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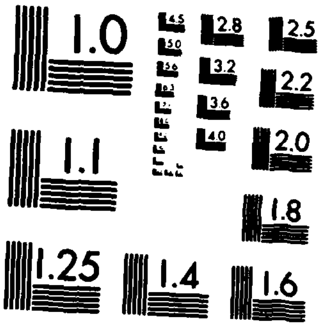


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ONE-SIDED TOLERANCE LIMITS FOR BALANCED ONE-WAY ANOVA RANDOM MODEL

by

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AUTHORS' FOOTNOTE

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gamma


## ABSTRACT

The authors ~~We~~ investigate various techniques for determining a tolerance limit L such that the probability is $\gamma$ that at least a proportion P of a population produced in batches exceeds L. First, they ~~we~~ evaluate the approach of Lemon (1977) for this problem and then present alternative approaches. If the variance ratio is known, one may obtain exact tolerance limits. For settings where the variance ratio is not necessarily known, they ~~we~~ describe a procedure, based on the Satterthwaite approximation, for obtaining conservative tolerance limits.



Key Words: Noncentral t-distribution; Satterthwaite Approximation; Cluster Sampling.

## 1. LEMON'S APPROACH FOR DETERMINING L

Lemon (1977) gives a method for setting tolerance limits on observations that vary in different batches. He sought to determine a lower tolerance limit L (where L is a function of the sample) such that the probability is $\gamma$ that at least a proportion P of the population is above L, i.e.,

$$\Pr\{\Pr_x[X \geq L|\text{sample}] \geq P\} = \gamma, \tag{1.1}$$

where X denotes an observation from the population of interest and where the outer probability in (1.1) is with respect to the sampling distribution of L.

Let $X_{ij}$ denote the $j^{\underline{th}}$ test observation from the $i^{\underline{th}}$ batch or cluster, and suppose that the test observations satisfy the random-effects linear model

$$X_{ij} = \mu + b_i + w_{ij} \qquad i = 1,\ldots,I,\ j = 1,\ldots,J\ ,$$

where $\mu$ is the overall mean, $\mu + b_i$ the mean of the $i^{\underline{th}}$ batch, and $w_{ij}$ a random deviation. We assume that the $b_i$'s and $w_{ij}$'s are independently distributed as normal variates with zero means and variances $\sigma_b^2$ and $\sigma_w^2$, respectively. Further, let $\hat{\mu}$ denote the sample mean $\sum_{i=1}^{I} \sum_{j=1}^{J} X_{ij}/IJ$ and let $s^2$ denote the "between groups" mean square (MS). The individual observations $X_{ij}$ have variance $\sigma_x^2 = \sigma_b^2 + \sigma_w^2$ while the sample mean $\hat{\mu}$ has variance $(J\sigma_b^2+\sigma_w^2)/(IJ)$, and $s^2$ estimates $\sigma^2 = J\sigma_b^2+\sigma_w^2$ .

Lemon showed that $L = \hat{\mu} - k_L s$ satisfies (1.1) for

$$k_L = T_{I-1}(\delta,\gamma)/(IJ)^{1/2} \tag{1.2}$$

where, for arbitrary constants $\alpha$, $\beta$ and $\nu$, $T_\nu(\alpha,\beta)$ denotes the $100\beta$ percentile for a noncentral t distribution with $\nu$ degrees of freedom (df) and noncentrality parameter $\alpha$, and where

$$\delta = BK_p(IJ)^{1/2} \tag{1.3}$$

$$B = \sigma_x/\sigma = \left(\frac{\sigma_b^2 + \sigma_w^2}{J\sigma_b^2 + \sigma_w^2}\right)^{1/2} = \left(\frac{R+1}{JR+1}\right)^{1/2} \tag{1.4}$$

$$R = \sigma_b^2/\sigma_w^2 \tag{1.5}$$

and $K_p$ = 100P percentile of the standard normal distribution. Generally, tolerance limits are defined in terms of an estimate of the population standard deviation. Lemon chose to follow this standard form, and so, computed tables for tolerance factors $k_L'$, where

$$k_L' = k_L(\sigma/\sigma_x) = T_{I-1}(\delta,\gamma)/[B\ (IJ)^{1/2}]. \tag{1.6}$$

The noncentrality parameter $\delta$ is a (monotone decreasing) function of R, so, both $k_L$ and $k_L'$ are functionally dependent on R. Since R is generally unknown, Lemon proposed taking

$$L = \hat{\mu} - k_L(\hat{R}) \cdot s \tag{1.7}$$

where $k_L(\hat{R})$ is obtained by substituting the sample moment estimator $\hat{R}$ (computed from the AOV) for R in expression (1.2), i.e.,

$$\hat{R} = \text{Maximum}\{0,(F-1)/J\},$$

where F denotes the MS ratio, $s^2/s_w^2$, and $s_w^2$ is defined to be the "within groups" MS. (If F < 1, one generally assumes $\sigma_b^2 = 0$ and then combines $s^2$ and $s_w^2$ to estimate $\sigma_w^2$.) Define $s_x^2 = J^{-1}s^2 + (1-J^{-1})s_w^2$, i.e., $s_x^2$ is a linear combination of the between and within MS's which estimates the population variance $\sigma_x^2 = \sigma_b^2 + \sigma_w^2$. The tolerance limit in (1.7) is equivalent to $L = \hat{\mu} - k_L'(\hat{R})s_x$, where $k'_L(\hat{R}) = k_L(\hat{R})(s/s_x)$.

Lemon's justification for using (1.7) was that the variability in $k_L(\hat{R})$ was insignificant. Although Lemon recognized the distribution of $k_L(\hat{R})$, his "numerical integration... over the rough grid" (p. 679) of 3 values for $\hat{R}$ did not adequately approximate the variability of $k_L(\hat{R})$. We obtained the mean and variance of $k_L(\hat{R})$, conditional on $F \geq 1$, for a variety of examples. The six cases given in Table 1 are those which Lemon mentioned investigating. We list the expected value (EV), standard deviation (SD), and coefficient of variation (CV) of $k_L(\hat{R})$. Lemon claimed a CV of less than 2% for $k_L(\hat{R})$, whereas we found values from 6 to 21% by careful integration, e.g., CV = 21.3% for Case 3.

In spite of this variability, we have found Lemon's procedure to be conservative, i.e., the probability

$$\gamma_L(R) = Pr_{\hat{\mu},s,\hat{R}}\{Pr_x[X \geq \hat{\mu} - k_L(\hat{R})\cdot s|\hat{\mu},s,\hat{R}] > P|F > 1\} \qquad (1.8)$$

generally exceeds $\gamma$. The probability $\gamma_L(R)$ is given in Table 1 for each case considered there. [We evaluate $\gamma_L(R)$ by computing

(1.8), conditional on $F = f$, and then numerically integrating these values with respect to the density of F (truncated at 1).] These few cases illustrate the conservativeness of Lemon's procedure. We found that $\gamma_L(R)$ appeared to be decreasing in R, e.g., for Case 2 in Table 1, $\gamma_L(1) = .9985$, whereas $\gamma_L(R)$ was computed to be .9999, .9799 and .9681 for R = .2, 5 and 10 respectively. Hence, Lemon's procedure appears to be the most conservative when J is large and I and R are small.

We offer two intuitive reasons for the fact that $\gamma_L(R) \geq \gamma$. First, note that $k_L(\hat{R})$ is a decreasing function of $\hat{R}$, while the EV of $s^2$, conditional on F, is an increasing function of $\hat{R}$. Thus, $k(\hat{R})$ tends to compensate for the variability in s, so that $\hat{\mu} - k(\hat{R})\cdot s$ is more stable than $\hat{\mu} - k(R)\cdot s$.

Second, the probability (1.8) is equivalent to

$$\Pr\left[\frac{Z + \delta}{s_x/\sigma_x} \leq k_L'(\hat{R}) \cdot (\sigma_x/\sigma) \cdot (IJ)^{1/2}\right], \tag{1.9}$$

where Z is a standard normal variate. Using the result of Satterthwaite (1946), $s_x^2/\sigma_x^2$ is approximately distributed as a $\chi_f^2/f$ variate, where

$$f = (R+1)^2/[(R+J^{-1})^2/(I-1) + (J-1)/IJ^2]. \tag{1.10}$$

Since f is greater than I-1 (though it approaches I-1 as R tends to infinity), the "df" in $s_x^2$ exceed the df for $s^2$. Therefore, the tolerance factors $k_L'$ which are based on $T_{I-1}(\delta,\gamma)$ tend to be larger than necessary. The fact that f is an increasing function of J and a decreasing function of R reinforces the observation made earlier that Lemon's procedure is more conservative for small R and large J.

## 2. AN ALTERNATIVE PROCEDURE BASED ON THE SATTERTHWAITE APPROXIMATION

In this section, we discuss a procedure for determining L which employs the Satterthwaite approximation mentioned at the close of Section 1. If $L = \hat{\mu} - k_S' s_x$ satisfies (1.1), this corresponds to $k_S'$ satisfying

$$\Pr\left[\frac{Z + \delta}{s_x/\sigma_x} \leq k_S' B(IJ)^{1/2}\right] = \gamma .$$

Since $s_x^2/\sigma_x^2$ is approximately distributed as a $\chi_f^2/f$ [where f is defined in (1.10)], we have (approximately)

$$k_S' = T_f(\delta,\gamma)/[B(IJ)^{1/2}]. \tag{2.1}$$

It is informative to investigate $k_S'$ as a function of R. As R tends to infinity, $k_S'$ approaches

$$k_S'(\infty) = T_{I-1}(K_p\sqrt{I},\gamma)/\sqrt{I} ,$$

which is the tolerance factor for a random sample of size I. Hence, when essentially all the variation is between groups, repeated measurements within a group provide no additional information. At $R = 0$, $k_S'(0) = T_{IJ-1-\varepsilon}(K_p(IJ)^{1/2},\gamma)/(IJ)^{1/2}$, where $\varepsilon = (J-1)/(IJ-J+1)$. (Note that $0 \leq \varepsilon \leq 1$ for $I > 1$.) Thus, except for the term $\varepsilon$, at $R = 0$ the approximation (2.1) corresponds to the tolerance factor for a random sample of size IJ. When $0 < R < \infty$, $k_S'$ is greater than $k_S'(0)$ and less than $k_S'(\infty)$. Selected values of $k_S'$ appear in Table 2.

If R were known, the tolerance factor $k_S'$ could be obtained from Table 2 or calculated using (2.2). When R is unknown, one might consider replacing R in (2.2) with $\hat{R}$. Let $k_S'(\hat{R})$ denote $k_S'$ evaluated at $R = \hat{R}$. We computed

$$\gamma_S(R) = \Pr_{\hat{\mu},s_x,\hat{R}}\{\Pr[\bar{\bar{X}} > \hat{\mu} - k_S'(\hat{R})s_x \mid \hat{\mu},s_x,\hat{R}] > P \mid F \geq 1\} \quad (2.3)$$

for a variety of examples in order to evaluate the procedure of taking $L = \hat{\mu} - k_S'(\hat{R})s_x$. [The computations were performed as described in Section 1 for $\gamma_L(R)$.] The function $\gamma_S(R)$ necessarily approaches $\gamma$ as R approaches infinity, and $\gamma_S(R)$ exceeds $\gamma$ for R sufficiently small. However, for intermediate values of R, $\gamma_S(R)$ is generally less than $\gamma$, e.g., for $I = J = 5$, $P = .9$ and $\gamma = .95$, $\gamma_S(R)$ is below .95 for $R \geq .5$ with infimum .91.

Since the probability $\gamma_S(R)$ can fall below $\gamma$ when using $L = \hat{\mu} - k_S'(\hat{R})$, we seek another procedure to replace it. We propose using $k_S'(R^*)$, where $R^*$ denotes an upper $\eta$ confidence bound for R, i.e.,

$$R^* = \max\{(FF_\eta - 1)/J, 0\},$$

where $F_\eta$ is the $100\eta$ percentile of an F distribution with degrees of freedom $\nu_1 = I(J-1)$ and $\nu_2 = I-1$ (Searle 1971, p. 414).

That is, we enter Table 2 with the upper confidence limit for R rather than the point estimate of R. Thus using $k_S'(R^*)$ instead of $k_S'(\hat{R})$ results in a more conservative procedure. The problem here is in choosing a reasonable value for $\eta$.

We found it necessary to vary $\eta$ according to the values of $\gamma$ and P that are being used. Let $\gamma_S^*(R)$ denote the probability obtained by replacing $\hat{R}$ with $R^*$ in (2.3). We found that $\gamma_S^*(R)$ is decreasing in J with a limiting value that may be computed using numerical integration. Thus, we were able to determine $\eta$, such that $\gamma_S^*(R) \geq \gamma$ for all J and R and for $I \geq 5$ (the limiting value was increasing in I). For the following combinations of $\gamma$ and P, the necessary values of $\eta$ are

| | $\gamma$ | | |
|---|---|---|---|
| P | .90 | .95 | .99 |
| .90 | .76 | .825 | .91 |
| .95 | .78 | .84 | .92 |
| .99 | .80 | .855 | .93 |

Lemon's procedure is always conservative, being most conservative for large J and small I and R. Our procedure above is most conservative for small J and large R. One practical solution here would be to choose in each situation (based on I, J and vague knowledge of R) the procedure which one expects will produce the smaller k value. However, if R is known or known to within a close approximation then the procedure given in Section 3 below should be used.

## 3. PROCEDURE WHEN R IS KNOWN

If the variance ratio is known, this additional information may be utilized to obtain a tolerance factor which is generally smaller than those obtained using either of the procedures described in Sections 1 and 2. Knowledge of R enables one to pool the two MS's and thus obtain an estimated SD which has IJ-1 df. The quantity

$$(s^2/\sigma^2) \cdot [(I-1) + I(J-1)(JR+1)/F]$$

is equivalent to $(I-1)(s^2/\sigma^2) + I(J-1)(s_w^2/\sigma_w^2)$, and, therefore is distributed as a chi-square variate with IJ-1 df. Hence, conditional on F, $(s^2/\sigma^2)$ is distributed as a known multiple of a $\chi^2_{IJ-1}$ variate. Using this result, the conditional probability,

$$\Pr_{\hat{\mu},s|F}[\Pr_X\{X > \hat{\mu} - ks|\hat{\mu},s,F\} \geq P|F] = \gamma \tag{3.1}$$

for

$$k = cT_{IJ-1}(\delta,\gamma)/(IJ)^{1/2},$$

with $c^2 = [I-1+I(J-1)(JR+1)/F]/(IJ-1)$.

The tolerance limit $L = \hat{\mu} - k \cdot s$ may be expressed in standard form as $L = \hat{\mu} - k's_x$, where $k' = c'k_R'$, with

$$k_R' = T_{IJ-1}(\delta,\gamma)/[B(IJ)^{1/2}] \tag{3.2}$$

$$c' = cBs/s_x =$$

$$\{[J(R+1)/(F+J-1)][I(J-1) + (I-1)F/(JR+1)]/(IJ-1)\}^{1/2}. \tag{3.3}$$

We chose to factor $k'$ in terms of $k'_R$ and $c'$, because $k'_R$ does not depend on F. In Table 3, we provide values for $k'_R$. The factor $c'$ is a decreasing function of the MS ratio (and hence, of $\hat{R}$) and equals 1 when $\hat{R} = R$. Thus $k'$ is somewhat smaller (larger) than the table value $k'_R$ if $\hat{R}$ is greater (less) than R.

Given I, J and F, $k'$ is a strictly increasing function of R. Thus, if one is certain that $R \leq r$, then one may enter Table 3 with $R = r$ to obtain $k'_R$ and then compute $c'$ from (3.3).

For settings where R is unknown, we considered the procedure of computing an upper $100(1-\beta)\%$ confidence bound $R^*$ for R based on F, computing $k'$ at $R = R^*$, and then combining the two probability statements to obtain an overall probability of at least $(\gamma-\beta)$. However, this approach produced extremely large (conservative) tolerance factors. This may be attributed to the fact that $c'k'_R$ increases without bound as R approaches infinity. Hence, the procedure based on (3.1) is not recommended unless precise knowledge about R is available apart from the sample.

## 4. DETERMINING $k'_S$ AND $k'_R$ BY INTERPOLATION

Tables 2 and 3 provide tolerance factors $k'_S$ and $k'_R$ respectively, for $\gamma = .95$, $P = .9$ and $.99$, $I \leq 10$, and for selected values of J and R. For combinations of J and R not appearing in the tables, the tolerance factor $k'_S$ or $k'_R$ may be obtained by linear interpolation in I/J and by linear interpolation in R for $0 < R < .2$, logarithmic interpolation for $.2 < R < 10$ and linear interpolation in $1/R$ for $R > 10$. (For $R = 0$, the appropriate tolerance factor is the factor for a random sample of size IJ.) This interpolation scheme is similar to one suggested by Lemon.

## 5. EXAMPLES

We illustrate the procedures discussed in Sections 2 and 3 for determining a tolerance limit, employing the example discussed by Lemon (1977, p. 680). Six samples from each of five independent batches of material composed the sample from which a lower tolerance limit for static strength is to be determined, with $P = .9$ and $\gamma = .95$. The summary statistics were $\hat{\mu} = 186$ ksi (thousand pounds per square inch), $\hat{R} = 1.37$ and $s_x = 9.04$. To determine $k_S'$ (as described in Section 2.2), we compute an 82.5% upper bound for R,

$$R^* = [9.22(2.67) - 1]/6 = 3.94.$$

The tolerance factor $k_S'(R^*)$ computed from Table 2a equals 2.83, and hence $L = 186 - 2.83(9.04) = 160.4$. Hence we can be at least 95% confident that at least 90% of the material in the population has static strength above 160.4 ksi. For comparison, we note that Lemon's procedure produces a tolerance limit of 156.3 ksi (based on $k_L'(\hat{R}) = 3.285$) which is more restrictive than is necessary.

To illustrate the procedure for computing k', suppose that, in addition to the sample information, it is known that $R \leq 1$.

Then, from Table 3a, we obtain $k'_R = 2.00$ and, using (3.3), $c' = .9385$. Hence $k' = c'k'_R = 1.877$, and $L = 169.0$ ksi. As mentioned in Section 3, $k'$ may be much smaller than $k'_L$ or $k'_S$ (as in the case here), yet the validity of the tolerance limit depends on the assumption about R.

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TABLE 1. Variability of $k_L(\hat{R})$ (P=.90, $\gamma$=.95, R=1)

| Case | I | J | k(R) | $k_L(\hat{R})$ EV | SD | CV | Pr[F > 1] | $\gamma_L(R)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 2 | 2.714 | 2.721 | .198 | .073 | .155 | .9913 |
| 2 | 5 | 5 | 1.889 | 2.010 | .317 | .158 | .047 | .9985 |
| 3 | 5 | 10 | 1.388 | 1.528 | .325 | .213 | .015 | .9995 |
| 4 | 10 | 2 | 1.878 | 1.895 | .121 | .064 | .057 | .9821 |
| 5 | 10 | 5 | 1.308 | 1.358 | .155 | .114 | .004 | .9925 |
| 6 | 10 | 10 | .961 | 1.005 | .130 | .130 | .000 | .9957 |

TABLE 2. ONE-SIDED TOLERANCE FACTORS $k'_S$

FOR ONE-WAY-RANDOM-EFFECTS-ANOVA

a. $\gamma = .95$ $P = .90$

| | J | I = 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| R=.2 | | | | | | | | | | |
| | 2 | 5.18 | 3.24 | 2.70 | 2.44 | 2.27 | 2.16 | 2.08 | 2.01 | 1.96 |
| | 4 | 3.08 | 2.42 | 2.17 | 2.04 | 1.95 | 1.88 | 1.83 | 1.79 | 1.76 |
| | 8 | 2.43 | 2.08 | 1.93 | 1.84 | 1.78 | 1.73 | 1.70 | 1.67 | 1.65 |
| | 16 | 2.15 | 1.91 | 1.80 | 1.74 | 1.69 | 1.65 | 1.63 | 1.60 | 1.55 |
| | ∞ | 1.89 | 1.74 | 1.67 | 1.62 | 1.59 | 1.57 | 1.55 | 1.53 | 1.52 |
| R=1 | | | | | | | | | | |
| | 2 | 7.68 | 3.89 | 3.06 | 2.69 | 2.47 | 2.33 | 2.22 | 2.14 | 2.08 |
| | 4 | 5.06 | 3.15 | 2.64 | 2.38 | 2.23 | 2.12 | 2.05 | 1.99 | 1.94 |
| | 8 | 4.18 | 2.84 | 2.44 | 2.24 | 2.11 | 2.03 | 1.96 | 1.91 | 1.87 |
| | 16 | 3.81 | 2.70 | 2.35 | 2.17 | 2.06 | 1.98 | 1.92 | 1.87 | 1.83 |
| | ∞ | 3.48 | 2.56 | 2.26 | 2.10 | 2.00 | 1.93 | 1.87 | 1.83 | 1.80 |
| R=5 | | | | | | | | | | |
| | 2 | 14.00 | 5.14 | 3.70 | 3.11 | 2.79 | 2.58 | 2.44 | 2.33 | 2.25 |
| | 4 | 11.66 | 4.72 | 3.48 | 2.97 | 2.69 | 2.50 | 2.37 | 2.27 | 2.19 |
| | 8 | 10.66 | 4.52 | 3.38 | 2.91 | 2.63 | 2.46 | 2.34 | 2.24 | 2.17 |
| | 16 | 10.20 | 4.42 | 3.34 | 2.87 | 2.61 | 2.44 | 2.32 | 2.22 | 2.15 |
| | ∞ | 9.77 | 4.33 | 3.29 | 2.84 | 2.59 | 2.42 | 2.30 | 2.21 | 2.14 |
| R=10 | | | | | | | | | | |
| | 2 | 16.56 | 5.56 | 3.89 | 2.24 | 2.88 | 2.66 | 2.50 | 2.39 | 2.29 |
| | 4 | 14.90 | 5.29 | 3.76 | 3.15 | 2.82 | 2.61 | 2.46 | 2.35 | 2.26 |
| | 8 | 14.14 | 5.16 | 3.70 | 3.11 | 2.79 | 2.59 | 2.44 | 2.33 | 2.25 |
| | 16 | 13.78 | 5.10 | 3.67 | 3.10 | 2.78 | 2.58 | 2.43 | 2.32 | 2.24 |
| | ∞ | 13.42 | 5.04 | 3.64 | 3.08 | 2.76 | 2.56 | 2.42 | 2.31 | 2.23 |
| R=∞[c] | | 20.58 | 6.16 | 4.16 | 3.41 | 3.01 | 2.76 | 2.58 | 2.45 | 2.36 |

Table 2 (Cont'd)

b. $\gamma$ = .95 P = .99

| J | I: 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| R=.2 | | | | | | | | | |
| 2 | 8.83 | 5.44 | 4.54 | 4.10 | 3.83 | 3.65 | 3.52 | 3.42 | 3.34 |
| 4 | 5.16 | 4.04 | 3.65 | 3.43 | 3.29 | 3.19 | 3.12 | 3.06 | 3.01 |
| 8 | 4.03 | 3.47 | 3.23 | 3.10 | 3.01 | 2.95 | 2.90 | 2.86 | 3.83 |
| 16 | 3.56 | 3.18 | 3.02 | 2.93 | 2.86 | 2.81 | 2.78 | 2.75 | 2.72 |
| ∞ | 3.11 | 2.89 | 2.79 | 2.73 | 2.69 | 2.66 | 2.63 | 2.61 | 2.60 |
| R=1 | | | | | | | | | |
| 2 | 13.28 | 6.54 | 5.12 | 4.50 | 4.14 | 3.90 | 3.74 | 3.61 | 3.51 |
| 4 | 8.58 | 5.24 | 4.38 | 3.97 | 3.73 | 3.56 | 3.44 | 3.35 | 3.27 |
| 8 | 7.02 | 4.70 | 4.05 | 3.73 | 3.53 | 3.39 | 3.29 | 3.22 | 3.15 |
| 16 | 6.37 | 4.45 | 3.89 | 3.61 | 3.43 | 3.31 | 3.22 | 3.15 | 3.09 |
| ∞ | 5.79 | 4.22 | 3.73 | 3.49 | 3.33 | 3.22 | 3.14 | 3.08 | 3.03 |
| R=5 | | | | | | | | | |
| 2 | 24.83 | 8.75 | 6.22 | 5.22 | 4.68 | 4.34 | 4.10 | 3.93 | 3.79 |
| 4 | 20.51 | 7.99 | 5.85 | 4.98 | 4.50 | 4.20 | 3.98 | 3.82 | 3.70 |
| 8 | 18.68 | 7.64 | 5.67 | 4.86 | 4.42 | 4.13 | 3.92 | 3.77 | 3.65 |
| 16 | 17.84 | 7.47 | 5.59 | 4.81 | 4.37 | 4.09 | 3.89 | 3.75 | 3.63 |
| ∞ | 17.05 | 7.30 | 5.50 | 4.75 | 4.33 | 4.06 | 3.86 | 3.72 | 3.61 |
| R=10 | | | | | | | | | |
| 2 | 29.58 | 9.49 | 6.57 | 5.44 | 4.84 | 4.47 | 4.21 | 4.02 | 3.87 |
| 4 | 26.49 | 9.01 | 6.34 | 5.30 | 4.74 | 4.39 | 4.14 | 3.96 | 3.82 |
| 8 | 25.08 | 8.78 | 6.23 | 5.23 | 4.69 | 4.35 | 4.11 | 3.93 | 3.79 |
| 16 | 24.41 | 8.67 | 6.18 | 5.19 | 4.66 | 4.33 | 4.09 | 3.92 | 3.78 |
| ∞ | 23.60 | 8.56 | 6.13 | 5.13 | 4.64 | 4.30 | 4.07 | 3.90 | 3.77 |
| R=∞[c] | | | | | | | | | |
| | 37.09 | 10.55 | 7.04 | 5.74 | 5.06 | 4.64 | 4.35 | 4.14 | 3.98 |

*c for all J

TABLE 3. ONE-SIDED TOLERANCE FACTORS $k'_R$

FOR ONE-WAY-RANDOM-EFFECTS-ANOVA

a. $\gamma = .95$ $P = .90$

| | J | I = 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| R=.2 | | | | | | | | | | |
| | 2 | 4.22 | 3.05 | 2.62 | 2.39 | 2.24 | 2.14 | 2.06 | 2.00 | 1.95 |
| | 4 | 2.69 | 2.30 | 2.11 | 1.99 | 1.91 | 1.86 | 1.81 | 1.77 | 1.74 |
| | 8 | 2.20 | 1.98 | 1.87 | 1.80 | 1.75 | 1.71 | 1.68 | 1.65 | 1.63 |
| | 16 | 1.98 | 1.83 | 1.75 | 1.70 | 1.66 | 1.63 | 1.60 | 1.58 | 1.57 |
| | ∞ | 1.76 | 1.67 | 1.62 | 1.58 | 1.56 | 1.54 | 1.52 | 1.51 | 1.49 |
| R=1 | | | | | | | | | | |
| | 2 | 4.34 | 3.14 | 2.69 | 2.45 | 2.30 | 2.19 | 2.11 | 2.04 | 1.99 |
| | 4 | 2.89 | 2.45 | 2.24 | 2.11 | 2.02 | 1.95 | 1.90 | 1.86 | 1.82 |
| | 8 | 2.45 | 2.19 | 2.04 | 1.95 | 1.89 | 1.84 | 1.80 | 1.76 | 1.74 |
| | 16 | 2.27 | 2.07 | 1.95 | 1.88 | 1.82 | 1.78 | 1.75 | 1.72 | 1.69 |
| | ∞ | 2.10 | 1.95 | 1.86 | 1.80 | 1.76 | 1.72 | 1.69 | 1.67 | 1.65 |
| R=5 | | | | | | | | | | |
| | 2 | 4.45 | 3.22 | 2.76 | 2.51 | 2.35 | 2.24 | 2.15 | 2.09 | 2.03 |
| | 4 | 3.06 | 2.58 | 2.35 | 2.21 | 2.11 | 2.03 | 1.98 | 1.93 | 1.89 |
| | 8 | 2.65 | 2.35 | 2.18 | 2.07 | 2.00 | 1.94 | 1.89 | 1.85 | 1.82 |
| | 16 | 2.49 | 2.24 | 2.10 | 2.01 | 1.95 | 1.89 | 1.85 | 1.82 | 1.79 |
| | ∞ | 2.34 | 2.15 | 2.03 | 1.95 | 1.89 | 1.85 | 1.81 | 1.78 | 1.76 |
| R=10 | | | | | | | | | | |
| | 2 | 4.48 | 3.24 | 2.78 | 2.53 | 2.36 | 2.25 | 2.16 | 2.10 | 2.04 |
| | 4 | 3.09 | 2.61 | 2.37 | 2.23 | 2.13 | 2.05 | 1.99 | 1.94 | 1.91 |
| | 8 | 2.69 | 2.38 | 2.21 | 2.10 | 2.02 | 1.96 | 1.91 | 1.87 | 1.84 |
| | 16 | 2.53 | 2.28 | 2.14 | 2.04 | 1.97 | 1.92 | 1.87 | 1.84 | 1.81 |
| | ∞ | 2.39 | 2.19 | 2.07 | 1.98 | 1.92 | 1.87 | 1.84 | 1.80 | 1.78 |
| R=∞ | | | | | | | | | | |
| | 2 | 4.51 | 3.26 | 2.80 | 2.54 | 2.38 | 2.26 | 2.18 | 2.11 | 2.05 |
| | 4 | 3.13 | 2.64 | 2.40 | 2.25 | 2.15 | 2.07 | 2.01 | 1.96 | 1.92 |
| | 8 | 2.74 | 2.42 | 2.24 | 2.13 | 2.05 | 1.98 | 1.94 | 1.89 | 1.86 |
| | 16 | 2.58 | 2.32 | 2.17 | 2.07 | 2.00 | 1.94 | 1.90 | 1.86 | 1.83 |
| | ∞ | 2.44 | 2.23 | 2.10 | 2.02 | 1.95 | 1.90 | 1.86 | 1.83 | 1.80 |

Table 3 (Cont'd)

b. $\gamma = .95 \quad P = .99$

| | J | I=2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| R=.2 | 2 | 7.08 | 5.09 | 4.38 | 4.00 | 3.77 | 3.60 | 3.48 | 3.39 | 3.31 |
| | 4 | 4.43 | 3.81 | 3.52 | 3.34 | 3.22 | 3.14 | 3.07 | 3.02 | 2.97 |
| | 8 | 3.59 | 3.28 | 3.12 | 3.02 | 2.94 | 2.89 | 2.85 | 2.81 | 2.79 |
| | 16 | 3.20 | 3.01 | 2.91 | 2.84 | 2.79 | 2.75 | 2.72 | 2.70 | 2.68 |
| | ∞ | 2.80 | 2.71 | 2.66 | 2.63 | 2.60 | 2.58 | 2.56 | 2.55 | 2.54 |
| R=1 | | | | | | | | | | |
| | 2 | 7.16 | 5.16 | 4.43 | 4.05 | 3.81 | 3.64 | 3.52 | 3.42 | 3.34 |
| | 4 | 4.58 | 3.93 | 3.62 | 3.43 | 3.31 | 3.21 | 3.14 | 3.08 | 3.04 |
| | 8 | 3.79 | 3.45 | 3.26 | 3.14 | 3.06 | 3.00 | 2.95 | 2.91 | 2.87 |
| | 16 | 3.46 | 3.22 | 3.09 | 3.00 | 2.93 | 2.89 | 2.85 | 2.81 | 2.79 |
| | ∞ | 3.15 | 3.00 | 2.91 | 2.85 | 2.80 | 2.77 | 2.74 | 2.71 | 2.69 |
| R=5 | | | | | | | | | | |
| | 2 | 7.24 | 5.21 | 4.48 | 4.09 | 3.85 | 3.68 | 3.55 | 3.45 | 3.37 |
| | 4 | 4.71 | 4.03 | 3.71 | 3.51 | 3.38 | 3.28 | 3.20 | 3.14 | 3.09 |
| | 8 | 3.97 | 3.59 | 3.38 | 3.25 | 3.16 | 3.09 | 3.03 | 2.99 | 2.95 |
| | 16 | 3.66 | 3.38 | 3.23 | 3.12 | 3.05 | 2.99 | 2.95 | 2.91 | 2.88 |
| | ∞ | 3.39 | 3.19 | 3.08 | 3.00 | 2.94 | 2.89 | 2.86 | 2.83 | 2.80 |
| R=10 | | | | | | | | | | |
| | 2 | 7.25 | 5.23 | 4.49 | 4.11 | 3.86 | 3.69 | 3.56 | 3.46 | 3.38 |
| | 4 | 4.74 | 4.06 | 3.73 | 3.53 | 3.39 | 3.29 | 3.22 | 3.15 | 3.10 |
| | 8 | 4.00 | 3.62 | 3.41 | 3.27 | 3.18 | 3.11 | 3.05 | 3.00 | 2.97 |
| | 16 | 3.70 | 3.42 | 3.26 | 3.15 | 3.07 | 3.01 | 2.97 | 2.93 | 2.89 |
| | ∞ | 3.44 | 3.23 | 3.11 | 3.03 | 2.97 | 2.92 | 2.88 | 2.85 | 2.82 |
| R=∞[c] | | | | | | | | | | |
| | 2 | 7.27 | 5.24 | 4.51 | 4.12 | 3.87 | 3.70 | 3.57 | 3.47 | 3.39 |
| | 4 | 4.78 | 4.08 | 3.75 | 3.55 | 3.41 | 3.31 | 3.23 | 3.17 | 3.12 |
| | 8 | 4.05 | 3.65 | 3.44 | 3.30 | 3.20 | 3.13 | 3.07 | 3.02 | 2.98 |
| | 16 | 3.75 | 3.46 | 3.29 | 3.18 | 3.10 | 3.04 | 2.99 | 2.95 | 2.92 |
| | ∞ | 3.49 | 3.28 | 3.15 | 3.06 | 3.00 | 2.95 | 2.91 | 2.87 | 2.85 |

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We investigate various techniques for determining a tolerance limit L such that the probability is $\gamma$ that at least a proportion P of a population produced in batches exceeds L. First, we evaluate the approach of Lemon (1977) for this problem and then present alternative approaches. If the variance ratio is known, one may obtain exact tolerance limits. For settings where the variance ratio is not necessarily known, we describe a procedure, based on the Satterthwaite approximation, for obtaining conservative tolerance limits.

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