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AN INVESTIGATION OF SPIN-ORBIT RESONANCE
EFFECTS ABOUT THE GEOSYNCHRONOUS ORBIT

THESIS

AFIT/GSO/AA/62D-2

Thomas Sean Kelsic
Capt USAF

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harmonic terms (J_2 , J_2^2 , J_4 , and J_{22}). The resulting Hamiltonian, which was valid for all inclinations and small eccentricities, was reduced to a single degree of freedom through a series of transformations to allow computer generation of phase portraits and analysis of the structure and librational stability.

Three resonance bands were discovered, two of which seemed practical for operational use. Although the second of these bands was contained within the primary resonance structure, its width (246 meters) and librational period (625 years) appear useful for satellite placement at the additional stable equilibrium points. Additionally, the general technique was demonstrated as a feasible alternative investigative tool to a purely analytic approach.

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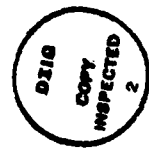
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AN INVESTIGATION OF SPIN-ORBIT RESONANCE
EFFECTS ABOUT THE GEOSYNCHRONOUS ORBIT

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

B



by

Thomas Sean Kelso, B.S., M.B.A.

Capt USAF

Graduate Space Operations

December 1982

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Notation

A, B, C	Geopotential coefficients
a	Semi-major axis
e	Eccentricity
F, F^*	Hamiltonian
F_0	Unperturbed Hamiltonian
F_1	J_2 Hamiltonian contribution
F_{2s}	Secular J_2^2 Hamiltonian contribution
F_{4s}	Secular J_4 Hamiltonian contribution
f	True anomaly
G, H, L	Delaunay elements
g	Argument of perigee
h	Longitude of the ascending node
i	Inclination
J_2, J_4, J_{22}	Geopotential harmonic coefficients
k	Constant
l	Mean anomaly
n	Index
n_0	Geocentric mean rotation rate
Q, R, S	Generalized momenta - second transformation
R_e	Geocentric mean equatorial radius
R_{22}	Geopotential perturbing function
r	Geocentric radial distance
S_2	Generating function
s	Generalized coordinate - second transformation
t	Time
u	Argument of latitude (f + g)

X, Y, Z	Generalized momenta - first transformation
x, y, z	Generalized coordinates - first transformation
α	Right ascension
β	Geocentric latitude
δ	Equatorial projection of the argument of latitude
μ	Geocentric gravitational parameter
λ	Geocentric longitude
λ_m	Mean longitude
λ_{22}	Longitude associated with J_{22} harmonic
θ	Greenwich sidereal time
T	Vernal equinox direction
ψ	Sum of $n_0 t + \lambda_{22}$
ω_1, ω_2	Frequency constants

Abstract

An investigation of the spin-orbit resonance effects about the geosynchronous orbit was undertaken to determine the existence and feasibility for use of additional stable equilibrium points in this regime for the placement of communications satellites. The Hamiltonian of the geopotential was developed in Delaunay elements using first and second order zonal and sectorial harmonic terms, (~~J_2 , J_2^2 , J_4 , and J_{22}~~). The resulting Hamiltonian, which was valid for all inclinations and small eccentricities, was reduced to a single degree of freedom through a series of transformations to allow computer generation of phase portraits and analysis of the structure and librational stability.

Three resonance bands were discovered, two of which seemed practical for operational use. Although the second of these bands was contained within the primary resonance structure, its width (246 meters) and librational period (625 years) appear useful for satellite placement at the additional stable equilibrium points. Additionally, the general technique was demonstrated as a feasible alternative investigative tool to a purely analytic approach.

I Introduction

With the growth of world-wide communications systems, the use of geosynchronous satellites has become progressively more important over the past decade. Within the Department of Defense the geosynchronous communications satellite is becoming a vital link in C³ -- Command, Control, and Communications. Unfortunately, the characteristics of the geosynchronous orbit present two major difficulties.

Because current communications satellites require a minimum beam-width separation, the fact that the nominal geosynchronous orbit is a circular, equatorial orbit, places an upper limit on the maximum number of satellites which can inhabit it, resulting in it being considered a scarce resource. Also, when basic resonance effects are considered, those arising from the triaxiality of the earth's figure, it is discovered that this orbit has only four primary equilibrium points, with only two of these points being stable. Therefore, considerable station-keeping is required to maintain the positioning necessary for adequate communication separation. Modern satellite lifetimes can be limited by these station-keeping requirements.

In an attempt to find a means to alleviate these problems, an investigation of the spin-orbit resonance effects arising from the interaction of the spin of the nonspherical earth with a satellite orbit was conducted. It will be shown that these resonance effects result in multiple equilibrium points at near geosynchronous orbit and that these equilibrium points, although not truly geosynchronous, may be useful in permitting an expanded use of this orbit for communications satellites.

Background

A review of the literature reveals that an extensive amount of research has been done concerning the effect of the earth's shape on satellite orbits. The majority of this research began with the advent of the artificial satellite in the 1950s. In particular, the effect of the earth's oblateness has been well established by several researchers. Many varied approaches were taken, as illustrated in the November 1959 issue of The Astronomical Journal where three independent treatments of the 'main problem' of satellite theory were published.

Selected investigations have attempted to determine the effects of the earth's longitude dependent harmonics on satellite motion. Because of its use for communications satellites, the geosynchronous orbit has proven to be of great interest.

Blitzer (Ref 7,8), using a linearized theory, discussed equilibrium solutions for a geosynchronous satellite in a circular, equatorial orbit under the influence of the principle longitude dependent term, $J_{2,2}$. Blitzer (Ref 4,5) later treated the effect, due to higher order tesseral harmonics, on a geosynchronous satellite of small eccentricity and inclination. He showed that, due to the dominance of the $J_{2,2}$ term, only a slight displacement of the equilibrium positions occurred, although there was a significant change in the librational periods.

Musen and Bailie (Ref 22), in 1962, studied the motion of synchronous satellites incorporating only the $J_{2,2}$ tesseral harmonic and the J_2 and J_4 zonal harmonics. Their analytic technique agrees with Blitzer for synchronous motion in the equatorial plane.

Allan (Ref 1), in 1965, discussed the motion of nearly circular but inclined synchronous orbits. In 1967 (Ref 2), he studied the effect of

resonance in inclination for synchronous satellites in nearly circular orbits when the positions of the nodes repeat relative to the rotating primary. He also studied the effect of resonance in eccentricity and inclination for a synchronous satellite in a nearly equatorial, eccentric orbit which occurs when the longitude of the line of apsides repeats relative to the primary (Ref 3).

In a book published in 1966, Kaula (Ref 19) developed expressions for the resonant disturbing function of the geopotential in terms of inclination and eccentricity functions and derived general expressions for the variation of orbital elements due to arbitrary zonal or tesseral harmonics.

And, in a series of articles over the years, Garfinkel (Ref 15) has developed a formulation known as the Ideal Resonance Problem which has found wide application in resonance theory by treating specific cases as perturbations of that problem.

From the preceding, it is apparent that considerable progress has been made in resonance theory, although it appears at this time that little hope exists for a general analytic solution except for certain specific cases. These include mostly satellites whose mean motion is strictly commensurate with the earth's rotation rate and whose orbits are at the critical inclination or have zero eccentricity. It should be noted that while Dallas and Diehl (Ref 12) claimed such a general solution in 1977, they were later shown by Jupp (Ref 17) to have made a serious error. Garfinkel (Ref 15), in his 1979 summary of the Ideal Resonance Problem, lists the synchronous satellite with non-zero eccentricity and the general $p:q$ resonance between the period of revolution of the satellite and the rotation of the earth as two of the outstanding

unsolved problems of resonance theory.

As a result, although the effort continues to find formal, global solutions to resonance problems, most recent efforts address themselves to specific satellite orbits or attack the problem via semi-analytic or numerical techniques such as used by Nacozy and Diehl (Ref 23).

One such technique is the computer plotting of phase portraits of the specific system Hamiltonian. Although this method has been used previously to provide some sort of physical feel, it would appear that little has been done to utilize the process as the primary investigative tool. While this technique provides information only in the regime of the particular inclination studied, it is valid for small eccentricities and permits investigation of all resonance terms associated with a particular commensurability ratio. It is also easily modified to incorporate additional harmonic terms or study different commensurability ratios. But most importantly, it circumvents most of the mathematically sophisticated and algebraically laborious techniques required in the search for more general analytic solutions.

Objectives and Scope

This investigation will undertake to determine whether multiple equilibrium points exist for the geosynchronous resonance and, if so, just what the characteristics of these equilibrium points are. These characteristics should indicate whether further investigations are justified to determine the usefulness of these points for the placement of communications satellites.

A secondary objective of this investigation is to demonstrate the feasibility of the general technique of computer generation of phase portraits for the examination of geopotential resonant structures.

The scope of this particular investigation will be limited to an examination of the spin-orbit resonances arising from the triaxiality of the earth's figure. The contributions from the first and second order zonal and sectorial harmonics, the J_2 , $(J_2)^2$, J_4 , and $J_{2,2}$ terms of the geopotential, will be examined for their affect on a near geosynchronous orbit.

General Approach

The Hamiltonian of the geopotential will be developed, using Delaunay's elements, by combining the secular first and second order zonal harmonic terms with the nominal two-body Hamiltonian, as extracted from Brouwer's 1959 article in The Astronomical Journal (Ref 9), and then developing the perturbing function arising from the primary sectorial harmonic term, $J_{2,2}$. Once obtained, this Hamiltonian will then be converted, by means of a contact transformation, to another set of elements which will facilitate work for the geosynchronous case. At this point, each resonance term will be treated independently and another canonical transformation will be used to allow reduction to a single degree of freedom. This form will allow the resonance effects to be analyzed by use of phase portraits of this single degree of freedom. From these phase portraits, stable equilibrium points can be determined and the stability analyzed.

Sequence of Presentation

Chapter II presents the theoretical development of the system Hamiltonian, the equations of motion for each resonance term, and the methods for determining librational stability. Chapter III discusses the computer application of these theoretical developments and Chapter

IV presents an analysis of the results and conclusions of the study as well as recommendations for further research.

II Theory

Geopotential Hamiltonian

In determining the spin-orbit resonance effects for a near geosynchronous orbit, initially the Hamiltonian for the geopotential must be obtained. The general form for the Hamiltonian may be written

$$F = F_0 + F_1 + F_{2_s} + F_{4_s} + R_{22}$$

where F_0 represents the contribution of the unperturbed potential, F_1 , F_{2_s} , and F_{4_s} are the secular contributions of the J_2 , J_2^2 , and J_4 terms, and R_{22} is the disturbing function resulting from the J_{22} term. The first four terms may be extracted from a 1959 Astronomical Journal article by Brouwer (Ref 9) and written ignoring long period variations as

$$F_0 = \frac{\mu^2}{2L^2}$$

$$F_1 = \frac{\mu^4 J_2 R^2}{L^3 G^3} e \left(-\frac{1}{4} + \frac{3}{4} \left(\frac{H}{G}\right)^2 \right)$$

$$F_{2_s} = \frac{\mu^6 J_2^2 R^4}{4L^{10}} e \left[\frac{15}{32} \left(\frac{L}{G}\right)^5 \left(1 - \frac{18}{5} \left(\frac{H}{G}\right)^2 + \left(\frac{H}{G}\right)^4 \right) \right. \\ \left. + \frac{3}{8} \left(\frac{L}{G}\right)^6 \left(1 - 6 \left(\frac{H}{G}\right)^2 + 9 \left(\frac{H}{G}\right)^4 \right) \right. \\ \left. - \frac{15}{32} \left(\frac{L}{G}\right)^7 \left(1 - 2 \left(\frac{H}{G}\right)^2 - 7 \left(\frac{H}{G}\right)^4 \right) \right]$$

$$F_{4_s} = \frac{3\mu^8 J_4 R^4}{8L^{10}} e \left(\frac{9}{16} \left(\frac{L}{G}\right)^5 - \frac{15}{16} \left(\frac{L}{G}\right)^7 \right) \left(1 - 10 \left(\frac{H}{G}\right)^2 + \frac{35}{3} \left(\frac{H}{G}\right)^4 \right)$$

where

$$L = \sqrt{\mu a}$$

ℓ = mean anomaly

$$G = L\sqrt{1-e^2}$$

g = argument of perigee

$$H = G(\cos i)$$

h = longitude of the ascending node

and μ is the geocentric gravitational parameter, R_e is the geocentric mean equatorial radius, a is the semi-major axis, e is the eccentricity, and i is the inclination.

The general form of the geopotential, as listed in Hagihara (Ref 16:459), is now used to determine the disturbing function due to the principal sectorial harmonic, $J_{2,2}$.

$$R_{2,2} = \frac{\mu R_e^2}{r^3} J_{2,2} P_2^2(\sin\beta) \sin(2\lambda - 2\lambda_{2,2})$$

Here β is the latitude, λ is the longitude, and $\lambda_{2,2}$ is the longitude associated with the $J_{2,2}$ coefficient. Since

$$P_2^2(\sin\beta) = 3\cos^2\beta$$

$$R_{2,2} = \frac{\mu R_e^2}{r^3} J_{2,2} (3\cos^2\beta) \sin(2\lambda - 2\lambda_{2,2})$$

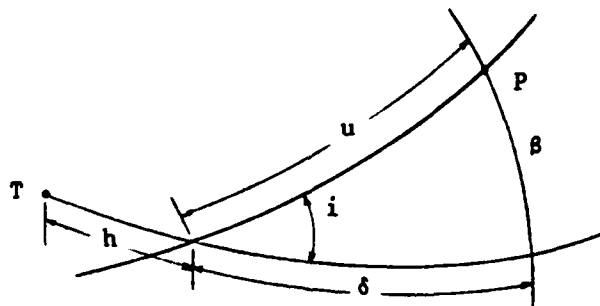


Figure 1. Orbit-Equator-Meridian Triangle

To express the disturbing function in terms of the Delaunay elements, β and λ must first be converted. Examination of Figure 1 indicates that the right ascension, α , of a satellite at point P would be expressed as

$$\alpha = h + \delta$$

and since the satellite's longitude is the difference between its right

ascension and the Greenwich sidereal time, θ , then

$$\lambda = \alpha - \theta = h + \delta - \theta$$

The Greenwich sidereal time may also be expressed as

$$\theta = n_0 t$$

so

$$\lambda = h + \delta - n_0 t$$

Letting

$$\psi = n_0 t + \lambda_{2,2}$$

so as to simplify the following expressions, and performing some trigonometric expansions

$$3\cos^2\beta\sin(2h-2\psi+2\delta) = 3\cos^2\beta[\sin(2h-2\psi)\cos 2\delta + \cos(2h-2\psi)\sin 2\delta]$$

$$\cos 2\delta = \cos^2\delta - \sin^2\delta$$

$$\sin 2\delta = 2\sin\delta\cos\delta$$

From spherical trigonometry,

$$\sin\delta = \sin u \cos i / \cos\beta$$

$$\cos\delta = \cos u / \cos\beta$$

where u is the argument of latitude, so

$$\cos 2\delta = (\cos^2 u - \sin^2 u \cos^2 i) / \cos^2 \beta$$

$$\sin 2\delta = 2\sin u \cos u \cos i / \cos^2 \beta$$

and therefore

$$\begin{aligned} 3\cos^2\beta\sin(2h-2\psi+2\delta) &= 3\sin(2h-2\psi)(\cos^2 u - \sin^2 u \cos^2 i) \\ &\quad + 6\cos(2h-2\psi)(\sin u \cos u \cos i) \end{aligned}$$

Note that the latitude dependence of the disturbing function has now disappeared and it is no longer necessary to express β in terms of the Delaunay elements.

Now, by substituting

$$\cos^2 u = \frac{1}{2} + \frac{1}{2} \cos 2u$$

$$\sin^2 u = \frac{1}{2} - \frac{1}{2} \cos 2u$$

and

$$\sin u \cos u = \frac{1}{2} \sin 2u$$

into the above equation and simplifying

$$\begin{aligned} 3\cos^2 \beta \sin(2h-2\psi+2\delta) &= \frac{3}{2}(1+\cos^2 i) \sin(2h-2\psi) \cos 2u \\ &+ \frac{3}{2} \sin^2 i \sin(2h-2\psi) \\ &+ 3\cos i \cos(2h-2\psi) \sin 2u \end{aligned}$$

Further, noting that

$$\sin(2h-2\psi) \cos 2u = \frac{1}{2} [\sin(2h-2\psi+2u) + \sin(2h-2\psi-2u)]$$

$$\cos(2h-2\psi) \sin 2u = \frac{1}{2} [\sin(2h-2\psi+2u) - \sin(2h-2\psi-2u)]$$

and substituting these expressions into the last equation yields

$$\begin{aligned} 3\cos^2 \beta \sin(2h-2\psi+2\delta) &= \frac{3}{4}(1+\cos i)^2 \sin(2h-2\psi+2u) \\ &+ \frac{3}{2} \sin^2 i \sin(2h-2\psi) \\ &+ \frac{3}{4}(1-\cos i)^2 \sin(2h-2\psi-2u) \end{aligned}$$

Finally, writing

$$u = f + g$$

where f is the true anomaly, and substituting back into the expression for the disturbing function results in

$$R_{22} = \frac{\mu R_e^2}{r^3} J_{22} \left[\frac{3}{4} (1+\cos i)^2 \sin(2g+2h+2f-2\psi) \right]$$

$$\begin{aligned}
& + \frac{3}{2} \sin^2 i \sin(2h-2\psi) \\
& + \frac{3}{4} (1-\cos i)^2 \sin(-2g+2h-2f-2\psi)
\end{aligned}$$

To express this term solely as a function of the Delaunay elements, r and f must be expanded in terms of the mean anomaly, ℓ , and substituted into the above expression. Note that up to this point in the development of the geopotential Hamiltonian, a completely general expression has been formed, valid for all eccentricities and inclinations and limited only by the number of zonal and sectorial harmonic terms included in the treatment.

Using r and f as listed by Brouwer and Clemence (Ref 10:76-77) and retaining terms to the seventh power in eccentricity,

$$\begin{aligned}
\frac{r}{a} &= 1 + e(-\cos \ell) + e^2\left(\frac{1}{2} - \frac{1}{2}\cos 2\ell\right) + e^3\left(\frac{3}{8}\cos \ell - \frac{3}{8}\cos 3\ell\right) \\
&+ e^4\left(\frac{1}{3}\cos 2\ell - \frac{1}{3}\cos 4\ell\right) + e^5\left(-\frac{5}{192}\cos \ell + \frac{45}{128}\cos 3\ell - \frac{125}{384}\cos 5\ell\right) \\
&+ e^6\left(-\frac{1}{16}\cos 2\ell + \frac{2}{5}\cos 4\ell - \frac{27}{80}\cos 6\ell\right) \\
&+ e^7\left(\frac{7}{9216}\cos \ell - \frac{567}{5120}\cos 3\ell + \frac{4375}{9216}\cos 5\ell - \frac{16807}{46080}\cos 7\ell\right) \\
f &= \ell + e(2\sin \ell) + e^2\left(\frac{5}{4}\sin 2\ell\right) + e^3\left(-\frac{1}{4}\sin \ell + \frac{13}{12}\sin 3\ell\right) \\
&+ e^4\left(-\frac{11}{24}\sin 2\ell + \frac{103}{96}\sin 4\ell\right) + e^5\left(\frac{5}{96}\sin \ell - \frac{43}{64}\sin 3\ell + \frac{1097}{960}\sin 5\ell\right) \\
&+ e^6\left(\frac{17}{192}\sin 2\ell - \frac{451}{480}\sin 4\ell + \frac{1223}{960}\sin 6\ell\right) \\
&+ e^7\left(\frac{107}{4608}\sin \ell + \frac{95}{512}\sin 3\ell - \frac{5957}{4608}\sin 5\ell + \frac{47273}{32256}\sin 7\ell\right)
\end{aligned}$$

Now, converting to canonical units ($R_e = \mu = 1$) and substituting these expressions into $R_{2,1}$

$$\begin{aligned}
R_{22} = & \frac{J_{22}}{a^2} \left(\frac{3}{4} (1 + \cos i)^2 \left[\left(-\frac{1}{2}e + \frac{1}{16}e^3 - \frac{5}{384}e^5 - \frac{143}{18432}e^7 \right) \sin(\ell + 2g + 2h - 2\psi) \right. \right. \\
& + \left(-\frac{1}{48}e^3 - \frac{11}{768}e^5 - \frac{313}{30720}e^7 \right) \sin(\ell - 2g - 2h + 2\psi) \\
& + \left(1 - \frac{5}{2}e^2 + \frac{13}{16}e^4 - \frac{35}{288}e^6 \right) \sin(2\ell + 2g + 2h - 2\psi) \\
& + \left(-\frac{1}{24}e^4 - \frac{7}{240}e^6 \right) \sin(2\ell - 2g - 2h + 2\psi) \\
& + \left(\frac{7}{2}e - \frac{123}{16}e^3 + \frac{489}{128}e^5 - \frac{1763}{2048}e^7 \right) \sin(3\ell + 2g + 2h - 2\psi) \\
& + \left(-\frac{81}{1280}e^5 - \frac{81}{2048}e^7 \right) \sin(3\ell - 2g - 2h + 2\psi) \\
& + \left(\frac{17}{2}e^2 - \frac{115}{6}e^4 + \frac{601}{48}e^6 \right) \sin(4\ell + 2g + 2h - 2\psi) \\
& + \left(-\frac{4}{45}e^6 \right) \sin(4\ell - 2g - 2h + 2\psi) \\
& + \left(\frac{845}{48}e^3 - \frac{32525}{768}e^5 + \frac{208225}{6144}e^7 \right) \sin(5\ell + 2g + 2h - 2\psi) \\
& + \left(-\frac{15625}{129024}e^7 \right) \sin(5\ell - 2g - 2h + 2\psi) \\
& + \left(\frac{533}{16}e^4 - \frac{13827}{160}e^6 \right) \sin(6\ell + 2g + 2h - 2\psi) \\
& + \left(\frac{228347}{3840}e^5 - \frac{3071075}{18432}e^7 \right) \sin(7\ell + 2g + 2h - 2\psi) \\
& + \left(\frac{73369}{720}e^6 \right) \sin(8\ell + 2g + 2h - 2\psi) \\
& + \left(\frac{12144273}{71680}e^7 \right) \sin(9\ell + 2g + 2h - 2\psi) \left. \right] \\
& + \frac{3}{2} \sin^2 i \left[\left(1 + \frac{3}{2}e^2 + \frac{15}{8}e^4 + \frac{35}{16}e^6 \right) \sin(2h - 2\psi) \right. \\
& + \left(\frac{3}{2}e + \frac{27}{16}e^3 + \frac{261}{128}e^5 + \frac{143097}{6144}e^7 \right) \sin(\ell + 2h - 2\psi) \\
& + \left(-\frac{3}{2}e - \frac{27}{16}e^3 - \frac{261}{128}e^5 - \frac{14309}{6144}e^7 \right) \sin(\ell - 2h + 2\psi) \\
& + \left(\frac{9}{4}e^2 + \frac{7}{4}e^4 + \frac{141}{64}e^6 \right) \sin(2\ell + 2h - 2\psi)
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{9}{4}e^2 - \frac{7}{4}e^4 - \frac{141}{64}e^6 \right) \sin(2l-2h+2\psi) \\
& + \left(\frac{53}{16}e^3 + \frac{393}{256}e^5 + \frac{24753}{10240}e^7 \right) \sin(3l+2h-2\psi) \\
& + \left(-\frac{53}{16}e^3 - \frac{393}{256}e^5 - \frac{24753}{10240}e^7 \right) \sin(3l-2h+2\psi) \\
& + \left(\frac{77}{16}e^4 + \frac{129}{160}e^6 \right) \sin(4l+2h-2\psi) \\
& + \left(-\frac{77}{16}e^4 - \frac{129}{160}e^6 \right) \sin(4l-2h+2\psi) \\
& + \left(\frac{1773}{256}e^5 - \frac{4987}{6144}e^7 \right) \sin(5l+2h-2\psi) \\
& + \left(-\frac{1773}{256}e^5 + \frac{4987}{6144}e^7 \right) \sin(5l-2h+2\psi) \\
& + \left(\frac{3167}{320}e^6 \right) \sin(6l+2h-2\psi) \\
& + \left(-\frac{3167}{320}e^6 \right) \sin(6l-2h+2\psi) \\
& + \left(\frac{432091}{30720}e^7 \right) \sin(7l+2h-2\psi) \\
& + \left(-\frac{432091}{30720}e^7 \right) \sin(7l-2h+2\psi)] \\
& + \frac{3}{4}(1-\cos i)^2 \left[\left(\frac{1}{2}e - \frac{1}{16}e^3 + \frac{5}{384}e^5 + \frac{143}{18432}e^7 \right) \sin(l+2g-2h+2\psi) \right. \\
& + \left(\frac{1}{48}e^3 + \frac{11}{768}e^5 + \frac{313}{30720}e^7 \right) \sin(l-2g+2h-2\psi) \\
& + \left(-1 + \frac{5}{2}e^2 - \frac{13}{16}e^4 + \frac{35}{288}e^6 \right) \sin(2l+2g-2h+2\psi) \\
& + \left(\frac{1}{24}e^4 + \frac{7}{240}e^6 \right) \sin(2l-2g+2h-2\psi) \\
& + \left(-\frac{7}{2}e + \frac{123}{16}e^3 - \frac{489}{128}e^5 + \frac{1763}{2048}e^7 \right) \sin(3l+2g-2h+2\psi) \\
& + \left(\frac{81}{1280}e^3 + \frac{81}{2048}e^5 \right) \sin(3l-2g+2h-2\psi) \\
& + \left(-\frac{17}{2}e^2 + \frac{115}{6}e^4 - \frac{601}{48}e^6 \right) \sin(4l+2g-2h+2\psi)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{4}{45}e^6\right)\sin(4\ell-2g+2h-2\psi) \\
& + \left(-\frac{845}{48}e^5 + \frac{32525}{768}e^5 - \frac{208225}{6144}e^7\right)\sin(5\ell+2g-2h+2\psi) \\
& + \left(\frac{15625}{129024}e^7\right)\sin(5\ell-2g+2h-2\psi) \\
& + \left(-\frac{533}{16}e^6 + \frac{13827}{160}e^6\right)\sin(6\ell+2g-2h+2\psi) \\
& + \left(-\frac{228347}{3840}e^5 + \frac{3071075}{18432}e^7\right)\sin(7\ell+2g-2h+2\psi) \\
& + \left(-\frac{73369}{720}e^6\right)\sin(8\ell+2g-2h+2\psi) \\
& + \left(-\frac{12144273}{71680}e^7\right)\sin(9\ell+2g-2h+2\psi)\}
\end{aligned}$$

Although this form of $R_{2,2}$ is now limited to small eccentricities, the practical limit, as noted by Kovalevsky (Ref 20:55), is of sufficient magnitude as to have no further affect on this treatment. Obviously, for small eccentricities, carrying the expansion to the seventh power in eccentricity should allow sufficient accuracy for this investigation.

Examination of the above expansion of $R_{2,2}$ indicates the existence of many resonance terms, permitting investigation of resonances other than the geosynchronous resonance merely by selecting the appropriate terms. For the geosynchronous case, however, resonance occurs when the mean motion of the satellite is commensurate with the mean rotation rate of the earth, leading to selection of terms including $2\ell-2\psi$ in the trigonometric argument.

Retaining only the near geosynchronous resonance terms and completing the transformation to Delaunay elements yields

$$R_{2,2} = \frac{J_{2,2}}{L^6} \left[\frac{3}{4} \left(1 + \frac{H}{G}\right)^2 \left(-\frac{233}{288} + \frac{119}{96} \left(\frac{G}{L}\right)^2 + \frac{43}{96} \left(\frac{G}{L}\right)^4 + \frac{35}{288} \left(\frac{G}{L}\right)^6 \right) \sin(2\ell+2g+2h-2\psi) \right]$$

$$\begin{aligned}
& + \frac{3}{2} \left(1 - \left(\frac{H}{G}\right)^2\right) \left(\frac{397}{64} - \frac{791}{64} \left(\frac{G}{L}\right)^2 + \frac{535}{64} \left(\frac{G}{L}\right)^4 - \frac{141}{64} \left(\frac{G}{L}\right)^6\right) \sin(2l+2h-2\psi) \\
& + \frac{3}{4} \left(1 - \frac{H}{G}\right)^2 \left(\frac{17}{240} - \frac{41}{240} \left(\frac{G}{L}\right)^2 + \frac{31}{240} \left(\frac{G}{L}\right)^4 - \frac{7}{240} \left(\frac{G}{L}\right)^6\right) \sin(2l-2g+2h-2\psi)
\end{aligned}$$

The geopotential Hamiltonian is now expressed completely as a function of the Delaunay elements and time as

$$F = F(l, g, h, L, G, H, t) = F_0 + F_1 + F_{2g} + F_{4g} + R_{22}$$

Canonical Transformation for the Geosynchronous Case

In order to simplify calculations later in this development, a new set of variables is selected which comprises a canonical transformation such that two of the new momenta variables are small for the geosynchronous equatorial case. These new momenta are defined as

$$\begin{aligned}
X &= L = \sqrt{\mu a} \\
Y &= L - G = \sqrt{\mu a} [1 - \sqrt{1-e^2}] \\
Z &= G - H = \sqrt{\mu a (1-e^2)} (1 - \cos i)
\end{aligned}$$

As a result, the generating function becomes

$$S_2 = Xl + (X - Y)g + (X - Y - Z)h$$

so

$$x = l + g + h \quad y = -g - h \quad z = -h$$

The Hamiltonian for the geopotential may now be expressed as

$$F = F(x, y, z, X, Y, Z, t) = F_0 + F_1 + F_{2g} + F_{4g} + R_{22}$$

where

$$F_0 = \frac{1}{2X^2}$$

$$F_1 = \frac{J_2}{X^6} A^3 \left(\frac{1}{2} - \frac{3}{2}B + \frac{3}{4}B^2\right)$$

$$F_{2s} = \frac{J_2^2}{4X^{10}} \left[\frac{15}{32} A^5 \left(-\frac{8}{5} + \frac{16}{5} B + \frac{12}{5} B^2 - 4B^3 + B^4 \right) \right. \\ \left. + \frac{3}{8} A^6 (4 - 24B + 48B^2 - 36B^3 + 9B^4) \right. \\ \left. - \frac{15}{32} A^7 (-8 + 32B - 44B^2 + 28B^3 - 7B^4) \right]$$

$$F_{4s} = \frac{3J_4}{8X^{10}} \left(\frac{9}{16} A^5 - \frac{15}{16} A^7 \right) \left(\frac{8}{3} - \frac{80}{3} B + 60B^2 - \frac{140}{3} B^3 + \frac{35}{3} B^4 \right)$$

$$R_{22} = \frac{J_{22}}{X^6} \left[\frac{3}{4} (4-4B+B^2) \left(1 - 5C + \frac{23}{4} C^2 - \frac{38}{9} C^3 \right) \right. \\ \left. + \frac{109}{48} C^4 - \frac{35}{48} C^5 + \frac{35}{288} C^6 \right] \sin(2x-2\psi) \\ + \frac{3}{2} (2B-B^2) \left(\frac{9}{2} C + \frac{19}{4} C^2 + \frac{85}{8} C^3 \right. \\ \left. - \frac{395}{16} C^4 + \frac{423}{32} C^5 - \frac{141}{64} C^6 \right) \sin(2x+2y-2z-2\psi) \\ \left. + \frac{3}{4} B^2 \left(\frac{1}{6} C^2 + \frac{1}{15} C^3 - \frac{37}{120} C^4 + \frac{7}{40} C^5 - \frac{7}{240} C^6 \right) \sin(2x+4y-4z-2\psi) \right]$$

with

$$A = \frac{X}{X-Y} \quad B = \frac{Z}{X-Y} \quad C = \frac{Y}{X}$$

Note that for the near geosynchronous case, the coefficients B and C are small while the coefficient A is approximately unity. From this observation it can be seen that the first resonance term is the strongest, followed by the second and third terms respectively.

Resonance Transformations

Now that a general form of the geopotential Hamiltonian has been developed for the geosynchronous case, each of the three resonance terms may be individually examined. That is, the geopotential Hamiltonian may be considered to consist of the first and second order zonal terms along

with a single resonance term from the disturbing function. The justification for this approach lies in the assumption that in the immediate region of each resonance band, the equations of motion are dominated by the corresponding resonance term. The relative strengths of the three terms should guarantee this result.

To facilitate this examination, phase portraits of the geopotential Hamiltonian will be used. In a phase portrait, each generalized coordinate and its associated momentum make up two of the dimensions of the phase space. The real advantage to this representation is that paths corresponding to particular unique solutions of the system Hamiltonian will be produced (Ref 21:173).

At this point, however, the geopotential Hamiltonian has six dimensions arising from the three generalized coordinates and their conjugate momenta. And since time is still explicit in the Hamiltonian, an additional dimension is introduced. These seven dimensions constitute what is referred to as a motion space. Because it is extremely difficult to conceptualize a trajectory in a seven-dimensional space, it would be extremely advantageous if the geopotential Hamiltonian could be limited to only two variable dimensions to permit plotting of these trajectories. This result may be accomplished by reducing the geopotential Hamiltonian to a single degree of freedom and eliminating the explicit time dependence through the use of a canonical transformation.

In each case, a new canonical transformation will be defined such that

$$F^* = F^*(_, _, s, Q, R, S, _) = F(x, y, z, X, Y, Z, t) - \frac{\partial S_2}{\partial t}$$

where the associated generating function S_2 will be of the form

$$S_2 = S_2(x, y, z, Q, R, S, t)$$

These transformations will yield a Hamiltonian for each resonance of the form

$$F^* = F_0 + F_1 + F_{2s} + F_{4s} + R_{22} + n_0 S$$

with

$$F_0 = \frac{1}{2S^2}$$

$$F_1 = \frac{J_2}{S^6} A^3 \left(\frac{1}{2} - \frac{3}{2} B + \frac{3}{4} B^2 \right)$$

$$F_{2s} = \frac{J_2^2}{4S^{10}} \left[\frac{15}{32} A^5 \left(-\frac{8}{5} + \frac{16}{5} B + \frac{12}{5} B^2 - 4B^3 + B^4 \right) \right. \\ \left. + \frac{3}{8} A^6 (4 - 24B + 48B^2 - 36B^3 + 9B^4) \right. \\ \left. - \frac{15}{32} A^7 (-8 + 32B - 44B^2 + 28B^3 - 7B^4) \right]$$

$$F_{4s} = \frac{3J_4}{8S^{10}} \left(\frac{9}{16} A^5 - \frac{15}{16} A^7 \right) \left(\frac{8}{3} - \frac{80}{3} B + 60B^2 - \frac{140}{3} B^3 + \frac{35}{3} B^4 \right)$$

These transformations will be done in such a way that the functional forms of F_0 , F_1 , F_{2s} , and F_{4s} will always be as listed above. The only change from resonance to resonance will be the functional dependence of the coefficients A , B , and C on the generalized momenta Q , R , and S . The particular resonance term selected from the disturbing function will be transformed so that its trigonometric argument will depend upon a single generalized coordinate.

By selecting a transformation of this form, three constants of the motion result. Since neither q or r appear explicitly in the resulting geopotential Hamiltonian, Q and R are constants. Also, because the geopotential Hamiltonian describes a natural system, it represents the

total system energy, which, due to the lack of an explicit time dependence, is also conserved. As a result, the new Hamiltonian is reduced to one degree of freedom, permitting the use of phase portraits.

For the first and primary resonance condition,

$$R_{22} = \frac{3J_{22}}{4S^6} (4-4B+B^2) \left(1 - 5C + \frac{23}{4}C^2 - \frac{38}{9}C^3 + \frac{109}{48}C^4 - \frac{35}{48}C^5 + \frac{35}{288}C^6\right) \sin(2s)$$

By noting that the trigonometric argument of this term expressed as a function of x , y , z , and t (as developed in the previous section) was

$$2x - 2n_0 t - 2\lambda_{22}$$

then

$$s = x - n_0 t - \lambda_{22}$$

By letting $q = y$ and $r = z$, the generating function becomes

$$S_2 = Qy + Rz + S(x - n_0 t - \lambda_{22})$$

and therefore

$$X = S \quad Y = Q \quad Z = R$$

and

$$A = \frac{S}{S-Q} \quad B = \frac{R}{S-Q} \quad C = \frac{Q}{S}$$

Equilibrium points exist where the time derivatives of the only remaining generalized coordinate in the Hamiltonian and its associated momentum are both zero. That is when

$$\frac{ds}{dt} = - \frac{\partial F^*}{\partial S} = 0 \quad \text{and} \quad \frac{dS}{dt} = \frac{\partial F^*}{\partial s} = 0$$

Therefore, for the first resonance term, the equilibrium points exist where

$$\frac{ds}{dt} = \frac{1}{S^3} - \frac{J_2}{S^7} \left[A^2 \left(-\frac{3}{2} + \frac{9}{2}B - \frac{9}{4}B^2 \right) + A^4 \left(-\frac{3}{2} + 6B - \frac{15}{4}B^2 \right) \right]$$

$$\begin{aligned}
& - \frac{J_1^2}{4S^4} \left[A^5 \left(\frac{15}{4} - \frac{15}{2}B - \frac{45}{8}B^2 + \frac{75}{8}B^3 - \frac{75}{32}B^4 \right) \right. \\
& \quad + A^6 \left(-\frac{9}{4} + 27B - \frac{639}{8}B^2 + 69B^3 - \frac{567}{32}B^4 \right) \\
& \quad + A^7 \left(-\frac{81}{4} + 108B - \frac{1647}{8}B^2 + \frac{1287}{8}B^3 - \frac{1395}{32}B^4 \right) \\
& \quad \left. + A^8 \left(-\frac{105}{4} + 120B - \frac{1485}{8}B^2 + \frac{525}{4}B^3 - \frac{1155}{32}B^4 \right) \right] \\
& - \frac{3J_2}{8S^4} \left[(A^5 - A^7) \left(-\frac{15}{2} + 75B - \frac{675}{4}B^2 + \frac{525}{4}B^3 - \frac{525}{16}B^4 \right) \right. \\
& \quad + A^6 \left(-\frac{15}{2} + 90B - \frac{945}{4}B^2 + 210B^3 - \frac{945}{16}B^4 \right) \\
& \quad \left. + A^8 \left(\frac{35}{2} - 200B + \frac{2025}{4}B^2 - \frac{875}{2}B^3 + \frac{1925}{16}B^4 \right) \right] \\
& - \frac{3J_{22}}{4S^7} \left[(4-4B+B^2) \left(-6 + 35C - 46C^2 + 38C^3 - \frac{545}{24}C^4 + \frac{385}{48}C^5 - \frac{35}{24}C^6 \right) \right. \\
& \quad \left. + A(4B-2B^2) \left(1 - 5C + \frac{23}{4}C^2 - \frac{38}{9}C^3 + \frac{109}{48}C^4 - \frac{35}{48}C^5 + \frac{35}{288}C^6 \right) \right] \sin(2s) \\
& - n_0 = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{dS}{dt} &= \frac{3J_{22}}{2S^8} (4-4B+B^2) \left(1 - 5C + \frac{23}{4}C^2 - \frac{38}{9}C^3 + \frac{109}{48}C^4 - \frac{35}{48}C^5 + \frac{35}{288}C^6 \right) \cos(2s) \\
&= 0
\end{aligned}$$

For the second resonance term,

$$R_{22} = \frac{3J_{22}}{2S^8} (2B-B^2) \left(\frac{9}{2}C + \frac{19}{4}C^2 + \frac{85}{8}C^3 - \frac{395}{16}C^4 + \frac{423}{32}C^5 - \frac{141}{64}C^6 \right) \sin(2s)$$

so,

$$s = x + y - z - n_0 t - \lambda_{22}$$

and choosing

$$q = y \quad r = z$$

yields

$$S_2 = Qy + Rz + S(x + y - z - n_0 t - \lambda_{22})$$

and therefore

$$X = S \quad Y = Q + S \quad Z = R - S$$

$$A = -\frac{S}{Q} \quad B = \frac{S-R}{Q} \quad C = \frac{Q+S}{S}$$

Now, the equilibrium condition becomes

$$\begin{aligned} \frac{ds}{dt} = & \frac{1}{S^3} - \frac{J_2}{S^7} [A^3(-\frac{3}{2} + \frac{9}{2}B - \frac{9}{4}B^2) + A^4(\frac{3}{2} - \frac{3}{2}B)] \\ & - \frac{J_2^2}{4S^{11}} [A^3(\frac{15}{4} - \frac{15}{2}B - \frac{45}{8}B^2 + \frac{75}{8}B^3 - \frac{75}{32}B^4) \\ & + A^4(-\frac{15}{2} + \frac{135}{4}B - \frac{531}{8}B^2 + \frac{417}{8}B^3 - \frac{27}{2}B^4) \\ & + A^7(-\frac{9}{4} + 9B - \frac{171}{8}B^2 + \frac{207}{8}B^3 - \frac{315}{32}B^4) \\ & + A^8(15 - \frac{165}{4}B + \frac{315}{8}B^2 - \frac{105}{8}B^3)] \\ & - \frac{3J_4}{8S^{11}} [(A^5 - A^7)(-\frac{15}{2} + 75B - \frac{675}{4}B^2 + \frac{525}{4}B^3 - \frac{525}{16}B^4) \\ & + (A^6 - \frac{5}{3}A^8)(15 - \frac{135}{2}B + \frac{315}{4}B^2 - \frac{105}{4}B^3)] \\ & - \frac{3J_{22}}{2S^7} [(2B-B^2)(\frac{9}{2} - 22C - \frac{49}{8}C^2 - \frac{1555}{8}C^3 + \frac{10015}{32}C^4 - \frac{1269}{8}C^5 + \frac{423}{16}C^6) \\ & - A(2-2B)(\frac{9}{2}C + \frac{19}{4}C^2 + \frac{85}{8}C^3 - \frac{395}{16}C^4 + \frac{423}{32}C^5 - \frac{141}{64}C^6)] \sin(2s) \\ & - n_0 = 0 \end{aligned}$$

and

$$\frac{dS}{dt} = \frac{3J_{22}}{S^6} (2B-B^2) (\frac{9}{2}C + \frac{19}{4}C^2 + \frac{85}{8}C^3 - \frac{395}{16}C^4 + \frac{423}{32}C^5 - \frac{141}{64}C^6) \cos(2s) = 0$$

For the final resonance term,

$$R_{22} = \frac{3J_{22}}{4S^6} B^2 (\frac{1}{6}C^2 + \frac{1}{15}C^3 - \frac{37}{120}C^4 + \frac{7}{40}C^5 - \frac{7}{240}C^6) \sin(2s)$$

$$s = x + 2y - 2z - n_0 t - \lambda_{22} \quad q = y \quad r = z$$

$$S_2 = Qy + Rz + S(x + 2y - 2z - n_0 t - \lambda_{22})$$

$$X = S \quad Y = Q + 2S \quad Z = R - 2S$$

$$A = -\frac{S}{S+Q} \quad B = \frac{2S-R}{S+Q} \quad C = \frac{2S+Q}{S}$$

and the resonance condition becomes

$$\begin{aligned} \frac{ds}{dt} = & \frac{1}{S^3} - \frac{J_2}{S^7} [A^3(-\frac{3}{2} + \frac{9}{2}B - \frac{9}{4}B^2) + A^4(\frac{9}{2} - 9B + \frac{15}{4}B^2)] \\ & - \frac{J_2^2}{4S^{11}} [A^5(\frac{15}{4} - \frac{15}{2}B - \frac{45}{8}B^2 + \frac{75}{8}B^3 - \frac{75}{32}B^4) \\ & + A^6(-\frac{51}{4} + \frac{81}{2}B - \frac{423}{8}B^2 + \frac{141}{4}B^3 - \frac{297}{32}B^4) \\ & + A^7(\frac{63}{4} - 90B + \frac{1305}{8}B^2 - \frac{873}{8}B^3 + \frac{765}{32}B^4) \\ & + A^8(\frac{225}{4} - \frac{405}{2}B + \frac{2115}{8}B^2 - \frac{315}{2}B^3 + \frac{1155}{32}B^4)] \\ & - \frac{3J_4}{8S^{11}} [(A^5-A^7)(-\frac{15}{2} + 75B - \frac{675}{4}B^2 + \frac{525}{4}B^3 - \frac{525}{16}B^4) \\ & + A^6(\frac{75}{2} - 225B + \frac{1575}{4}B^2 - \frac{525}{2}B^3 + \frac{945}{16}B^4) \\ & + A^8(-\frac{135}{2} + 425B - \frac{3075}{4}B^2 + 525B^3 - \frac{1925}{16}B^4)] \\ & - \frac{3J_{22}}{4S^7} [B^2(\frac{2}{3}C - \frac{14}{15}C^2 - \frac{46}{15}C^3 + \frac{29}{6}C^4 - \frac{91}{40}C^5 + \frac{7}{20}C^6) \\ & - A(4B-2B^2)(\frac{1}{6}C^2 + \frac{1}{15}C^3 - \frac{37}{120}C^4 + \frac{7}{40}C^5 - \frac{7}{240}C^6)] \sin(2s) \\ & - n_s = 0 \end{aligned}$$

$$\frac{dS}{dt} = \frac{3J_{22}}{2S^6} B^2 (\frac{1}{6}C^2 + \frac{1}{15}C^3 - \frac{37}{120}C^4 + \frac{7}{40}C^5 - \frac{7}{240}C^6) \cos(2s) = 0$$

Obviously, for all three resonances, the equilibrium points will occur where

$$s = \frac{(2n+1)\pi}{4} \quad n = 0, 1, 2, 3$$

This observation can be made from a brief examination of the first and

second partial derivatives of the geopotential Hamiltonians. The general form of the first partial derivatives is

$$\frac{\partial F^*}{\partial s} = k \cos(2s)$$

where k is a constant for some fixed value of S corresponding to the value at the equilibrium point. Now, the second partial derivatives have the form

$$\frac{\partial^2 F^*}{\partial s^2} = -2k \sin(2s)$$

indicating a negative value when n is even and a positive value when n is odd. Because $\partial^2 F^* / \partial S^2$ is of order $1/S^4$, which is always greater than zero, then, by the Second Derivative Test, a local minimum exists when n is odd and a saddle point exists when n is even. The local minima correspond to stable equilibrium points and the saddle points to unstable equilibrium points. This treatment assumes that the cross partial terms are negligible, which, due to the system geometry, should be expected. This assumption was subsequently verified through the use of numerical techniques.

Although each resonance term has identical values of s defining the equilibrium points, their interpretations are slightly different. For the first term,

$$s = l + g + h - n_0 t - \lambda_{2,2}$$

which for the nearly circular, nearly equatorial case of interest to this investigation

$$l \approx f$$

$$l + g \approx \delta$$

which means that the stable equilibrium points exist where the 'mean

longitude' of the satellite, λ_m , is

$$\lambda_m \approx \frac{(2n+1)\pi}{4} + \lambda_{22} \quad n = \text{odd}$$

Since the values of s for the other two resonance terms vary only by subtracting differing multiples of the argument of perigee, g , the stable points for these resonance bands exist where

$$\lambda_m \approx \frac{(2n+1)\pi}{4} + g + \lambda_{22} \quad n = \text{odd} \quad (\text{Second Resonance})$$

$$\lambda_m \approx \frac{(2n+1)\pi}{4} + 2g + \lambda_{22} \quad n = \text{odd} \quad (\text{Third Resonance})$$

Unstable points occur for even values of n .

The corresponding values for S are somewhat more difficult to determine. Computer analysis will be used to determine these values and to plot the structure of the resonances.

Librational Analysis

Once the stable equilibrium points have been located for each resonance, there are basically two methods for obtaining values for the period of libration about them. The first method would involve the evaluation of a closed line integral along a contour of constant energy. Since the equations of motion were developed for the remaining generalized coordinate and momentum, the period of libration for any given energy contour may be expressed as the closed line integral of either

$$\int \frac{dS}{\dot{S}} \quad \text{or} \quad \int \frac{ds}{\dot{s}}$$

where \dot{S} and \dot{s} are the time derivatives of S and s , respectively, and are each functions only of S and s . To attempt to determine the period of libration analytically using this method would be extremely difficult,

and yet, a numerical evaluation, while computationally intensive, should yield very good results.

The second method involves the use of linear analysis to estimate the basic oscillatory period in the immediate region of the stable equilibrium points. While not yielding as specific a result as the previous method, it is much less computationally intensive and should result in an answer sufficient for the purposes of this investigation.

Using the Chain Rule of calculus

$$\delta \dot{s} = \frac{\partial \dot{s}}{\partial s} \delta s + \frac{\partial \dot{s}}{\partial S} \delta S$$

$$\delta \dot{S} = \frac{\partial \dot{S}}{\partial s} \delta s + \frac{\partial \dot{S}}{\partial S} \delta S$$

Now, since

$$\dot{s} = - \frac{\partial F^*}{\partial S} \quad \dot{S} = \frac{\partial F^*}{\partial s}$$

therefore

$$\frac{\partial \dot{s}}{\partial S} = - \frac{\partial^2 F^*}{\partial S^2}$$

$$\frac{\partial \dot{S}}{\partial s} = \frac{\partial^2 F^*}{\partial s^2}$$

and

$$\frac{\partial \dot{S}}{\partial S} = \frac{\partial \dot{s}}{\partial s} = \frac{\partial^2 F^*}{\partial s \partial S}$$

Assuming again that the cross partial terms are small compared to the other terms, the resulting system of equations becomes

$$\delta \dot{s} = - \frac{\partial^2 F^*}{\partial S^2} \delta S$$

$$\delta \dot{S} = \frac{\partial^2 F^*}{\partial s^2} \delta s$$

or

$$\delta \dot{s} = -\omega_1 \delta S$$

$$\delta \dot{S} = \omega_2 \delta s$$

(ω_1 and ω_2 are constants, as all partial derivatives are evaluated at the equilibrium points). Taking the time derivative of the first equation and substituting into it the second equation yields

$$\delta \ddot{s} = -(\omega_1 \omega_2) \delta s$$

which is the equation of motion for a harmonic oscillator with frequency $\sqrt{\omega_1 \omega_2}$.

While this method is somewhat more involved analytically, the resulting frequency should prove very easy to evaluate numerically. Using an approximation of the limit definition of a second partial derivative and the fact that the first partials are identically zero at the equilibrium points yields

$$\omega_1 \approx \frac{\partial F^*(S+\Delta S, s) / \partial S}{\Delta S}$$

$$\omega_2 \approx \frac{\partial F^*(S, s+\Delta s) / \partial S}{\Delta s}$$

Since these partials have already been calculated above, it is a simple matter (once these equations are programmed) to approximate the frequency of oscillation by using small values for Δs and ΔS .

III Computer Application

In implementing the theory described in Chapter II, four basic programs were developed. This chapter will describe those computer algorithms used for the determination of resonance values, plotting of the phase portrait, and the two methods of determination of the libration period about the stable equilibrium points. Listings of the computer algorithms employed are provided in Appendix A.

Crucial to the implementation of all the programs used in this investigation was the programming of the equations of motion developed in Chapter II. These equations were located in the subroutine FDF and, when passed the values of s and S along with the value of the resonance of interest, provided the values of the Hamiltonian and its two partial derivatives.

Resonance Value Determination

The initial step taken to ascertain the utility of each resonance structure for use in the placement of satellites was to find the location of the equilibrium points, both stable and unstable, and the width of the resonance structure. The program VALUE was written for this purpose.

Since the equilibrium values of s were readily determined from the equations of motion, finding the location of the equilibrium points required only that the value of S , for the zero point of the time derivative of s , be determined along these equilibrium values of s . To accomplish this goal, a binary search for the zero point of the time derivative was employed in the subroutine RESON. Once the equilibrium values for S had been determined, the location of the four equilibrium

points for each resonance structure would be known. To give more of a physical feel for their location, their radial distance from the nominal two-body geosynchronous orbit was also determined.

The width of the resonance structure was found by determining the roots of the Hamiltonian at the value of the unstable equilibrium points along the azimuths of the stable equilibrium values of s . Initially, the function values of the geopotential Hamiltonian for the stable and unstable equilibrium points were determined. Since the function value of the unstable equilibrium points defines the boundary of the stable region, the roots of these values along the azimuth of the stable equilibrium points describes the maximum width of the resonance. In the subroutine ROOTS, Newton's method

$$x_{i+1} = x_i - f(x_i)/f'(x_i)$$

for finding the roots of an equation was employed. Once the two roots were determined, the distance between them was found by converting back to distance units (DUs) and finally to meters.

The results of VALUE should serve to give an idea of the physical location of the equilibrium points along with a feeling as to whether the size of the resonance structure is adequate for satellite placement.

Plotting of Phase Portraits

Once the general characteristics of a given resonance structure are determined, the actual structure of the resonance may be plotted as a phase portrait. It should first be noted, that since the basic structure of each of the three resonance bands considered in this investigation is identical, the technique used, with some modifications, is the same for all three bands.

In order to generate the phase portraits, a contour following

subroutine, furnished by Dr. William E. Wiesel and modified slightly by the author, was used. This subroutine, CONTUR, which utilizes Newton's method for finding roots generalized to functions with two independent variables, was employed in conjunction with a plotting routine for the CALCOMP 1038 plotter, in the subroutine DRAW, in generating the phase portrait in the program RSPLIT.

Librational Periods

Each of the two methods described in Chapter II for determining the period of libration about the stable equilibrium points was implemented. In the application of the line integral approach in the program TIME, the subroutine DRAW was modified to create the subroutine PERIOD. This approach was used in order to take advantage of the already developed algorithm for generating the points for a given energy contour. As these points were generated, an elementary numerical integration technique was employed to determine the value of the integral. To simplify the calculation of the distance between successive points, an approximation was made to use the largest of the two step sizes of s or S . This method is actually an adaptation of the technique used in CONTUR to find the next point along the contour. Switching of which step size was used necessarily meant switching of which form of line integral was being used, but this switching was also easily performed using the existing structure of CONTUR. Once the value of the line integral was determined, the result, expressed in time units (TUs) was converted to mean solar days.

The second approach, using the method of linearization, proved as easy to implement as expected. In the program TIMES, values of the second partials of the Hamiltonian with respect to both s and S

$(\partial^2 F^* / \partial s^2 \text{ and } \partial^2 F^* / \partial S^2)$ were calculated numerically from the values of the first partials provided through the subroutine FDF. These values were then multiplied together and the square root taken to provide the frequency of oscillation. This frequency was then converted to a period in mean solar days.

IV Analysis

With the design of the computer algorithms completed, the remaining portion of this investigation now centers on an analysis of methods used and their validity and implications. Before turning to the actual examination of the results, it is first necessary to stipulate the assumptions made in their generation.

Assumptions

Although the theory, as developed in Chapter II, for the investigation of the geosynchronous spin-orbit resonances is quite general, certain assumptions had to be made during the computer application of this theory in order to obtain the results. None of these assumptions, however, will cause any significant changes in the conclusions to be drawn from this study. Throughout the development of the computer algorithms, an effort was made to permit these assumptions to easily and readily changed to allow further investigation of this subject.

Necessarily, a certain set of constants to be used for definition of the geopotential was required. These constants, as derived from the SAO Standard Earth III (Ref 13), are listed in Table I below.

TABLE I

SAO Standard Earth III Constants

$$J_2 = 1082.637 \times 10^{-6}$$

$$J_4 = -1.617999 \times 10^{-6}$$

$$J_{2,2} = 2.7438636 \times 10^{-6}$$

$$n_p = 5.86729371 \times 10^{-2} \text{ rad/TU}$$

$$R_e = 6.378140 \times 10^6 \text{ meters}$$

Another assumption involved the setting of the values of the two constant momenta, Q and R. In the development of the computer algorithms, it was assumed that it would be advantageous to initially specify some particular nominal eccentricity and inclination for investigation. Careful examination of the equations of motion developed in Chapter II along with their implementation for plotting of the phase portrait in Chapter III indicated that, as the satellite moves along a contour of constant energy, the requirement that Q and R are constant dictates that the eccentricity and inclination be constantly changing. Since these changes should be small, the values of Q and R were determined by using the nominal values of eccentricity and inclination of interest at the nominal two-body geosynchronous radius. Admittedly, this determination is somewhat arbitrary and was selected primarily for its ease of computation. Alternative methods of calculating the values used for Q and R, however, should cause minimal changes in the results.

Results

Resonance Values. With the specification of the necessary assumptions clearly stated, the first step in obtaining the results was to determine the locations of the equilibrium points and the width of the resonance structure for each resonance for a particular nominal eccentricity and inclination. Initially the structure was examined for zero eccentricity and inclination. Although the primary resonance band displays structure for these conditions, examination of the geopotential Hamiltonian for the remaining two bands indicates that no structure will result with either zero eccentricity or inclination. Table II lists the results of these conditions for the primary resonance (Resonance 1) but not for the remaining resonances since they will have zero width and are

of little practical interest. Positions of the equilibrium points are given relative to a nominal geosynchronous radius.

TABLE II

Primary Resonance Values -- Zero Eccentricity and Inclination

	<u>Stable</u>	<u>Unstable</u>	<u>Width</u>
Resonance 1	+ 2053.63 m	+ 2117.04 m	84513.86 m

Note that the effect of the nonspherical nature of the earth's figure is to move the location of both the stable and unstable equilibrium points out by a distance of approximately 2 kilometers from a nominal two-body geosynchronous orbit. This result is in good agreement with previous studies. More detailed results are contained in Appendix B.

To demonstrate the effect of a case of non-zero eccentricity and inclination of more practical interest for use with geosynchronous communications satellites, the approximate eccentricity and inclination for FltSatCom V (International Designation 1981 073A, NASA Catalog Number 12635) were used. This particular satellite was chosen as the geosynchronous satellite having both the largest eccentricity and inclination listed in the NASA Satellite Situation Report (Ref 24). Characteristics for the three resonance structures for $e = 0.025$ and $i = 0.11$ radians are listed in Table III.

TABLE III

Resonance Values -- Non-Zero Eccentricity and Inclination

	<u>Stable</u>	<u>Unstable</u>	<u>Width</u>
Resonance 1	+ 2012.19 m	+ 2075.03 m	84192.80 m
Resonance 2	- 12.87 m	- 13.14 m	246.04 m
Resonance 3	- 2069.46 m	- 2069.46 m	0.28 m

Some obvious conclusions may be drawn from these results. First, changing the eccentricity and inclination does have some effect on the location of the stable and unstable equilibrium points and the width of the resonance structure, although not very much. Second, with a width of over 84 kilometers, the primary resonance engulfs the equilibrium points of both the second and third resonances. And finally, for any reasonable eccentricity and inclination, the width of the third resonance structure seems to preclude its use for any practical satellite placement. As a result of this final conclusion, the third resonance band was dropped from further consideration.

Phase Portraits. Once the structural characteristics of the three resonance bands had been determined, the next step was to generate the phase portraits of these bands. It was decided that the only phase portrait generated would be that of the primary resonance. The reasoning behind this decision lay in the fact that the general structure of the two resonances being considered was basically identical, with only the scale being different. Since the scale has already been specified in Table II, it was felt to be unnecessary to generate more than one phase portrait. Since the secondary objective of this investigation was to demonstrate the feasibility of this technique as an investigative tool, the decision to investigate other eccentricities and inclinations or to generate additional phase portraits was left to future researchers interested in more specific cases.

The results of the generation of the phase portrait for the primary resonance with the values of eccentricity and inclination specified in use for generating Table II are illustrated in Figure 2. Due to restrictions imposed by the scaling of the phase portrait, a conformal

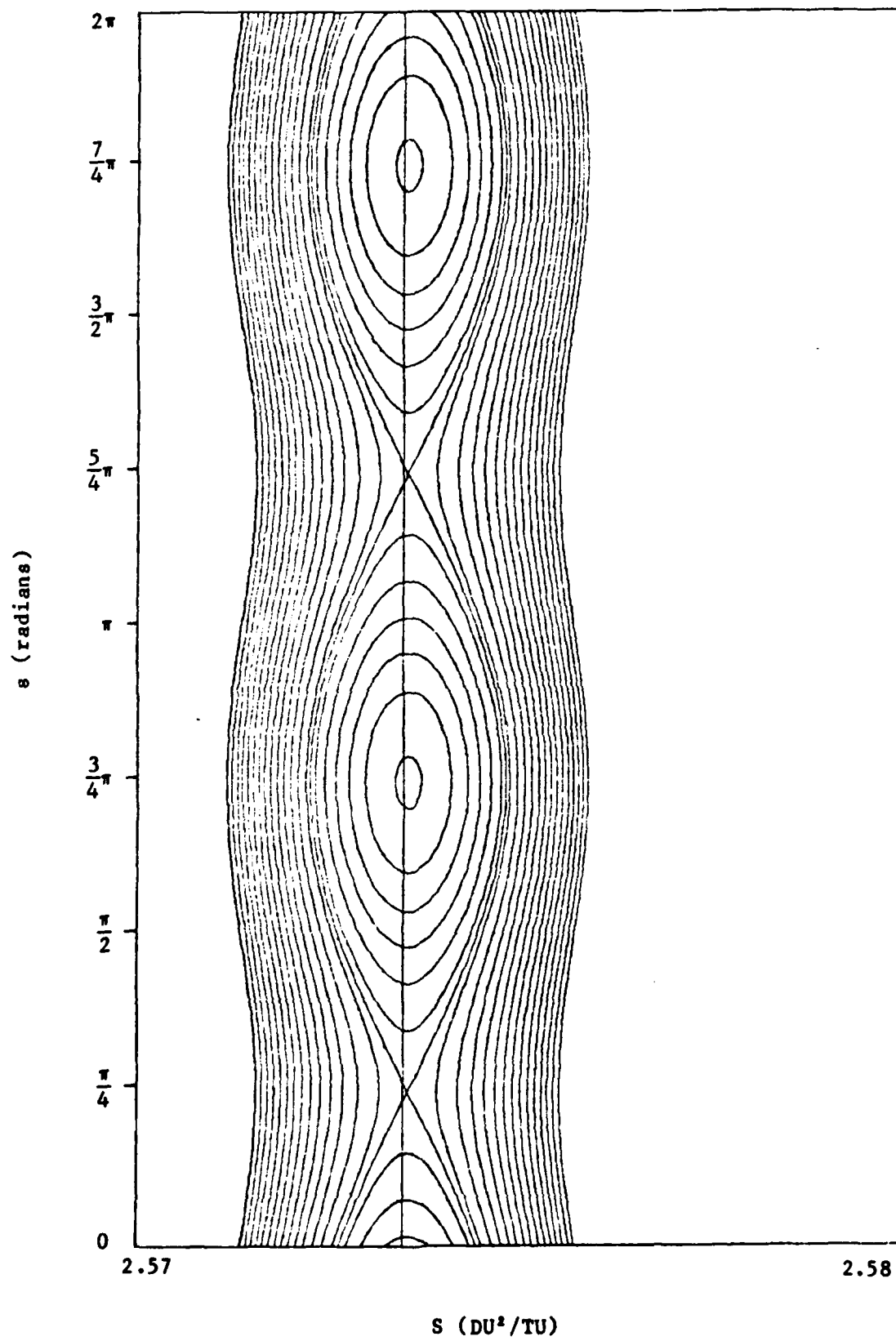


Figure 2. Primary Resonance Structure

mapping was used in the plotting of the results rather than plotting the results in polar coordinates. What appears to be a straight line slightly in from the stable and unstable equilibrium points is actually the secondary resonance's unstable equilibrium contours drawn to the same scale. The result gives a feel for the relative locations of the two resonance structures and their relative sizes. The single line width is an upper limit for the width of this structure on the scale of the primary resonance.

Librational Periods. To demonstrate the more involved, and presumably more precise method of determining the period of libration about the stable equilibrium points, the line integral method was applied to the primary resonance.

TABLE IV

Librational Periods -- Primary Resonance

Contour	Period	Contour	Period	Contour	Period
1	670.94 days	9	440.81 days 440.25 days	15	293.33 days 292.77 days
2	703.93 days	10	398.96 days 398.41 days	16	281.00 days 280.44 days
3	747.38 days	11	368.08 days 367.53 days	17	270.13 days 269.58 days
4	806.99 days	12	343.85 days 343.30 days	18	260.47 days 259.91 days
5	900.46 days	13	324.09 days 323.53 days	19	251.79 days 251.24 days
6	1109.36 days	14	307.51 days 306.96 days	20	243.94 days 243.38 days
7	638.64 days 638.08 days				
8	504.43 days 503.88 days				

Table IV lists the librational period for each of the energy contours shown in Figure 2, working from the inside of the resonance structure

out (this, of course, excludes the unstable equilibrium contour which would have an infinite librational period). Use of the line integral method allows a means for determining the specific librational period associated with a given energy or displacement from the stable equilibrium points. It should be noted that decreasing the step size for the integration as a means of verification produced minor changes (less than one day) in the resulting periods.

Occurrence of double entries in Table IV are the result of the existence of dual trajectories for a given energy level once outside of the stable equilibrium region. The librational period of each of the two contours of equal energy is listed, with the outer of the two bands listed first. These periods may be more appropriately considered as a type of synodic period arising from the relative drift of satellites in different orbits.

For Contours 1 through 6, which are within the stable equilibrium region, the basic period of libration is within the range of 700 to 900 days often seen cited in the literature (Ref 1,4,5,6,7,14,16,18). As it turns out, the period of libration drops off quickly as a satellite librates closer to the stable point and approaches infinity as it nears the boundary of the stable region. This result is expected from the theory of unstable point analysis.

Turning to the linearization method for determining the period of libration yields the comparable result of 667.10 days. Since this result is in good agreement with the existing literature and the results of the previous method and is much less computationally intensive to obtain, this method was used to obtain the period of libration for the secondary resonance's stable region. It should be clear that it is

unnecessary to obtain precise periods of libration for either resonance band for the purposes of this study, that is, to determine whether other practical stable equilibrium regions exist for the placement of communications satellites. The result of the determination of the libration period for the secondary resonance band is 228,289.46 days or approximately 625 years.

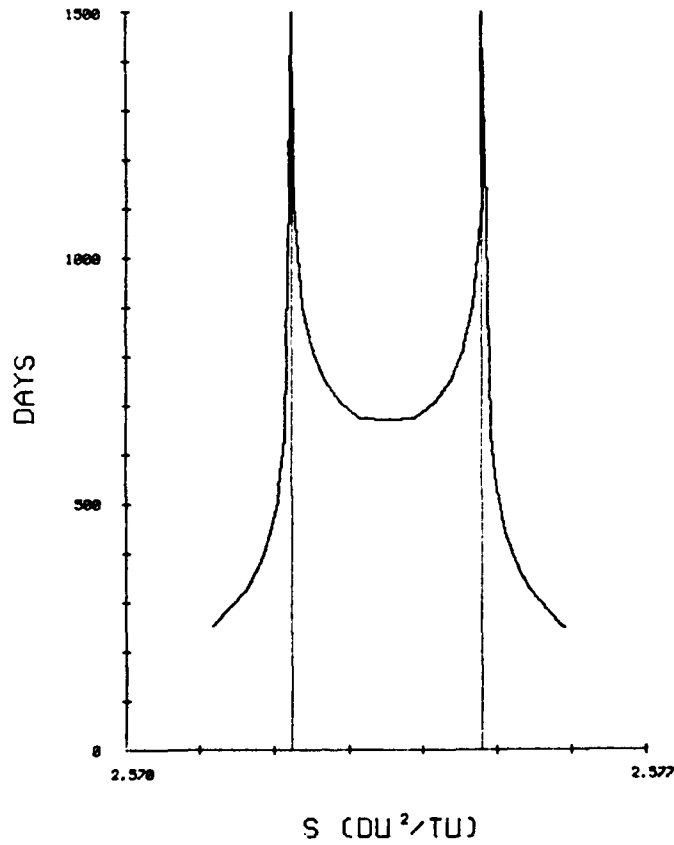


Figure 3. Librational Period -- Primary Resonance

Combining the values listed in Table IV with the value obtained through the linearization method, Figure 3 illustrates how the librational period varies with changes in the generalized momentum S for a

fixed value of the generalized coordinate s corresponding to that of either of the stable equilibrium points. A minimum oscillatory period of 667.10 days occurs at the stable points. Moving radially away from these points results in an rapid increase in the librational period as the unstable contour is approached, with the period asymptotically approaching infinity. Once outside the stable region indicated by the two vertical lines, the period of oscillation quickly drops off, reflecting the relative drift of satellites in different orbits.

It should be noted at this point that the assumption made in Chapter II on the development of this method was verified numerically. That is, the cross partial terms did indeed prove to be negligible, being at least 10 orders of magnitude smaller than the other term in each equation.

Conclusions

Although some of the conclusions to be drawn from this study have already been noted, there remain several more to be elucidated. In regard to the first objective, it has been demonstrated that additional stable equilibrium points do exist in the geosynchronous regime. The two being used in the primary resonance band have been well studied and this investigation serves merely to reinforce previous conclusions. More significantly, however, are the results pertaining to the other two resonance bands. Although the third resonance band appears to be too small to be of any practical use in the cases considered, the second resonance band appears to be more than adequate in this regard.

The second resonance band is not only wide enough for placement of communications satellites, its period of libration is sufficiently long enough not to cause significant problems for operational use. The only

tracking problem for ground stations for these orbits would result from the daily oscillation of the orbit due to its inclination.

The stable points of the second resonance band and their span are more than adequately separated spatially from those of the primary resonance, and, it appears, can be separated in longitude by adjusting the argument of perigee. If this separation is indeed possible, many equilibrium points exist, permitting placement of many satellites at the stable points within this resonance region and allowing for a reduction in station-keeping. One problem, however, may prevent their practical use.

This limitation involves an issue not directly addressed in this investigation. Since the secondary resonance band lies entirely within the much broader primary resonance band, in order to determine whether or not these new stable points are of any practical value would require an analysis of the combined effects of the primary and secondary resonance structures and the amount of station-keeping necessary to maintain positioning within the secondary resonance's stable region.

Turning to the secondary objective, this investigation should have adequately demonstrated the feasibility of this general technique as a primary investigative tool for the study of resonance structures. Anyone familiar with the analytical methods required for an investigation of this magnitude will appreciate the relative simplicity of this numerical approach to the problem. It is only hoped that this method will be useful for further research in this area.

Recommendations

The primary avenues for further research on this topic center on the limitations listed in the Conclusions section above. It is the

author's belief that the major thrust of future research on this topic should involve developing a technique to analyze the stability of the secondary resonance's stable points in conjunction with the primary resonance's effects in determining the station-keeping requirements necessary to maintain a satellite at one of these points.

Should it prove feasible to maintain such orbits, further consideration must necessarily be given to the problem of directional separation. It may turn out that the additional capacity for satellite placement in these orbits along with communications frequency separation will provide for expanded use of the geosynchronous orbit. And finally, it may be desirable to determine the effect of the inclusion of additional zonal and tesseral harmonic terms on the conclusions of this and future studies. Although such inclusions should not drastically alter the overall structures, they may affect the stability of these structures.

One further recommendation should also be made in regard to the methods used for analytical techniques. This author found the recently developed forms of computer algebra programs of immense help in simplifying the tedious expansions and developing the partial derivatives in Chapter II. The use of these programs, which are capable of doing algebra, trigonometric substitutions and simplifications, differentiation, and integration, should prove extremely valuable in pushing forward the analytical methods without increasing the amount of human effort required.

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Appendix A

Computer Programs and Subroutines

The following is a listing of the computer programs used in the process of this investigation. The four main programs are listed first followed by the major subroutines employed.

```

PROGRAM VALUE
DOUBLE PRECISION J2, J4, J22, NO, RAD, SR, ECC, INC, X, Y, Z, PI,
1ST1, UNST1, RCS, RESMN, RCU, RESMX, DF, ROOT1, ROOT2
INTEGER RES
COMMON/SAO/J2, J4, J22, NO, RAD, SR, Y, Z, RES
ECC = 2.5D-2
INC = 1.1D-1
C *****
C *** SAO III CONSTANTS ***
C *****
J2 = 1082.637D-6
J4 = -1.617999D-6
J22 = 2.7438636D-6
NO = 5.86729371D-2
RAD = 6.378140D6
SR = (1D0/NO)**(2D0/3D0)
PRINT ('("NOMINAL RADIUS",F30.26," DU"/)'), SR
X = DSQRT(SR)
Y = X*(1D0 - DSQRT(1D0 - ECC**2))
Z = X*DSQRT(1D0 - ECC**2)*(1D0 - DCOS(INC))
PI = 4D0*DATAN(1D0)
ST1 = 3D0*PI/4D0
UNST1 = PI/4D0
C *****
C *** MAIN PROGRAM ***
C *****
DO 1 RES=1,3
PRINT ('(" RESONANCE ",I1/)'), RES
PRINT ('(" STABLE")')
CALL RESON(RCS,ST1,RESMN)
PRINT ('(" UNSTABLE")')
CALL RESON(RCU,UNST1,RESMX)
DF = RESMX - RESMN
CALL ROOTS(RCS,ST1,DF,ROOT1,ROOT2,RESMN)
PRINT ('(" ROOT 1 =",F30.26)'), ROOT1
PRINT ('(" ROOT 2 =",F30.26)'), ROOT2
PRINT ('(" WIDTH =",F20.5," METERS)'), (ROOT2**2 - ROOT1**2)*RAD
1 PRINT ('(" DF =",D34.26/)'), DF
STOP
END

```

The job control for program RSPLLOT is included due to the machine dependent nature of the plotting routines used. Also note that the resonance of interest and scale factors must be specified in the FACTORS block.

TSK,T200. T820730,KELSO,4289,10/26/82
ATTACH,CCPLOT,CCPLOT1038,ID=LIBRARY,SN=ASD.
LIBRARY,CCPLOT.
FTN5.
LGO.
ROUTE,TAPE1,DC=PU,TID=AF,ST=CSA,FID=TSK.
*EOR

```
PROGRAM RSPLLOT
DOUBLE PRECISION J2, J4, J22, NO, SR, X, Y, Z, ECC, INC, PI, ST1, ST2,
1UNST1, ZERO, DF, DFS, DFM, RCS, RESMN, RCU, RESMX, DEL, ROOT1, ROOT2
REAL XMIN, XMAX, YMIN, YMAX, XSCALE, YSCALE
INTEGER RES
COMMON/SAO/J2, J4, J22, NO, SR, Y, Z, RES
COMMON/SIZE/XMIN, XMAX, YMIN, YMAX, XSCALE, YSCALE
CALL PLOTS(0,0,1)
ECC = 2.5D-2
INC = 1.1D-1
```

```
C *****
C *** SAO III CONSTANTS ***
C *****
      J2 = 1082.637D-6
      J4 = -1.617999D-6
      J22 = 2.7438636D-6
      NO = 5.86729371D-2
      SR = (1D0/NO)**(2D0/3D0)
      X = DSQRT(SR)
      Y = X*(1D0 - DSQRT(1D0 - ECC**2))
      Z = X*DSQRT(1D0 - ECC**2)*(1D0 - DCOS(INC))
      PI = 4D0*DATAN(1D0)
      ST1 = 3D0*PI/4D0
      ST2 = 7D0*PI/4D0
      UNST1 = PI/4D0
      ZERO = 0D0
```

```
C *****
C *** FACTORS - RESONANCE N ***
C *****
C      RES = N
C      XMIN = *****
C      XMAX = *****
C      DF = *****
C      DFS = *****
C      DFM = *****
```

RSPLIT (Continued)

```
C *****
C *** MAIN PROGRAM ***
C *****
  YMIN = 0.0
  YMAX = 2.0*SNGL(PI)
  XSCALE = (XMAX - XMIN)/5.0
  YSCALE = (YMAX - YMIN)/8.0
  CALL BOX
  CALL RESON(RCS,ST1,RESMN)
  CALL RESON(RCU,UNST1,RESMX)
  DEL = PI/1D2
10 IF (DF .LT. DFM) THEN
  CALL ROOTS(RCS,ZERO,DF,ROOT1,ROOT2,RESMN)
  CALL DRAW(ROOT1,ZERO,DEL)
  CALL DRAW(ROOT2,ZERO,DEL)
  IF (RESMN + DF .LT. RESMX) THEN
    CALL ROOTS(RCS,ST1,DF,ROOT1,ROOT2,RESMN)
    CALL DRAW(ROOT2,ST1,DEL)
    CALL ROOTS(RCS,ST2,DF,ROOT1,ROOT2,RESMN)
    CALL DRAW(ROOT2,ST2,DEL)
    CALL DRAW(ROOT2,ST2,-DEL)
  END IF
  DF = DF + DFS
  GO TO 10
END IF
DF = RESMX - RESMN
CALL ROOTS(RCS,ZERO,DF,ROOT1,ROOT2,RESMN)
CALL DRAW(ROOT1,ZERO,DEL)
CALL DRAW(ROOT2,ZERO,DEL)
CALL ROOTS(RCS,ST1,DF,ROOT1,ROOT2,RESMN)
CALL DRAW(ROOT1,ST1,DEL)
CALL DRAW(ROOT1,ST1,-DEL)
CALL DRAW(ROOT2,ST1,DEL)
CALL DRAW(ROOT2,ST1,-DEL)
CALL ROOTS(RCS,ST2,DF,ROOT1,ROOT2,RESMN)
CALL DRAW(ROOT1,ST2,DEL)
CALL DRAW(ROOT1,ST2,-DEL)
CALL DRAW(ROOT2,ST2,DEL)
CALL DRAW(ROOT2,ST2,-DEL)
CALL PLOTE(0)
STOP
END
```

```

PROGRAM TIME
DOUBLE PRECISION J2, J4, J22, NO, SR, ECC, INC, X, Y, Z, PI, ST1, UNST1, RCS,
1RESMN, RCU, RESMX, DF, DFS, DFM, DEL, ROOT1, ROOT2
INTEGER RES
COMMON/SAO/J2, J4, J22, NO, SR, Y, Z, RES
ECC = 2.5D-2
INC = 1.1D-1
C *****
C *** SAO III CONSTANTS ***
C *****
      J2 = 1082.637D-6
      J4 = -1.617999D-6
      J22 = 2.7438636D-6
      NO = 5.86729371D-2
      SR = (1D0/NO)**(2D0/3D0)
      X = DSQRT(SR)
      Y = X*(1D0 - DSQRT(1D0 - ECC**2))
      Z = X*DSQRT(1D0 - ECC**2)*(1D0 - DCOS(INC))
      PI = 4D0*DATAN(1D0)
      ST1 = 3D0*PI/4D0
      UNST1 = PI/4D0
C *****
C *** FACTORS - RESONANCE N ***
C *****
      RES = N
      DF = *****
      DFS = *****
      DFM = *****
C *****
C *** MAIN PROGRAM ***
C *****
      CALL RESON(RCS, ST1, RESMN)
      CALL RESON(RCU, UNST1, RESMX)
      DEL = PI/1D3
10 IF (DF .LT. DFM) THEN
      CALL ROOTS(RCS, ST1, DF, ROOT1, ROOT2, RESMN)
      CALL PERIOD(ROOT2, ST1, DEL)
      IF (DF .GT. RESMX-RESMN) CALL PERIOD(ROOT1, ST1, DEL)
      PRINT '("0")'
      DF = DF + DFS
      GO TO 10
END IF
STOP
END

```

```

PROGRAM TIMES
DOUBLE PRECISION J2, J4, J22, NO, SR, X, Y, Z, ECC, INC, PI, ST1,
1RCS, RESMN, XX, YY, DX, DY, DXX, DYY, W1, W2, SID, CON, PER
INTEGER RES
COMMON/SAO/J2, J4, J22, NO, SR, Y, Z, RES
ECC = 2.5D-2
INC = 1.1D-1

```

```

C *****
C *** SAO III CONSTANTS ***
C *****
  J2 = 1082.637D-6
  J4 = -1.617999D-6
  J22 = 2.7438636D-6
  NO = 5.86729371D-2
  SR = (1D0/NO)**(2D0/3D0)
  X = DSQRT(SR)
  Y = X*(1D0 - DSQRT(1D0 - ECC**2))
  Z = X*DSQRT(1D0 - ECC**2)*(1D0 - DCOS(INC))
  PI = 4D0*DATAN(1D0)
  SID = 1.00273790931D0
  CON = NO/SID
  ST1 = 3D0*PI/4D0
C *****
C *** MAIN PROGRAM ***
C *****
C   RES = N
  CALL RESON(RCS, ST1, RESMN)
  DXX = 1D-9
  DYY = 1D-9
  XX = RCS + DXX
  YY = ST1 + DYY
  CALL FDF(RCS, YY, F, DX, DY)
  W1 = DSQRT(DABS(DY/DYY))
  CALL FDF(XX, ST1, F, DX, DY)
  W2 = DSQRT(DABS(DX/DXX))
  W = W1 * W2
  PER = CON/W
  PRINT '(" PERIOD =", F10.2, " DAYS")', PER
  STOP
  END

```

Below is a sample FACTORS block, the one used for this study.

```

C *****
C *** FACTORS - RESONANCE 1 ***
C *****
  RES = 1
  XMIN = 2.57
  XMAX = 2.58
  DF = 5D-9
  DFS = 1D-8
  DFM = 2D-7

```

The remainder of Appendix A is a listing of major subroutines.

```
SUBROUTINE BOX
REAL P(1:7),Q(1:7),XMIN,XMAX,YMIN,YMAX,XSCALE,YSCALE
COMMON/SIZE/XMIN,XMAX,YMIN,YMAX,XSCALE,YSCALE
CALL PLOT(0.0,-0.5,-3)
CALL PLOT(0.0,1.5,-3)
P(1) = XMIN
P(2) = XMIN
P(3) = XMAX
P(4) = XMAX
P(5) = XMIN
P(6) = XMIN
P(7) = XSCALE
Q(1) = YMIN
Q(2) = YMAX
Q(3) = YMAX
Q(4) = YMIN
Q(5) = YMIN
Q(6) = YMIN
Q(7) = YSCALE
CALL LINE(P,Q,5,1,0,0)
RETURN
END
```

C
C

```
SUBROUTINE RESON(S,SY,F)
DOUBLE PRECISION S,SY,F,J2,J4,J22,NO,RAD,SR,Y,Z,DS,DX,DY
INTEGER RES
COMMON/SAO/J2,J4,J22,NO,RAD,SR,Y,Z,RES
S = 2D0
DS = 2D0
30 IF (DS .GT. 1D-28) THEN
    DS = DS/2D0
    CALL FDF(S,SY,F,DX,DY)
    S = S - DS * DSIGN(1D0,DX)
    GO TO 30
END IF
PRINT ('" RADIUS =",F30.26," DU")', S**2
PRINT ('" DISTANCE FROM NOMINAL =",F12.5," METERS")',
1 (S**2 - SR)*RAD
PRINT ('" S =",F30.26/)', S
RETURN
END
```

The above listing of RESON was used only in VALUE. The following listing was used in all other programs.

```
SUBROUTINE RESON(S,SY,F)
DOUBLE PRECISION S,SY,F,DS,DX,DY
S = 2D0
DS = 2D0
30 IF (DS .GT. 1D-28) THEN
    DS = DS/2D0
    CALL FDF(S,SY,F,DX,DY)
    S = S - DS * DSIGN(1D0,DX)
    GO TO 30
END IF
RETURN
END
```

C
C

```
SUBROUTINE ROOTS(X,Y,DF,ROOT1,ROOT2,FMIN)
DOUBLE PRECISION X,Y,DF,ROOT1,ROOT2,FMIN,F,FN,F1,F2,DX,DY
F = FMIN + DF
CALL FDF(X,Y,FN,DX,DY)
IF (FN .GT. F) THEN
    ROOT1 = 0D0
    ROOT2 = 0D0
ELSE
    ROOT1 = 2D0
41 CALL FDF(ROOT1,Y,F1,DX,DY)
    ROOT1 = ROOT1 - (F1-F)/DX
    IF (DABS(F-F1) .GT. 1D-28) GO TO 41
    ROOT2 = 3D0
42 CALL FDF(ROOT2,Y,F2,DX,DY)
    ROOT2 = ROOT2 - (F2-F)/DX
    IF (DABS(F-F2) .GT. 1D-28) GO TO 42
END IF
RETURN
END
```

```

SUBROUTINE DRAW(ROOT,SY,DLT)
DOUBLE PRECISION ROOT,SY,DLT,PI2,DEL,X,Y,XN,YN,F
REAL P(1:2500),Q(1:2500),XMIN,XMAX,YMIN,YMAX,XSCALE,YSCALE
INTEGER N,IOK,ICAS
COMMON/SIZE/XMIN,XMAX,YMIN,YMAX,XSCALE,YSCALE
IF (ROOT .EQ. 0D0) RETURN
PI2 = 8D0*DATAN(1D0)
DEL = DLT
N = 1
P(N) = SNGL(ROOT)
Q(N) = SNGL(SY)
ICAS = 0
CALL CONTUR(ROOT,SY,DEL,XN,YN,IOK,ICAS,F)
X = ROOT
51 IF (IOK .NE. 0) THEN
    IF (YN .GT. PI2+DEL/2D0 .OR. YN .LT. 0D0) GO TO 52
    N = N + 1
    P(N) = SNGL(XN)
    Q(N) = SNGL(YN)
    IF (DABS(YN-SY) .LT. DABS(DEL/2D0) .AND. X .LT. XN) GO TO 52
    X = XN
    Y = YN
    CALL CONTUR(X,Y,DEL,XN,YN,IOK,ICAS,F)
    GO TO 51
END IF
IF (N .EQ. 1) RETURN
52 P(N+1) = XMIN
P(N+2) = XSCALE
Q(N+1) = YMIN
Q(N+2) = YSCALE
CALL LINE(P,Q,N,1,0,0)
RETURN
END

```

```

SUBROUTINE PERIOD(ROOT,SY,DLT)
DOUBLE PRECISION ROOT,SY,DLT,J2,J4,J22,N0,SR,Y,Z,PI2,SID,CONV,
1DEL,XO,YO,XN,YN,F,DX,DY,PER
COMMON/SAO/J2,J4,J22,N0,SR,Y,Z,RES
INTEGER RES,IOK,ICAS
PI2 = 8D0*DATAN(1D0)
SID = 1.00273790931D0
CONV = N0/PI2/SID
DEL = DLT
XO = ROOT
YO = SY
PER = 0D0
ICAS = 0
CALL CONTUR(XO,YO,DEL,XN,YN,IOK,ICAS,F)
51 IF (IOK .NE. 0) THEN
  IF (YN .GT. PI2+DEL/2D0) GO TO 52
  CALL FDF(XN,YN,F,DX,DY)
  IF (ICAS .EQ. 1) THEN
    PER = PER + DABS((YN-YO)/DX)
  ELSE
    PER = PER + DABS((XN-XO)/DY)
  END IF
  IF (DABS(YN-SY) .LT. DABS(DEL/2D0) .AND. XO .LT. XN) GO TO 52
  XO = XN
  YO = YN
  CALL CONTUR(XO,YO,DEL,XN,YN,IOK,ICAS,F)
  GO TO 51
END IF
52 PRINT (' PERIOD =",F9.2," DAYS"', PER*CONV)
RETURN
END

```

```

SUBROUTINE CONTUR(X, Y, DEL, XNEW, YNEW, IOK, ICAS, FF)
DOUBLE PRECISION X, Y, DEL, XNEW, YNEW, FF, TOLXY, TOLF, FDG, F, DFDX, DFDY,
1DX, DY, DFPDX, DFPDY, RES, FP, ERRF1, ERRF2, ERRF, ERRXY1, ERRXY2, ERRXY
INTEGER NL, IOK, ICAS, I
TOLXY = 1D-22
TOLF = 1D-28
FDG = 5D2
IOK = 1
NL = 50
CALL FDF(X, Y, F, DFDX, DFDY)
IF (ICAS .EQ. 0) FF = F
IF (DABS(DFDY) .GT. DABS(DFDX/FDG)) THEN

```

```

C *****

```

```

C *** SHALLOW SLOPE ***

```

```

C *****

```

```

IF (ICAS .EQ. 0) ICAS = -1
IF (ICAS .EQ. 1) THEN
DEL = -DEL * DSIGN(1D0, DFDX) * DSIGN(1D0, DFDY)
ICAS = -1

```

```

END IF

```

```

XNEW = X + DEL/FDG

```

```

YNEW = Y

```

```

DY = -DFDX*DEL/DFDY/FDG

```

```

DO 61 I = 1, NL

```

```

YNEW = YNEW + DY

```

```

CALL FDF(XNEW, YNEW, FP, DFPDX, DFPDY)

```

```

RES = FP - FF

```

```

DY = -RES/DFPDY

```

```

ERRF1 = DABS(RES)

```

```

ERRF2 = DABS(ERRF1/FF)

```

```

ERRF = DMIN1(ERRF1, ERRF2)

```

```

ERRXY1 = DABS(DY)

```

```

ERRXY2 = 1D0

```

```

IF (YNEW .NE. 0D0) ERRXY2 = DABS(DY/YNEW)

```

```

ERRXY = DMIN1(ERRXY1, ERRXY2)

```

```

61 IF (ERRXY .LT. TOLXY .AND. ERRF .LT. TOLF) GO TO 63

```

CONTUR (Continued)

```
C *****
C *** STEEP SLOPE ***
C *****
      ELSE
        IF (ICAS .EQ. 0) ICAS = 1
        IF (ICAS .EQ. -1) THEN
          DEL = -DEL * DSIGN(1D0,DFDX) * DSIGN(1D0,DFDY)
          ICAS = 1
        END IF
        XNEW = X
        YNEW = Y + DEL
        DX = -DFDY*DEL/DFDX
        DO 62 I = 1,NL
          XNEW = XNEW + DX
          CALL FDF(XNEW,YNEW,FP,DFPDX,DFPDY)
          RES = FP - FF
          DX = -RES/DFPDX
          ERRF1 = DABS(RES)
          ERRF2 = DABS(ERRF1/FF)
          ERRF = DMIN1(ERRF1,ERRF2)
          ERRXY1 = DABS(DX)
          ERRXY2 = DABS(DX/XNEW)
          ERRXY = DMIN1(ERRXY1,ERRXY2)
        62 IF (ERRXY .LT. TOLXY .AND. ERRF .LT. TOLF) GO TO 63
        END IF
        IOK = 0
        63 RETURN
      END
```

The COMMON and DOUBLE PRECISION blocks of FDF should contain the variable RAD when run with program VALUE.

```

SUBROUTINE FDF(X,XX,F,DFDX,DFDXX)
DOUBLE PRECISION X,XX,F,DFDX,DFDXX,J2,J4,J22,NO,RAD,SR,Y,Z,Q,R,S,
1A,B,C,F0,F1,F2,F4,R22,DFDX0,DFDX1,DFDX2,DFDX4,DFDX22
INTEGER RES
COMMON/SAO/J2,J4,J22,NO,RAD,SR,Y,Z,RES
Q = Y - DBLE(RES-1)*DSQRT(SR)
R = Z + DBLE(RES-1)*DSQRT(SR)
S = X
GO TO (71,72,73) RES

```

```

C *****
C *** 1ST RESONANCE ***
C *****
  71 A = S/(S - Q)
     B = R/(S - Q)
     C = Q/S
C *****
C *** J22 TERM ***
C *****
  R22 = J22/(X**6)*(4D0 - 4D0*B + B**2)*(288D0 - 144D1*C
1+ 1656D0*C**2 - 1216D0*C**3 + 654D0*C**4 - 21D1*C**5 + 35D0*C**6)
2*DS IN(2D0*XX)/384D0
C *****
C *** MOMENTUM DERIVATIVE ***
C *****
  DFDX1 = J2/(X**7)*(A**3*(- 6D0 + 18D0*B - 9D0*B**2) + A**4*(- 6D0
1+ 24D0*B - 15D0*B**2))/4D0
  DFDX2 = J2**2/(X**11)*(A**5*(12D1 - 24D1*B - 18D1*B**2
1+ 3D2*B**3 - 75*B**4) + A**6*(- 72D0 + 864D0*B - 2556D0*B**2
2+ 2208D0*B**3 - 567D0*B**4) + A**7*(- 648D0 + 3456D0*B
3- 6588D0*B**2 + 5148D0*B**3 - 1395D0*B**4) + A**8*(- 84D1
4+ 384D1*B - 594D1*B**2 + 42D2*B**3 - 1155*B**4))/128D0
  DFDX4 = 3D0*J4/(X**11)*((A**5 - A**7)*(- 12D1 + 12D2*B
1- 27D2*B**2 + 21D2*B**3 - 525D0*B**4) + A**6*(- 12D1
2+ 144D1*B - 378D1*B**2 + 336D1*B**3 - 945D0*B**4) + A**8*(28D1
3- 32D2*B + 81D2*B**2 - 7D3*B**3 + 1925D0*B**4))/128D0
  DFDX22 = 3D0*J22/(X**7)*((4D0 - 4D0*B + B**2)*(- 288D0
1+ 168D1*C - 2208D0*C**2 + 1824D0*C**3 - 109D1*C**4 + 385D0*C**5
2- 7D1*C**6)/48D0 + A*(4D0*B - 2D0*B**2)*(288D0 - 144D1*C
3+ 1656D0*C**2 - 1216D0*C**3 + 654D0*C**4 - 21D1*C**5 + 35D0*C**6)
4/288D0)*DS IN(2D0*XX)/4D0
C *****
C *** POSITION DERIVATIVE ***
C *****
  DFDXX = J22/X**6*(4D0 - 4D0*B + B**2)*(288D0
1- 144D1*C + 1656D0*C**2 - 1216D0*C**3 + 654D0*C**4 - 21D1*C**5
2+ 35D0*C**6)*DCOS(2D0*XX)/192D0
GO TO 74

```

FDF (Continued)

```

C *****
C *** 2ND RESONANCE ***
C *****
  72 A = -S/Q
      B = (S - R)/Q
      C = (Q + S)/S
C *****
C *** J22 TERM ***
C *****
R22 = J22/(X**6)*(3D0/128D0*(2D0*B - B**2)*(288D0*C + 304D0*C**2
1+ 68D1*C**3 - 158D1*C**4 + 846D0*C**5 - 141D0*C**6))*DSIN(2D0*XX)
C *****
C *** MOMENTUM DERIVATIVE ***
C *****
DFDX1 = J2/(X**7)*(A**3*(- 6D0 + 18D0*B - 9D0*B**2) + A**4*(6D0
1- 6D0*B))/4D0
DFDX2 = J2**2/(X**11)*(A**5*(12D1 - 24D1*B - 18D1*B**2
1+ 3D2*B**3 - 75D0*B**4) + A**6*(- 24D1 + 108D1*B - 2124D0*B**2
2+ 1668D0*B**3 - 432D0*B**4) + A**7*(- 72D0 + 288D0*B - 684D0*B**2
3+ 828D0*B**3 - 315D0*B**4) + A**8*(48D1 - 132D1*B + 126D1*B**2
4- 42D1*B**3))/128D0
DFDX4 = 3D0*J4/(X**11)*((A**5 - A**7)*(- 12D1 + 12D2*B
1- 27D2*B**2 + 21D2*B**3 - 525D0*B**4)/16D0 + (3D0*A**6
2- 5D0*A**8)*(6D1 - 27D1*B + 315D0*B**2 - 105D0*B**3)/12D0)/8D0
DFDX22 = 3D0*J22/(X**7)*((2D0*B - B**2)*(144D0 - 704D0*C
1-196D0*C**2 - 622D1*C**3 + 10015D0*C**4 - 5076D0*C**5
2+ 846D0*C**6)/64D0 - A*(2D0 - 2D0*B)*(288D0*C + 304D0*C**2
3+ 68D1*C**3 - 158D1*C**4 + 846D0*C**5 - 141D0*C**6)/128D0)
4*DSIN(2D0*XX)
C *****
C *** POSITION DERIVATIVE ***
C *****
DFDXX = J22/(X**6)*(3D0/64D0*(2D0*B - B**2)*(288D0*C + 304D0*C**2
1+ 68D1*C**3 - 158D1*C**4 + 846D0*C**5 - 141D0*C**6))*DCOS(2D0*XX)
GO TO 74

```

FDF (Continued)

```

C *****
C *** 3RD RESONANCE ***
C *****
  73 A = -S/(S + Q)
      B = (2D0*S - R)/(S + Q)
      C = (2D0*S + Q)/S
C *****
C *** J22 TERM ***
C *****
  R22 = J22/(X**6)*B**2*(4D1*C**2 + 16D0*C**3 - 74D0*C**4
1+ 42D0*C**5 - 7D0*C**6)*DS IN(2D0*XX)/32D1
C *****
C *** MOMENTUM DERIVATIVE ***
C *****
  DFDX1 = J2/(X**7)*(A**3*(- 6D0 + 18D0*B - 9D0*B**2) + A**4*(18D0
1- 36D0*B + 15D0*B**2))/4D0
  DFDX2 = J2**2/(X**11)*(A**5*(12D1 - 24D1*B - 18D1*B**2
1+ 3D2*B**3 - 75D0*B**4) + A**6*(- 408D0 + 1296D0*B - 1692D0*B**2
2+ 1128D0*B**3 - 297D0*B**4) + A**7*(504D0 - 288D1*B + 522D1*B**2
3- 3492D0*B**3 + 765D0*B**4) + A**8*(18D2 - 648D1*B
4+ 846D1*B**2 - 504D1*B**3 + 1155D0*B**4))/128D0
  DFDX4 = 3D0*J4/(X**11)*((A**5 - A**7)*(- 12D1 + 12D2*B
1- 27D2*B**2 + 21D2*B**3 - 525D0*B**4) + A**6*(6D2 - 36D2*B
2+ 63D2*B**2 - 42D2*B**3 + 945D0*B**4) + A**8*(- 108D1
3+ 68D2*B - 123D2*B**2 + 84D2*B**3 - 1925D0*B**4))/128D0
  DFDX22 = 3D0*J22/(X**7)*(B**2*(8D1*C - 112D0*C**2 - 368D0*C**3
1+ 58D1*C**4 - 273D0*C**5 + 42D0*C**6) - A*(2D0*B - B**2)
2*(4D1*C**2 + 16D0*C**3 - 74D0*C**4 + 42D0*C**5 - 7D0*C**6))
3*DS IN(2D0*XX)/48D1
C *****
C *** POSITION DERIVATIVE ***
C *****
  DFDXX = J22/(X**6)*B**2*(4D1*C**2 + 16D0*C**3 - 74D0*C**4
1+ 42D0*C**5 - 7D0*C**6)*DCOS(2D0*XX)/16D1
C *****
C *** HAMILTONIAN ***
C *****
  74 F0 = 1D0/(X**2)/2D0
      F1 = J2/(X**6)*A**3*(2D0 - 6D0*B + 3D0*B**2)/4D0
      F2 = J2**2/(X**10)*(3D0/32D0*A**5*(- 8D0 + 16D0*B + 12D0*B**2
1- 2D1*B**3 + 5D0*B**4) + 3D0/8D0*A**6*(4D0 - 24D0*B + 48D0*B**2
2- 36D0*B**3 + 9D0*B**4) - 15D0/32D0*A**7*(- 8D0 + 32D0*B
3- 44D0*B**2 + 28D0*B**3 - 7D0*B**4))/4D0
      F4 = J4/(X**10)*(9D0*A**5 - 15D0*A**7)*(8D0 - 8D1*B
1+ 18D1*B**2 - 14D1*B**3 + 35D0*B**4)/128D0
      F = F0 + F1 + F2 + F4 + R22 + NO*X
      DFDX0 = - 1D0/X**3
      DFDX = DFDX0 + DFDX1 + DFDX2 + DFDX4 + DFDX22 + NO
      RETURN
      END

```

Appendix B

Computer Output

Appendix B contains a summary of the output generated for the program VALUE. All other relevant computer output is summarized within the text of the thesis. The first set of values, for Resonance 1 only, was computed for zero eccentricity and inclination. All other calculations assume $e = 0.25$ and $i = 0.11$ radians.

NOMINAL RADIUS 6.62279705974354939872109534 DU

RESONANCE 1

STABLE

RADIUS = 6.62311903956047332338676051 DU
DISTANCE FROM NOMINAL = 2053.63235 METERS
S = 2.57354211925130017355733216

UNSTABLE

RADIUS = 6.62312898069230474566012610 DU
DISTANCE FROM NOMINAL = 2117.03828 METERS
S = 2.57354405066093724823231373

ROOT 1 = 2.57225535879742894051568446
ROOT 2 = 2.57482973825684011213949358
WIDTH = 84513.86393 METERS
DF = .56666332874386549500224905D-07

RESONANCE 1

STABLE

RADIUS = 6.62311254138901482663038207 DU
DISTANCE FROM NOMINAL = 2012.18610 METERS
S = 2.57354085675534104714050865

UNSTABLE

RADIUS = 6.62312239460014021257026258 DU
DISTANCE FROM NOMINAL = 2075.03126 METERS
S = 2.57354277108427716384806879

ROOT 1 = 2.57225898236003338751447668
ROOT 2 = 2.57482358318862921027817112
WIDTH = 84192.80152 METERS
DF = .56236552674965968019912343D-07

RESONANCE 2

STABLE

RADIUS = 6.62279504161728455154854446 DU
DISTANCE FROM NOMINAL = -12.87189 METERS
S = 2.57347917062044326769576973

UNSTABLE

RADIUS = 6.62279499906509052836411341 DU
DISTANCE FROM NOMINAL = -13.14330 METERS
S = 2.57347916235299844488103640

ROOT 1 = 2.57347542322313551926585699
ROOT 2 = 2.57348291802502672216083357
WIDTH = 246.03951 METERS
DF = .48024587235371029523549358D-12

RESONANCE 3

STABLE

RADIUS = 6.62247259764080133997643103 DU
DISTANCE FROM NOMINAL = -2069.46472 METERS
S = 2.57341652237658591978112241

UNSTABLE

RADIUS = 6.62247259764079875539562401 DU
DISTANCE FROM NOMINAL = -2069.46472 METERS
S = 2.57341652237658541761196624

ROOT 1 = 2.57341652195460740223585979
ROOT 2 = 2.57341652279856443771039969
WIDTH = .02770 METERS
DF = .60909469144020180570086397D-20

Vita

Thomas Sean Kelso was born on 18 October 1954 in Denver, Colorado. He graduated from high school in 1972 from Miami Carol City, Florida. He was appointed to the United States Air Force Academy where he received the degree of Bachelor of Science in Physics and Mathematics and a regular commission in the United States Air Force in 1976. He performed duty as a Minuteman ICBM Missile Combat Crew Member at Whiteman AFB, Missouri where he also received the degree of Master of Business Administration in Quantitative Methods from the University of Missouri-Columbia in 1979. He was selected as a Minuteman ICBM Operations Training Instructor for the 4315 CCTS at Vandenberg AFB, California in 1979, where he performed instructor duties until entering the graduate program in Space Operations at the School of Engineering, Air Force Institute of Technology, in June 1981.

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