1.21).

Assume that

$$\beta_n^2 \le K\alpha_n^2 \tag{5.24}$$

where K is defined in (5.4), and also suppose that

$$\lim_{n\to\infty} \alpha_n = \infty \ . \tag{5.25}$$

But (5.3), (5.24), (5.25) imply that the right-hand side of (5.23) is bounded above by $\frac{\delta}{2}\alpha_n^2$, where δ is the constant in (1.21). Therefore, as $Q(a,Au_n,t_n) \ge 0$, we arrive at a violation of (1.21). Thus either (5.24) or (5.25) is false. First, assume that for some subsequence

$$\alpha_n < K^{-1/2}\beta_n \quad ; \tag{5.26}$$

then (5.25) implies

$$\lim_{n\to\infty} \beta_n = \infty \ . \tag{5.27}$$

Using (5.4), (5.26), (5.27) and the fact that $u \in L^\infty(\mathbb{R}^+,H)$ to estimate the right-hand side of (5.19) yields (for n sufficiently large)

$$Q(a,Au_n,t_n) \ge \mu(c_0\beta_n)^2 \ . \tag{5.28}$$

Now using (5.3), (5.25), (5.28) in (5.23) again leads to a violation of (1.21). Thus we must have $\liminf_{n\to\infty} \alpha_n < \infty$, and, without loss of generality, we let

$$\sup_n \alpha_n < \infty \ . \tag{5.29}$$

Therefore, also by (5.24),

$$\sup_n \beta_n < \infty \quad . \tag{5.30}$$

By (5.29) we may obviously strengthen (5.10) to

$$\lim_{n\to\infty} |Au(t_n - T_n)| = \lim_{n\to\infty} |u(t_n - T_n)| = 0 \quad , \tag{5.31}$$

Thus $e > 0$ in (5.22).

To complete the proof use $e > 0$ in (5.23), and recall that $Q(a, Au_n, t_n) > 0$. By (1.21), (5.3) this gives

$$\alpha_n^2 = \frac{4}{3}\frac{\mu}{\delta}(c_0\beta_n)^2 \quad . \tag{5.32}$$

But the assumption $p > 0$, together with (5.2), (5.32) implies

$$\inf_n \beta_n > 0 \quad . \tag{5.33}$$

If (5.4), (5.31), (5.32), (5.33) are used in (5.19), one again obtains (5.28). Substituting (5.28) in (5.23), and using (5.3), $e > 0$, one obtains a contradiction of (1.21). We thus conclude that the assumption $p > 0$ is false which yields the desired *conclusion (1.31) of Theorem* 5.

To prove conclusion (1.32) we begin by defining $F_1 = F - a^*Au - u^*db$. By assumptions (1.15), (1.26), (1.30) and by conclusion (1.31) one has

$$\lim_{t\to\infty} \int_{t-1}^{t} |F_1(\tau)|^2 d\tau = 0 \quad . \tag{5.34}$$

Next form the inner product of (1.1) with u and integrate over the interval $[t-T_1, t]$, $|T_1| < 1$, to obtain (using (1.20), (1.28), (5.34))

$$\lim_{t\to\infty} \sup_{|T_1|<1} (|u(t)|^2 - |u(t-T_1)|^2) = 0 \quad . \tag{5.35}$$

Finally, combining assumption (1.26), conclusion (1.31), and (5.35) yields conclusion (1.32) which completes the proof of Theorem 5.

6. <u>Application to Nonlinear Heat Flow in Materials with Memory</u>. We begin with a formulation of the mathematical model based on the consideration of energy balance for heat transfer in a body B in R^n ($n = 1,2,3$); for

simplicity we restrict ourselves to the case $n = 1$ and only comment on the more general situation. If $\varepsilon(t,x)$ represents the internal energy, $\bar{q}(t,x)$ the heat flux, and $h(t,x)$ the external heat supply at time t and position $x \in B$, the energy balance states that

$$\varepsilon_t = -\operatorname{div} \bar{q} + h \quad (t > 0,\ x \in B) \quad .$$

Consider nonlinear heat flow in a homogeneous bar of unit length of a material of "fading memory" type with the temperature $u = u(t,x)$ maintained at zero at the ends $x = 0$ and $x = 1$. According to the theory for such materials developed by Coleman, Gurtin, Noll, Pipkin, MacCamy and Nunziato (see e.g., Coleman and Gurtin [6], Coleman and Mizel [7], Gurtin and Pipkin [12], MacCamy (14), (15), Nunziato [18] - also Nohel [16] for a recent summary) we assume that the history of temperature $v(t,x)$ is prescribed for $t \leq 0$ and $0 < x < 1$ with $v(t,0) = v(t,1) \equiv 0$, $t \leq 0$, and we assume that the internal energy ε and the heat flux q are functionals (rather than functions for heat flow in ordinary materials) respectively of the temperature u and of the gradient of u. A reasonable realization of these functionals is

$$\varepsilon(t,x) = \varepsilon_0 + b_0 u + \int_{-\infty}^{t} b(t-\tau)u(\tau,x)d\tau \quad , \tag{6.1}$$

$$q(t,x) = -\chi(u_x) - \int_{-\infty}^{t} a(t-\tau)\sigma(u_x(\tau,x))d\tau \quad , \tag{6.2}$$

where $-\infty < t < \infty$, $0 < x < 1$. We assume $u(t,x) = v(t,x)$ is the prescribed history of the temperature for $t \leq 0$, $0 < x < 1$, and that u satisfies prescribed boundary conditions at $x = 0$ and $x = 1$ for $-\infty < t < \infty$; in (6.1), (6.2) $\varepsilon_0 > 0$, $b_0 > 0$ are given constants, $a,b : [0,\infty) \to R$ are given, sufficiently smooth functions, $\chi, \sigma : R \to R$ are assigned, nondecreasing sufficiently smooth constitutive functions normalized so that $\chi(0) = \sigma(0) = 0$.

In the physical literature (see e.g., Nunziato [18]) it is customary to define

$$\beta(t) = b_0 + \int_0^t b(\tau)d\tau, \ \kappa(t) = a_0 + \int_0^t a(\tau)d\tau$$

as the internal energy and heat flux relaxation functions respectively; thus $b(t) = \beta'(t)$, $a(t) = \kappa'(t)$. It is then argued, partly on physical grounds, that the equilibrium heat capacity $\beta(\infty) > \beta(0) = b_0 > 0$, and that $\kappa(0)$ and $\kappa(\infty)$ are positive; is also usually assumed that

$$b(t) = \sum_{k=1}^{m} b_k e^{-\beta_k t}, \ a(t) = \sum_{k=1}^{n} a_k e^{-\alpha_k t} , \tag{6.3}$$

$b_k, \beta_k, a_k, \alpha_k > 0$. As will be seen the specific forms (6.3) are not needed for the applications of the mathematical theory.

Letting $h : \mathbb{R} \times [0,1] \to \mathbb{R}$ denote the external heat supply, and using energy balance $(\varepsilon_t = -\mathrm{div}\, \overline{q} + h)$, where ε, q are given by (6.1), (6.2), shows that the temperature u is governed by the nonlinear Volterra history-value problem:

$$b_0 \frac{\partial u}{\partial t} + \frac{\partial}{\partial t} \left(\int_{-\infty}^{t} b(t-\tau)u(\tau,x)d\tau\right) = \chi(u_x)_x$$
$$+ \int_{-\infty}^{t} a(t-\tau)\sigma(u_x(\tau,x))_x d\tau + h(t,x) \tag{6.4}$$

for $-\infty < t < \infty$, $0 < x < 1$, where

$$u(t,x) = v(t,x), \ -\infty < t \leq 0, \ 0 \leq x \leq 1 , \tag{6.5}$$

where it is assumed that the history function v satisfies equation (6.4) in some precise sense for $t \leq 0$. If the ends of the rod are maintained at zero temperature, we adjoin to (6.4), (6.5) the boundary conditions

$$u(t,0) = u(t,1) \equiv 0, \ -\infty < t < \infty . \tag{6.6}$$

To study the evolution of the temperature in the rod for $t > 0$ means to find a global extension of the history v such that (6.4) - (6.6) are satisfied under physically reasonable assumptions.

Upon setting

$$F(t,x) = h(t,x) + \int_{-\infty}^{0} a(t-\tau)\sigma(v_x(\tau,x))_x d\tau$$

$$- \int_{-\infty}^{0} b'(t-\tau)v(\tau,x)d\tau \quad (0 < t < \infty,\ 0 < x < 1) \tag{6.7}$$

$$u_0(x) = v(0,x),\ 0 < x < 1 \quad , \tag{6.8}$$

the history-value problem (6.4) - (6.6) reduces to the boundary-initial value problem

$$b_0 \frac{\partial u}{\partial t} + \frac{\partial}{\partial t}(b*u) = \chi(u_x)_x + a*\sigma(u_x)_x + F \ (0 < t < \infty,\ 0 < x < 1) \ , \tag{6.9}$$

$$u(0,x) = u_0(x) \quad , \quad 0 < x < 1 \tag{6.10}$$

$$u(t,0) = u(t,1) \equiv 0 \quad , \quad 0 < t < \infty \ . \tag{6.11}$$

We shall next apply the abstract global existence, boundedness and asymptotic results (Theorems 2-7) to the model problem (6.9) - (6.11). Without loss of generality we take the constant $b_0 = 1$ in (6.9).

<u>Remark 6.1</u> While we will restrict the details to one space dimension, we comment on the situation in two or three dimensions. Let Ω be a bounded domain in $\mathbb{R}^n$ (for heat flow $n = 2$ or 3) with smooth boundary Γ and let $u(t,x)$ denote the temperature at time t and $x \in \Omega$. In the formulation the internal energy functional ε remains unchanged; the heat flux functional q (6.2) (now a vector in $\mathbb{R}^n$) becomes

$$q(t,x) = -\lambda(|\nabla u|)\nabla u - \int_{-\infty}^{t} a(t-\tau)\nu(|\nabla u(\tau,x)|)\nabla u(\tau,x)d\tau \tag{6.2^n}$$

where $\lambda, \nu : \mathbb{R}^+ \to \mathbb{R}$ are given smooth functions normalized so that $\lambda(0) > 0$, $\nu(0) > 0$, ∇u is the gradient of u, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, and the relaxation function a is as before. Applying the energy balance to (6.1), (6.2^n) and proceding as before, the mathematical model for heat flow for $n > 1$ corresponding to (6.9) - (6.11) becomes

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial t}(b*u) = \nabla\cdot[\lambda(|\nabla u|)\nabla u] + a*(\nabla\cdot[\nu(|\nabla u|u)\nabla u) \tag{6.9^n}$$
$$+ F \quad (0 < t < \infty,\ x \in \Omega)$$

$$u(0,x) = u_0(x),\ x \in \Omega \tag{6.10^n}$$

$$u(t,x) = 0,\ x \in \Gamma \quad , \quad 0 < t < \infty \quad . \tag{6.11^n}$$

The next step is to show that the problem (6.9) - (6.11) can be written in the abstract form (1.1) and then apply the abstract theory. For this purpose assume that the constitutive functions χ, σ satisfy the assumptions:

$$\chi, \sigma \in C^1(\mathbb{R}),\ \chi(0) = \sigma(0) = 0 \quad ; \tag{6.12}$$

there exist constants $\beta > 0$, $M > 0$ such that

$$0 < \sigma'(\xi) < \lambda\chi'(\xi) < M < \infty,\ \xi \in \mathbb{R} \quad ; \tag{6.13}$$

there exist constants $c_1 > 0$, $c_2 > 0$ such that

$$\xi\chi(\xi) \geq c_1\xi^2,\ \xi\sigma(\xi) \geq c_2\xi^2,\ \xi \in \mathbb{R} \quad . \tag{6.14}$$

Define the functions ζ, $\Sigma : \mathbb{R} \to \mathbb{R}$ by

$$\zeta(r) = \int_0^r \chi(\xi)d\xi,\ \Sigma(r) = \int_0^r \sigma(\xi)d\xi,\ r \in \mathbb{R} \quad , \tag{6.15}$$

and the functions ϕ, $\psi : L^2(0,1) \to (-\infty,\infty]$ by

$$\phi(u) = \begin{cases} \int_0^1 \zeta(\frac{du}{dx})dx & \text{if} \quad u \in H_0^1(0,1) \\ +\infty & \text{otherwise} \quad , \end{cases} \tag{6.16}$$

$$\psi(u) = \begin{cases} \int_0^1 \Sigma(\frac{du}{dx})dx & \text{if} \quad u \in H_0^1(0,1) \\ +\infty & \text{otherwise} \quad . \end{cases} \tag{6.17}$$

It is clear that by (6.14)

$$\zeta(r) \geq \frac{c_1}{2} r^2,\ \Sigma(r) \geq \frac{c_2}{2} r^2,\ r \in \mathbb{R} \tag{6.18}$$

and ϕ, ψ are well defined, proper, and convex by (6.13) and l.s.c. by (6.18). Moreover, it is standard that

$$\partial\phi(u) = -\frac{d}{dx}\chi(\frac{du}{dx}),\ u \in D(\partial\phi) = \{u \in H_0^1(0,1);\ \frac{d}{dx}\chi(\frac{du}{dx}) \in L^2(0,1)\}\ , \tag{6.19}$$

$$\partial\psi(u) = -\frac{d}{dx}\,\sigma(\frac{du}{dx}),\ u \in D(\partial\psi) = \{u \in H_0^1(0,1);\ \frac{d}{dx}\,\chi(\frac{du}{dx}) \in L^2(0,1)\}\ . \qquad (6.20)$$

Thus the heat flow problem (6.9) - (6.11) is of the abstract form (1.1) on the Hilbert space $H = W = W' = L^2(0,1)$ provided we take Au, Bu as respectively $\partial\psi(u)$, $\partial\phi(u)$.

Remark 6.2. For the multidimensional problem (6.9^n) - (6.11^n) formulated in Remark 6.1 assume that the constitutive functions λ, ν satisfy

$$\lambda(0) > 0,\ \text{there exists}\ p_0 > 0\ \text{such that}\ \lambda(\xi) \geq p_0\ \text{and}$$
$$\xi\lambda'(\xi) + \lambda(\xi) \geq p_0 \quad (\xi \in R)\quad ,$$

and similarly for ν. Letting $H = L^2(\Omega)$ and defining

$$\phi(u) = \begin{cases} \int_\Omega \Lambda(|\nabla u|)dx & \text{if}\quad u \in H_0^1(\Omega) \\ +\infty & \text{otherwise}\quad , \end{cases}$$

where $\Lambda(r) = \int_0^r \xi\lambda(\xi)d\xi$, $r \in \mathbb{R}$, we find (see e.g., [16, Remark 2.4])

$$Bu = \partial\phi(u) = -\nabla\cdot(\lambda(|\nabla u|))\quad \text{where}$$
$$D(\partial\phi) = \{u \in H_0^1(\Omega)\ :\ \nabla\cdot(\lambda(|\nabla u|)) \in L^2(\Omega)\}\quad ;$$

the operator A is defined in the same way using the primitive of ν. Thus the problem (6.9^n) - (6.11^n) is also of the abstract form (1.1).

It will be shown next how to apply Theorem 2 to deduce existence of solutions of the model problem (6.9) - (6.11) using assumptions (6.12) - (6.14). For this purpose we first check the General Assumptions. The conditions (1.2) - (1.6) are satisfied with the above choice of W, H, ϕ and ψ. To check that condition (1.7) is satisfied observe that

$$|B(u)|^2 = \int_0^1\Big(\chi'(\frac{du}{dx})\,\frac{d^2u}{dx^2}\Big)^2dx \geq \frac{1}{\beta^2}\int_0^1\Big(\sigma'(\frac{du}{dx})\,\frac{d^2u}{dx^2}\Big)^2dx = \frac{1}{\beta^2}\,|Au|^2\ , \qquad (6.21)$$

where we have used (6.13). Since $|A_\lambda u| \leq |Au|$, $\lambda > 0$, (recall that A and also B are assumed single-valued),

$$|(Bu,A_\lambda u)| \leq |Bu|\ |A_\lambda u| \leq |Bu|\ |Au|\quad ,$$

and this, together with (6.21) implies

$$(Bu, A_\lambda u) \geq -\beta |Bu|^2$$

which is of the form (1.7), where β is the constant in (6.13).

Remark 6.3. In Example 2 of [8] which is also a special case of (1.1) with $b \equiv 0$ the condition (1.7) was shown to hold with $\beta = 0$. Although B was then linear the demonstration of this was far from trivial. The above consideration does however show that provided we satisfy ourselves with $\beta > 0$ (which is permitted in (1.7)) then the verification of (1.7) is almost trivial even if B is nonlinear. In fact, it is not obvious to us how (1.7) with $\beta = 0$ could be verified in the case when both A and B are nonlinear.

The compactness condition (1.8) is clearly satisfied in $L^2(0,1)$ by (6.16), (6.18), from which it follows that $|\phi(u)|$ bounded implies $|\frac{du}{dx}|_{L^2}$ bounded.

To see that the condition (1.13) is satisfied under our assumptions observe that (6.13) implies

$$(Au, Bu) = \int_0^1 \sigma'(\frac{du}{dx})\chi'(\frac{du}{dx})(\frac{d^2u}{dx^2})^2 dx \geq \frac{1}{\beta}\int_0^1 [\sigma'(\frac{du}{dx})]^2(\frac{d^2u}{dx^2})^2 dx \tag{6.21}$$

$$= \frac{1}{\beta}|Au|^2 .$$

Also

$$b(0)(Au,u) \geq b(0)c_2\pi^2|u|^2 \quad , \tag{6.22}$$

by using integration by parts, (6.14), and the Poincaré inequality. A routine calculation now shows that (1.13) is satisfied with $v = Au$, $w = Bu$ if $b(0) < 2^{-1}$.

If all the above assumptions are satisfied, if the kernel a satisfies conditions (a), if the kernel b satisfies assumption (1.11) (which is the case for the special case of a, b in (6.3) - see Proposition a in [8], also in more general cases than (6.3)), if $b(0) > 0$, and if $F \in W^{1,1}_{\ell oc}(\mathbb{R}^+, H)$, $u_0 \in D(\phi) \cap D(\psi)$, then according to Theorem 2 the problem (6.9) - (6.11) has a solution u satisfying the conclusions of Theorem 2 with $v = Au$, $w = Bu$. No claim is made that this solution is unique.

To verify the applicability of Theorem 3 to the physical problem we observe first that (1.20) is satisfied with $c = c_1\pi^2$ by (6.14). From (6.21), (6.22) now follows that (1.21) is satisfied if

(i) $\mu(c_1\pi^2)^2 - \frac{b(0)}{2} + b(0)c_2\pi^2 > 0$.

and

(ii) $b(0) < 2\beta^{-1}$

hold. Concerning the condition (i) we note that if $c_2\pi^2 > \frac{1}{2}$ then, as $b(0) > 0$, it is trivially satisfied. If $c_2\pi^2 < \frac{1}{2}$ then (i) requires μ to be sufficiently large compared to $b(0)$.

Then under the above conditions, the conclusions of Theorem 3 hold for solutions of (6.9) - (6.11), provided the kernels a, b satisfy (1.15) - (1.17) (trivially true for the special kernels (6.3), but also true for large classes of other kernels), and provided $F \in L^2(\mathbb{R}^+, H)$.

To check the hypotheses and applicability of Theorem 4 to (6.9) - (6.11) we note that (1.24) is trivially satisfied for the special kernels (6.3), but is also true for many other kernels also satisfying (1.15) - (1.17). Thus one only has to check (1.26). For this purpose we add the hypothesis

$$\sigma'(\xi) > \varepsilon > 0 \quad \text{for some} \quad \varepsilon > 0, \ \xi \in \mathbb{R} \tag{6.23}$$

to (6.13); then

$$|Au|^2 = \int_0^1 [\sigma'(\frac{du}{dx})]^2 (\frac{d^2u}{dx^2})^2 dx > \varepsilon^2 \int_0^1 \frac{d^2u}{dx^2} dx \quad .$$

By an easy variant of Lemma A.2 in [17] (here u satisfies zero boundary conditions at $x = 0, 1$ instead of periodic boundary conditions; the mean value of $u = 0$ in [17] is not used - instead use the Poincaré inequality) one concludes

$$\int_0^1 \frac{d^2u}{dx^2}\,dx \geq 2\int_0^1 \left(\frac{du}{dx}\right)^2 dx \geq 2\pi^2 \int_0^1 u^2(x)\,dx \quad .$$

Thus $|Au| \geq \sqrt{2}\,\varepsilon\pi\,|u|$ if (6.23) holds, and (1.26) is satisfied with $\rho = (\sqrt{2}\,\varepsilon\pi)^{-1}$! Thus under all of our assumptions the conclusions of Theorem 4 hold for solutions of (6.9) - (6.11) if one takes $F \in L^2_\infty(\mathbb{R}^+, H)$ in (6.9).

For the application of Theorem 5 we only require that F in (6.9) satisfy the weak hypothesis (1.30).

For the application of Theorems 6 and 7 to the problem (6.9) - (6.11) define $F(\infty) = \overline{F}(x) = \lim_{t\to\infty} F(t,x)$ in (6.9); we remark that for the special case of F defined by (6.7) arising from the history-value problem (6.4), (6.5)

$$F(\infty) = \overline{h}(x) = \lim_{t\to\infty} h(t,x) \quad ,$$

under our assumptions concerning the kernels a and b, where $h(t,x)$ represents external heat supply. Since assumption (6.14) implies that both of the single-valued operators A, B defined by (6.19) and (6.20) are coercive and since $\int_0^\infty a(t)dt > 0$, the limit equation (1.33) has a unique solution $u(\infty)$, provided $\overline{F}(x) \in L^2(0,1)$. To apply Theorem 6 we only impose assumptions (1.37); these are trivial for the special cases of a, b in (6.3) (but satisfied for more general kernels). The application of Theorem 7 is equally routine. This completes our discussion.

Remark 6.4. It is clear from the above analysis of the model problem (6.9) - (6.11), that a similar application of the general theory can be made to the multidimensional problem (6.9^n) - (6.11^n) described in Remarks 6.1, 6.2.

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We study the nonlinear Volterra integrodifferential equation

$$\frac{du}{dt} + Bu(t) + a*Au(t) + \frac{d}{dt}(b*u(t)) \ni F(t) \quad \text{a.e. on } \mathbb{R}^+$$

$$u(0) = u_0 \;;$$

A, B are nonlinear operators, a, b, F are functions defined on $[0,\infty)$, * denotes the convolution on $[0,t]$, and u_0 is a given element. Under

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ABSTRACT (continued)

various assumptions motivated by heat flow in materials with memory results on existence of solutions are obtained, followed by various results on boundedness and the asymptotic behaviour of solutions as $t \to \infty$, with applications to such heat flow problems.

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