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ADVANCES IN TECHNIQUES FOR TRACKING AND TERMINAL STATE
ESTIMATION OF A GO... (U) CALIFORNIA UNIV LOS ANGELES
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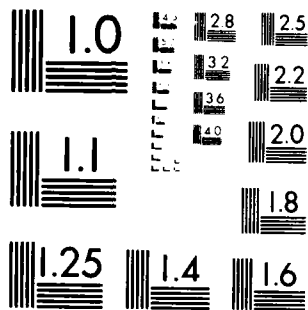
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ADVANCES IN TECHNIQUES FOR TRACKING AND TERMINAL STATE ESTIMATION OF A GOAL SEEKING VEHICLE (THE TARGET RV) VIA AN OPTIMAL BMD INTERCEPT VEHICLE FILTER		5. TYPE OF REPORT & PERIOD COVERED Final, 11/29/79-11/28/81
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ABSTRACT

→ A common problem which arises in tactical situations is the tracking of a non-cooperative vehicle on the basis of noisy measurements received at a tracking station. The standard estimation procedures used in such circumstances assume that the vehicle can be modeled as a dynamic system subject to a white noise disturbance. In many situations, such a model may be inappropriate since the vehicle may be moving toward a specific goal, guided by some form of intelligence. The tracker should be able to incorporate knowledge of this fact into his algorithm for computing his optimal estimate.

This report proposes a simple model for such competitive situations. The vehicle's equations of motions are described by a system of linear differential equations with a guidance law which minimizes a quadratic cost functional and moves the vehicle toward a goal unknown to the tracker. Both a fixed endpoint and free endpoint formulation appropriate to this problem are considered. The tracker's optimal estimation algorithm for both cases is derived for the case where the tracker has a discrete measurement system. It is shown that the tracker can use a coupled set of linear difference equations to form his optimal estimate of both the vehicle's current state and goal.

An example is presented of the fixed endpoint variety. The performance of the proposed algorithm under a variety of initial assumptions is compared to the standard estimation algorithm.

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An example is presented of the fixed endpoint variety. The performance of the proposed algorithm under a variety of initial assumptions is compared to the standard estimation algorithm.

NOMENCLATURE

Upper case letters denote matrices.

Lower case letters denote vectors.

- M_{ij} - ij^{th} component of M
- m_i - i^{th} component of m
- M^T, m^T - Transpose of M or m respectively
- \dot{M}, \dot{m} - Time derivative of M or m respectively
- M^{-1} - Inverse of M
- $\delta(t-s)$ - Dirac delta function
- $\delta_k(k-n)$ - Kronecker delta function
- $E[x]$ - Expected value of x

SECTION 1. INTRODUCTION

The problem of tracking a non-cooperative vehicle is one which frequently arises in tactical situations. The standard tracking and estimation procedures often assume that the vehicle being tracked behaves as a dynamic system with white noise disturbances and unknown initial conditions. Based on these assumptions, the tracker processes a series of noisy measurements to obtain an estimate of the vehicle's current state (i.e., position, velocity, etc.). In this formulation, no account is taken of any potential intelligence in the vehicle which may be guiding it toward a specific goal which, depending on the problem at hand, may be known or partially known to the tracker. It is logical to expect that the tracker can improve his performance through the incorporation of knowledge about the vehicle's goal and guidance law (intelligence) into his tracking procedure. In this report one method of incorporating such knowledge, and the resultant tracking algorithm, will be developed.

Prior to developing the tracking algorithm outlined above, it is instructive to consider an example where the proposed algorithm potentially offers a significant advantage over the standard tracking procedures. Assume that an enemy vehicle is penetrating the airspace in which the tracker is operating. At the time tracking is initiated, the tracker estimates that three widely separated potential targets can be reached by the enemy vehicle in thirty minutes. The tracker's problem is not only to track the enemy vehicle but to scramble interceptor vehicle from one of several airfields and vector them to a good intercept position. It is assumed to take fifteen minutes to scramble the

interceptors and for them to fly to the potential target nearest their base. If the interceptors are forced to fly to one of the target areas further away, there is a high probability that the enemy will reach the target prior to the interceptors. Obviously, in order to maximize the probability of intercept and avoid scrambling unnecessary vehicle's, the tracker should scramble only interceptors on the base nearest the enemy vehicle's goal. Thus, the tracker needs to estimate the enemy vehicle's goal as well as its current state.

A thorough analysis of tactical situations such as that described above quickly leads to a complex formulation in the realm of dynamic game theory for which general solution techniques do not yet exist. The approach taken herein is a preliminary attempt to study this problem utilizing optimal control and estimation theory. The algorithm developed is attractive in that it is, in theory, no more difficult to implement than the standard tracking and estimation algorithms. It does, however, require that the tracker possess considerable knowledge about the vehicle to be tracked.

SECTION 2. PROBLEM DEFINITION

In tactical situations, such as the one described previously, it is convenient to view the problem from two perspectives; that of the attacker, and that of the tracker. The attacker's problem is to determine an optimal guidance law which will take him from some initial state to his goal. The tracker's problem is to estimate the attacker's current state and terminal state (or goal) based on a series of noisy measurements. Let us now develop a rigorous statement of each of these problems.

2.1 ATTACKER'S OPTIMAL CONTROL PROBLEM

The form of the optimal control problem developed in this section is of the deterministic fixed endpoint, fixed final time variety. It is felt that this represents the most appropriate formulation with respect to the type of problem being considered. It is recognized that in certain instances it may be desirable to formulate the control problem as a free endpoint problem. This formulation is treated subsequently.

It is assumed that the attacker has perfect knowledge of his position. Prior to reaching some initial point, at which time the attacker will begin following the optimal guidance law, the attacker will have selected his goal. The guidance law being followed prior to reaching the initial point and the method of selecting the goal are not considered here and will depend on the nature of the tactical

problem. However, it is assumed that the goal can be reasonably chosen prior to reaching the initial point and that once selected, it is not changed enroute.

The attacker has control over his vehicle governed by the dynamic system described by the set of first order, linear differential equations

$$\dot{x}(t) = F(t)x(t) + D(t)u(t)$$

where

$x(t)$ is an $n \times 1$ state vector

$F(t)$ is an $n \times n$ system matrix

$D(t)$ is an $n \times m$ control matrix

$u(t)$ is an $m \times 1$ control vector

The system is initially at the state $x(0) = x_0$. Prior to this time, the terminal state (or goal) of the vehicle $x(T) = x_F$ has been selected as has the time required for this transition, T .

The attacker's problem is to determine the optimal guidance law (control) that will move his vehicle from x_0 to x_F in time T . It is assumed that his guidance law is chosen such as to minimize the quadratic cost functional

$$J = \frac{1}{2} \int_0^T \left\{ x^T(t)Q(t)x(t) + (x(t) - x_F)^T E(t) (x(t) - x_F) + u^T(t)R(t)u(t) \right\} dt.$$

as suggested by Luneberger [1968]. The nxn matrices Q(t) and E(t) are assumed to be positive semi-definite and the mxm matrix R(t) is positive definite.

The cost functional J presumably represents the optimal strategy chosen by the attacker in the sense that it represents operational and tactical costs which the strategy requires be minimized. Operational costs consist of factors such as vehicle wear, fuel consumption, operator fatigue, etc. Tactical costs include noise, vehicle vulnerability, etc. The choice of the parameters Q, E, and R are beyond the scope of this report and will depend on the specific tactical situation being considered. However, they are assumed to be unique within the context of a given situation.

2.2 TRACKING PROBLEM

This paper is primarily concerned with the problem from the tracker's perspective. The tracker's problem is to develop estimates of the attacker's current and terminal states based on a series of noisy measurements. Initially it is assumed that the tracker has estimates of the attacker's initial state x_0 and terminal state x_F : denoted \hat{x}_0 and \hat{x}_F respectively. Associated with these estimates are covariances

$$P_1(0) = E \left[(\hat{x}_0 - x_0) (\hat{x}_0 - x_0)^T \right]$$

$$P_3(0) = E \left[(\hat{x}_F - x_F) (\hat{x}_F - x_F)^T \right]$$

The tracker is assumed to obtain discrete noisy measurements. At each sampling interval, the tracker obtains a measurement from the measurement system

$$z(t_k) = H(t_k) x(t_k) + v(t_k)$$

where

$z(t_k)$ is an $r \times 1$ measurement vector

$H(t_k)$ is an $r \times n$ measurement matrix

$v(t_k)$ is an $r \times 1$ vector of zero mean white noise with covariance

$$E[v(t_k) v^T(t_n)] = V(t_k) \delta_k(t_k - t_n)$$

t_k is the k^{th} sampling time

For a discussion of the case where the tracker has a continuous measurement system, the reader is referred to Luneberger [1963].

The tracker utilizing the set of noisy measurements and his knowledge of the attacker's system and objectives must formulate his estimates of the attacker's current and terminal state. It is assumed that the tracker knows the dynamic system equations governing the attacker, the form and parameters associated with the attacker's cost functional, and the terminal time T . Given this knowledge, the tracker knows the attacker's movement (i.e., state trajectory) to within the $2n$ variables x_0 and x_T . Thus, one may view this as a special type of linear regression problem.

At this point, it is worthwhile to make some general observations concerning the tracker's problem. Upon cursory examination, it may

appear that the set of constraints imposed upon the tracker will be unsatisfiable in the real world. While this may be true in many cases, there are numerous examples where it is reasonable to assume that the tracker possesses the required knowledge. Consider once again the case of enemy vehicles penetrating into the tracker's airspace, with the goal of attacking a specific ground target. In general, specific targets are preassigned to vehicles, or groups of vehicles, as are specific ingress/egress corridors and weapon delivery profiles. In addition, vehicles fly according to well defined doctrine and tactics which determine flight speeds, altitudes, evasive maneuvers, etc. In addition, terminal time-On-target may often be specified to avoid multiple vehicles appearing over the target simultaneously, to coincide with jamming of the tracker's air defense radars, etc. Thus, the tracker may, if provided with good intelligence data, possess the required information.

In the preceding discussion a situation in which it is plausible for the tracker to possess the knowledge required for implementation of the tracking algorithm to be developed has been described. No further attempts will be made to justify the possession of sufficient knowledge by the tracker. However, it should be noted that the collection of the required data will remain a significant, and perhaps the tracker's greatest problem.

SECTION 3. DEVELOPMENT OF THE TRACKING ALGORITHM

3.1 SOLUTION OF THE FIXED ENDPOINT OPTIMAL CONTROL PROBLEM

In order to develop our tracking algorithm an explicit solution to the attacker's optimal control problem, in a state feedback form, is required. In developing the solution to the problem presented in Section 2.1 it is therefore necessary to choose a solution method which yields a control law in the desired form.

We now proceed to solve the optimal control problem.

Given:

$$\dot{x} = Fx + Du$$

$$x(0) = x_0, x(T) = x_F$$

minimize:

$$J = \frac{1}{2} \int_0^T \{x^T Q x + (x - x_F)^T E (x - x_F) + u^T R u\} dt$$

The time parameter in the above expressions has been suppressed. Throughout the remainder of this section, this parameter will be used only where ambiguities would otherwise arise.

The necessary conditions for optimality are easily derived utilizing methods from the calculus of variations, and are

$$\dot{x} = Fx + Du \tag{3.1}$$

$$-\dot{\lambda} = F^T \lambda + Qx + E(x - x_F) \tag{3.2}$$

$$u = -R^{-1} D^T \lambda \tag{3.3}$$

with $x(0) = x_0$ and $x(T) = X_F$. Here, $\lambda(t)$ is an $n \times 1$ dimensional adjoint variable. Together these conditions define the optimal control law and state trajectory. Sufficiency conditions will be developed subsequently.

Using the concepts developed in Bullock 1966 we seek a control law of the form

$$x(t) = S(t)\lambda(t) + b(t) \quad (3.4)$$

where S is an $n \times n$ matrix and b is an $n \times 1$ vector to be determined.

Differentiating (3.4) with respect to time yields

$$\dot{x} = \dot{S}\lambda + S\dot{\lambda} + \dot{b},$$

upon substitution of (3.1) and (3.2) this becomes

$$Fx + Du = \dot{S}\lambda - S \left[F^T \lambda + Qx + E(x - x_F) \right] + \dot{b}$$

Substituting for u using (3.3) and x using (3.4)

$$\begin{aligned} F(S\lambda + b) - DR^{-1}D^T\lambda &= \dot{S}\lambda - SF^T\lambda - SQ(S\lambda + b) \\ &\quad - SE(S\lambda + b) + SEx_F + \dot{b} \end{aligned} \quad (3.5)$$

Obviously (3.5) can be satisfied by choosing b and S to satisfy the set of differential equations

$$\dot{S} = FS + SF^T + S(Q + E)S - DR^{-1}D^T \quad (3.6)$$

and

$$\dot{b} = Fb + S(Q + E)b = SEx_F \quad (3.7)$$

It is possible to select the boundary conditions for (3.6) and (3.7) such that the boundary condition $x(T) = x_F$ is satisfied. From inspection of (3.4) if

$$b(T) = x_F \quad (3.8)$$

$$S(T) = 0 \quad (3.9)$$

the required boundary condition is satisfied. We note here that S is symmetric since $S(T)$ is, and equation (2.6) is a symmetric Riccati equation.

The desired state feedback control formulation can now be stated. From (3.3) and (3.4) the control is given by

$$u = -R^{-1} D^T S^{-1}(x - b) \quad (3.10)$$

where S and b are found by integrating (2.6) and (2.7) backward from $t = T$ to $t = 0$ using the boundary conditions (3.8) and (3.9).

If the solution described by equation (2.10) is to exist, S must be a nonsingular bounded matrix on the interval $0, T$. Let us first consider the conditions under which S is nonsingular on this interval. These conditions are presented in Lemma 2.1 which is attributable to Bullock 1966 .

Lemma 2.1. For the fixed endpoint problem the matrix Riccati equation

$$\dot{S} = SF^T + FS + S(Q + E)S - DR^{-1}D^T$$

with $S(T) = 0$ has a nonsingular solution on $[0, T)$ if and only if there are no conjugate points in the interval $[0, T)$.

Where a conjugate point is defined as:

Definition 2.1. If two neighboring extremals $x_1(t)$ and $x_2(t)$ cross at $t = t^*$, i.e., $x_1(t^*) = x_2(t^*)$ then the extremal is said to have a conjugate point at $t = t^*$.

The existence, or absence, of conjugate points has significant implications as to the existence of a solution of the given form as shown by Breakwell and Ho 1965. Specifically, there exists a unique feedback control law if and only if there are no conjugate points.

Furthermore, the absence of a conjugate point is necessary to the existence of an extremal which is also a minimizing trajectory. Thus, provided that no conjugate points exist and that S remains bounded on $[0, T)$, then equation (2.10) represents a unique feedback control law. These conditions on S are in fact sufficient to the existence of a solution.

The behavior of the solution as $t \rightarrow T$ deserves consideration. In general, fixed endpoint problem formulations have the undesirable property of calling for large control gains as $t \rightarrow T$, resulting from S^{-1} becoming singular. However, since x and b both approach x_F in the limit, $(x-b) \rightarrow 0$ and therefore control gains may remain bounded and well defined except at $t = T$. In any event, it has been found that in practice it is usually possible to approximate the optimal

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There are several interesting properties exhibited by this solution which have significance in terms of the development of the tracking algorithm. First, S is independent of the state trajectory $x(t)$. Thus, equation (2.6) can be integrated by both the attacker and tracker without regard to the attacker's initial or terminal state. Secondly, the vector b depends only upon the attacker's terminal state. Thus, the attacker can integrate equation (2.7) once he has chosen his destination, with the tracker's knowledge of b at any time being purely a reflection of his estimate of x_F .

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3.2 SOLUTION TO THE TRACKING PROBLEM

Prior to the initiation of tracking, the tracker must prepare to accept measurements by first determining the attacker's state equation system incorporating the feedback control law. To do this, he must integrate equations (3.6) and (3.7) backward in time, just as the attacker must do. As previously noted, the tracker can integrate (3.6) exactly.

$$\dot{S} = FS + SF^T + S(Q + E)S - DR^{-1}DT \quad (3.11)$$

$$S(T) = 0$$

Since S is symmetric, this requires integrating only $\frac{1}{2}n(n+1)$ linear simultaneous differential equations. Since the tracker does not

know x_F precisely, he must integrate equation (3.7) for all possible x_F . Equation (3.7) is linear, hence, this can be done by integrating the single matrix differential equation

$$\dot{B} = FB + S(Q + E)B + SE \quad (3.12)$$

$$B(T) = I$$

where $B(t)$ is an $n \times n$ square matrix. If x_F were known to the tracker then b could be obtained from

$$b = Bx_F.$$

The tracker can now describe the motion of the attacker by the coupled set of homogenous differential equations

$$\dot{x} = Fx - DR^{-1}D^T S^{-1} [x - Bx_F] \quad (3.13)$$

$$\dot{x}_F = 0$$

where the explicit expressions for the control from equations (3.10) and (3.12) have been substituted into equation (3.1). This can now be written in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_F \end{bmatrix} = \begin{bmatrix} F - DR^{-1}D^T S^{-1} & DR^{-1}D^T S^{-1} B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_F \end{bmatrix} \quad (3.14)$$

The original optimal control problem has now been transformed into an initial value problem albeit at the expense of doubling the size of the state vector.

The tracker cannot use (3.14) as it stands. The tracker is assumed to have a discrete measurement system and thus it is necessary to convert (3.14) into a discretized form. This can be accomplished by several means, the selection of which will depend upon the complexity of the system, accuracy desired, etc.

Regardless of the method used for the discretization process, the development to follow holds. To simplify the problem notationally, we introduce a $2n$ - dimensional state vector

$$\underline{x}(k) = \begin{bmatrix} x(k) \\ x_F \end{bmatrix}$$

and the discrete system matrix $\underline{A}(k)$

The discrete form of (3.14)

can now be written

$$\underline{x}(k+1) = \underline{A}(k) \underline{x}(k)$$

The tracker can now view his problem as one of estimating the $2n$ - dimensional vector $\underline{x}(k)$ from the noisy measurement system

$$z(k) = \underline{H}(k) \underline{x}(k) + v(k)$$

where

$$\underline{H}(k) = \begin{bmatrix} H(k) & 0 \end{bmatrix}$$

and from his initial estimate $\hat{\underline{x}}(0)$. In this formulation, the tracker can simultaneously estimate both the current state $x(k)$ and the attacker's terminal state or goal. The problem as stated is amenable to solution by the standard Kalman filtering techniques.

The required estimation equation is

$$\hat{\underline{x}}(k+1) = \underline{A}(k)\hat{\underline{x}}(k) + \underline{K}(k+1)\left[z(k+1) - \underline{H}(k+1)\underline{A}(k)\hat{\underline{x}}(k)\right] \quad (3.16)$$

where

$\hat{\underline{x}}(k)$ is the $2n \times 1$ estimation vector

$\underline{K}(k+1)$ is the $2n \times r$ Kalman filter at step $k+1$

The Kalman filter and estimation covariances are defined by

$$\underline{K}(k+1) = \underline{P}(k+1|k)\underline{H}^T(k+1)\left[\underline{H}(k+1)\underline{P}(k+1|k)\underline{H}^T(k+1) + \underline{V}(k+1)\right]^{-1} \quad (3.17)$$

$$\underline{P}(k+1|k) = \underline{A}(k)\underline{P}(k|k)\underline{A}^T(k) \quad (3.18)$$

$$\underline{P}(k+1|k+1) = \left[\underline{I} - \underline{K}(k+1)\underline{H}(k+1)\right]\underline{P}(k+1|k) \quad (3.19)$$

where

$\underline{P}(k+1|k+1)$ is a $2n \times 2n$ symmetric error covariance matrix equal to $E\left[\left\{\hat{\underline{x}}(k+1) - \underline{x}(k+1)\right\}\left\{\hat{\underline{x}}(k+1) - \underline{x}(k+1)\right\}^T\right]$

$\underline{P}(k+1|k)$ is a $2n \times 2n$ symmetric one stage predicted error covariance matrix

$\underline{V}(k+1)$ is an $r \times r$ covariance matrix defined by

$$E\left[\underline{v}(k+1)\underline{v}(k+1)^T\right].$$

The initial conditions for the estimation algorithm are

$$\hat{\underline{x}}(0) = \begin{bmatrix} \hat{\underline{x}}_0 \\ \hat{\underline{x}}_F \end{bmatrix} \quad (3.20)$$

and the associated covariance

$$\underline{P}(0|0) = \begin{bmatrix} \underline{P}_\gamma(0) & 0 \\ 0 & \underline{P}(0) \end{bmatrix} \quad (3.21)$$

While in general, equations (3.16) through (3.20) cannot be simplified in any meaningful way, their behavior in certain limiting cases is readily ascertained. In particular, consider the case where x_F is known exactly by the tracker. In this case, $P_3(0) = 0$. Examination of (3.17) to (3.19) reveals that P_3 will then remain zero for all time. Hence, once the tracker knows x_F , it remains known for all time. However, it should be noted that if the tracker assumes he knows x_F , when he, in fact, does not, and thus sets $P_3(0) = 0$, P_3 still remains zero for all time. Therefore, if the tracker makes an invalid assumption as to his knowledge of x_F , his tracking accuracy should be degraded.

SECTION 4. TRACKING OF A VEHICLE UTILIZING A FREE ENDPOINT OPTIMAL CONTROL LAW

In this section, a tracking and terminal state estimation algorithm for the case where the attacker is utilizing a free endpoint optimal control formulation will be developed. The development of this algorithm directly parallels the development in the previous sections for the fixed endpoint formulation, hence, some details will be omitted in the development to follow.

The attacker is assumed to have control over his vehicle described by the dynamic system

$$\dot{x}(t) = F(t)x(t) + D(t)u(t) \quad (4.1)$$

where the parameters are as previously described. The attacker's problem is to determine his optimal control law which will move his vehicle from its initial state, $x(0) = x_0$, to an unspecified terminal state $x(T)$ in time T . T is assumed to be known at time $t = 0$. The optimal control law is such that the cost functional

$$J = x(T) - x_F \quad M \quad x(T) - x_F + \frac{1}{2} \int_0^T \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \quad (4.2)$$

is minimized. The $n \times n$ matrix $Q(t)$ is assumed to be positive semi-definite and the $m \times m$ matrix $R(t)$ is positive definite. The $n \times 1$ state vector x_F is assumed to an intended goal which the attacker wishes to approach at time $t = T$.

Utilizing the concepts from the calculus of variations leads to the set of necessary conditions for optimality.

$$\dot{x}(t) = F(t)x(t) + D(t)u(t) \quad (4.3)$$

$$-\dot{\lambda}(t) = F^T(t)\lambda(t) + Q(t)x(t) \quad (4.4)$$

$$u(t) = -R^{-1}(t)D^T(t)\lambda(t) \quad (4.5)$$

$$\lambda(T) = M(x(T) - x_F) \quad (4.6)$$

with the initial condition $x(0) = x$. $\lambda(t)$ here is an $n \times 1$ adjoint variable.

It is necessary from the tracker's point of view to convert this problem into a form embodying explicit state feedback. Following Bullock [1966] a solution is sought of the form

$$\lambda(t) = S(t)x(t) + b(t) \quad (4.7)$$

Differentiating (4.7) with respect to time, gives

$$\dot{\lambda}(t) = \dot{S}(t)x(t) + S(t)\dot{x}(t) + \dot{b}(t)$$

Substituting from (4.3), (4.4), and (4.5) for $\dot{\lambda}(t)$, $\dot{x}(t)$ and $u(t)$ yields

$$-F^T Sx = F^T b - Qx = SFx - SDR^{-1}D^T Sx - SDR^{-1}D^T b + \dot{S}x + \dot{b} \quad (4.8)$$

where the time parameter has been suppressed. Equation (4.8) can be satisfied exactly if S and b are chosen such that

$$\dot{S} = -SF - F^T S - Q + SDR^{-1}D^T S \quad (4.9)$$

and

$$\dot{b} = -F^T b + SDR^{-1}D^T S b \quad (4.10)$$

It is possible to select the boundary conditions for (4.9) and (4.10) such that (4.6) is satisfied. Inspection of (4.7) shows that

$$S(t) = M \quad (4.11)$$

$$b(t) = -Mx_F \quad (4.12)$$

are the required conditions.

From equation (4.5) and (4.7), the desired feedback control law is given by

$$u = -R^{-1}D^T \{ S(t)x(t) + b(t) \} \quad (4.13)$$

Since M is known to both the attacker and tracker, both can integrate (4.9) backward in time with the boundary condition (4.11). The tracker is assumed to have imperfect knowledge of x_F , hence, he must integrate (4.10) for all possible x_F . Since (4.10) is linear, this can be done by integrating the single matrix differential equation

$$\dot{B} = -F^T B + SDR^{-1}D^T S B \quad (4.14)$$

with the boundary condition $B(T) = I$. If the attacker's goal were known exactly to the tracker, then he could determine $b(t)$ from

$$b(t) = -B(t)Mx_F \quad (4.15)$$

As for the fixed endpoint case, the tracker's knowledge of $S(t)$ is independent of his knowledge of the attacker's state at any time, and only his knowledge of x_F affects his estimate of b . \dot{b} :

The sufficient conditions under which a solution of the form of equation (4.13) exists are given in Lemma 4.1 attributable to Bullock.

Lemma 4.1. For the free endpoint problem, the Matrix Riccati equation

$$\dot{S} = -SF - F^T S - Q + SDR^{-1}D^T S$$

with $S(T) = M$ has a solution on $[0, T)$ if and only if there are no conjugate points in $[0, T)$.

The comments concerning conjugate points made in Section 3.2 still apply, hence, if there is a solution to (4.9), then (4.13) in fact represents the unique optimal feedback control law.

The tracker may now express the attacker's motion in terms of the initial value problem

$$\dot{x} = Fx - DR^{-1}D^T [Sx - BMx_F] \quad (4.16)$$

$$\dot{x}_F = 0$$

with the initial condition that $x(0) = x_0$. In matrix form (4.16) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_F \end{bmatrix} = \begin{bmatrix} F - DR^{-1}D^T S & DR^{-1}D^T BM \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_F \end{bmatrix} \quad (4.17)$$

The tracker is assumed to have a discrete measurement system of the form

$$z(k) = H(k)x(k) + v(k)$$

where the parameters are the same as those in Section 2. Therefore, the tracker must convert (4.17) into a discrete form to incorporate it into his estimation procedures.

The discrete form of (4.17) can be written

$$\underline{x}(k+1) = \underline{A}(k)\underline{x}(k) \quad (4.18)$$

where $\underline{x}(k) = [x(k) \ x_F]^T$

$\underline{A}(k)$ is a $2n \times 2n$ discrete system matrix.

The tracker's measurement system may now be written as

$$\underline{x}(k) = \underline{H}(k)\underline{x}(k) + v(k)$$

where $\underline{H}(k) = [H(k) \ 0]$

The tracker's estimation equations are

$$\hat{\underline{x}}(k+1) = \underline{A}(k)\hat{\underline{x}}(k) + K(k+1) [z(k+1) - \underline{H}(k+1)\underline{A}(k)\hat{\underline{x}}(k)] \quad (4.19)$$

The Kalman filter is given by

$$K(k+1) = P(k+1|k)\underline{H}^T(k+1) [\underline{H}(k+1)P(k+1|k)\underline{H}^T(k+1) + V(k+1)]^{-1} \quad (4.20)$$

and the error covariance is given by

$$P(k+1|k) = \underline{A}(k)P(k|k)\underline{A}^T(k) \quad (4.21)$$

$$P(k+1|k+1) = [I - K(k+1)\underline{H}(k+1)] P(k+1|k) \quad (4.22)$$

where $\hat{x}(k)$, $K(k)$, $P(k+1|k)$, $P(k|k)$ are as defined in Section 3.

The initial conditions for the estimation algorithm is

$$\hat{x}(0) = \begin{bmatrix} \hat{x}_0 & \hat{x}_F \end{bmatrix}^T \quad (4.23)$$

where \hat{x}_0 is an estimate of $x(0)$ and \hat{x}_F is an estimate of x_F at $t = 0$.

The initial condition for the error covariance is

$$P(0|0) = \begin{bmatrix} P_1(0) & 0 \\ 0 & P_3(0) \end{bmatrix} \quad (4.24)$$

The estimation algorithm, (4.19) through (4.24), is identical to that developed in Section 3. The only difference lies in the derivation of the matrix $\underline{A}(k)$ and in the interpretation of the estimator \hat{x}_F . For the fixed endpoint case, we note that \hat{x}_F is an exact estimator of the state of the attacker at the terminal time. Specifically, if x_F is known exactly, then the attacker's state at T is $x(T) = x_F$. For the free endpoint case, \hat{x}_F is not an exact estimator of the attacker's terminal state. Rather, \hat{x}_F estimates a set of terminal conditions which the attacker will tend to approach as $t \rightarrow T$. The closeness with which the attacker approaches x_F will, in general, depend not only upon the parameters in his control formulation, but also the initial state x_0 .

SECTION 5. CASE STUDY

5.1 PROBLEM DEFINITION

The tracking algorithms developed in Sections 3 and 4 promise superior tracking performance compared to the standard estimation procedures. In addition, they have the added advantage of encompassing terminal state tracking. In pursuing a case study, it is hoped insight into a number of issues surrounding the nature of this superiority can be gained. While a single case study can never provide conclusive answers, it will help to answer such questions as the relative difficulty of implementing the proposed algorithm, and the improvement in performance obtained.

The optimal control problem which shall be considered is of the fixed endpoint variety. Given:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (5.1)$$

$$x(0) = [0.5 \quad 0.25]^T = x_0, \quad x(20) = [20.0 \quad 5.0]^T = x_F$$

Minimize:

$$J = \frac{1}{2} \int_0^{20} \left\{ x^T(t) \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} x(t) + [x(t) - x_F]^T \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} [x(t) - x_F] + 20 u^2(t) \right\} dt$$

Using the techniques of Section 3, this problem may be transformed into an equivalent initial value problem. First, the matrices $S(t)$ and

$B(t)$ are determined. $S(t)$ is found by integrating the matrix Riccati equation

$$\dot{S} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} S + S \begin{bmatrix} 0 & 0 \\ 1 & -0.1 \end{bmatrix} + S \begin{bmatrix} 10+t & 0 \\ 0 & 10+t \end{bmatrix} S - \begin{bmatrix} 0 & 0 \\ 0 & .05 \end{bmatrix} \quad (5.2)$$

backward in time with the boundary condition

$$S(20) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$B(t)$ is computed by integrating the matrix differential equation

$$\dot{B} = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix} B + S \begin{bmatrix} 10+t & 0 \\ 0 & 10+t \end{bmatrix} B - S \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \quad (5.3)$$

backward in time with the boundary condition

$$B(20) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The desired initial value problem is then given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_F \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & .05 \end{pmatrix} S^{-1}(t) & \begin{pmatrix} 0 & 0 \\ 0 & .05 \end{pmatrix} S^{-1}(t) B(t) \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix} \begin{bmatrix} x(t) \\ x_F \end{bmatrix} \quad (5.4)$$

with the boundary conditions $x(0) = x_0$. The solution of this equation represents the optimal state trajectory of the attacker.

From the tracker's point of view, it is now necessary to discretize (5.4).

We note that (5.4) is time variant due to the presence of $S(t)$ and $B(t)$, and hence, determining the state transition matrix for (5.4) is non-trivial. Therefore, we shall develop an approximation to the transition matrix.

The exact solution to (5.1) is

$$x(t_{k+1}) = e^{\begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \Delta t} x(t_k) + \int_{t_k}^{t_{k+1}} e^{\begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} (t_{k+1}-\tau)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau) d\tau$$

where $\Delta t = (t_{k+1} - t_k)$. This solution is approximated as

$$x(t_{k+1}) = \begin{bmatrix} 1 & 10(1 - e^{-0.1\Delta t}) \\ 0 & e^{-0.1\Delta t} \end{bmatrix} x(t_k) + \int_{t_k}^{t_{k+1}} \begin{bmatrix} 1 & 10[1 - e^{-0.1(t_{k+1}-\tau)}] \\ 0 & e^{-0.1(t_{k+1}-\tau)} \end{bmatrix} d\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t_k)$$

Upon taking the integral and replacing t_k by k , the desired form is

$$x(k+1) = \begin{bmatrix} 1 & 10(1 - e^{-0.1\Delta t}) \\ 0 & e^{-0.1\Delta t} \end{bmatrix} x(k) + \begin{bmatrix} 10[\Delta t - 10(1 - e^{-0.1\Delta t})] \\ 10[1 - e^{-0.1\Delta t}] \end{bmatrix} u(k)$$

The discretized form of (5.4) can be written

$$\begin{bmatrix} x(k+1) \\ x_F \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & a_2 \end{pmatrix} - \begin{pmatrix} 0 & a_3 \\ 0 & a_1 \end{pmatrix} S^{-1}(k) & \begin{pmatrix} 0 & a_3 \\ 0 & a_1 \end{pmatrix} S^{-1}(k) B(k) \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} x(k) \\ x_F \end{bmatrix} \quad (5.5)$$

where

$$\alpha_1 = 10(1 - e^{-0.1\Delta t})$$

$$\alpha_2 = e^{-0.1\Delta t}$$

$$\alpha_3 = 10 \left[\Delta t - 10(1 - e^{-0.1\Delta t}) \right]$$

The tracker's measurement system is

$$z(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k)$$

where $z(k)$ and $v(k)$ are scalar quantities and $E[v(k)v(n)] = V(k)\delta(k-n) = 1$. In terms of the augmented state vector $[x(k) \ x_F]^T$ the measurement equation is

$$z(k) = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x(k) \\ x_F \end{bmatrix} + v(k). \quad (5.6)$$

Using the results of Section 3 and equation (5.5) and (5.6), the estimation equations are

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{x}_F(k+1) \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 & \alpha_1 \\ 0 & \alpha_2 \end{pmatrix} - \begin{pmatrix} 0 & \alpha_3 \\ 0 & \alpha_1 \end{pmatrix} S^{-1}(k) & \begin{pmatrix} 0 & \alpha_3 \\ 0 & \alpha_1 \end{pmatrix} S^{-1}(k)B(k) \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{x}_F(k) \end{bmatrix} + K(k+1) \left[z(k+1) - \left\{ (1 \ \alpha_1) - (0 \ \alpha_3)S^{-1}(k) \ (0 \ \alpha_3)S^{-1}(k)B(k) \right\} \begin{bmatrix} \hat{x}(k) \\ \hat{x}_F(k) \end{bmatrix} \right] \quad (5.7)$$

where $K(k+1)$ is the 4×1 dimensional vector:

$$K(k+1) = \begin{bmatrix} K_1(k+1) \\ K_2(k+1) \\ K_3(k+1) \\ K_4(k+1) \end{bmatrix} = \begin{bmatrix} P_{11}(k+1|k) \\ P_{21}(k+1|k) \\ P_{31}(k+1|k) \\ P_{41}(k+1|k) \end{bmatrix} [P_{11}(k+1|k) + V(k+1)]^{-1} \quad (5.8)$$

$P(k+1|k)$ is

of the form

$$P(k+1|k) = A(k)P(k|k)A^T(k) \quad (5.9)$$

where $A(k)$ is the system matrix of equation (5.5). The expression for the covariance $P(k+1|k+1)$ is easily expressed in compact form as

$$P_{ij}(k+1|k+1) = P_{ij}(k+1|k) - K_i(k+1)P_{ij}(k+1|k). \quad (5.10)$$

$$i, j = 1, 2, 3, 4$$

The initial conditions for the estimations equations are

$$\hat{x}(-1) = (0 \quad 0)^T$$

$$\hat{x}_F(-1) = (\text{See Section 5.2, individual cases considered})$$

$$P(-1|-1) = \begin{bmatrix} P_1(-1) & 0 \\ 0 & P_3(-1) \end{bmatrix}$$

where $P_1(-1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P_3(-1)$ is selected as appropriate to the individual cases considered. Note that here it is assumed that the tracker's first measurement occurs at $k = 0$, therefore, the tracker's initial estimates are at $k = -1$. The value of k for this example runs from $k = -1$ to $k = 320$. The time increment $\Delta t = 0.0625$.

Prior to solving the optimal control and estimation equations derived previously, it will be useful to formulate this problem in terms of the standard tracking and estimation procedures. The

tracking performance of the algorithms described by equations (5.7) through (5.10) may then be compared to the nominal performance exhibited by the standard procedures.

To develop the estimation equations, a discrete description of the attacker's dynamic system with a white noise disturbance is required. This is given by

$$x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & a_1 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} a_3 \\ a_1 \end{bmatrix} w(k) \quad (5.11)$$

where $w(k)$ is a scalar white noise disturbance with covariance

$$E [w(k)w(n)] = W(k)\delta_k(k-n) = 1$$

$$a_1 = 10(1 - e^{-0.1\Delta t})$$

$$a_2 = e^{-0.1\Delta t}$$

$$a_3 = 10(\Delta t - 10(1 - e^{-0.1\Delta t}))$$

$W(k)$ was selected to reflect the magnitude of the control in the optimal control problem as shown in Section 5.2.

The tracker's measurement system is taken to be

$$z(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + v(k) \quad (5.12)$$

where $v(k)$ has the associated covariance $E[v(k)v(n)] = V(k)\delta_k(k-n) = 1$.

This is, of course, identical to equation (5.6). The estimation

equations are

$$\hat{x}(k+1) = \begin{bmatrix} 1 & \alpha_1 \\ 0 & \alpha_2 \end{bmatrix} \hat{x}(k) + K(k+1) \left[z(k+1) - \hat{x}_1(k) - \alpha_1 \hat{x}_2(k) \right] \quad (5.13)$$

where $K(k+1)$ is defined by

$$K(k+1) = \begin{bmatrix} K_1(k+1) \\ K_2(k+1) \end{bmatrix} = \begin{bmatrix} P_{11}(k+1|k) \\ P_{21}(k+1|k) \end{bmatrix} \left(P_{11}(k+1|k) + V(k+1) \right)^{-1} \quad (5.14)$$

with $P_{ij}(k+1|k)$ being the ij^{th} component of $P(k+1|k)$. The single stage predicted error covariance $P(k+1|k)$ is

$$P(k+1|k) = \begin{bmatrix} 1 & \alpha_1 \\ 0 & \alpha_2 \end{bmatrix} P(k|k) \begin{bmatrix} 1 & 0 \\ \alpha_1 & \alpha_2 \end{bmatrix} + \begin{bmatrix} \alpha_3 \\ \alpha_1 \end{bmatrix} W(k) [\alpha_3 \quad \alpha_1] \quad (5.15)$$

The error covariance at step $k+1$ is given by

$$P(k+1|k+1) = \begin{bmatrix} 1 - K_1(k+1) & 0 \\ -K_2(k+1) & 1 \end{bmatrix} P(k+1|k) \quad (5.16)$$

The initial conditions for (5.13) through (5.16) are

$$\hat{x}(-1) = [0 \quad 0]^T$$

$$P(-1|-1) = P_1(-1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where it is assumed that the first measurement occurs at $k = 0$ as before, and k runs from $k = -1$ to $k = 320$ with $\Delta t = 0.0625$.

5.2 RESULTS

The test case simulation was carried out on an IBM System/360 Computer. The differential equations were solved via a modified

Hamming's method, as described in IBM[1970], with a nominal step size of 0.03125.

The solution to the optimal control problem was first generated. Equations (5.2) and (5.3) were integrated backward in time from $t = 20$ to $t = -0.03125$ using the given boundary conditions. The optimal state trajectory was then found by integrating equation (5.4) with $x(0) = x_0$. The optimal control law corresponding to this trajectory is given by

$$u(t) = - \begin{bmatrix} 0 & 0 \\ 0 & .05 \end{bmatrix} S^{-1}(t) [x(t) - B(t)x_F] .$$

The State trajectory and optimal control law are portrayed in Figure 1.

The standard (Kalman) estimation algorithm was next employed to establish a benchmark against which the optimal estimation algorithm developed in this paper could be compared. The measurements $z(k)$ (5.12) for $k = 0$ to 320 were generated using the state trajectory from the optimal control problem and noise values $v(k)$ taken from the table of random normal numbers presented in Hoel, Port and Stone [1971].

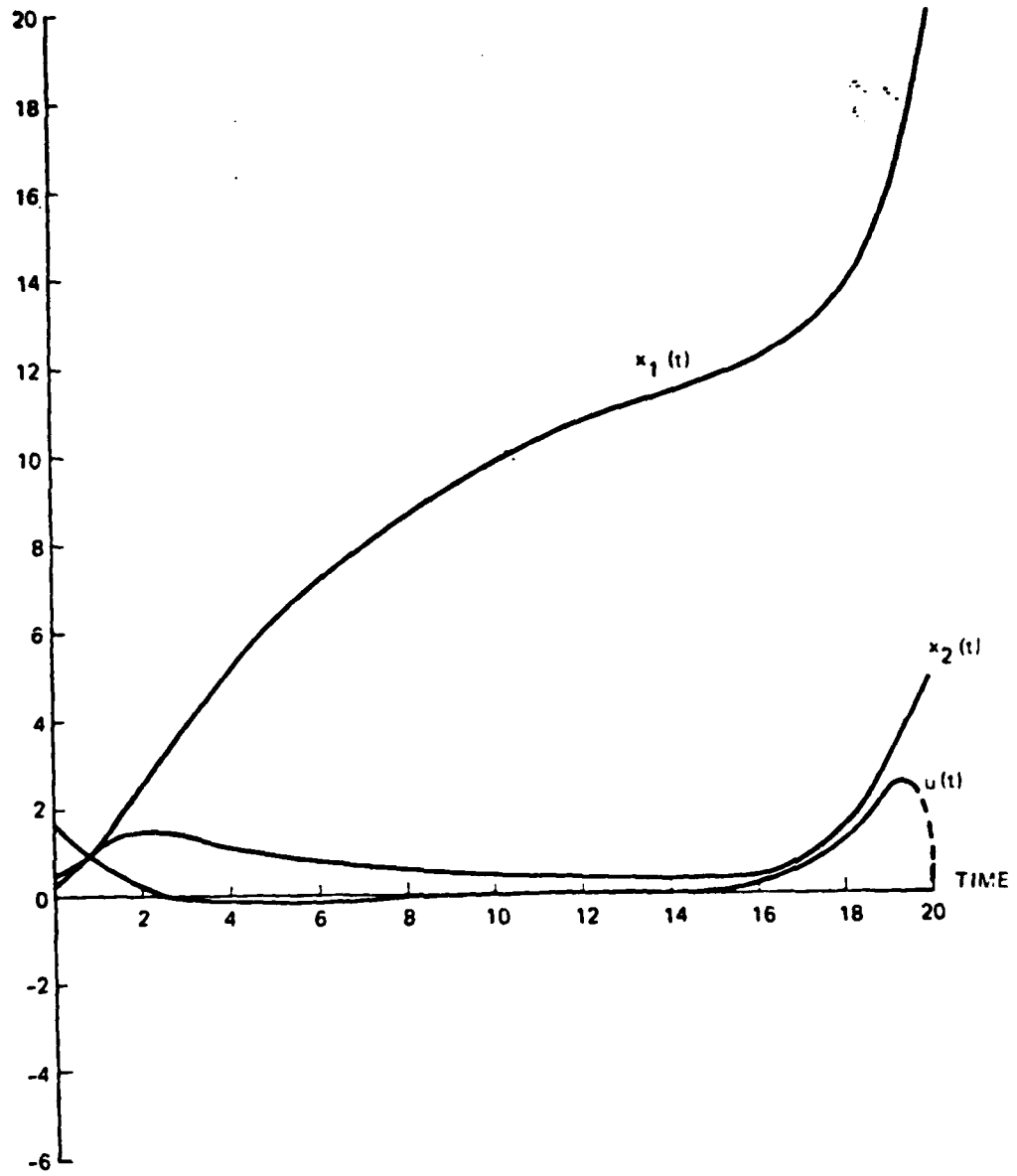


Figure 1. State Trajectory and Optimal Control Values for Case Study

The estimated state vector $\hat{x} = x_1 x_2^T$ and the associated covariance matrices and Kalman filter were generated using (5.13) through (5.16). The initial conditions are

$$\hat{x}(-1) = [0 \ 0]^T$$

$$P(-1|-1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The outputs from this simulation are summarized in Table 1 and Figure 1. The error covariance ($P_u(k)$) and measurement ($z(k)$) values for selected k values are shown in Table 1. The measurement values ($z(k)$) shown are the same as those used in the optimal situation algorithm simulations presented subsequently. Figure 1 presents a smoothed plot of the absolute error values $\hat{x}_f(k) - x_f(k)$ for a sampling interval of $\Delta t=0.625$.

The optimal estimation algorithm simulations were carried out using initial conditions covering a variety of possible circumstances. In the first case simulated, it was assumed that x_F was known exactly. The initial conditions corresponding to this case are

$$\hat{x}(-1) = [0 \ 00 \ 20 \ 5]^T$$

$$P(-1|-1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The estimated state vector $\hat{x} = \hat{x}_1 \hat{x}_2 \hat{x}_{F1} \hat{x}_{F2}^T$ and the associated

Table 1. Standard Estimation Algorithm Errors for Selected Increments

INCREMENT (k)	Z	P_{11}	P_{12}^2, P_{22}^2	P_{22}
0	-.8290	.5010	.0310	.9895
1	.2668	.3372	.0611	.9755
2	-.3222	.2585	.0899	.9564
3	-.4553	.2147	.1168	.9310
4	.1003	.1889	.1410	.8988
5	1.5472	.1735	.1619	.8598
10	2.0770	.1556	.2069	.6012
20	2.0851	.1381	.1392	.2253
30	2.0357	.1109	.0848	.1134
40	3.4083	.0933	.0623	.0841
50	4.3674	.0842	.0544	.0770
60	6.2264	.0805	.0523	.0758
70	5.7883	.0794	.0520	.0757
80	5.9870	.0791	.0521	.0757
90	7.5093	.0791	.0521	.0756
100	6.9318	.0791	.0521	.0755
110	8.2205	.0791	.0521	.0755
120	7.2707	.0791	.0521	.0755
130	8.7200	.0790	.0521	.0755
140	8.5930	.0791	.0521	.0755
150*	9.1058	.0791	.0521	.0755
160	9.9452			
170	9.3990			
180	9.1990			
190	10.1702			
200	11.7018			
210	11.0577			
220	12.9071			
230	11.2639			
240	10.0449			
250	11.0284			
260	12.1858			
270	12.0494			
280	11.9063			
290	14.5229			
300	15.8203			
310	15.7467			
319	19.1849			
320	20.3890			

*ERROR COVARIANCE ATTAINS STEADY STATE VALUE AT INCREMENT k = 144.

465010

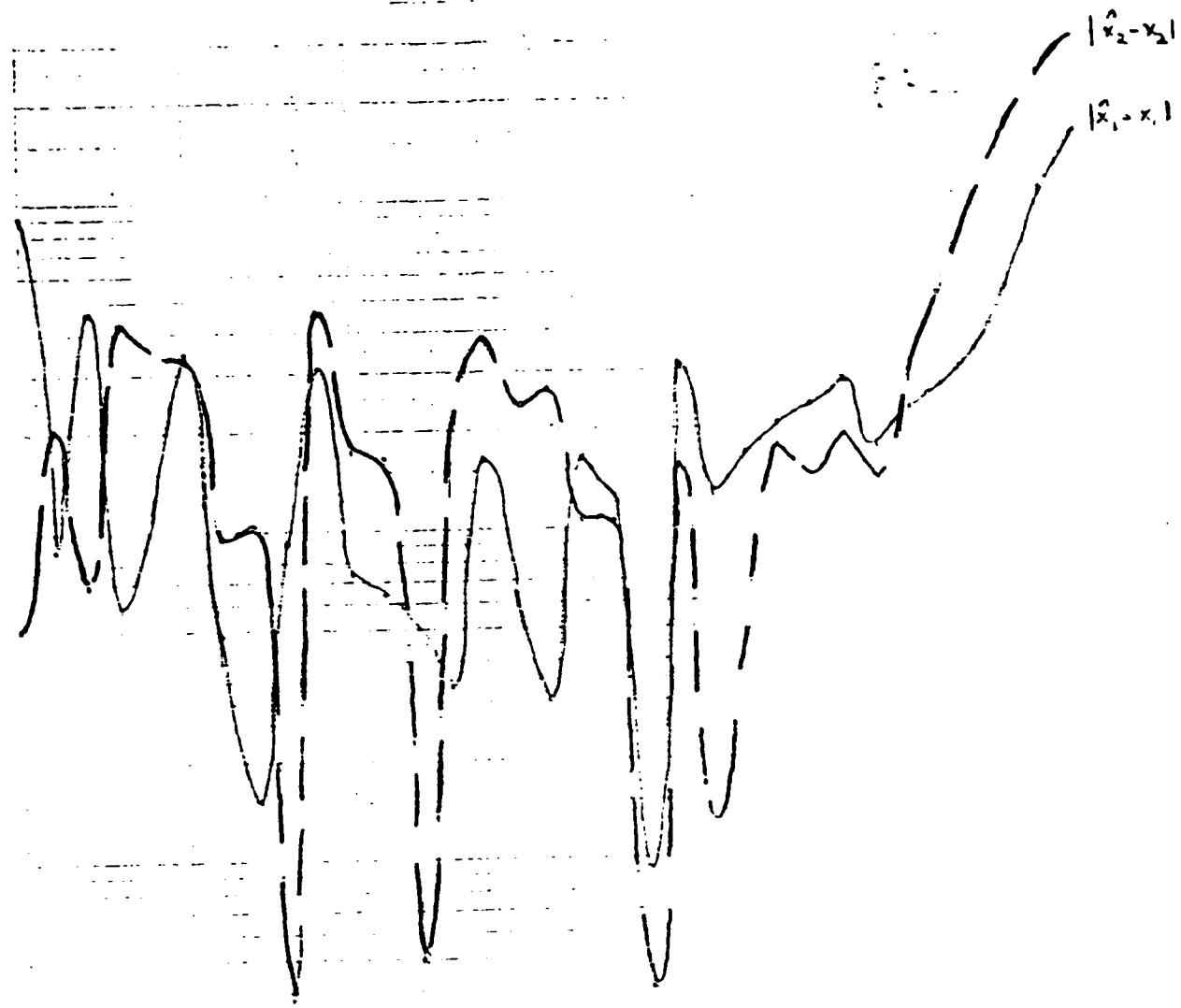


FIGURE 1. ABSOLUTE STATE
ESTIMATION ERRORS FOR KALMAN
FILTER (0.025 SAMPLING INTERVAL)

0 50 100 150 200 250 300

error covariances and Kalman filter were generated using (5.7) through (5.10). The measurement $z(k)$ (5.6) are identical to those generated for the standard estimation simulation. The error covariance for selected increments is presented in Table 2 and the absolute state estimation errors are presented in Figure 2.

Comparison of the results presented in Table 2 and Figure 2 with those in Table 1, indicates the superiority of the optimal estimation algorithm as compared to the standard estimation procedure. The error covariance is one to two orders of magnitude smaller after the 20th iteration and four orders of magnitude less by the 150th iteration, when the standard estimation covariance has reached steady state, resulting in a greater confidence in the estimated values. This result is reflected in the state estimation errors. The optimal estimation algorithm is seen to have state estimation errors of an order of magnitude over most of the range. It is apparent from Figure 2 that the error associated with x_2 becomes large during the final increments. The importance of this is debatable, since x_F is known and hence the tracker can detect this deviation. However, this result occurs in all of the cases simulated and is thus worthwhile to investigate. This result is caused by the nature of the optimal control law as reflected in the system matrix (5.4). We note that the control acts as a forcing function on x_2 . Near $t = 20$, the control function has a large negative slope (see Figure 1) which causes a negative predicted estimate of \hat{x}_2 given by $\hat{x}_2(k+1|k) = A_2(k) \hat{x}_1(k) \hat{x}_2(k) \hat{x}_{F1}(k) \hat{x}_{F2}(k)^T$.

Table 2. Error Covariance* for the Optimal Estimation Algorithm at Selected Increments when x_F Assumed Known

INCREMENT (k)	P_{11}	$P_{12} = P_{21}$	P_{22}
0	.5002	.0036	.8313
1	.5048	.2790	.8313
2	.2524	.2073	.5694
3	.2044	.0347	.4683
4	.1733	.0389	.3834
5	.1517	.0406	.3124
10	.0989	.0276	.1060
20	.0543	-.0020	.0129
30	.0288	-.0094	.0055
40	.0135	-.0072	.0044
50	.0058	-.0040	.0029
60	.0021	-.0018	.0016
70	.68E-03	-.66E-03	.67E-03
80	.18E-03	-.20E-03	.25E-03
90	.37E-04	-.51E-04	.76E-04
100	.62E-05	-.91E-05	.18E-04
110	.13E-05	-.10E-05	.35E-05
120	.89E-06	-.19E-06	.55E-06
130	.55E-06	-.22E-06	.17E-06
140	.29E-06	-.17E-06	.12E-06
150	.12E-06	-.86E-07	.71E-07
160	.43E-07	-.39E-07	.37E-07
170	.13E-07	-.13E-07	.15E-07
180	.31E-08	-.37E-08	.47E-08
190	.59E-09	-.84E-09	.13E-08
200	.90E-10	-.14E-10	.28E-09
210	.15E-10	-.16E-10	.47E-10
220	.70E-11	-.24E-11	.67E-11
230	.44E-11	-.20E-11	.18E-11
240	.22E-11	-.14E-11	.11E-11
250	.86E-12	-.70E-12	.61E-12
260	.28E-12	-.27E-12	.27E-12
270	.78E-13	-.86E-13	.96E-13
280	.18E-13	-.22E-13	.29E-13
290	.33E-14	-.48E-14	.72E-14
300	.43E-15	-.80E-15	.15E-14
310	.22E-16	-.72E-16	.24E-15
315	.14E-17	-.88E-17	.56E-16
320	.37E-19	-.18E-17	.84E-16

*FOR x_F ASSUMED KNOWN, ALL $P_{ij} = 0$, EXCEPT P_{11} , P_{12} , P_{21} , P_{22} .

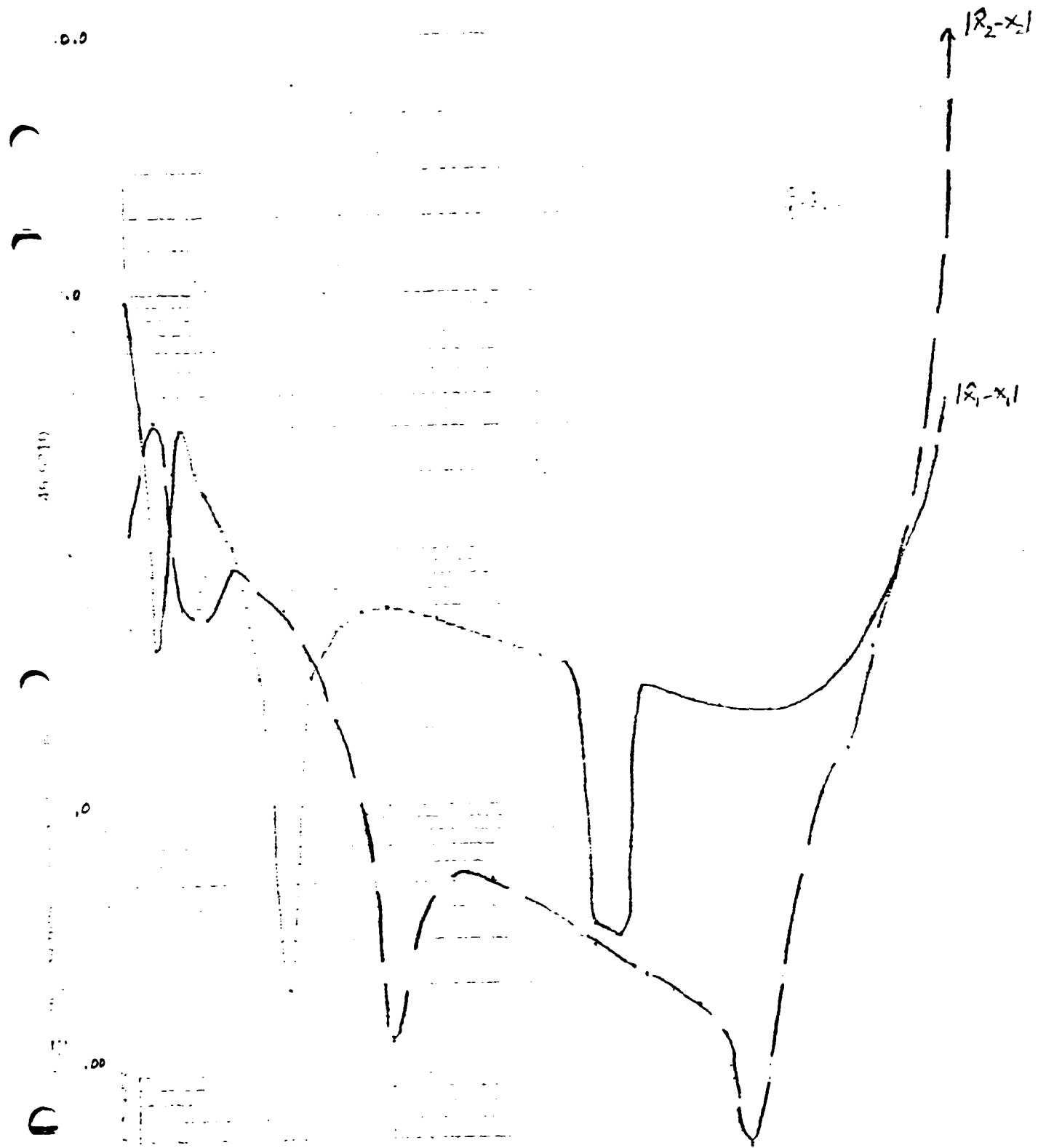


FIGURE 2. ABSOLUTE STATE ESTIMATION ERRORS FOR X_F KNOWN (0.5% SAMPLING INTERVAL)

.000 0 50 100 150 200 250 300

With the small Kalman filter values at this time (same order of magnitude as the error covariance) this predicted estimate dominates the state estimate. This problem is, of course, a specific reflection of the nature of the example being considered and the time constant used. It cannot be concluded that such problems will arise in general.

A comparison example to the one just described is the case where x_F is assumed known when it is in fact not known. For this case, the initial conditions are

$$\hat{x}(-1) = [0 \quad 0 \quad 10 \quad 10]^T$$

$$P(-1|-1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The absolute state estimation errors for this case are indicated in Figure 3. The terminal state error is constant with

$$\hat{x}_{F1} - x_{F1} = -10.0$$

$$x_{F2} - x_{F2} = 5.0$$

The error covariances are those given in Table 2. Since invalid assumptions have been made, one would expect the state estimation errors to be large relative to the preceding case, as is the case. Thus, it is evident that a large penalty is paid for incorrectly assuming knowledge about the terminal state.

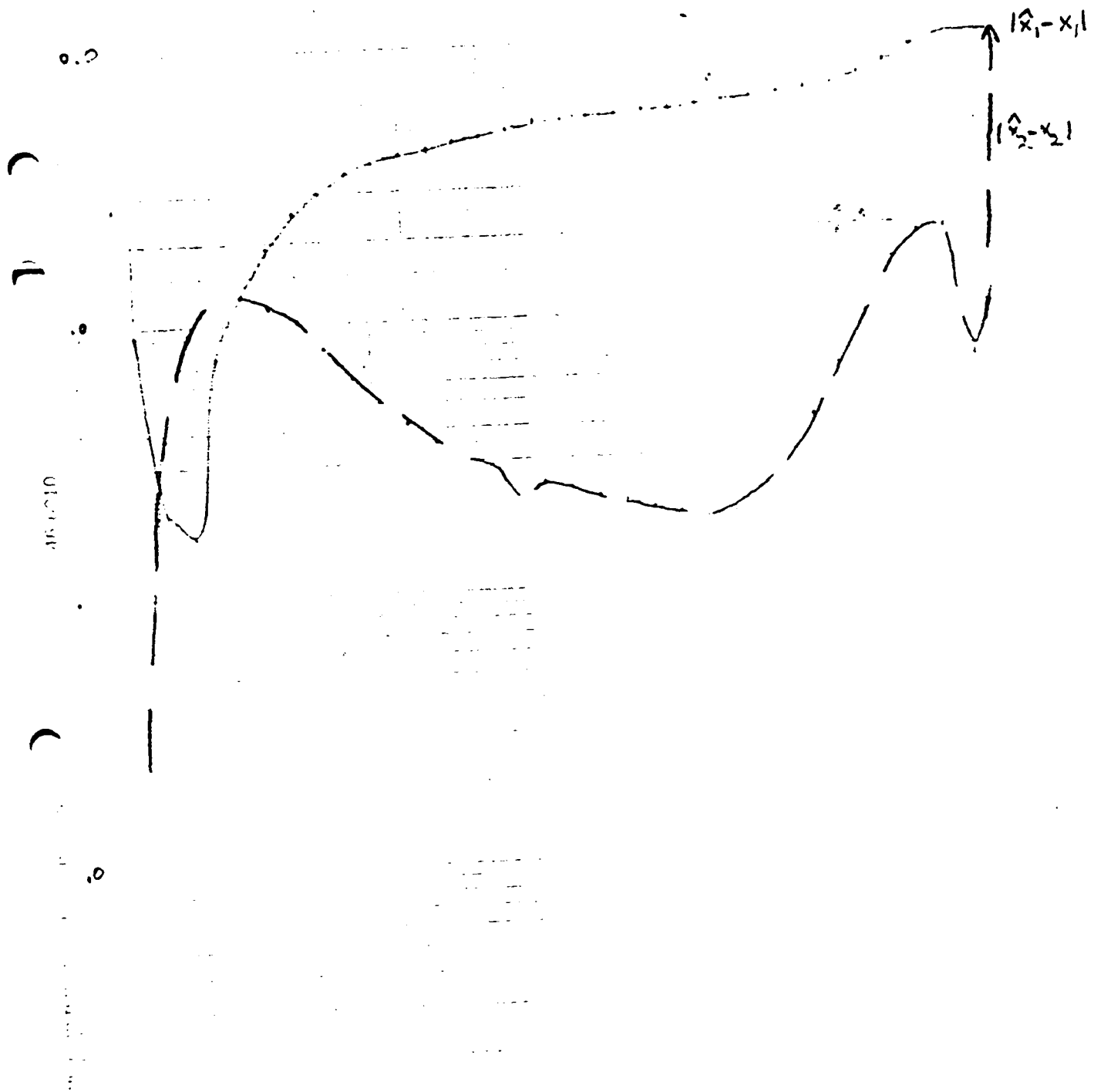


FIGURE 3. ABSOLUTE STATE ESTIMATION
 ERRORS FOR x_F INCORRECTLY
 ASSUMED KNOWN (CLOSE SAMPLING
 INTERVAL)

.000 0 50 100 150 200 250 300

There is a high probability that the tracker would discover his mistaken assumptions far in advance of the terminal time. This can be seen from the difference in the actual measurements and the predicted measurements, or measurement error

$$\hat{z}(k) = \{z(k) - H(k)A(k-1)x(k-1)\}$$

This difference for the two cases already described at selected increments are shown below

INCREMENT (k)	\hat{z} FOR ASSUMED VALUE OF TERMINAL STATE	
	$\hat{x}_F = [20 \quad 5]^T$	$\hat{x}_F = [10 \quad 10]^T$
50	.2513	2.2523
100	.3386	3.7433
150	.2548	4.8220
200	1.6143	6.6125
250	.8668	5.5247
300	.7063	18.9356

From this data, it is evident that the invalid assumptions lead to a large measurement error. One use for this result would be to provide a means of discriminating against an attacker having a goal other than that of interest to the tracker.

It is far more likely that the tracker will assume he possesses imperfect knowledge of the attacker's terminal state. The initial condition for the error covariance matrix under this assumption is

$$P(-1|-1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The error covariance matrix value at selected increments is given in Table 5. The estimation simulations were carried out for two initial conditions for \hat{x} ;

$$\hat{x}(-1) = [0 \quad 0 \quad 20 \quad 5]^T$$

and

$$\hat{x}(-1) = [0 \quad 0 \quad 10 \quad 10]^T.$$

The estimation errors for these cases are given in Tables 6 and 7 respectively.

Comparing the error covariance values in Table 5 with those in Tables 1 and 2 allows one to gauge the performance of the optimal estimation algorithm under the assumption that x_F is unknown. Let us first consider the error variances $E\left\{\left(\hat{x}_1 - x_1\right)\left(\hat{x}_1 - x_1\right)\right\}$ and $E\left\{\left(\hat{x}_2 - x_2\right)\left(\hat{x}_2 - x_2\right)\right\}$. These error variances in Table 5 are essentially an order of magnitude lower than the values in Table 1 after the 10th iteration. They are significantly higher than the error variances under the assumption that x_F is known (Table 2). Thus, the tracking performance should be superior (higher confidence) to the standard procedure, but not as good as for the case where x_F is known. Next, consider the terminal state estimation variances $E\left\{\left(\hat{x}_{F1} - x_{F1}\right)\left(\hat{x}_{F1} - x_{F1}\right)\right\}$ and $E\left\{\left(\hat{x}_{F2} - x_{F2}\right)\left(\hat{x}_{F2} - x_{F2}\right)\right\}$. From the values in Table 5 (P_{33} and P_{44}) it is evident that the error variance associated with x_{F2} is significantly higher than that for any other

Table 5. Error Covariance for the Optimal Estimation Algorithm at Selected Increments when x_F Assumed Unknown

INCREMENT (i)	P_{11}	$P_{12} - P_{21}$	P_{22}	$P_{11} - P_{11}$	$P_{12} - P_{12}$	$P_{22} - P_{22}$	$P_{11} - P_{11}$	$P_{12} - P_{12}$	$P_{22} - P_{22}$	$P_{11} - P_{11}$	$P_{12} - P_{12}$	$P_{22} - P_{22}$	$P_{11} - P_{11}$	$P_{12} - P_{12}$	$P_{22} - P_{22}$
0	.5001	.0015	.8310	.0001	.0075	1.0000	-.0001	.0040	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.3337	.0164	.6392	.0005	.0145	1.0000	-.0001	-.0076	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	.2524	.0272	.5195	.0012	.0211	1.0000	-.0006	-.0108	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	.2011	.0317	.4607	.0022	.0272	.9999	-.0011	-.0137	.9999	.9999	.9999	.9999	.9999	.9999	.9999
4	.1711	.0290	.3911	.0031	.0329	.9999	-.0017	-.0163	.9999	.9999	.9999	.9999	.9999	.9999	.9999
5	.1517	.0206	.3116	.0037	.0382	.9999	-.0024	-.0186	.9999	.9999	.9999	.9999	.9999	.9999	.9999
10	.0990	.0234	.1093	.0117	.0492	.9993	-.0045	-.0262	.9993	.9993	.9993	.9993	.9993	.9993	.9993
20	.0560	.0099	.0214	.0306	.0559	.9981	-.0161	-.0311	.9981	.9981	.9981	.9981	.9981	.9981	.9981
30	.0316	.0010	.0166	.0229	.0636	.9554	-.0257	-.0238	.9554	.9554	.9554	.9554	.9554	.9554	.9554
40	.0279	.0053	.0152	.1060	.0916	.8678	-.0153	-.0394	.8678	.8678	.8678	.8678	.8678	.8678	.8678
50	.0293	.0096	.0107	.1246	.0731	.7206	-.0324	-.0199	.7206	.7206	.7206	.7206	.7206	.7206	.7206
60	.0321	.0097	.0060	.1131	.0495	.5628	-.0262	-.0004	.5628	.5628	.5628	.5628	.5628	.5628	.5628
70	.0323	.0076	.0030	.1093	.0309	.4233	-.0173	-.0057	.4233	.4233	.4233	.4233	.4233	.4233	.4233
80	.0304	.0053	.0014	.0924	.0192	.3109	-.0087	-.0006	.3109	.3109	.3109	.3109	.3109	.3109	.3109
90	.0276	.0034	.0007	.0773	.0126	.2472	-.0015	-.0007	.2472	.2472	.2472	.2472	.2472	.2472	.2472
100	.0247	.0027	.0004	.0651	.0087	1.963	-.0042	-.0007	1.963	1.963	1.963	1.963	1.963	1.963	1.963
110	.0227	.0021	.0003	.0556	.0056	1.598	-.0005	-.0001	1.598	1.598	1.598	1.598	1.598	1.598	1.598
120	.0199	.0016	.0002	.0481	.0049	1.323	-.0113	-.0004	1.323	1.323	1.323	1.323	1.323	1.323	1.323
130	.0182	.0013	.0001	.0422	.0039	1.123	-.0183	-.0007	1.123	1.123	1.123	1.123	1.123	1.123	1.123
140	.0166	.0011	.0001	.0374	.0032	.0965	-.0162	-.0010	.0965	.0965	.0965	.0965	.0965	.0965	.0965
150	.0153	.0007	.0001	.0335	.0026	.0839	-.0176	-.0004	.0839	.0839	.0839	.0839	.0839	.0839	.0839
160	.0142	.0004	.0001	.0303	.0022	.0736	-.0187	-.0005	.0736	.0736	.0736	.0736	.0736	.0736	.0736
170	.0131	.0002	.0001	.0276	.0018	.0652	-.0196	-.0004	.0652	.0652	.0652	.0652	.0652	.0652	.0652
180	.0121	.0001	.0001	.0252	.0015	.0583	-.0203	-.0003	.0583	.0583	.0583	.0583	.0583	.0583	.0583
190	.0117	.0001	.0001	.0232	.0014	.0524	-.0211	-.0003	.0524	.0524	.0524	.0524	.0524	.0524	.0524
200	.0110	.0001	.0001	.0215	.0011	.0474	-.0211	-.0003	.0474	.0474	.0474	.0474	.0474	.0474	.0474
210	.0103	.0001	.0001	.0200	.0009	.0403	-.0211	-.0003	.0403	.0403	.0403	.0403	.0403	.0403	.0403
220	.0093	.0001	.0001	.0184	.0006	.0304	-.0210	-.0003	.0304	.0304	.0304	.0304	.0304	.0304	.0304
230	.0082	.0001	.0001	.0163	.0006	.0207	-.0214	-.0003	.0207	.0207	.0207	.0207	.0207	.0207	.0207
240	.0076	.0001	.0001	.0153	-.0003	.0160	-.0213	-.0003	.0160	.0160	.0160	.0160	.0160	.0160	.0160
250	.0062	.0001	.0001	.0142	-.0003	.0122	-.0203	-.0003	.0122	.0122	.0122	.0122	.0122	.0122	.0122
260	.0046	.0001	.0001	.0102	.0013	.0302	-.0225	-.0003	.0302	.0302	.0302	.0302	.0302	.0302	.0302
270	.0117	.0005	.0001	.0069	.0075	.0206	-.0168	-.0003	.0206	.0206	.0206	.0206	.0206	.0206	.0206
280	.0202	.0176	.0116	-.0003	-.0106	.0205	-.0204	-.0003	.0205	.0205	.0205	.0205	.0205	.0205	.0205
290	.0375	.0173	.0110	-.0061	-.0101	.0293	-.0165	-.0003	.0293	.0293	.0293	.0293	.0293	.0293	.0293
300	.0365	.0067	.0012	-.0076	-.0015	.0212	-.0173	-.0003	.0212	.0212	.0212	.0212	.0212	.0212	.0212
310	.0205	.0161	.0145	.0001	.0251	.0235	-.0632	-.0003	.0235	.0235	.0235	.0235	.0235	.0235	.0235
315	.0116	-.0001	.0332	.0083	.0161	.0211	-.0175	-.0003	.0211	.0211	.0211	.0211	.0211	.0211	.0211
320	-.0018	-.0037	.0216	.0043	-.0157	.0188	-.0051	-.0003	.0188	.0188	.0188	.0188	.0188	.0188	.0188

state. The error variance associated with x_{F1} is approximately an order of magnitude higher than that for x_1 and x_2 . Hence, the terminal state estimates will have lower confidence than the current state estimates.

The estimation errors from the simulations are presented in Figures 4 and 5 and bear out the results expected from the error variances. For the case where the initial estimate is the true terminal state (Figure 4), the current and terminal state estimates are of the same order of magnitude as for the standard estimation algorithm over most of the increments. In general, the estimates of x have an absolute error of the same order of magnitude as for the standard estimation algorithm, with those for x_2 being an order of magnitude less. As expected, when the initial guess as to the terminal state is good, the estimate remains relatively good. Figure 5 presents the state estimation errors for the case where the initial guess of the terminal state is relatively inaccurate. From Figure 5, we see that the current state estimation error for x_1 is again of the same magnitude as for the standard estimation algorithm of most of the range and better over the remainder, with that for x_2 being an order of magnitude better.

From increment 100 onward, it can be seen that the terminal state estimation error is consistent with the calculated variances. The estimate of x_{F1} being of the same order of magnitude as the standard algorithm errors, and for x_2 an order of magnitude greater.

0.0

$|\hat{x}_2 - x_2|$

FIGURE 4. ABSOLUTE STATE ESTIMATION

ERRORS FOR x_F UNKNOWN, $\hat{x}_F(-1) = [20 \ 5]$

(0.025 SAMPLING INTERVAL)

0.6

0.0

0.00

0.000

50

100

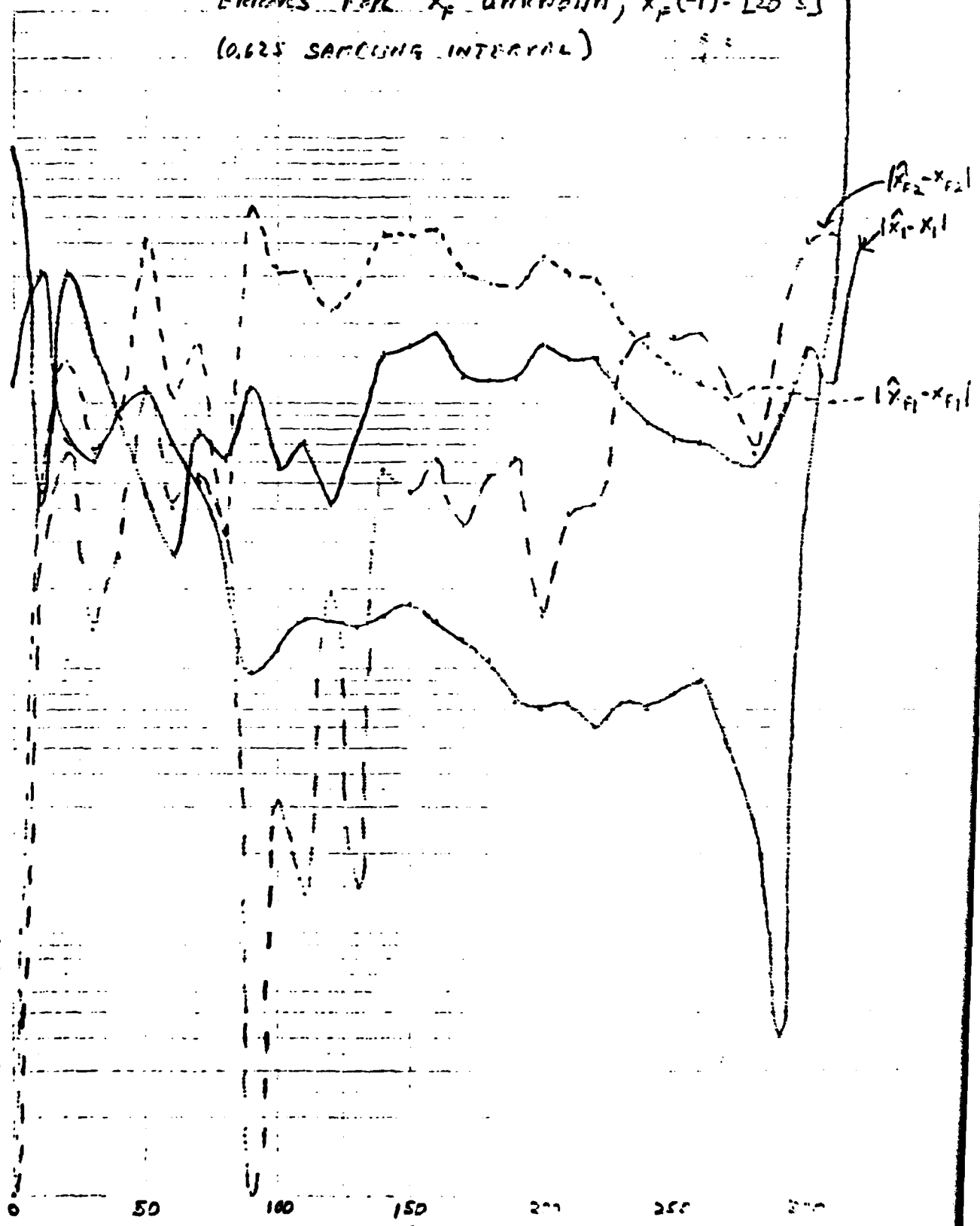
150

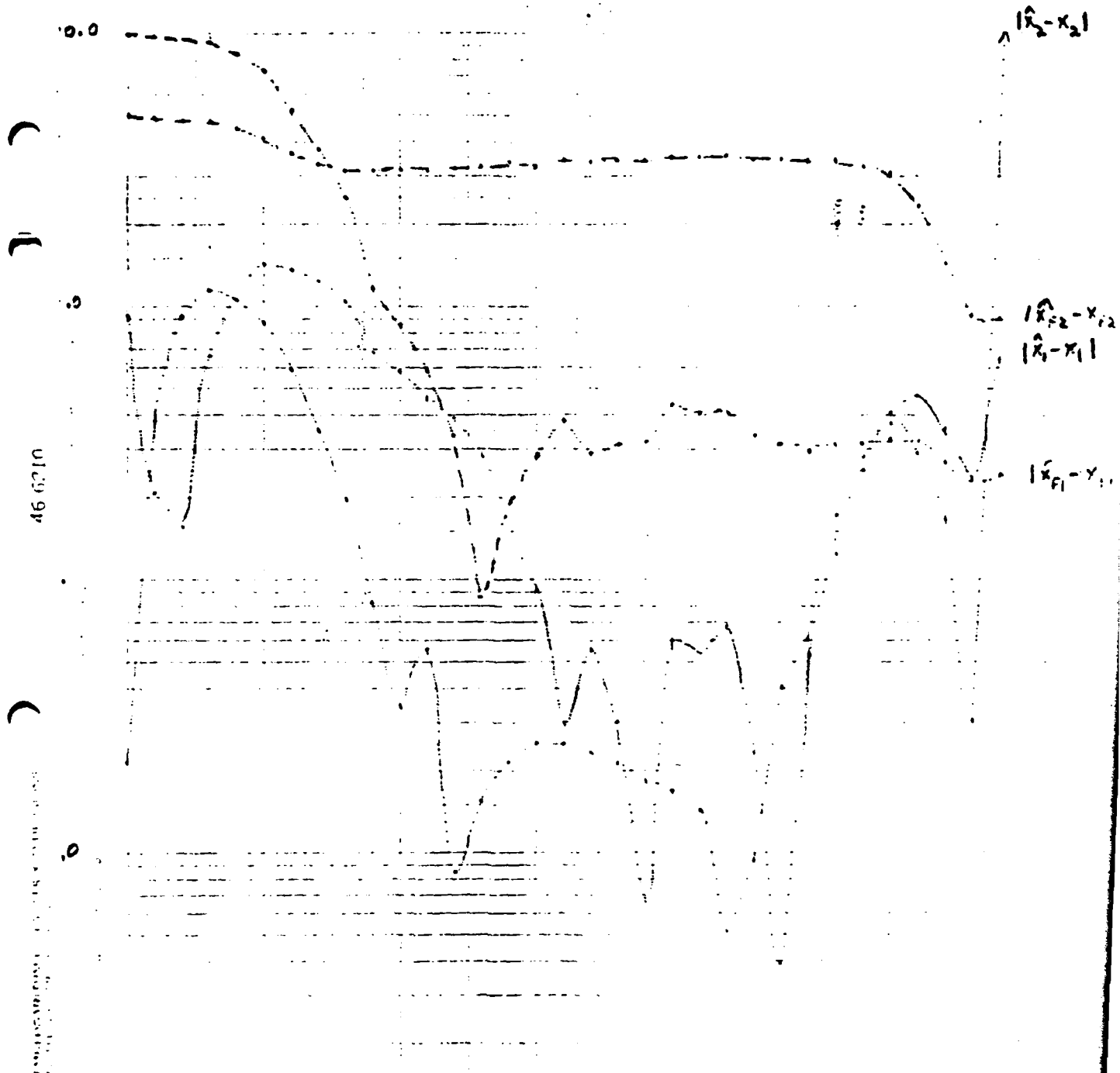
200

250

300

44





46 6210

FIGURE 5. ABSOLUTE STATE ESTIMATION
 ERRORS FOR y_F UNKNOWN, $\hat{x}_F(-) = [10 \ 10]$
 (0.625 SAMPLING INTERVAL)

The relative quality of the state estimates required is naturally a function of the problem at hand. For some problems, the improved estimation performance of the algorithm developed here may be superfluous. In such cases, the added difficulty of implementing this algorithm make its use questionable. For other problems, the improved performance may be highly desirable. In either case, the incorporation of terminal state estimation provides valuable information not obtainable from the standard estimation algorithm. This fact will in many cases make the optimal estimation algorithm desirable even when the current state estimates are no better than for the standard estimation algorithm.

SECTION 6. CONCLUSIONS

The optimal estimation algorithm developed in this paper promises to provide superior tracking performance than the standard estimation algorithm. In addition, terminal state estimation is incorporated providing useful information to the tracker within the context of the problem as outlined in the introduction. There are several limitations of the proposed method. Foremost of these are the assumption that the attacker moves in an optimal fashion toward a specific goal via a fixed deterministic criteria known to both the attacker and tracker. However, in many cases, these assumptions may be valid.

Despite the limitations of the proposed algorithm, it represents one of the few attempts to incorporate the goal seeking nature of a vehicle to be tracked into the estimation procedure. While the method is based on a somewhat complicated formulation, the resulting algorithm is straightforward and computationally is not significantly more complex than the standard estimation algorithm. The biggest problem would seem to be the increase in dimensionality required.

The utility of the proposed algorithm rest primarily on the validity of the given problem formulation as a model of the tactical problem being considered.

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