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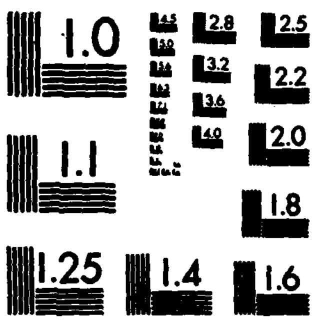

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A survey is given of nonlinear phenomena in classical electromagnetic theory with the emphasis on singularity development in nonlinear materials and electromagnetic shock wave formation in nonlinear distributed parameter transmission lines. As influences on shock wave formation we [the author] consider the mechanisms of anisotropy, nonlinear conduction, dispersion and relaxation.

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Systems of Nonlinear Hyperbolic Equations Associated with Problems of Classical Electromagnetic Theory†

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Contents

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I. Introduction: Nonlinear Phenomena in Electromagnetic theory

The subject matter of this survey is the manifestation of nonlinear phenomena in a wide variety of problems of electromagnetic theory. The specific problems which we intend to focus on include shock formation and propagation for plane electromagentic waves propagating into an isotropic nonlinear dielectric half-space, the possible dissipative effects of anisotropy, nonlinear conduction, dispersion, and relaxation on such wave propagation, and the development and propagation of singularities in electromagnetic wave propagation in nonlinear distributed parameter transmission lines.

In earlier work we have studied the problem of shock formation in non-dissipative, non-dispersive nonlinear dielectric half-spaces and singularity development in rigid, infinite, nonlinear dielectric rods; these problems have been treated in [1] and [2] with some of the results described in §II below. The basic achievement in [1] was to place the problem of shock formation within the context of the general theory of homogeneous quasilinear systems of hyperbolic conservation laws and thus the approach differs from that taken in earlier work by De Martini, Townes, Gustafson, and Kelley [3], Broer [4], Jeffrey [5], and Korobeinikov [6], Kataev [7] and Donato and Fusco [8]. In particular, setting the problem within the context of the theory of quasilinear systems of hyperbolic conservation laws allows for a relatively simple transition to the cases where the media is either anisotropic, and exhibits a small nonlinear conduction current, or exhibits dispersive effects or relaxation phenomena. It is shown in §II that in each of these cases our problem may again be set within the theory of hyperbolic conservation laws, the difference with the isotropic, non-dispersive case being that inhomogeneous terms which are possibly dissipative in nature make their appearance; a chief goal of the work described in §IIb will be to examine conditions under which shocks do or do not form and in this regard recent work of Nishida [9] and Slemrod [10] on dissipative quasilinear systems and of Liu [11], Dafermos [12], and Dafermos and Hsiao [13] on hyperbolic conservation laws with inhomogenecity and dissipation may prove useful. Some interesting recent numerical studies by Fisher and Bishel [14] at Los Alamos on the interplay between dispersion and nonlinearity for intense plane wave laser pulses are also described in §IIb.

In §III we consider the problem of shock formation and propagation in distributed parameter nonlinear transmission lines where one of the parameters C, the capacitance, depends on the voltage drop v accross the capacitor; this problem has been considered previously by Landauer [15], Jeffrey [16], Riley [42] and Kataev [7]. However, these authors considered primarily the formation and propagation of jump discontinuities in the first derivatives of current and voltage in the transmission line; nore were these problems previously set within the context of the theory of quasilinear systems of conservation laws and such a formulation is effected in §III where it is conjectured that for sufficiently small initial gradients of current and voltage shocks will not form in a nonlinear transmission line with either nonzero linear resistance or leakage conductance, whereas such shocks will form if these initial gradients are sufficiently large. Shock formation and propagation in nonlinear transmission lines is of some considerable importance for as Kataev [7] points out experimental tests have shown that such electromagnetic shock waves have led to technically feasible methods of producing short current pulses with a fast rise time.

Maxwell's Equations, in the absence of free current, free charge, and and external magnetization, have the standard form:

$$
\text{(I.1)} \quad \begin{cases} \dfrac{\partial \underset{\sim}{B}}{\partial t} = -\operatorname{curl} \underset{\sim}{E}, \quad \operatorname{div} \underset{\sim}{B} = 0 \\[2ex] \dfrac{\partial \underset{\sim}{D}}{\partial t} = \operatorname{curl} \underset{\sim}{H}, \quad \operatorname{div} \underset{\sim}{D} = 0 \end{cases}
$$

in some bounded, open domain $\Omega \subseteq R^3$, where $\underset{\sim}{B}$ is the magnetic field, $\underset{\sim}{H}$ the magnetic intensity, $\underset{\sim}{E}$ the electric field, and $\underset{\sim}{D}$ the electric induction field (or electric displacement). The latter field is defined by $\underset{\sim}{D} \equiv \varepsilon_0 \underset{\sim}{E} + \underset{\sim}{P}(\underset{\sim}{E})$ with $\underset{\sim}{P}(\underset{\sim}{E})$ being the polarization and ε_0 the permittivity of free space. In a vacuum, $\underset{\sim}{P} = 0$, $H = \mu_0^{-1} B$ where μ_0 is the permeability of free space and $\varepsilon_0 > 0$, $\mu_0 > 0$ satisfy $\varepsilon_0 \mu_0 = c^{-2}$. In most standard texts on electromagnetic theory only linear constitutive relations between $\underset{\sim}{P}$ and $\underset{\sim}{E}$ (and, hence, between $\underset{\sim}{D}$ and $\underset{\sim}{E}$) are considered, i.e., $\underset{\sim}{P}(\underset{\sim}{E}) = (\varepsilon - \varepsilon_0)\underset{\sim}{E}$, $\varepsilon > \varepsilon_0$, or perhaps, $\underset{\sim}{D} = \underset{\sim}{\varepsilon}\, \underset{\sim}{E}$ where $\underset{\sim}{\varepsilon}$ is a tensor of rank four.

In [1912] Voterra propose the set of constitutive

$$
\text{(I.2)} \quad \begin{cases} \underset{\sim}{D}(x,t) = \underset{\sim}{\varepsilon}\, \underset{\sim}{E}(x,t) + \underset{-\infty}{\overset{t}{\underset{\sim}{D}}}(E(x,t)) \\[2ex] \underset{\sim}{B}(x,t) = \underset{\sim}{\mu}\, \underset{\sim}{H}(x,t) + \underset{-\infty}{\overset{t}{\underset{\sim}{B}}}(\underset{\sim}{H}(x,t)) \end{cases}
$$

which, though still representing an a priori separation of electric and magnetic effects, are sufficiently broad enough to include as special cases the linear, rigid non-conductors of Maxwell [1973], the Maxwell-Hopkinson Dielectrics [1897]:

$$
\text{(I.3)} \quad \begin{cases} \underset{\sim}{D}(x,t) = \varepsilon\, \underset{\sim}{E}(x,t) + \displaystyle\int_{-\infty}^{t} \phi(t-\tau) \underset{\sim}{E}(x,\tau)\, d\tau \\[2ex] \underset{\sim}{H}(x,t) = \mu^{-1} \underset{\sim}{B}(x,t) \end{cases}
$$

$\varepsilon > 0$, $\mu > 0$, $\phi(t)$, $t \geq 0$, continuous and monotonically decreasing, and special cases of the holohedral and hemihedral isotropic dielectrics introduced by Toupin and Rivlin [21] in order to explain the phenomena of absorbtion and dispersion of electromagnetic waves in dielectric materials. All of these special cases refer to linear theories and are fully discussed in the recent monograph by this author [22].

As is clearly indicated in Bloembergen [23], nonlinear properties of the constitutive relations

$$D = \epsilon(E)E, \quad B = \mu(H)H \tag{I.4}$$

have been recognized for some time, i.e., that the dielectric constant and magnetic permeability can be functions of the field strengths. The importance of nonlinearly in problems of electromagnetic theory is also emphasized in Kataev [7] who points out that "linearization of the parameters of a media is only a first approximation. For example, a linear relationship between the conduction current and the voltage (Ohm's law) exists, generally speaking, only if a number of conditions are observed: constancy of temperature and composition of the medium, and absence of magnetic fields".

The monograph [23] begins with the constitutive hypothesis (I.4) and most recent attempts to prove focusing of polarized electromagnetic waves propagating through a nonlinear dielectric media have been based on constitutive hypotheses of this form with μ = const. and $\epsilon(E) = \epsilon_0 + \epsilon_2||E||^2$, $\epsilon_0 > 0$, $\epsilon_2 > 0$. Similar constitutive relations have been employed in the work on shock formation cited in [3]-[8], e.g., in [3] the nonlinearity of the constitutive relations is reflected in the consideration of a material with an intensity-dependent index of refraction and it is shown that such dependence tends to distort an optical pulse along its direction of propagation thus giving rise to pulse self-steepening (optical shock development). Other interesting non-linear problems for dielectrics which have been treated in recent years, and which are based on constitutive relations of the form (I.4), include the work of Kazakia and Venkataraman [24] who use the method of characteristics to study the early phases of propagation of a large amplitude electromagnetic disturbance in a nonlinear dielectric slab embedded between two linear dielectric media, the work of Venkataraman and Rivlin [25] who present a method for calculating the change in amplitudes and phases of harmonics of all orders for a plane electromagnetic wave propagating in a nonlinear, non-dissipative, isotropic dielectric, and the work of Rogers, Cekirge, and Askar [26] who apply a modification of the Bergman integral operator method to a hodograph system describing linearly polarized plane electromagnetic pulse propagation in a nonlinear dielectric, thus obtaining closed form solutions which are valid up to the time of shock formation.

II. Singularity Development in Nonlinear Materials

a. Shock Formation in Non-dissipative, Non-dispersive Isotropic Media

We assume Maxwell's equations have the form (I.1) with conduction vector $\underset{\sim}{J} = \underset{\sim}{0}$ and free charge density $\rho = 0$. Our constitutive equations have the form (I.4). We consider a linearly polarized plance wave of the form $\underset{\sim}{E} = (0, E_y(x,t), 0)$, $\underset{\sim}{D} = (0, D_y(x,t), 0)$, $\underset{\sim}{B} = (0, 0, B_z(z,t))$ and $\underset{\sim}{H} = (0,0,H_z(x,t))$ propagating into the region $x > 0$ occuppied by the nonlinear medium; in what follows we will drop the subscripts and just write $E = E_y$, $D = D_y$, etc. We also set $\tilde{\varepsilon}(\zeta) = \varepsilon(0,\zeta,0)$, $\tilde{\mu}(\zeta) = \mu(0,0,\zeta)$, $\forall \zeta \in R^1$ and assume that

(II.1) $\qquad (\zeta\varepsilon(\zeta))' > 0, \quad (\zeta\tilde{\mu}(\zeta))' > 0,$ at least $\forall \zeta \in R^1$, $|\zeta|$ sufficiently small.

(II.1) suffices to yield local hyperbolicity of the resulting quasilinear system. Substituting the assumed form of the propagating plane wave into (I.1), Maxwell's equations reduct to the quasilinear, homogeneous, hyperbolic system

$$\text{(II.2)} \qquad \begin{pmatrix} E \\ H \end{pmatrix}_{,t} + \begin{pmatrix} 0 & a(E) \\ b(H) & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}_{,x} = 0$$

with $a(E) = 1/(\tilde{\varepsilon}(E)E)'$, $b(H) = 1/(\tilde{\mu}(H)H)'$. In all of the previous works which dealt with this problem, i.e., [4]-[8], the system of evolution equations has been expressed in terms of E and H; the system (II.2), however is not in conservation form. To rewrite it in conservation form we note that

$$\text{(II.3)} \qquad D = \int_0^E \frac{d\zeta}{a(\zeta)}, \qquad B = \int_0^H \frac{d\zeta}{b(\zeta)}$$

so that $D_t = E_t/a(E)$, $B_t = H_t/b(H)$. Furthermore, in view of (II.1), the integrals in (II.3) are monotone in E and H, respectively, and thus invertible. Therefore, $\exists \mathcal{E}, \mathcal{H}$ such that $E = \mathcal{E}(D)$, $H = \mathcal{H}(B)$. The system (II.2) then assumes the form

$$\text{(II.4)} \qquad \begin{pmatrix} D \\ B \end{pmatrix}_{,t} \begin{pmatrix} 0 & \mathcal{H}'(B) \\ \mathcal{E}'(D) & 0 \end{pmatrix} \cdot \begin{pmatrix} D \\ B \end{pmatrix}_{,x} = 0$$

where $\mathcal{E}(D) = \lambda(D)D$, $\mathcal{H}(B) = \gamma(B)B$ and $\lambda(D) = 1/\tilde{\varepsilon}(\mathcal{E}(D))$, $\gamma(B) = 1/\tilde{\mu}(\mathcal{H}(B))$. (II.4) has the form of a hyperbolic conservation law of the form

$$\frac{\partial}{\partial t} u_i + \frac{\partial}{\partial x} f_i(\underset{\sim}{u}) = 0 \quad \text{with} \quad \underset{\sim}{u} = \begin{pmatrix} D(x,t) \\ B(x,t) \end{pmatrix}, \quad \underset{\sim}{f}(\underset{\sim}{u}) = \begin{pmatrix} \mathcal{H}(B) \\ \mathcal{E}(D) \end{pmatrix}.$$

For nonmagnetic materials $(\mu(\underset{\sim}{H}) = \mu_o)$, $\mathcal{H}'(B) = 1/\mu_o$ and the system (II.4) has associated real, distinct eigenvalues

$$\lambda_{1,2} = \pm \frac{1}{\sqrt{\mu_o}} \sqrt{\mathcal{E}'(D)}$$

and associated characteristics defined by the equation $\frac{dx}{dt} = \lambda_{1,2}(x,t)$. With (II.4) we associate initial data of the form $D(x,0) = D_o(x)$, $B(x,0) = B_o(x)$ and we remark that a priori estimates of the type proven by Nishida [9] and Slemrod [10] may be used to show that if $\mathcal{E}'(\zeta) > 0$, $\forall \zeta$ s.t. $|\zeta|$ is sufficiently small (a consequence of the definition of $\mathcal{E}$ and (II.1)) and $|\sup_{R^+} D_o(x)|$, $|\sup_{R^+} B_o(x)|$ are sufficiently small, then $\mathcal{E}'D(x,t)) > 0$ for as long as C^1 solution of the IVP for (II.4) exists. Thus the characteristics, as well as the Riemann Invariants given by

(II.5)

$$r(D,B) = B + \frac{1}{\sqrt{\mu_0}}\int_0^D \sqrt{E'(\zeta)}\,ds, \qquad s(D,B) - \frac{1}{\sqrt{\mu_0}}\int_0^D \sqrt{E'(\zeta)}\,ds,$$

are well-defined. It is a simple matter to show that r, s are constant along their respective characteristics so that (II.4) is equivalent to the diagonal system.

$$\text{(II.6)} \quad \begin{cases} r'(x,t) = \frac{\partial r}{\partial t} - \lambda_1(x,t)\frac{\partial r}{\partial x} = \frac{d}{dt}\, r(x,(\gamma,t),t) = 0 \\ s'(x,t) = \frac{\partial r}{\partial t} + \lambda_1(x,t)\frac{\partial s}{\partial x} = \frac{d}{dt}\, s(x_2(\delta,t),t) = 0 \end{cases}$$

$$x_1(\gamma,0) = \gamma, \quad x_2(\delta,0) = \delta, \quad \gamma,\delta \in R^+$$

To this system with associated initial data $r(x,0) = B_0(x) + \frac{1}{\sqrt{\mu_0}}\int_0^{D_0(x)}\sqrt{E'(\zeta)}\,ds$ etc. we may apply the following results on singularity development

(i) [Lax, [27]. If $D_0(\cdot)$ is periodic and $B_0 = 0$ (where we extend from R^+ to R^1 by periodicity) and $E''(0) \neq 0$ (genuine nonlinearity) then finite-time blow up must occur for

$$r_x(x,t) = B_x(x,t) + \frac{1}{\sqrt{\mu_0}}\sqrt{E'(D)}\quad D_x(x,t)$$

$$= -\mu_0 D_t(x,t) + \frac{1}{\sqrt{\mu_0}}\sqrt{E'(D)}\quad D_x(x,t)$$

i.e. $\nabla_{(x,t)}D = (D_x, D_t)$ must blow-up in finite time and a shock develops. Also, if the initial values differ little from a constant r_0, s_0, then the time beyond which a smooth solution cannot be continued is given by

$$\text{(II.7)} \qquad t_{max} \simeq \mu_0\sqrt{E'(0)} \,/\, (\max_{R^+} D_0'(x)\,|E''(0)|).$$

(ii) ([Klainerman and Majda [29]) Suppose that $D_0(\cdot)$, $B_0(\cdot)$ (and hence $r_0(\cdot)$, $s_0(\cdot)$) have compact support; then any C^1 solution of

$\kappa' = \delta' = 0$, s.t. $\kappa_0, \delta_0 \in C^1$ must develop singularities in finite time if $\mathcal{E}'(\zeta)$ is not constant on any open interval.

We now describe briefly two results obtained by this author in [2] which are based on the above theorems. For a nonmagnetic material of the form

$$D = (\epsilon_0 + \chi_0)E + \chi_1 E^2, \quad B = \mu_o H$$

we have $E = \lambda_0 D + \lambda_1 D^2 + O(D^3) \equiv \mathcal{E}(D)$ where λ_0, λ_1 are related to the linear optical susceptibility χ_0 and the first nonlinear optical susceptibility χ_1 by $\lambda_0 = \chi_1/(\epsilon_0+\chi_0)$, $\lambda_1 = -\chi_1^2/(\epsilon_0+\chi_0)^{3/2}$. As $|\mathcal{E}''(0)| = 2|\lambda_1| \neq 0$ we may take $B_0 = 0$ and $E_0(\cdot)$ periodic and apply the results of Lax [27]. In the MKS system ϵ_0, χ_0, χ_1 are all of order of magnitude 10^{-13} while μ_0 is of order of magnitude 10^{-7}. Assuming $E_0 = \sup_{R^+}|E(x,0)|$ to be of order of magnitude 10^9 volts/meter (as it would be in an intense lightwave or laser beam) we find the approximations

$$t_{max} \simeq \frac{\mu_0(\epsilon_0+\chi_0)}{\chi_1{}^{5/2}} \frac{1}{(\max_{R^+} E_0)(\max_{R^+} E_0')}$$

$$v_{beam} \simeq (\mu_0\chi_1)^{-1/2}(\max_{R^+}|E_0|)^{-1/2}$$

and $$s_{max} = C_0(\max_{R^+}|E_0|)^{-3/2}(\max_{R^+}|E_0'|)^{-1}$$

where s_{max} is the maximum distance travelled by the beam into the media until shock formation and $C_0 = \mu_0^{1/2}(\epsilon_0+\chi_0)/\chi_1^3$ is a characteristic constant of the particular material. The above expression for s_{max} compares quite favorably with that derived in [3] especially in as much as the two results show that s_{max} is inversely proportional to the largest initial gradient of the beam (the results in [3] are derived in an entirely different manner). One of the chief benefits of writing our system (II.4) in conservation form is that we may now apply standard results for hyperbolic conservation laws to study the formation and evolution of shocks. The Rankine-Hugoniomnt conditions imply, in particular, the possible existence of two shocks with speeds s_r, s_ℓ given by

(II.8) $$s_r, s_\ell = \pm \frac{1}{\sqrt{\mu_0}} \sqrt{\frac{[E]}{[D(E)]}}$$

while the Lax k-shock conditions [30] (which serve as an admissibility criteria to single out a unique physically realizable weak solution of IVP for (II.4)) reduce to statements that for $s = s_r$, $s = s_\ell$ either

(II.9) $$\frac{1}{\sqrt{(\tilde{\varepsilon}(E_-)E_-)'}} > \sqrt{\mu_0}\, s > \frac{1}{\sqrt{(\tilde{\varepsilon}(E_+)E_+)'}}$$

or

$$-\frac{1}{\sqrt{(\tilde{\varepsilon}(E_-)E_-)'}} > \sqrt{\mu_0}\, s > -\frac{1}{\sqrt{(\tilde{\varepsilon}(E_+)E_+)'}}$$

where E_-, E_+ are the respective values of E just in back of and in front of the shock. For $s = s_r$ it is shown in [1] that only the first of the inequalities is applicable while for $s = s_\ell$ only the second inequality is applicable. Furthermore, it is shown in [1] that in the important special case

$$\tilde{\varepsilon}(E) = \varepsilon_0 + \varepsilon_2 E^2, \qquad \varepsilon_0 > 0, \qquad \varepsilon_2 > 0,$$

where $\mathcal{E}''(D(\zeta)) = -\frac{6\varepsilon_2\zeta}{(\varepsilon_0+3\varepsilon_2\zeta^2)} \cdot \mathcal{E}'(D(\zeta))$, so that $\mathcal{E}''(0) = 0$ but $\mathcal{E}''(D(\zeta)) \neq 0$ for $\zeta \neq 0$, the results of Klainerman and Majda [29] may be applied and for

$$s = s_r = \mu_0^{-1/2}\, (\varepsilon_0+\varepsilon_2(E_+^2 + (E_+ + E_-)E_-))^{-1/2}$$

the admissibility criteria imply that

$$2E_+^2 - E_-^2 > E_+E_- > 2E_-^2 - E_+^2$$

from which it follows that $E_+^2 > E_-^2$ for the shock moving to the right. In an analogous fashion we show in [1] that the admissibility criteria imply that $E_-^2 > E_+^2$ for the shock moving to the left. As the local electric field energy density in the wave given by

$$\text{(II.10)} \qquad \frac{1}{2}\underset{\sim}{D}(\underset{\sim}{E}) \cdot \underset{\sim}{E} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\varepsilon_2 E^4,$$

and energy must be dissipated accross the shock,(Kataev [7]),we conclude that the shock moving to the right with speed $s = s_r$ is not physically realizable.

For the system in nonconservation form, i.e., (II.2), it can be shown (Kataev [7], Jeffrey and Korobeinikov [6]) that there exist simple wave solutions of the form

$$\text{(II.11)} \quad E = E_0(x \pm\sqrt{a(D)b(H)}t), \qquad H = H_0(x \pm\sqrt{a(E)b(H)}t).$$

For the nonmagnetic material these simple waves propagating in the media assume the form

$$\text{(II.12)} \quad E = E_0(x \pm \mu_0^{-1/2}\sqrt{a(E)}t), \quad H = H_0(x \pm \mu_0^{-1/2}\sqrt{a(E)}t).$$

It was essentially deduced by Broer [4] by a direct computation based on (II.12$_1$) that $E_x(x,t) \to \infty$ as $t \to t^* = 2\sqrt{\mu_0}\,\varepsilon^*$ if $\exists\, \varepsilon^* > 0$ s.t. $\frac{\sqrt{a(\zeta)}}{a'(\zeta)} < \frac{1}{\varepsilon^{*'}}$ $\forall \zeta \in R^1$, $|\zeta|$ sufficiently small; this condition is essentially equivalent to the genuine non-linearity assumption. Shock development in the more general case(II.11) may be deduced directly from the work of Jeffrey and Korobeinikov [6], i.e., it is possible to deduce conditions under which E_x, H_x blow up along the characteristics defined by $\frac{dx}{dt} = \pm\sqrt{a(E)b(H)}$ by direct differentiation of (II.11) and use of the fact that

$$\text{(II.13)} \qquad \left.\begin{matrix} r(E,H) \\ s(E,H) \end{matrix}\right\} = \left\{\int_0^E \frac{d\zeta}{\sqrt{a(\zeta)}}\right\} \pm \left\{\int_0^H \frac{d\zeta}{\sqrt{b(\zeta)}}\right\},$$

the Riemann invariants for this problem, are constant along their respective characteristics.

b. Mechanisms for Dissipation: Anisotropy and Nonlinear Conduction, Dispersion, and Relaxation

With the exception of some rather vague remarks in Broer [4], concerning the effects of dispersion on plane wave propagation in isotropic nonlinear dielectrics, the numerical studies of Fisher and Bishel [14] and Shimizu [38], on the interplay between nonlinearity and dispersion, and the effort made in the paper of DeMartini, Townes, Gustafson, and Kelley [3] to build an exponential relaxation process into the light pulse equation, most treatments of wave propagation in nonlinear media, to date, have ignored the effects on shock formation and propagation of anisotropy, nonlinear conduction, dispersion, and relaxation. In this section we will outline an exact mathematical formulation of some of these phenomena emphasizing chiefly the role of anisotropy and nonlinear conduction and indicate how recent work on dissipative and inhomogeneous quasilinear hyperbolic systems can be used to study the models obtained. Electromagnetic shock waves are usually quite difficult to observe; even in the case of ideal isotropy, with no conduction, no dispersion, and no relaxation (instantaneous response) the calculations in [1] and [3], e.g. the expression for s_{max} in §IIa, indicate that whereas shocks always form the physical observation of these shocks depends on the magnitude of the largest initial gradient of the pulse. Gradients which are extremely large, but still within the range of applicability of classical physics, are needed in order that shock formation occur within physically observable distances of several meters (or less). In [3] the authors consider an initial Gaussian pulse of the form

$$\rho(0,t) = \rho_0 \exp(-4t^2/t_\ell^2)$$

where $\rho = n_0^2E^2/8\pi$ is the approximate energy density in the wave and n_0

the nonlinear refractive index; t_ℓ is the initial width of the Gaussian in time. The expression for the maximum distance travelled until shock formation which is derived in [3] is of the form

$$s_{max}^{-1} = 3v_2\left(\frac{d\rho}{dt}\right)_{min}/v_0^2$$

where v_0 is the linear velocity of propagation of the wave. Thus, s_{max} varies inversely with $\left(\frac{d\rho}{dt}\right)_{min}$ which is the largest negative slope of the initial pulse in time. For Ω-switched pulses in CS_2 the authors [3] indicate that t_ℓ is about 10 nsec yielding an s_{max} of about 5 meters; for a mode-locked laser, however, they [3] indicate that t_ℓ can be less than 10^{-11} sec and that pulse steepening (shock formation) can occur over propagation path lengths of less than a centimeter. The basic conjecture which we now put forth is that anisotropy, nonlinear conduction, dispersion and relaxation may act as dissipative mechanisms in the governing evolution equations. That dispersion and relaxation may interfere with shock formation has been touched upon in [3], [4], and [14] via appropriate (numerical) computations. No study, as far as we can tell, has ever touched upon the influence of anisotropy or nonlinear conduction on shock formation in nonlinear electromagnetic materials, even though no material is perfectly isotropic and certainly no dielectric is a perfect nonconductor (especially when interacting with an intense light wave of the type generated by a laser); we therefore begin by formulating a mathematical approach to the influence of anisotropy and conduction on plane wave propagation in nonlinear dielectrics.

We again consider plane waves of the form $\underset{\sim}{E} = (0,E(x,t),0)$, $H = (0,0,H(x,t))$ and assume the material to be nonmagnetic so that $\underset{\sim}{B} = \mu_0\underset{\sim}{H}$ and thus $\underset{\sim}{B} = (0,0,B,(x,t))$. Our assumption of anisotropy takes the form $\underset{\sim}{P} = \underset{\sim}{P}(x,\underset{\sim}{E}(x,t))$, i.e., $\underset{\sim}{P}$ and $\underset{\sim}{E}$ are not parallel in the media; specifically we take

(II.15) $\underset{\sim}{P}(x,\underset{\sim}{E}(x,t)) = (\delta(x),\ P(E(x,t)),0), \quad \delta \in C^1(R^+)$

with

(II.16) $\sup_{x \in R^+}|\delta(x)| \le \delta_0, \quad \left|\sup_{x\in R^+}\right| \frac{\partial\delta}{\partial x} < \delta_1; \quad \delta_0 > 0, \quad \delta_1 > 0$

Then by the definition of the electric induction field

$$\text{(II.17)} \qquad \begin{aligned} \underset{\sim}{D}(x,t) &= \varepsilon_0\underset{\sim}{E} + \underset{\sim}{P}(x,\underset{\sim}{E}(x,t)) \\ &= (\delta(x),\varepsilon_0 E(x,t) + P(E(x,t)),0) \\ &\equiv (\delta(x),\ D(E(x,t)),0). \end{aligned}$$

Clearly div $\underset{\sim}{B} = 0$ and div $\underset{\sim}{D} = \partial\delta/\partial x$ so that $\partial\delta/\partial x$ represents the free charge density. In response to the $\underset{\sim}{E}$ field, and by virtue of the existence of an effective free charge density $\partial\delta(x)/\partial x$, the conduction vector $\underset{\sim}{J}$ can be expected to be nonzero in the dielectric with $\underset{\sim}{J} = \sigma(x,\underset{\sim}{E})\underset{\sim}{E}$ (nonlinear Ohm's Law; the explicit dependence of the conductivity σ on position is consistent with inhomogeneity). In view of our constitutive assumptions

(II.18) $\underset{\sim}{J} = (0,\tilde{\sigma}(x,E(x,t))E(x,t),0)$

where $\tilde{\sigma}(\cdot,\zeta) \equiv \sigma(\cdot,(0,\zeta,0)), \quad \zeta \in R^1$

and $\partial/\partial t(\frac{\partial\delta(x)}{\partial x}) + \nabla\cdot\underset{\sim}{J} = 0$ so that the equation expressing conservation of charge is trivially satisfied. As $\underset{\sim}{J} \neq \underset{\sim}{0}$ the third of Maxwell's equations must be modified so as to read

$$\underset{\sim}{J} + \partial\underset{\sim}{D}/\partial t = \nabla \times \underset{\sim}{H}$$

while the fourth equation is, of course, just div $D = \partial\delta/\partial x$. With the assumption of plane wave propagation these equations then reduce to the scalar quasilinear system

$$\text{(II.19)} \qquad \begin{cases} \tilde{\sigma}(x,E)E + \dfrac{\partial D(E)}{\partial t} = -\dfrac{\partial H}{\partial x} \\[2ex] \dfrac{\partial B}{\partial t} = -\dfrac{\partial E}{\partial x}. \end{cases}$$

Assuming again that $\partial D/\partial E > 0$ so that $\exists \varepsilon$ with $E(X,t) = \varepsilon(D(x,t))$ we may rewrite (II.19) in conservation form as

$$\text{(II.20)} \qquad \begin{cases} \dfrac{\partial D}{\partial t} + \dfrac{1}{\mu_0}\dfrac{\partial B}{\partial x} = -\tilde{\sigma}(x,E(D))E(D) \equiv -\sum(x,D) \\[2ex] \dfrac{\partial B}{\partial t} + E'(D)\,\dfrac{\partial D}{\partial x} = 0 \end{cases}$$

where $\sum(x,D) \equiv \tilde{\sigma}(x,E(D))E(D)$. Clearly (II.20) is an inhomogeneous conservation law of the form

$$\text{(II.21)} \qquad \frac{\partial u_i}{\partial t} + \frac{\partial f_i(\underset{\sim}{u})}{\partial x} = g_i(x,\underset{\sim}{u})$$

with $\underset{\sim}{u} = \binom{D}{B}$, $\underset{\sim}{f}(\underset{\sim}{u}) = \begin{pmatrix} 1/\mu_0\ \underset{\sim}{B} \\ E(\underline{D}) \end{pmatrix}$, $\underset{\sim}{g}(x,\underset{\sim}{u}) = -\begin{pmatrix} \sum(x,D) \\ 0 \end{pmatrix}$.

The system (II.20) may be compared with that considered by Nishida [9], and Slemrod [10], i.e.,

$$\text{(II.22)} \qquad \begin{cases} w_t - v_x = 0 \\ v_t - \Gamma(w)_x = -av \end{cases} \qquad (a \geq 0)$$

which yields the damped nonlinear wave equation

$$w_{tt} + aw_t = \Gamma(w)_{xx}.$$

Initial value problems for (II.22) have been shown in [9] to have global C^1 solutions (w,v) provided the initial gradients $w_x(x,0)$, $v_x(x,0)$ are sufficiently small in the C^1 norm while shocks have been shown to develop, in spite of the damping factor a, if these gradients are sufficently large pointwise [10]. Our system (II.20) leads to the wave equation

$$\text{(II.23)} \qquad D_{tt} + \Sigma'(x,D)D_t = E(D)_{xx}.$$

As $\Sigma'(\cdot,\zeta) = E'(\zeta)(\tilde{\sigma}(\cdot,E(\zeta)) - E(\zeta)\tilde{\sigma}'(\cdot,E(\zeta))$, and $E'(\zeta) > 0$, $\forall\zeta \in R^1$ (hyperbolicity), we <u>conjecture</u> that for sufficiently small initial gradients

$D_x(x,0)$, $B_x(x,0)$ solutions of IVP for (II.20) will also be globally smooth if $\Sigma'(\cdot,\zeta) > 0, \forall\zeta \in R^1 \Leftrightarrow \tilde{\sigma}(\cdot,\gamma) > \gamma\tilde{\sigma}'(\cdot,\gamma), \forall\gamma \in R^1$, but that shocks will develop if these initial gradients are, pointwise, sufficiently large. The analysis in [9], [10] is based on the equivalent system of equations satisfied by the Riemann invariants associated with (II.21). For our system (II.20) the Riemann invariants are again given by (II.5) where now, for the sake of simplicity, we normalize and take $\mu_0 = 1$. Then (II.20) is easily seen to be equivalent to

$$(II.24) \qquad r' = -\sqrt{E'(D)}\ \Sigma(x,D), \quad s^{`} = \sqrt{E'(D)}\ \Sigma(x,D)$$

where $' = \partial/\partial t + \lambda\,\partial/\partial x$, $^{`} = \partial/\partial t - \lambda\partial/\partial x$, $\lambda = +\sqrt{E'(D)}$.

As $\frac{r-s}{2} = \int_0^D \sqrt{E'(\zeta)}\ d\zeta$, $\exists\,\beta^*$ such that

$$D = \beta^*(r-s) \quad (\text{i.e. } \frac{a}{2} = \int_0^\beta \sqrt{E'(\zeta)}\ d\zeta \rightarrow \beta = \beta^*(a)).$$

Thus (II.24) assumes the form

$$(II.25) \qquad \begin{aligned} r' &= -\rho(x,r-s), \quad ' = \partial/\partial t + \lambda^*(r-s)\partial/\partial x \\ s^{`} &= \rho(x,r-s), \quad ^{`} = \partial/\partial t - \lambda^*(r-s)\partial/\partial x \end{aligned}$$

where

$$\begin{cases} \rho(x,r-s)) = \sqrt{E'(\beta^*(r-s)} \cdot \Sigma(x,\beta^*(r-s)) \\ \Sigma(x,\beta^*(r-s)) = \tilde{\sigma}(x,E(\beta^*(r-s)))E(\beta^*(r-s)) \\ \lambda^*(r-s) = \lambda(\beta^*(r-s)) = \sqrt{E'(\beta^*(r-s))}. \end{cases}$$

Thus, one goal of further work in this area should be to investigate the existence or nonexistence of global smooth solutions for IVP for the system (II.25) beginning with the case where the conductivity is homogeneous so the ρ does not depend explicitly on position. In this regard an attempt should be made to generalize and extend the results of Nishida [9] and Slemrod [10] for systems of the form

$$r' = -\frac{a}{2}(r+s), \qquad s` = -\frac{a}{2}(r+s)$$

by examining the behavior of r, s along their respective characteristics (in our problem the curves defined by $\frac{dx}{dt} = \pm\sqrt{E'(\beta^*(r-s))} = \pm\lambda(\beta^*(r-s)) = \lambda^*(r-s)$. Such work is now in progress.

We note here that we could equally well work with E, H instead of D, B. If we set $a(E) = \partial D/\partial E$, $b(H) = \partial B/\partial H$ (we do not now begin with the assumption that the media is nonmagnetic), $A(E) = 1/a(E)$, $B(H) = 1/b(H)$, then (II.19) assumes the form

$$\text{(II.26)} \qquad \begin{pmatrix} E \\ H \end{pmatrix}_{,t} + \begin{pmatrix} 0 & A(E) \\ B(H) & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}_{,x} = -\begin{pmatrix} \hat{\Sigma}(x,E) \\ 0 \end{pmatrix}$$

where $\hat{\Sigma}(x,E) = -\tilde{\sigma}(x,E)A(E)E$

The characteristics associated with (II.26) are given by

$$\frac{dx}{dt} = \pm\sqrt{A(E)B(H)}$$

and the left-eigenvectors associated with $\begin{pmatrix} 0 & A(E) \\ B(H) & 0 \end{pmatrix}$ are $\underset{\sim}{\ell} = (1, \pm\sqrt{A(E)/B(H)})$.

Using multiplication by the componenets of the first left eigenvector to combine the equations in (II.26) in the standard fashion we find that

$$\frac{E'}{\sqrt{A(E)}} + \frac{H'}{\sqrt{B(H)}} = -\frac{\tilde{\Sigma}'(x,E)}{\sqrt{A(E)}} = -\sqrt{A(E)}\tilde{\sigma}(x,E)E \equiv -\Lambda(x,E)$$

where $' = \partial/\partial t + \sqrt{A(E)B(H)}\ \partial/\partial x$ denotes differentiation along the characteristic $\frac{dx}{dt} = \sqrt{A(E)B(H)}$. Thus if we define the Riemaniann invariants

$$\begin{cases} r(E,H) = \int_0^E \sqrt{a(\zeta)}\, d\zeta + \int_0^H \sqrt{b(\zeta)}\, d\zeta \\ s(E,H) = \int_0^E \sqrt{a(\zeta)}\, d\zeta - \int_0^H \sqrt{b(\zeta)}\, d\zeta \end{cases}$$

we have

$$r' = -\Lambda(x,E), \quad s` = -\Lambda(x,E)$$

In the nonmagnetic case

$$\text{(II.27)} \qquad r = H + \int_0^E \sqrt{a(\zeta)}\, d\zeta, \qquad s = -H + \int_0^E \sqrt{a(\zeta)}\, d\zeta$$

and

$$\text{(II.28)} \qquad r' = -\Lambda(x,E), \qquad s` = -\Lambda(x,E)$$

with $' = \partial/\partial t + \sqrt{A(E)}\ \partial/\partial x$, $` = \partial/\partial t - \sqrt{A(E)}\ \partial/\partial x$.

But, by (II.27), $\frac{r+s}{2} = \int_0^E \sqrt{a(\zeta)}\, d\zeta$ so $\exists g^*$ s.t. $E = g^*(r+s)$.

Setting $\lambda(E) = \sqrt{A(E)}$, we then have as a consequence of (II.28)

$$\text{(II.29)} \qquad \begin{cases} r_t + \lambda^*(r+s) r_x = -\Psi(x, r+s) \\ s_t - \lambda^*(r+s) s_x = -\Psi(x, r+s) \end{cases}$$

where

$$\begin{cases} \lambda^*(r+s) = \lambda(g^*(r+s)) = \sqrt{A(g^*(r+s)} \\ \Psi(x, r+s) = +\Lambda(x, g^*(r+s)) \\ \quad = g^*(r+s)\sqrt{A(g^*(r+s))}\,\tilde{\sigma}(x, g^*(r+s)). \end{cases}$$

Finally, if we do not specialize to the case of a nonmagnetic material then

$$\frac{r+s}{2} = \int_0^E \sqrt{a(\zeta)}\, d\zeta, \qquad \frac{r-s}{2} = \int_0^H \sqrt{b(\zeta)}\, d\zeta$$

so that we may set

$$E = g^*(r+s), \qquad H = f^*(r-s)$$

$$\lambda(E) = \sqrt{A(E)} = \sqrt{A(g^*(r+s))} = \lambda^*(r+s)$$

$$\phi(H) = \sqrt{B(H)} = \sqrt{B(f^*(r-s))} = \phi^*(r-s)$$

and in place of (II.29) we obtain

$$\text{(II.30)}\quad \begin{aligned} r_t + \lambda^*(r+s)\phi^*(r-s)r_x &= -\Psi(x,r+s) \\ s_t - \lambda^*(r+s)\phi^*(r-s)\,s_x &= -\Psi(x,r+s) \end{aligned}$$

with $\Psi(x,r+s)$ as in (II.29).

We could also approach these problems in terms of trying to apply and/or extend results on the existence and behavior os solutions to inhomogeneous hyperbolic conservation laws of the form (II.21), where in our problem $\underset{\sim}{f}(\underset{\sim}{u}) = \binom{A(B)}{E(D)}$, when $\mu(H) = \text{const}$. With regard to such problems the existing literature is quite small but some important recent work (Liu [11], Liu and Li [31], Dafermos and Hsiao [13], and Dafermos [12]) should prove to be of interest for our particular problem. Liu [11], in fact, constructs weak solution of (II.21) for the case where $\underset{\sim}{u} = \underset{\sim}{u}(x,t)$ is an n-vector, $\underset{\sim}{f}$ is a smooth n-vector-valued function of $\underset{\sim}{u}$ and $\underset{\sim}{g}$ and $\partial\underset{\sim}{g}/\partial\underset{\sim}{u}$ are piecewise continuous n-vector-valued functions of x which are continuous in $\underset{\sim}{u}$, and studies their asymptotic behavior as $t \to +\infty$. It is assumed that $\partial\underset{\sim}{f}(\underset{\sim}{u})/\partial\underset{\sim}{u}$ has real and distinct eigenvalues $\lambda_1(\underset{\sim}{u}) < \lambda_2(\underset{\sim}{u}) < \cdots < \lambda_n(\underset{\sim}{u})$ for each $\underset{\sim}{u}$. The analysis in [11] is based on a numerical scheme which generalizes the Glimm scheme [32] for hyperbolic conservation laws. Among the results proven in [11] are the following:

(i) when the $\lambda_i(\underset{\sim}{u}(x))$ $i = 1,2,\ldots,n$ are nonzero and the L^1 norm of $g(x,\underset{\sim}{u}(x))$ is small for $\underset{\sim}{u}(x)$ uniformly close to the initial data $\underset{\sim}{u}_0(x)$, a global solution of the IVP for (II.21) exists and tends pointwise to a steady state solution of $\frac{\partial\underset{\sim}{f}(u)}{\partial x} = \underset{\sim}{g}(x,\underset{\sim}{u})$.

(ii) when each characteristic field is either genuinely nonlinear or linearly degenerate (Lax [30]) the solution tends uniformly to a linear superposition of shock waves, refraction waves, travelling waves and a steady state solution; however, these waves are determined by the values of the initial data $\underset{\sim}{u}_0(x)$ at $x = \pm\infty$. It is assumed, in the general case, that the initial data have small total variation.

In [31], Liu and Li study IVP for (II.21) under the assumption that $g(x.u)$ has compact support in x, with particular emphasis on constructing non-interacting wave patterns for conservation laws with a general moving source term of the form

$$\text{(II.31)} \qquad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(x-ct,u).$$

This system can be reduced to (II.21) under the change of variables $f(u) \to f(u) - cu$ and $x \to x - ct$. In [13] Dafermos and Hsiao consider IVP for systems of the form (II.21) with $g(x\ u(x,t)) \to g(x,t,u(x,t))$ and establish the existence of locally defined solutions with shock waves. They prove the existence of globally defined solutions by introducing an appropriate definition of dissipativeness for the source $g(x,t,u)$ and showing (via a combination of Glimm's scheme for conservation laws and the method of fractional steps) that its'effect counterbalances the wave amplification due to inhomogeneity.

Many of the existing results for systems of the for (II.21) in both R^1 and R^n, $n > 1$, with an emphasis on asymptotic behavior, are summarized in Dafermos [12] while problems for particular forms of either the inhomogeneity g or the function $f(u)$ have been treated by Liu [33] and Ying and Wang [34].

Concerning the possible dissipative effects of relaxation on the formation and evolution of shock waves in nonlinear electromagnetic materials, we should be interested in replacing constitutive relations of the form (I.41) by assumptions of either the form $D(x,t) = \underset{-\infty}{\overset{t}{E}}(E(x,\tau))E(x,t)$ with

$$\text{(II.32)} \qquad \underset{-\infty}{\overset{t}{E}}(E(x,\tau)) = \varepsilon_0 + \int_{-\infty}^{t} \varepsilon_2(t-\tau)\,|E|^2(x,\tau)\ d\tau.$$

and $E = (0,E(x,t),0)$, $H = (0,0,H(x,t))$, $B = H$

or $\underset{\sim}{P} = (0, P(x,t), 0)$ with

$$\text{(II.33)} \qquad P(x,t) = \overset{t}{\underset{-\infty}{P}} (E(x,t)) = \int_{-\infty}^{t} \psi(t-\tau)E^2(x,\tau)d\tau.$$

In this latter case we would have $\underset{\sim}{D} = (0, D(x,t), 0)$ and

$$\text{(II.34)} \qquad D(x,t) = \varepsilon_0 E(x,t) + \int_{-\infty}^{t} \psi(t-\tau)E^2(x,\tau)d\tau.$$

It then follows that Maxwell's equations reduce to the system of scalar quasilinear hyperbolic integrodifferential equations

$$\text{(II.35)} \quad \begin{cases} \varepsilon_0 \dfrac{\partial E}{\partial t} + \dfrac{\partial H}{\partial x} = -\psi(0)E^2(x,t) - \displaystyle\int_{-\infty}^{t} \psi_t(t-\tau)D^2(x,\tau)d\tau \\ \dfrac{\partial H}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \end{cases}$$

With Riemann invariants

$$r = E + \frac{1}{\sqrt{\varepsilon_0}} H, \qquad s = E - \frac{1}{\sqrt{\varepsilon_0}} H,$$

and $' = \partial/\partial t + \frac{1}{\sqrt{\varepsilon_0}} \partial/\partial x, \qquad ` = \partial/\partial t - \frac{1}{\sqrt{\varepsilon_0}} \partial/\partial x,$

we obtain the following system of nonlinear integrodifferential equations along the characteristics $\frac{dx}{dt} = \pm \frac{1}{\sqrt{\varepsilon_0}}$.

$$\text{(II.36)} \quad \begin{cases} r' = -\frac{1}{4}\psi(0) \; (r+s)^2 - \frac{1}{4} \displaystyle\int_{-\infty}^{t} \psi_t(t-\tau) \; (r+s)^2 \; (x,\tau)d\tau \\ s` = -\frac{1}{4}\psi(0) \; (r+s)^2 - \frac{1}{4} \displaystyle\int_{-\infty}^{t} \psi_t(t-\tau) \; (r+s)^2 \; (x,\tau)d\tau. \end{cases}$$

If we follow the constitutive hypothesis (II.32), instead of (II.33), then we obtain, in place of (II.38) the system

$$
\text{(II.37)} \quad \begin{cases} [\varepsilon_0 + \int_{-\infty}^{t} \varepsilon_2(t-\tau)E^2(x,\tau)d\tau] \frac{\partial E}{\partial t} + \frac{\partial H}{\partial x} = \\ -\varepsilon_2(0)E^3(x,t) - \left(\int_{-\infty}^{t} \frac{\partial}{\partial t} \varepsilon_2(t-\tau)E^2(x,\tau)d\tau\right) E(x,t) \\ \frac{\partial H}{\partial t} + \frac{\partial E}{\partial x} = 0 \end{cases}
$$

whose characteristics are defined by the equations

$$
\frac{dx}{dt} = \pm[\varepsilon_0 + \int_{-\infty}^{t} \varepsilon_2(t-\tau)E^2(x,\tau)d\tau]^{-1/2}
$$

The system (II.36) has the advantage of possessing families of parallel straight lines as characteristic curves in the x,t plane but distortion of propagating waves may occur due to the nonlinear source terms on the right hand sides of these equations; an analysis paralleling the work of MacCamy [35], [36] and Hattori [37] on materials with memory may be possible. The form of the constitutive hypothesis (II.32) more closely follows the formulation in [3] than does (II.33) and a numerical treatment in [3] indicates that a rarefaction shock may still build up. The importance of considering relaxation effects in electromagnetic media is emphasized in Kataev [7] who indicates that equations (I.4) "can be invalid for material media (if there is) a lag in the change in induced fields which occurs in response to a rapid change in the field intensities in the media. This lag in the reactions may be attributable to the magnetic or dielectric viscosity of the medium"

Up to this point we have totally ignored the role that dispersion might play in propagation of waves in nonlinear electromagnetic media. In a linear dielectric ($\underset{\sim}{P} = \varepsilon_0\chi E$) where the dimensionless coefficient χ is the polarisibility of the medium and $\underset{\sim}{E}$ satisfies the wave equation

$$
\text{(II.38)} \qquad \Delta E - \varepsilon_0\mu_0(1+\chi)\partial^2 E/\partial t^2 = 0,
$$

so that the wave speed is given by $\tilde{c} = \{\varepsilon_0\mu_0(1+\chi)^{-1/2}\}$ and the refractive index $n = c/\tilde{c} = (1+\chi)^{1/2}$, dispersion and relaxation are usually introduced into the linear theory by replacing the above constitutive relation between $\underset{\sim}{P}$ and $\underset{\sim}{E}$ by the Lorentz equation

$$\text{(II.39)} \qquad \frac{1}{\omega_0^2} \frac{\partial^2 \underset{\sim}{P}}{\partial t^2} + \underset{\sim}{P} = \varepsilon_0 \chi \underset{\sim}{E}.$$

For a wave having an $\underset{\sim}{E}$ field with monochromatic frequency ω (II.41) leads to the well known result that $n^2 = 1 + \chi\omega_0^2/(\omega_0^2 - \omega^2)$. One way of introducing dispersion into the nonlinear optics relations (I.4) for an isotropic media (which we also assume to be nonmagnetic) would be to take

$$\text{(II.40)} \qquad \varepsilon = \varepsilon(\omega;E) = \varepsilon_0(\omega) + \varepsilon_2(\omega)E^2$$

and consider the propagation of plane waves in which E has the form (0,E,0) with,

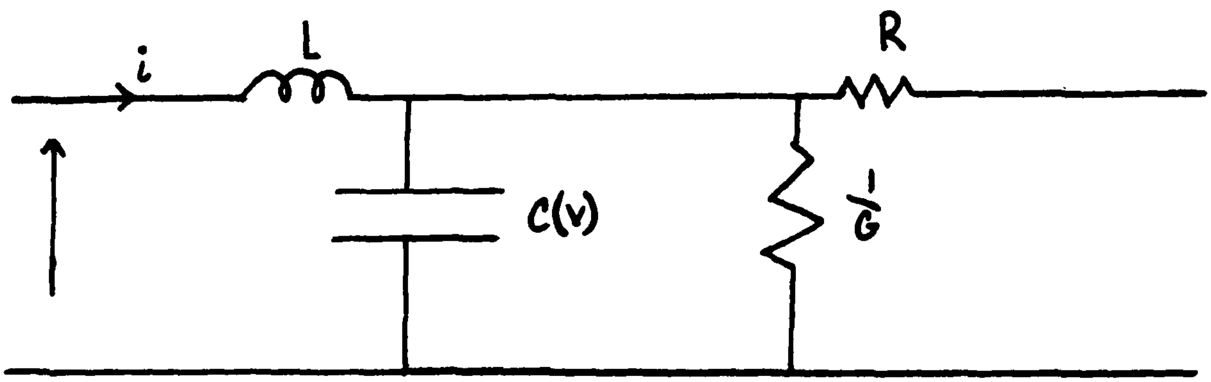
$$\text{(II.41)} \qquad E(x,t) = \mathrm{Re}(E_0(x)e^{-i\omega t}e^{ikx}), \quad k \text{ the wave number.}$$

The corresponding reduction of Maxwell's equations is a straight-forward matter and will not be pursued here (although it is being pursued by this author within the scope of the general research program described above).
We conclude this subsection by noting that both the numerical studies of Fisher and Bishel [14] as well as the earlier numerical studies of Shimizu [38] have shown that shocks can form on the leading edge of pulses in a Cs_2 laser when dispersion and nonlinearity are considered simultaneously.

III. Electromagnetic Shock Wave Formation in Nonlinear Distributed Parameter Transmission Lines

Shock formation and propagation in nonlinear transmission lines has been studied by a number of authors ([7], and the references to some of the Russian literature cited therein, [15], [16], [42], and [43]). As Jeffrey [16] points out, the method of analysis that is used for linear systems does not easily generalize to the study of nonlinear distributed parameter transmission lines since the customary procedure in the linear theory is to convert the two first order transmission line equations into a single second order equation, the well-known equation of telegraphy. When the transmission line is nonlinear, the voltage and current are no longer solutions to the linear telegrapher's equation and superposition of solutions no longer applies. (Further developments in this vein require that the nonlinearity be simplified via approximations of various kinds, as in Ostrovskii [43]).

Landauer [15] and Riley [42] both exploited the formal equivalence between the nonlinear transmission line equations and the one-dimensional isentropic gas flow equations to study nonlinear transmission line behavior. In fact, Landauer's work [15] represents a direct application of the methods of gas dynamics [41] while Riley's [42] contains an application of the work in [15] to a nonlinear transmission line involving a voltage dependence capacitance. but Jeffrey [16] notes that neither of the author's ([15], [42]) "considered the problem of when a continuous wave propagating down a nonlinear transmission line first becomes discontinuous and forms an electromagnetic shock wave"; he considers in [16] an idealized transmission line with distributed parameters C(v) the capacitance, R the resistance, L the self-inductance, and G the leakage conductance per unit length of the line (see figure below)



Here $v(x,t)$ is the voltage at a point which is x units distant from an origin taken in the line; we also denote by $i(x,t)$ the current at (x,t) and note that by definition $C(v) = dQ/dv$ where $Q(v)$ is the voltage dependent charge per unit length of the line. Ohm's and Kirchoff's laws when applied to the element of the transmission line depicted above yield the system of equations

$$\text{(III.1)} \quad \begin{cases} L \dfrac{\partial i}{\partial t} + \dfrac{\partial v}{\partial x} + Ri = 0 \\ C(v) \dfrac{\partial v}{\partial t} + \dfrac{\partial i}{\partial x} + Gv = 0 \end{cases}$$

In (III.1) the analysis in [16] assumes that L, R, and G are constant. It is clear that (III.1) has the equivalent form

$$\text{(III.2)} \quad \begin{pmatrix} i \\ v \end{pmatrix}_{,t} + \begin{pmatrix} 0 & 1/L \\ 1/C(v) & 0 \end{pmatrix} \begin{pmatrix} i \\ v \end{pmatrix}_{,x} = \begin{pmatrix} -\frac{R}{L} i \\ -\frac{Gv}{C(v)} \end{pmatrix}$$

of an inhomogeneous quasilinear hyperbolic system if $C(v) = \frac{dQ}{dv} > 0$. In fact, the associated characteristics are clearly defined by $\frac{dx}{dt} = \frac{1}{\sqrt{LC(v)}}$. For C independent of v, $C = C_0 = \text{const.}$, Jeffrey concludes [16] that

$$\text{(III.3a)} \quad \frac{d}{dt} \left(i\left(\frac{L}{C_0}\right)^{1/2} + v\right) = -\left(\frac{R}{\sqrt{LC_0}} i + \frac{G}{C_0} v\right)$$

$$\text{along solutions of } \frac{dx}{dt} = -\frac{1}{\sqrt{LC_0}}$$

$$\text{(III.3b)} \quad \frac{d}{dt} \left(i\left(\frac{L}{C_0}\right) 1/2 - v\right) = -\left(\frac{R}{\sqrt{LC_0}} i - \frac{G}{D_0} v\right)$$

$$\text{along solutions of } \frac{dx}{dt} = -\frac{1}{\sqrt{LC_0}}$$

and shows that i, v both propagate along the linear line with attenuation factor $-G/C_0$ if $R/L = G/C_0$. In attempting, however, to show that an electromagnetic shock forms in the line with $C = C(v) > 0$, $K = G = 0$ he incorrectly (his eqs. (4. 1a), (4.1b) claims that along solutions of $\frac{dx}{dt} = \pm \frac{1}{\sqrt{LC(v)}}$, respectively,

$$\begin{cases} \frac{d}{dt}\left(i\left(\frac{L}{C(v)}\right)^{1/2} + v\right) = 0 & \text{and} \\ \frac{d}{dt}\left(i\left(\frac{L}{C(v)}\right)^{1/2} - v\right) = 0 \end{cases}$$

whereas the appropriate Riemann invariants must now be taken as

$$\text{(III.4)} \qquad \begin{cases} r(i,v) = i + \frac{1}{\sqrt{L}} \int_0^v \sqrt{C(\zeta)}\, d\zeta \\ s(i,v) = i - \frac{1}{\sqrt{L}} \int_0^v \sqrt{C(\zeta)}\, d\zeta. \end{cases}$$

That r, s as defined by (III.4) are constant along the respective characteristics defined by $\frac{dx}{dt} = \pm \frac{1}{\sqrt{LC(v)}}$ follows directly from (III.2) with $R = G = 0$; shock formation for IVP associated with (III.2), with $R = G = 0$, is then a consequence of the results of Lax [27] for the genuinely nonlinear situation $Q''(0) \neq 0$ or Klainerman and Majda [29] for the case of compactly supported initial data $i(x,0)$, $v(x,0)$ with $Q''(\zeta) \neq 0$ on every open interval of R^1. Even allowing for the incorrect form for the Riemann invariants in [16], shock formation is obtained in this paper only for waves propagating into a constant state i_0, v_0 and follows from the fact that characteristics of one family adjacent to the constant state have constant slope and are thus straight lines. Moreover, the shocks obtained are weak in the sense that the fields are required to be everywhere continuous so that "shock formation" involves the development of jump discontinuities in the first derivatives of the fields. For the model (III.2) with $R^2 + G^2 \neq 0$, Jeffrey shows [16, §5] that disconinuities form if either

(i) $\gamma_x \left(\frac{\partial C}{\partial v}\right)_0 < 0$ or

(ii) $\tilde{\gamma}_x(\frac{\partial C}{\partial v})_0 > 0$ and $\frac{C_0^2}{\tilde{\gamma}_x(\frac{\partial C}{\partial v})_0}(\frac{R}{L} + \frac{G}{C_0}) < 1$

where the zero subscript indicates evaluation at the constant state ahead of the wavefront trace and the superposed $\sim$ indicates the limiting value along the wave front trace. However, as $Q = Q(v)$ and $C(v) = \frac{dQ}{dv} > 0$ the expression $Q(v) = \int_{v_0}^{v} C(\zeta)d\zeta$ may be inverted to yield $v = V(Q)$ and the inhomogeneous system (III.2) may be rewritten in conservation form as

$$\text{(III.5)} \qquad \begin{cases} \frac{\partial i}{\partial t} + \frac{1}{L}(V(Q))_{,x} = -Ri \\ \frac{\partial Q}{\partial t} + \frac{\partial i}{\partial x} = -GV(Q). \end{cases}$$

If $G = 0$ (no leakage current between the conductors) then by direct analogy with the damped Quasilinear system (II.21) $[w \to Q,\ v \to i,\ \Gamma \to \frac{1}{L}v$ and $a \to R]$ it should follow from the work of Nishida [9] and Slemrod [10] that C^1 solutions of IVP for (III.5), and thus also for (III.1)), should exist globally if $i_x(x,0)$ and $Q_x(x,0)$ are sufficiently small in the C^1 norm,but that shocks should develop if either of these initial gradients is sufficiently large at some point in the transmission line; this contrasts sharply with the results of Jeffrey [16] cited above where discontinuities are proven to form (albeit, weak shocks) if $\tilde{\gamma}_x Q(v)_0 < 0$ and the wave is propagating into a constant state. Work is now in progress which involves analyzing the system (III.5), with $G \neq 0$, by re-writing (III.5) in terms of the appropriate Riemann invariants and studying the behavior of these Riemann invariants along their respective characteristics; in this sense the study of (III.5) is expected to have close connections with the study of the systems (II.25), (II.29), and (II.30).

In the model considered in [7], Kataev begins with the equations

$$\text{(III.6)} \quad \begin{cases} \frac{\partial \Phi}{\partial t} + \frac{\partial v}{\partial x} = 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial i}{\partial x} = -J \end{cases}$$

where J is the leakage current and Φ the magnetic flux per unit length between the conductors. The connection with (III.1) is that $\Phi = \Phi(i)$, $L(i) = \partial\Phi/\partial i$ (the differential inductance), $J = J(i,v) = G(i,v)\cdot v$ and, as previously assumed, $Q = Q(v)$; in (III.1), where $R \neq 0$, it has been assumed that $L(i)$ and the conductance $G(v,i)$ are constant. Kataev [7] then specializes to the case where $L(i) = L_0$ = const., $C(v) = C_0$ = const. and thus obtains from (III.6) the system

$$\text{(III.7)} \quad \begin{cases} L_0 \frac{\partial i}{\partial t} + \frac{\partial v}{\partial x} = 0 \\ C_0 \frac{\partial v}{\partial t} + \frac{\partial i}{\partial x} = -G(i,v)\cdot v \end{cases}$$

whose characteristics, defined by $\frac{dx}{dt} = \pm \frac{1}{\sqrt{L_0C_0}}$, are straight lines (we assume that $L_0C_0 > 0$). He looks for solutions of (III.7) in the form of stationary shocks, i.e., for solutions which are functions of $w = x - \beta t$ where β is the velocity of propagation; in this case $i = i(w)$, $v = v(w)$ and (II.7) becomes

$$\text{(III.8)} \quad \begin{cases} v - v_0 = \beta L_0(i-i_0) \\ \frac{di}{dw} = \beta C_0 \frac{dv}{dw} - G(i(w),v(w))\cdot v(w) \end{cases}$$

after integration of the first equation. In (III.8), i_0, v_0 refer to the initial values of the current and voltage. Solutions of (III.8) are then sought for which $i \to i_0$ as $w \to +\infty$ and $i \to i_0 + \hat{i}_0$ as $w \to +\infty$, where i_0 is the amplitude of the stationary wave; this is achieved by integrating (III.8):

$$\left(\frac{1}{1-\beta^2L_0C_0}\right) = \int_{i_0}^{i} \frac{d\zeta}{G[\zeta,\beta L_0(\zeta-i_0) + v_0]\cdot[L_0\beta(\zeta-i) + v_0]}$$

and then imposing the boundary conditions so as to conclude that

(a) for $i_0 < i < i_0 + \hat{i}_0$ and $v = \beta L_0(i-i_0) + v_0$, $G(i,v) > 0$

(b) $\int_{i_0}^{i} \frac{di}{G(i,v)v}$ is divergent for $v = \beta L_0(i-i_0) + v_0$

when $i = i_0$ and $i = i_0 + \hat{i}_0$.

A further analysis then shows that (a), (b) require that as we approach points on the profile of the wave $G(i,v) \to 0$ yielding (in Kataev's notation) at each point on the profile a relation of the form $v \approx \gamma(i)$. Taking this last relation into account, Kataev then rewrites the first equation in the system (III.7) in the form

(III.9) $$\frac{\partial i}{\partial x} \simeq -(L_0/\frac{d\gamma(i)}{di}) \frac{\partial i}{\partial t}$$

for which (approximate)simple wave solutions of the form

(III.10) $$i \simeq I(x - \frac{1}{L_0}\frac{d\gamma(i)}{di} \cdot t)$$

are immediate. The development of discontinuities in the current i now follows immediately, under obvious conditions on $\gamma(i)$, via a direct computation of the gradient i_x. A more satisfactory approach to the study for (III.7) in our opinion, consists of introducing the Riemann invariants

(III.11) $r(i,v) = \sqrt{L_0}\ i + C_0\ v$, $s(i,v) = \sqrt{L_0}\ i - \sqrt{C_0}\ v$

which satisfy (along their respective characteristics defined by $\frac{dx}{dt} = \frac{1}{\sqrt{L_0 C_0}}$)

the equations

(III.12) $$\begin{cases} r' = \frac{1}{-2\sqrt{L_0C_0}} G_0(r+s, r-s)\cdot(r-s) \\ s` = +\frac{1}{2\sqrt{L_0C_0}} G_0(r+s, r-s)\cdot(r-s) \end{cases}$$

where, $G_0(r+s, r-s) = G(\frac{1}{2\sqrt{L_0}}(r+s), \frac{1}{2\sqrt{C_0}}(r-s))$. The inhomogeneous terms on the right hand side of (III.12) simplify considerably if the conductance $G(i,v)$ is independent of the current i.

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