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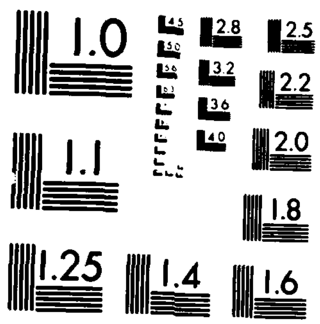
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The Logarithmic Poisson Gamma Distribution: A Model for Leadtime Demand

by

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We consider a process in which demand occurences obey a stationary Poisson process and at each occurence a random number of units are demanded. Furthermore, the leadtime for replenishment is assumed to be random as well. Assuming a logarithmic compounding distribution and a gamma distribution of leadtime we derive the exact distribution of total demand in a leadtime. We call this the Logarithmic-Poisson-Gamma (LPG) distribution. An approximation is derived which is equivalent to a scaled version of the negative binomial distribution. In the final section we derive the first four central moments,

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1. Introduction

Many of the inventory models which are used in practice rely upon knowing the probability distribution of demand over a leadtime. The common assumption is that this distribution is normal. However, in certain circumstances, the normality assumption may be inappropriate. The purpose of this paper is to derive the exact distribution of leadtime demand under the following assumptions: customer requisitions occur according to a stationary Poisson process, requisition sizes follow a logarithmic distribution and leadtime is a random variable with the gamma distribution. In addition to deriving the exact distribution of leadtime demand, we compare our results to actual operational data and discuss a variety of approximations.

A number of researchers have considered the problem of determining inventory operating policies when requisition size exceeds one. For example, Hausman [6] extends Hadley and Whitin's [5] heuristic while Archibald and Silver [1] derive optimal (s, S) policies. These studies differ from ours in two ways. First, in every case leadtime was assumed to be deterministic. Second, they focus primarily on describing optimal and suboptimal ordering policies. Our interest is in a detailed examination of the distribution of demand over leadtime.

2. The Logarithmic Distribution

The logarithmic (or log series) distribution was originally derived by Fisher et. al. [4] and has been discussed by Sherbrooke [11] in connection with inventory problems. It can be derived as a limiting case of the negative binomial distribution and has the form

$$(1)\quad f(x) = \frac{\Theta^x}{-x \ln(1-\Theta)} \quad \text{for } x = 1, 2, \ldots$$

where $0 < \Theta < 1$. Chakrauarti et. al. [3] recommend the method of moments be used to estimate Θ. It is easy to show that

(2) $$E(\chi) = \frac{\Theta}{-(1-\Theta)\ln(1-\Theta)}$$

which means that an estimator for Θ, say $\hat{\Theta}$, solves the trancendental equation

(3) $$\overline{\chi} = \frac{\Theta}{-(1-\Theta)\ln(1-\Theta)}$$

where $\overline{\chi}$ is the observed sample mean. Since the right hand side is an increasing function of Θ, this equation can be solved very efficiently by interval bisection.

We have collected data describing the requisition size distribution for a number of Air Force EOQ-type (i.e. consumable) items. For many of these items, the logarithmic distribution appears to be a useful approximation to the observed data. An example of a specific item is presented in Table 1. In this case, the observed sample mean is 3.94, which results in $\hat{\Theta} = .901$. Notice the very close agreement between the observed and the predicted cumulative distribution functions for this item.

3. The LPG Distribution

Let us now assume that requisitions are generated by a Poisson process and the requisition size has a logarithmic distribution. (That is, the demand process is a compound Poisson process with logarithmic compounding distribution). It is well known that the total number of units demanded in any fixed time, t, say Z(t), has the negative binomial distribution. In particular, we obtain

(4) $$P\{Z(t) = x\} = \frac{(ct + x - 1)!}{x!\,(ct-1)!}(1-\Theta)^{ct}\,\Theta^{x} \quad \text{for } x = 0, 1, 2, \ldots$$

where $c = -\lambda/\ln(1-\Theta)$ and λ is the requisition arrival rate.

Table 1. Comparison of observed frequencies and those predicted by logarithmic distribution for a typical EOQ type item.

x	Number of Observations	Observed Frequency	Theoretical Frequency	Observed Cumulative	Theoretical Cumulative
1	93	.4247	.3896	.4247	.3896
2	31	.1416	.1755	.5663	.5651
3	13	10594	.1054	.6257	.6705
4	15	.0685	.0712	.6942	.7417
5	10	.0457	.0514	.7399	.7931
6	15	.0685	.0386	.8084	.8317
7	8	.0365	.0298	.8449	.8615
8	8	.0365	.0235	.8814	.8850
9	3	.0137	.0188	.8951	.9031
10	4	.0183	.0152	.9134	.9190
11	7	.0320	.0125	.9454	.9315
12	3	.0137	.0103	.9591	.9418
13	0	.0000	.0086	.9591	.9504
14	1	.0046	.0072	.9637	.9576
15	2	.0091	.0061	.9728	.9637
16	1	.0046	.0052	.9774	.9689
17	0	.0000	.0043	.9774	.9732
18	1	.0046	.0037	.9820	.9769
19	0	.0000	.0031	.9820	.9800
20	2	.0091	.0027	.9911	.9827
⋮	⋮	⋮	⋮	⋮	⋮
25	2	.0091	.0008	1.0000	.9915

This result appears to be due to Quenouille [10]. Baswell and Patil [2] give fifteen different derivations of the negative binomial distribution, thus accounting for its power in describing many common phenomena.

Now let us assume that the procurement leadtime, τ, is a continuous non-negative random variable with probability density $g(\tau)$. In general, the number of units demanded in time τ is a random variable with probability function $h(x)$ given by

$$(5) \quad h(x) = \int_0^\infty f(x|\tau)\, g(\tau)\, d\tau$$

where $f(x|\tau)$ is the probability function of the number of units demanded in a time τ. Under our assumptions, $f(x|\tau)$ has the negative binomial distribution. Since $c\tau$ is in general not an integer, we use the gamma function representation for the factorials, so that

$$(6) \quad h(x) = \frac{\Theta^x}{x!} \int_0^\infty \frac{\Gamma(c\tau + x)}{\Gamma(c\tau)} (1-\Theta)^{c\tau}\, g(\tau)\, d\tau.$$

Using the fact that $\Gamma(\alpha) = (\alpha-1)\ \Gamma\ (\alpha-1)$ we have

$$(7) \quad \frac{\Gamma(c\tau + x)}{\Gamma(c\tau)} = \prod_{j=0}^{x-1} (c\tau + j) = \sum_{k=1}^{x} (c\tau)^k S_{xk},$$

where the coefficients S_{xk} are known as Stirling numbers of the first kind and can be computed from the recursion

$$(8) \quad S_{xk} = S_{x-1,k-1} + (x-1)\, S_{x-1,k},$$

for $k = 1, 2, \ldots, x$ and $x = 1, 2, \ldots$,

with $S_{xo} = 0$ for all x.

Furthermore, from the definition of c,

$$(9) \quad (1-\Theta)^{c\tau} = \exp\{c\tau \ln (1-\Theta)\} = e^{-\lambda\tau},$$

so that we may now write

$$(10)\quad h(x) = \frac{\Theta^x}{x!} \sum_{k=1}^{x} c^k S_{xk} \int_0^\infty \tau^k e^{-\lambda\tau} g(\tau)\, d\tau.$$

We now specialize to the case where $g(\tau)$ has the gamma distribution with parameters α and β so that

$$(11)\quad g(\tau) = \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\beta\tau} \quad \text{for } \tau > o.$$

Since leadtimes must be non-negative, the gamma distribution should provide sufficient flexibility to model leadtime variability in many operating environments.

Using the fact that

$$(12)\quad \int_0^\infty \tau^{k+\alpha-1} e^{-(\lambda+\beta)\tau} = \frac{\Gamma(k+\alpha)}{(\lambda+\beta)^{k+\alpha}}$$

and that, as above,

$$(13)\quad \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} = \sum_{j=1}^{k} \alpha^j S_{kj},$$

we obtain the following as the probability function for the number of units demanded in a leadtime:

$$(14)\quad h(x) = \left(\frac{\beta}{\lambda+\beta}\right)^\alpha \frac{\Theta^x}{x!} \sum_{k=1}^{x} \left(\frac{c}{\lambda+\beta}\right)^k S_{xk} \sum_{j=1}^{k} \alpha^j S_{kj} \quad \text{for } x = 1, 2, 3, \ldots$$

$$\text{and } h(o) = \left(\frac{\beta}{\lambda+\beta}\right)^\alpha.$$

We call this the LPG distribution (for Logarithmic-Poisson-Gamma). Its four parameters are α, β, Θ and λ (Θ and λ determine c). An example of the LPG distribution is presented in Table 2 for $\alpha = 1$, $\beta = 1$, $\Theta = .8$ and $\lambda = 1$.

4. A Recursion for Integer α

For numeric calculations, we found the following recursion to be useful. Let us assume that α is integer, and let $C_1 = (\beta/(\lambda+\beta))^{\alpha}$ and $C_2 = c/(\lambda+\beta)$. Further, using (13) let us define

$$(15) \quad T_{x,k} = C_1 \cdot \frac{\Theta^x}{x!} C_2^k \; S_{x,k} \frac{(\alpha + k - 1)!}{(\alpha - 1)!}$$

Hence, h(x) may be computed as a sum of $T_{x,k}$:

$$(16) \quad h(x) = \Sigma_{k=1}^{x} T_{x,k}$$

Note that $T_{x,o} = 0$ since $S_{x,o} = 0$

and that

$$T_{x,x} = C_1 \cdot \frac{(\Theta C_2)^x}{x!} \frac{(\alpha + x - 1)!}{(\alpha - 1)!}$$

$$(17) \quad = \frac{\Theta C_2}{x} (\alpha + x - 1) \; T_{x-1,x-1}$$

since $S_{x,x} = 1$. Using (8), we may now write $T_{x,k}$ in terms of $T_{x-1,k}$; specifically, we obtain:

$$(18) \quad T_{x,k} = \frac{\Theta}{x} [C_2 \; (\alpha + k - 1) \; T_{x-1,k-1} + (x - 1) \; T_{x-1,k}]$$

Thus, h(x) may be evaluated using only $T_{x-1,k}$ terms. This provides significant reductions in computer memory and calculation requirements compared to a direct evaluation of (14) for each x.

5. Approximations

Many inventory models require computing reorder points from fractiles of the leadtime demand distribution. Finding exact fractiles of the LPG distribution might be too demanding computationally for many real applications. In

Table 2. The L'G Distribution

Parameters $\alpha=1$, $\beta=1$, $\Theta=.8$, $\lambda=1$.

x	h(x)	H(x)
0	.5000	.5000
1	.1243	.6243
2	.0806	.7049
3	.0589	.7638
4	.0451	.8089
5	.0355	.8444
6	.0283	.8727
7	.0228	.8955
8	.0185	.9140
9	.0151	.9291
10	.0124	.9415
11	.0101	.9516
12	.0083	.9599
13	.0069	.9668
14	.0057	.9725
15	.0047	.9772
16	.0039	.9811
17	.0032	.9843
18	.0027	.9869
19	.0022	.9892
20	.0018	.9910

this section we consider an approximation which uses a scaled version of the Poisson distribution to approximate the negative binomial distribution.

The mean and variance of $Z(t)$, the number of units demanded in time t, are respectively

(19) $E(Z(t)) = ct\Theta/(1-\Theta)$.

(20) $VAR(Z(t)) = ct\Theta/(1-\Theta)^2$,

which gives $VAR(Z(t))/E(Z(t)) = (1-\Theta)^{-1}$. In certain circumstances, one may have knowledge of the variance to mean ratio of the demand which can then be used to estimate Θ directly.

The approximation is based upon replacing the negative binomial distribution of $Z(t)$ with a scaled Poisson distribution. Let Y be a Poisson random variable with parameter μt and let W be defined by $W = kY$ for some $k > o$. We may think of W as a random variable which assumes values $0, k, 2k, \ldots$ and whose distribution depends upon the two parameters μt and k.

Since

(21) $E(W) = k\mu t$

(22) $VAR(W) = k^2\mu t$

we have $VAR(W)/E(W) = k$. Thus, we set $k = (1 - \Theta)^{-1}$ to achieve the same variance to mean ratio. Comparing the mean and variances of W and $Z(t)$ we see that $\mu = c\Theta$ (recall that $c = -\lambda/\ln(1-\Theta)$).

Since the negative binomial distribution is defined on all non-negative integers, we would like the approximation to be defined on the non-negative integers as well. We have found the following procedure works well. Assume that the scaled Poisson probabilities are shifted to $k/2, 3k/2, 5k/2, \ldots$ so that

(23) $$P\{W = (n+1)\,k/2\} = \frac{e^{-\mu}\mu^{n}}{n!} \qquad n=0,\ 1,\ 2, \ldots$$

We then assume that the cumulative distribution function is linear between nk/2 and (n+1)k/2. As an example, suppose $\theta = .75$, $t = 1$, $c = 2$ (that is $\lambda = 2.77$). Then $\mu = 1.5$, $k = 4$ and

$$P\{W = 2\} = e^{-\mu t} = .2231$$

$$P\{W = 6\} = e^{-\mu t}\mu t = .3347$$

$$P\{W = 10\} = e^{-\mu t}(\mu t)^{2}/2 = .2510$$

$$P\{W = 14\} = e^{-\mu t}(\mu t)^{3}/3! = .1255$$

etc.

The comparison of the exact negative binomial probabilities and the scaled Poisson approximation is presented in Table 3 for this case.

We now obtain an approximation to the LPG distribution by averaging the scaled Poisson approximation of the negative binomial with the gamma distribution of leadtime. That is,

(24) $$P\{Z(\tau) = x\} \simeq \int_0^\infty \frac{e^{-\mu\tau}(\mu\tau)^{x/k}}{(x/k)!} \cdot \frac{\beta^{\alpha}\tau^{\alpha-1}e^{-\beta\tau}}{\Gamma(\alpha)}\, d\tau.$$

But this integral is exactly a Poisson mixture with a gamma distribution which is still another way that the negative binomial distribution can be derived (see Baswell and Patil [2]). Hence, the approximation for the LPG distribution is a scaled version of the negative binomial distribution. The approximation therefore is:

(25) $$P\{Z(\tau) = kx\} \simeq \frac{(\alpha+x-1)!}{x!\,(\alpha-1)!}\left(\frac{\beta}{\mu+\beta}\right)^{\alpha}\left(\frac{\mu}{\mu+\beta}\right)^{x}$$

for $x = 0, 1, 2, \ldots$.

Table 3. Comparison of Negative Binomial and Scaled Poisson Approximations

(θ=.75, t=1, c=2, λ=2.77)

x	Negative Binomial Probabilities	Negative Binomial Cumul. Probabilities	Scaled Poisson Cumul. Probabilities	Scaled Poisson Probabilities
0	.0625	.0625		.0744
1	.0938	.1563		.0744
2	.1055	.2618	.2231	.0744
3	.1055	.3673		.0837
4	.0989	.4662		.0837
5	.0890	.5552		.0837
6	.0779	.6331	.5578	.0837
7	.0667	.6998		.0628
8	.0563	.7561		.0628
9	.0469	.8030		.0628
10	.0387	.8417	.8088	.0628
11	.0312	.8729		.0314
12	.0257	.8986		.0314
13	.0208	.9134		.0314
14	.0167	.9361	.9343	.0314
15	.0134	.9495		.0118
16	.0106	.9601		.0118
17	.0085	.9685		.0118
18	.0667	.9753	.9814	.0118
19	.0053	.9806		.0035
20	.0040	.9846		.0035
21	.0033	.9879		.0035
22	.0026	.9905	.9955	.0035

Note that these probabilities are defined on 0, k, 2k, ... As with the scaled Poisson we suggest shifting these probabilities to k12, 3k12, ... and approximating the probability function by assuming the cumulative distribution function is linear between these fractile points. We tested a variety of cases and found the fit to be excellent, especially in the tails. In Table 4 we compare the exact LPG probabilities for the parameter set considered in Table 2 with the scaled negative binomial approximation. Note that since $\Theta=.8$, we have $k=5$ and the approximate cumulative probabilities (labelled $\bar{H}(x)$ in the table) are defined at the points 2.5, 7.5, 12.5, etc. The final column gives the approximate cumulative distribution function defined on the positive integers obtained from a linear interpolation between the fractiles. Notice the close agreement between the exact and approximate cumulative probabilities in the tail of the distribution.

6. The First Four Moments of the LPG Distribution

Knowledge of the moments of a complex distribution can be utilized in a variety of ways. The moments can be used to estimate the distribution parameters or to approximate the distribution itself. We derive the first four central moments (moments about the mean) of the LPG distribution.

The distribution of $Z(t)$, the number of units demanded in time t, is negative binomial with parameters $q=\Theta$, $p=1-\Theta$ and $n=ct$. From Kendall and Stuart [7], the first four cumulants of the negative binomial distribution are given by

$$K_1 = nq/p, \quad K_2 = nq/p^2, \quad K_3 = nq(1+q)/p^3 \quad \text{and} \quad K_4 = nq(1+4q+q^2)/p^4.$$

The first three cumulants are equal to the first three central moments, respectively, while the fourth central moment, μ_4, is given by

Table 4. The Scaled Negative Binomial Approximation to the LPG Distribution (Parameters are the same as those of Table 2).

	Exact Probabilities		Approximate Probabilities		
x	h(x)	H(x)	$\overline{H}(x)$	h(x)	Approximate Cumulative
0	.5000	.5000		.1905	.1905
1	.1243	.6243		.1905	.3810
2	.0806	.7049	.6667	.1905	.5715
3	.0589	.7638		.1174	.6889
4	.0451	.8089		.0444	.7333
5	.0355	.8444		.0444	.7777
6	.0283	.8727		.0444	.8221
7	.0228	.8955	.8889	.0444	.8665
8	.0185	.9140		.0296	.8961
9	.0151	.9291		.0148	.9109
10	.0124	.9415		.0148	.9257
11	.0101	.9516		.0148	.9405
12	.0083	.9599	.9630	.0148	.9553
13	.0069	.9668		.0099	.9652
14	.0057	.9725		.0049	.9701
15	.0047	.9775		.0049	.9750
16	.0039	.9811		.0049	.9799
17	.0032	.9843	.9877	.0049	.9848
18	.0027	.9869		.0033	.9881
19	.0022	.9892		.0016	.9897
20	.0018	.9910		.0016	.9913

$$\mu_4 = K_4 + 3K_2^2.$$

Hence, the first four central moments (f.f.m.) of Z(t), say μ_i, $1 \le i \le 4$, are

(26) $\mu_1 = ct\Theta/(1-\Theta)$

(27) $\mu_2 = ct\Theta/(1-\Theta)^2$

(28) $\mu_3 = ct\Theta(1+\Theta)/(1-\Theta)^3$

(29) $\mu_4 = ct\Theta(1+4\Theta+\Theta^2+3ct\Theta)/(1-\Theta)^4$

In order to derive the f.f.m. of the LPG distribution, we use the following relationships which can be found in Parzen [8], p. 55: Let X and Y be two (dependent) random variables. Then

(30) $E(Y) = E[E(Y|X)]$

(31) $VAR(Y) = E[VAR(Y|X)] + VAR[E(Y|X)]$

(32) $\mu_3(Y) = E[\mu_3(Y|X)] + \mu_3[E(Y|X)]$

(33) $\mu_4(Y) = E[\mu_4(Y|X)] + 6E[VAR(Y|X)] \cdot VAR[E(Y|X)]$

$$+ \mu_4[E(Y|X)]$$

where

(34) $\mu_3(Y) = E[(Y-E(Y))^3]$

(35) $\mu_4(Y) = E[[Y-E(Y)]^4]$

In the context of our problem, we interpret Y as $Z(\tau)$ and X as τ. It follows that

$$E(Z(\tau)) = E[E[Z(\tau)|\tau]]$$

$$= E[c\tau\Theta/(1-\Theta)]$$

(36) $$= c\alpha\Theta/\beta(1-\Theta).$$

Similarly,

$$\mathrm{VAR}(Z(\tau)) = E[\mathrm{VAR}(Z(\tau)|\tau)] = \mathrm{VAR}[E(Z(\tau)|\tau)]$$

$$= E[c\tau\Theta/(1-\Theta)^2] + \mathrm{VAR}[c\tau\Theta/(1-\Theta)]$$

$$= c\alpha\Theta/(\beta(1-\Theta)^2) + (c\Theta)^2\ \alpha/(\beta^2(1-\Theta)^2)$$

(37) $$= c\alpha\Theta/[\beta(1-\Theta)]^2\ \{\beta+c\Theta\}.$$

Following the same kinds of arguments, one eventually obtains

(38) $$\mu_3(Z(\tau)) = \frac{c\alpha\Theta}{[\beta(1-\Theta)]^3}\ \{\beta^2(1+\Theta) + 2c^2\Theta^2\}$$

(39) $$\mu_4(Z(\tau)) = \frac{c\alpha\Theta}{[\beta(1-\Theta)]^4}\ \{\beta^3(1+4\Theta+\Theta^2) + \beta^2 c\Theta(3\alpha+1) + 6\beta c^2\Theta^2\alpha + c^3\Theta^3(3\alpha+6)\}$$

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