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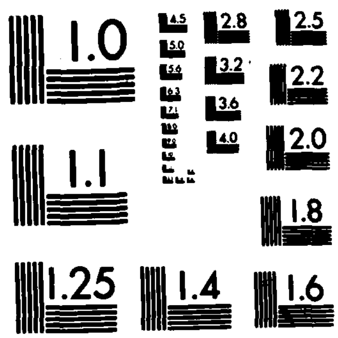

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LINEAR PARABOLIC EQUATIONS WITH A SINGULAR LOWER ORDER COEFFICIENT

Klaus Höllig

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

Let a,b belong to the Hölder class $H^{\alpha,\alpha/2}([0,1]\times[0,T])$ with $\alpha \in (0,1)$ and $a > a_0 > 0$. It is shown that for the solution of the problem

$$u_t - au_{xx} - t^{-1/2}bu_x = t^{-1/2}f, \quad (x,t) \in [0,1]\times[0,T] \quad ,$$

$$u(\cdot,0) = \phi$$

$$u(\nu,\cdot) = \psi_\nu \quad , \quad \nu = 0,1 \quad ,$$

the estimate

$$\|u,u_x,t^{1/2}u_{xx}\|_{\alpha,\alpha/2} <$$

$$c\{\|f\|_{\alpha,\alpha/2} + \|\phi,\phi'\|_\alpha + \sum_{\nu=0}^{1} \|\psi_\nu,t^{1/2}\psi'_\nu\|_{\alpha/2}\}$$

holds if the data f, ϕ, ψ_ν satisfy the appropriate compatibility conditions. Here $\| \ \|_{\alpha,\alpha/2}$, $\| \ \|_\alpha$, $\| \ \|_{\alpha/2}$ denote the norms of the Hölder classes $H^{\alpha,\alpha/2}([0,1]\times[0,T])$, $H^\alpha([0,1])$ and $H^{\alpha/2}([0,T])$ respectively and the constant c depends on α, a_0, $\|a,b\|_{\alpha,\alpha/2}$ and T. The result extends to quasilinear problems with a, b and f depending on x, t and u.

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Key Words: parabolic equations, linear, singular, regularity

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SIGNIFICANCE AND EXPLANATION

This report was motivated by the study of free boundary value problems related to the Stefan problem. The precise description of the smoothness of the free boundary requires sharp regularity results for linear parabolic equations with singular coefficients.

These results are of independent interest. They can be applied to parabolic equations on domains with curved boundaries that touch the x-axis. As a particular example consider the heat equation;

$$u_t - u_{xx} = t^{-1/2} f \quad \text{on} \quad \Omega$$

$$u = 0 \quad \text{on} \quad \partial\Omega$$

on the domain $\Omega := \{(x,t) : t > 0,\ x > -t^{1/2}\}$. Our results imply that u and u_x are Hölder continuous up to the boundary if f is and $f(0,0) = 0$. This example arises in the ~~convexification~~ convexity of the nonlinear parabolic problem studied in MRC Technical Summary Report #2354.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

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LINEAR PARABOLIC EQUATIONS WITH A SINGULAR LOWER ORDER COEFFICIENT

Klaus Höllig

1. Statement of the result

We consider the linear parabolic initial boundary value problem

$$(1.1)\qquad \begin{cases} u_t(x,t) - a(x,t)u_{xx}(x,t) - t^{-1/2}b(x,t)u_x(x,t) = t^{-1/2}f(x,t), \\ \qquad\qquad\qquad\qquad (x,t) \in \Omega_T := [0,1]\times[0,T], \\ u(x,0) = \phi(x)\ ,\ x \in [0,1], \\ u(\nu,t) = \psi_\nu(t),\ t \in [0,T],\ \nu = 0,1\ . \end{cases}$$

Problems of this type, with a singular coefficient of u_x, may arise when transforming parabolic equations from a domain with curved boundaries to the standard domain Ω_T. Consider, e.g., the heat equation,

$$(1.2)\qquad \begin{cases} v_t - v_{yy} = t^{-1/2}g & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

on the domain $\Omega = \{(y,t) : 0 < y < 1 + t^{1/2},\ t \in [0,T]\}$. By the change of variables $x = y/(1 + t^{1/2})$, (1.2) is equivalent to (1.1) with $u(x,t) = v(y,t)$, $f(x,t) = g(y,t)$, $\phi = \psi_\nu = 0$, $a = (1 + t^{1/2})^{-2}$ and $b(x,t) = \frac{1}{2}\,x/(1 + t^{1/2})$. More generally, any parabolic equation on a domain with smooth vertical boundaries that touch the x-axis, but have nonzero curvature as $t \to 0$, leads, after a change of variables, to a problem of the form (1.1).

We shall show that, under the assumptions on the coefficients and the data specified in (1.3) - (1.5) below, the solution of problem (1.1) and its partial derivative with respect to x are Hölder continuous up to the

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boundary. This is, in general, no longer true for a singularity of the form $t^{-1/2-\varepsilon}$, $\varepsilon > 0$, in the coefficient of u_x.

We use the Hölder norms

$$|\chi|_{\alpha,I} := \sup_{z,z' \in I} |\chi(z) - \chi(z')|/|z - z'|^{\alpha}$$

$$\|\chi\|_{\alpha,I} := \|\chi\|_{\infty,I} + |\chi|_{\alpha,I}$$

$$|w|_{\alpha,x,\Omega} := \sup_{(x,t),(x',t) \in \Omega} |w(x,t)-w(x',t)|/|x-x'|^{\alpha}$$

$$|w|_{\beta,t,\Omega} := \sup_{(x,t),(x,t') \in \Omega} |w(x,t)-w(x,t')|/|t-t'|^{\beta}$$

$$|w|_{\alpha,\beta,\Omega} := |w|_{\alpha,x,\Omega} + |w|_{\beta,t,\Omega}$$

$$\|w\|_{\alpha,\beta,\Omega} := \|w\|_{\infty,\Omega} + |w|_{\alpha,\beta,\Omega}$$

where $\alpha,\beta \in (0,1)$ and denote by $H^{\alpha}(I)$ and $H^{\alpha,\beta}(\Omega)$ the Hölder classes corresponding to the norms $\|\ \|_{\alpha,I}$ and $\|\ \|_{\alpha,\beta,\Omega}$. The subscripts I and Ω are omitted if the domains are clear from the context. We also need the subspaces $\overset{\circ}{H}{}^{\alpha}([0,T]) := \{\chi \in H^{\alpha}([0,T]) : \chi(0) = 0\}$ and $\overset{\circ}{H}{}^{\alpha,\beta}(I\times[0,T]) := \{w \in H^{\alpha,\beta}(I\times[0,T]) : w(x,0) = 0,\ x \in I\}$. For simplicity of notation we write $t^{\gamma}\chi$ and $t^{\gamma}w$ for the functions $t \mapsto t^{\gamma}\chi(t)$ and $(x,t) \mapsto t^{\gamma}w(x,t)$.

We assume that the coefficients a, b and the data f, ϕ, ψ_{ν} for the problem (1.1) satisfy

$$\left\{\begin{array}{l} a,b \in H^{\alpha,\alpha/2} \\ a > a_0 > 0 \end{array}\right. \tag{1.3}$$

$$\left\{\begin{array}{l} f \in H^{\alpha,\alpha/2} \\ \phi,\phi' \in H^{\alpha} \\ \psi_{\nu}, t^{1/2}\psi_{\nu}' \in H^{\alpha/2} \end{array}\right. \tag{1.4}$$

for some $\alpha \in (0,1)$ and the compatibility conditions

$$
(1.5) \qquad \begin{cases} \phi(\nu) = \psi_\nu(0) , \\ \lim_{t\to 0} t^{1/2}\psi_\nu'(t) = f(\nu,0) + b(\nu,0)\phi'(\nu), \ \nu = 0,1 . \end{cases}
$$

Theorem 1.1. Under the assumptions (1.3) - (1.5) the solution of the problem (1.1) satisfies

$$
(1.6) \qquad \begin{aligned} &\|u,u_x,t^{1/2}u_{xx}\|_{\alpha,\alpha/2,\Omega_T} < \\ &c\{\|f\|_{\alpha,\alpha/2,\Omega_T} + \|\phi,\phi'\|_{\alpha,[0,1]} + \\ &\sum_{\nu=0}^{1} \|\psi_\nu,t^{1/2}\psi_\nu'\|_{\alpha/2,[0,T]}\} \end{aligned}
$$

where the constant c depends on α, a_0, $\|a,b\|_{\alpha,\alpha/2,\Omega_T}$ and T. Moreover we have

$$
(1.7) \qquad \lim_{t\to 0} t^{1/2}u_{xx}(x,t) = 0, \ x \in [0,1] .
$$

The Theorem remains valid if we replace in problem (1.1) $t^{-1/2}b(x,t)$ by $t^{-\gamma}b(x,t)$ with $\gamma < 1/2$. In this case $t^{-\gamma}bu_x$ can be regarded as a minor term and the result can be obtained from Theorem 1.1 by iteration. The Theorem also extends to quasilinear problems of the type (1.1) with $a = a(x,t,u)$, $b = b(x,t,u)$, $f = f(x,t,u)$ if we assume, e.g., that a, b, f are C^1. We stated the result in the simplest setting to focus on its essential feature, the singularity in the coefficient of u_x.

The difficult part of the proof of Theorem 1.1 is to show (1.6) for the constant coefficient problem on the domain $R_+\times[0,T]$ (section 2.3). We then use a standard technique [L, pp. 295-341] to extend the result to the case of variable coefficients (section 3). To keep the report self contained, the proofs of estimates for the Cauchy problem (section 2.2) are included although they are similar to the corresponding estimates for the heat equation.

2. The constant coefficient problem

2.1. Auxiliary lemmas

We list in this section some elementary inequalities and properties of Hölder norms. In the sequel c denotes generic constants which may depend on α, γ, a_0, $\|a,b\|_{\alpha,\alpha/2}$.

(2.1) For $z,j > 0$,

$$z^j \exp(-z) \le c \exp(-cz) \quad .$$

(2.2) For $t > s > 0$, $z \in \mathbb{R}$,

$$- c + c|z|/(t-s)^{1/2} \le |z + t^{1/2} - s^{1/2}|/(t-s)^{1/2} \le c + c|z|/(t-s)^{1/2} \quad .$$

(2.3) For $t,z > 0$, $j > 1$,

$$\int_0^t s^{-j/2}(t-s)^{-1/2}\exp(-\frac{z}{s})ds \le cz^{(1-j)/2} \quad .$$

Lemma 2.1. For $\chi \in H^{\gamma}([0,T])$, $\gamma \in (0,1)$,

$$|\chi|_{\gamma} \le c \sup_{j\in\mathbb{N}} |\chi|_{\gamma,[2^{-j}T,2^{-j+1}T]} \quad . \tag{2.4}$$

When estimating differences $\chi(t) - \chi(t')$ it will be sometimes convenient to assume that $|t-t'| \le c \min(t,t')$. In view of the Lemma this is no loss of generality when estimating $|\chi|_{\gamma}$.

Lemma 2.2. Let $v(y,t) := u(x + ct^{1/2},t)$, then

$$|v|_{\alpha/2,t} \le c|u|_{\alpha,\alpha/2} \quad . \tag{2.5}$$

Lemma 2.3. For $t^{1/2}\chi \in \overset{o}{H}{}^{\gamma}([0,T])$, $\gamma \in (0,1)$, the norms $\|t^{1/2}\chi\|_{\gamma}$, $|t^{1/2}\chi|_{\gamma}$ and $(\|t^{1/2-\gamma}\chi\|_{\infty} + \sup_{t\in(0,T)} t^{1/2}|\chi|_{\gamma,[t,T]})$ are equivalent.

An analogous version of this Lemma holds for $\overset{\circ}{H}^{\alpha,\alpha/2}$.

__Proof__. Assume that $\|t^{1/2-\gamma}\chi\|_\infty + \sup t^{1/2}|\chi|_{\gamma,[t,T]} < 1$ and let $0 < s < t < T$. We have

$$|t^{1/2}\chi(t) - s^{1/2}\chi(s)| <$$

$$|(t^{1/2} - s^{1/2})\chi(t)| + |s^{1/2}(\chi(t) - \chi(s))| <$$

$$(t-s)t^{-1/2}t^{-1/2+\gamma} + (t-s)^\gamma < 2(t-s)^\gamma$$

and therefore $|t^{1/2}\chi|_\gamma < 2$.

Now assume that $|\bar{\chi}|_\gamma < 1$ with $\bar{\chi} := t^{1/2}\chi$ and let $0 < s/2 < t < s < T$. We have

$$|s^{-1/2}\,\bar{\chi}(s) - t^{-1/2}\,\bar{\chi}(t)| <$$

$$|s^{-1/2}(\bar{\chi}(s) - \bar{\chi}(t))| + |(s^{-1/2} - t^{-1/2})\bar{\chi}(t)| <$$

$$s^{-1/2}(s-t)^\gamma + t^{-3/2}(s-t)t^\gamma <$$

$$2t^{-1/2}(s-t)^\gamma$$

and therefore $|\chi|_{\gamma,[t,T]} < ct^{-1/2}$.

__Lemma 2.4__. For $\gamma \in (0,1/2)$ define

$$Q^\gamma := \{\chi : \chi(0) = \lim_{t\to 0} t^{1/2}\chi'(t) = 0 \quad ,$$

(2.6)

$$\|\chi\|_Q := |t^{1/2}\chi'|_{\gamma,[0,T]} < \infty\} \quad .$$

Then we have, for $\chi \in Q^\gamma$,

(2.7) $$|\chi|_{\gamma+1/2},\ |t^{-1/2}\chi|_\gamma < c\|\chi\|_Q \quad .$$

2.2. The Cauchy problem

In this section we consider the problem

$$(2.8)\qquad \begin{cases} u_t - au_{xx} - t^{-1/2}bu_x = t^{-1/2}f, \ (x,t) \in \mathbb{R}\times[0,T] \ , \\ u(x,0) = \phi(x), \ x \in \mathbb{R} \ , \end{cases}$$

where $a > 0$ and b are constants.

Theorem 2.1. Let $f \in H^{\alpha,\alpha/2}(\mathbb{R}\times[0,T])$, $\phi,\phi' \in H^{\alpha}(\mathbb{R})$. Then the solution of problem (2.8) satisfies

$$(2.9)\qquad \|u,u_x,t^{1/2}u_{xx}\|_{\alpha,\alpha/2} \leq c\{\|f\|_{\alpha,\alpha/2} + \|\phi,\phi'\|_{\alpha}\}$$

where c depends on a and b. Moreover we have

$$(2.10)\qquad \lim_{t\to 0} t^{1/2}u_{xx}(x,t) = 0, \ x \in \mathbb{R} \ .$$

The change of variables $y := x + 2bt^{1/2}$, $v(y,t) := u(x,t)$, $g(y,t) := f(x,t)$, transforms the problem (2.8) into the heat equation

$$(2.11)\qquad \begin{cases} v_t - av_{yy} = t^{-1/2}g, \ (y,t) \in \mathbb{R}\times[0,T] \ , \\ v(y,0) = \phi(y), \ y \in \mathbb{R} \ . \end{cases}$$

Therefore the analysis of the Cauchy problem is fairly simple. We merely have to take into account the singular behavior of $t^{-1/2}g$ as $t \to 0$. However, for convenience of the reader, we give a complete proof of Theorem 2.1.

By Lemma 2.2 it suffices to prove the assertion (2.9) and (2.10) for the solution v of problem (2.11), i.e.

$$(2.12)\qquad \|v,v_y,t^{1/2}v_{yy}\|_{\alpha,\alpha/2} \leq c\{\|g\|_{\alpha,\alpha/2} + \|\phi,\phi'\|_{\alpha}\}$$

$$(2.13)\qquad \lim_{t\to 0} t^{1/2}v_{yy}(y,t) = 0, \ y \in \mathbb{R} \ .$$

Denote by $\Gamma(x,t) := (4\pi t)^{-1/2}\exp(-\frac{x^2}{4t})$ the fundamental solution of the heat equation. We have the estimate [L, p. 274]

$$(2.14) \qquad |D_x^j D_t^k \Gamma(x,t)| \le ct^{-j/2-k}\exp(-c\frac{x^2}{t}) \quad .$$

Also note that

$$(2.15) \qquad \int_{\mathbb{R}} D_x^j \Gamma(x,t)dx = \begin{cases} 1 & , \quad j = 0 \\ 0 & , \quad j > 0 \quad . \end{cases}$$

$$(2.16) \qquad \int_0^\infty \Gamma_x(x,t)dt = -\frac{1}{2} \quad .$$

To prove (2.12) and (2.13) we assume that $a = 1$ in problem (2.11) and consider two cases

(i) For $\underline{g = 0}$ we have

$$\|v,v_y\|_{\alpha,\alpha/2} \le c\|\phi,\phi'\|_\alpha$$

$$\|v_{yy}\|_{\alpha,\alpha/2,R\times[t,T]} \le c\|v_{yy}(\cdot,t)\|_\alpha \le ct^{-1/2}|\phi'|_\alpha \quad .$$

By Lemma 2.3, this proves (2.12) once we show

$$\|t^{1/2-\alpha/2}v_{yy}\|_\infty \le c|\phi'|_\alpha$$

which, in turn, yields (2.13). To this end we note that by (2.14) and (2.15),

$$|v_{yy}(y,t)| = |\int_{\mathbb{R}} \Gamma_y(y-z,t)(\phi'(z) - \phi'(y))dz| \le ct^{-1/2+\alpha/2}|\phi'|_\alpha \quad .$$

(ii) Now assume that $\underline{\phi = 0}$.

<u>Lemma 2.5</u>.

$$(2.17) \qquad |v_y|_{\alpha,y} \le c|g|_{\alpha,y} \quad .$$

<u>Proof</u>. Assume that $|g|_{\alpha,y} \le 1$ and let $y-y' =: h > 0$. Using (2.15) we have

$$v_y(y,t) - v_y(y',t) =$$
$$\int_0^t \int_{|y-z|<2h} \Gamma_y(y-z,t-s)s^{-1/2}(g(z,s) - g(y,s))dzds$$

$$- \int_0^t \int_{|y-z|<2h} \Gamma_y(y'-z,t-s)s^{-1/2}(g(z,s) - g(y',s))dzds$$

$$+ \int_0^t \int_{|y-z|>2h} (\Gamma_y(y-z,t-s) - \Gamma_y(y'-z,t-s))s^{-1/2}(g(z,s) - g(y,s))dzds$$

$$+ \int_0^t \int_{|y-z|>2h} \Gamma_y(y'-z,t-s)s^{-1/2}(g(y',s) - g(y,s))dzds$$

$$=: \sum_{\nu=1}^{4} I_\nu \ .$$

Estimating these terms, using (2.14), we get

$$|I_1| < c \iint (t-s)^{-1}\exp\left(-c\,\frac{(y-z)^2}{t-s}\right)s^{-1/2}|y-z|^\alpha dzds$$

$$< c \int_{|y-z|<2h} |y-z|^{-1+\alpha}dz < ch^\alpha \ .$$

For the second inequality we have used (2.3) with $j = 2$. $|I_2|$ is estimated similarly.

$$|I_3| < h \iint |\Gamma_{yy}(\xi-z,t-s)|s^{-1/2}|z-y|^\alpha dzds$$

with $\xi = \xi(z,s) \in (y',y)$. Since for $|y-z| > 2h = 2(y-y')$, $|y-z|/2 <$ $|\xi-z| < 2|y-z|$, we obtain, using (2.3) with $j = 3$,

$$|I_3| < ch \iint (t-s)^{-3/2}\exp\left(-c\,\frac{(y-z)^2}{t-s}\right)s^{-1/2}|y-z|^\alpha dzds$$

$$< ch \int_{|z-y|>2h} |y-z|^{-2+\alpha}dz < ch^\alpha \ .$$

$$|I_4| < \int_0^t |\Gamma(y'-z,t-s)\Big|_{z=y-2h}^{z=y+2h}|s^{-1/2}h^\alpha ds$$

$$< ch^\alpha \int_0^t (t-s)^{-1/2}s^{-1/2}ds < ch^\alpha \ .$$

<u>Lemma 2.6.</u>

(2.18) $$|v_y|_{\alpha/2,t} < c|g|_{\alpha,y} \ .$$

Proof. Assume that $|g|_{\alpha,y} < 1$ and let $t-t' =: h \in (0,t/2)$. Using (2.15) we have

$$v_y(y,t) - v_y(y,t') =$$

$$\int_{t-2h}^{t} \int_R \Gamma_y(y-z,t-s)s^{-1/2}(g(z,s) - g(y,s))dzds$$

$$- \int_{t-2h}^{t'} \int_R \Gamma_y(y-z,t'-s)s^{-1/2}(g(z,s) - g(y,s))dzds$$

$$+ \int_0^{t-2h} \int_R (\Gamma_y(y-z),t-s) - \Gamma_y(y-z,t'-s))s^{-1/2}(g(z,s) - g(y,s))dzds$$

$$=: \sum_{\nu=1}^{3} I_\nu \quad .$$

Estimating these terms, using (2.14), we get

$$|I_1| < c \iint (t-s)^{-1} \exp\left(-c \frac{(y-z)^2}{t-s}\right)s^{-1/2}|y-z|^{\alpha}dzdx$$

$$< c \int_{t-2h}^{t} (t-s)^{-1/2+\alpha/2} s^{-1/2}ds < ch^{\alpha/2} \quad .$$

$|I_2|$ is estimated similarly.

$$|I_3| < h \iint |\Gamma_{yt}(y-z,\xi-s)|s^{-1/2}|y-z|^{\alpha}dzds$$

$$< ch \iint (t-s)^{-2} \exp\left(-c \frac{(y-z)^2}{t-s}\right)s^{-1/2}|y-z|^{\alpha}dzds$$

$$< ch \int_0^{t-2h} (t-s)^{-3/2+\alpha/2} s^{-1/2}ds < ch^{\alpha/2} \quad .$$

Lemma 2.7.

$$\|t^{1/2-\alpha/2}v_{yy}\|_\infty < c|g|_{\alpha,y} \quad . \tag{2.19}$$

Proof. Assuming $|g|_{\alpha,y} < 1$ we have, using (2.14) and (2.15),

$$|v_{yy}(y,t)| = \left|\int_0^t \int_R \Gamma_{yy}(y-z,t-s)s^{-1/2}(g(z,s) - g(y,s))dzds\right|$$

$$< c \iint (t-s)^{-3/2} \exp\left(-c \frac{(y-z)^2}{t-s}\right)s^{-1/2}|y-z|^{\alpha}dzds$$

$$\leq c \int_0^t (t-s)^{-1+\alpha/2} s^{-1/2} ds \leq ct^{-1/2+\alpha/2} .$$

This Lemma proves (2.13). In combination with Lemma 2.6 and the fact that $v(\cdot,0) = 0$ it also shows that

$$|v|_{1/2,t} \leq \|t^{1/2} v_t\|_\infty \leq c\|g\|_{\alpha,\alpha/2}$$

and it follows from Lemma 2.5 that

$$\|v,v_y\|_{\alpha,\alpha/2} \leq c\|g\|_{\alpha,\alpha/2} .$$

To complete the proof of assertion (2.12), we note that

$$|v_{yy}|_{\alpha,\alpha/2,\mathbb{R}\times[s,T]} \leq$$

$$c(|v_{yy}(\cdot,s)|_\alpha + |t^{-1/2}g|_{\alpha,y,\mathbb{R}\times[s,T]}) \leq$$

$$c(s^{-1/2}|v_y(\cdot,s/2)|_\alpha + s^{-1/2}|g|_{\alpha,y,\mathbb{R}\times[s/2,T]}) \leq$$

$$c\, s^{-1/2}|g|_{\alpha,y} .$$

In view of Lemmas 2.3 and 2.7 this yields

$$\|t^{1/2} v_{yy}\|_{\alpha,\alpha/2} \leq c|g|_{\alpha,y} .$$

2.3. The boundary value problem

In this section we consider the problem

$$(2.20) \qquad \begin{cases} u_t - au_{xx} - t^{-1/2}bu_x = t^{-1/2}f, & (x,t) \in \mathbb{R}_+\times[0,T] \\ u(x,0) = \phi(x), & x \in \mathbb{R}_+ \\ u(0,t) = \psi(t), & t \in [0,T] \end{cases}$$

where $a > 0$ and b are constants.

Theorem 2.2. Let $f \in H^{\alpha,\alpha/2}(\mathbb{R}_+\times[0,T])$, $\phi,\phi' \in H^{\alpha}(\mathbb{R}_+)$, $\psi, t^{1/2}\psi' \in H^{\alpha/2}([0,T])$ and assume that the compatibility conditions

$$(2.21)\qquad \begin{cases} \phi(0) = \psi(0) \\ \lim\limits_{t\to 0} t^{1/2}\psi(t) = f(0,0) + b\phi'(0) \end{cases}$$

hold. Then the solution of problem (2.20) satisfies

$$(2.22)\qquad \begin{aligned} &\|u,u_x,t^{1/2}u_{xx}\|_{\alpha,\alpha/2} \le \\ &c\{\|f\|_{\alpha,\alpha/2} + \|\phi,\phi'\|_{\alpha} + \|\psi,t^{1/2}\psi'\|_{\alpha/2}\} \end{aligned}$$

where c depends on a and b. Moreover we have

$$(2.23)\qquad \lim_{t\to 0} t^{1/2}u_{xx}(x,t) = 0,\ x \in \mathbb{R}_+ \ .$$

Denote by $\bar{f}$, $\bar{\phi}$ smooth extensions of the function f, ϕ to the domains $\mathbb{R}\times[0,T]$ and $\mathbb{R}$ respectively. Subtracting from u the solution of the Cauchy problem with right hand side $t^{-1/2}\,\bar{f}$ and initial values $\bar{\phi}$ and using Theorem 2.1, we see that we have to prove Theorem 2.2 only for $f = \phi = 0$, an assumption we make throughout this section. In this case, the compatibility conditions (2.21), together with the assumption $\psi,t^{1/2}\psi' \in H^{\alpha/2}([0,T])$, can be stated in the form

$$(2.24)\qquad \psi \in Q^{\alpha/2}$$

where Q has been defined in (2.6). The change of variables $y := x + 2bt^{1/2}$ is of no help for the proof of Theorem 2.2 since $\mathbb{R}_+\times[0,T]$ is transformed to the domain $\{(y,t) : t \in [0,T],\ y > 2bt^{1/2}\}$. However, we may assume $a = 1$, by a linear change of the t-variable.

Let us first obtain a representation for the solution of problem (2.20) (with $a = 1$, $f = \phi = 0$) in terms of the fundamental solution $K(x-y,t,s)$ for the equation

$$(2.25)\qquad u_t - u_{xx} - t^{-1/2}bu_x = 0 \ .$$

By taking Fourier-transforms we see that

$$K(x,t,s) = \Gamma(x + 2b(t^{1/2} - s^{1/2}),t-s)$$

$$(2.26) \qquad = (4\pi)^{-1/2}(t-s)^{-1/2}\exp\left(- \frac{(x+2b(t^{1/2} - s^{1/2}))^2}{4(t-s)}\right) .$$

Using (2.1), (2.2), (2.14) and the fact that K satisfies equation (2.25) for $x,\ t-s \neq 0$ one can easily check that K satisfies the same estimates as the fundamental solution of the heat equation

$$(2.27) \qquad |D_x^j D_t^k K(x,t,s)| \le c(t-s)^{-j/2-k} \exp\left(-c \frac{x^2}{t-s}\right) .$$

Also note, that

$$(2.28) \qquad \frac{\partial}{\partial s} K(x,t,s) + K_{xx}(x,t,s) + bs^{-1/2}K_x(x,t,s) = 0, \quad x,\ t-s \neq 0 ,$$

which follows from (2.26).

<u>Proposition 2.1</u>. Let $\psi \in Q^{\alpha/2}$. The solution of problem (2.20) (with $f = \phi = 0$) can be represented in the form

$$(2.29) \qquad u(x,t) = -2 \int_0^t K_x(x,t,s)\chi(s)ds$$

where $\chi \in Q^{\alpha/2}$ is the solution of

$$(2.30) \qquad \chi(t) = \psi(t) + 2 \int_0^t K_x(0,t,s)\chi(s)ds$$

and

$$(2.31) \qquad |t^{1/2}\chi'|_{\alpha/2} \le c|t^{1/2}\psi'|_{\alpha/2} .$$

<u>Proof</u>. We claim that the operator R defined by

$$(2.32) \qquad (R\chi)(t) := -2 \int_0^t K_x(0,t,s)\chi(s)ds$$

is a strict contraction on the space Q with respect to the norm $\|\chi\|_Q = |t^{1/2}\chi'|_{\alpha/2}$. We have

$$-2 K_x(0,t,s) = \frac{b(t^{1/2} - s^{1/2})}{\pi^{1/2}(t-s)^{3/2}} \exp\left(- \frac{(b(t^{1/2} - s^{1/2}))^2}{t-s}\right)$$

and we rewrite $R\chi$ in the form

$$(R\chi)(t) = \int_0^1 \frac{b}{\pi^{1/2}} \frac{1}{(1-s)^{1/2}(1+s^{1/2})} \exp\left(-\frac{(b(1-s^{1/2}))^2}{1-s}\right)\chi(ts)\,ds$$

$$=: \int_0^1 r(b,s)\chi(ts)\,ds \quad .$$

Substituting $z := (b(1-s^{1/2}))^2/(1-s)$ we see that

$$\|r(b,\cdot)\|_1 = \pi^{-1/2}\int_0^{b^2} \frac{b^2-z}{b^2+z} z^{-1/2}\exp(-z)\,dz < 1 \quad . \tag{2.33}$$

Since

$$t^{1/2}(R\chi)'(t) = \int_0^1 (r(b,s)s^{1/2})(ts)^{1/2}\chi'(ts)\,ds$$

this implies that R is a contraction on Q.

It remains to show that u, given by (2.29), satisfies the boundary condition $u(0,\cdot) = \psi$. We write (2.29) in the form

$$u(x,t) =$$

$$\int_0^t \frac{x}{2\pi^{1/2}(t-s)^{3/2}} \exp\left(-\frac{(x+2b(t^{1/2}-s^{1/2}))^2}{4(t-s)}\right)\chi(t)\,ds$$

$$+ \int_0^t \frac{x}{2\pi^{1/2}(t-s)^{3/2}} \exp\left(-\frac{(x+2b(t^{1/2}-s^{1/2}))^2}{4(t-s)}\right)(\chi(s) - \chi(t))\,ds$$

$$+ \int_0^t \frac{b(t^{1/2}-s^{1/2})}{\pi^{1/2}(t-s)^{3/2}} \exp\left(-\frac{(x+2b(t^{1/2}-s^{1/2}))^2}{4(t-s)}\right)\chi(s)\,ds$$

$$=: \sum_{\nu=1}^{3} I_\nu(x)$$

and obtain

$$\lim_{x\to 0} I_1(x) =$$

$$\chi(t) \lim_{x\to 0} \int_0^{t/x^2} \frac{1}{2\pi^{1/2}} s^{-3/2} \exp\left(-\frac{(1+2b((t/x^2)^{1/2} - (t/x^2-s)^{1/2}))^2}{4s}\right)ds =$$

$$\chi(t)\int_0^\infty \frac{1}{2\pi^{1/2}} s^{-3/2}\exp(-\frac{1}{4s})ds = \chi(t) \quad .$$

$$|I_2(x)| \le c\int_0^t x(t-s)^{-3/2}\exp\left(-c\frac{x^2}{t-s}\right)(t-s)^{\alpha/2}|\chi|_{\alpha/2}ds \le cx^\alpha|\chi|_{\alpha/2} \quad .$$

$$\lim_{x\to 0} I_3(x) = (R\chi)(t) = \psi(t) - \chi(t) \quad .$$

We now analyse the smoothness of the solution of problem (2.20) via the representation (2.29).

<u>Proposition 2.2</u>. Let u be defined by (2.29) with $\chi \in \overset{\circ}{H}{}^\gamma$, $\gamma < 1/2$, then

(2.34) $$|u|_{\gamma,t} \le c|\chi|_\gamma \quad .$$

<u>Proof</u>. Assume $|\chi|_\gamma \le 1$. In particular, since $\chi(0) = 0$, $|\chi(t)| \le t^\gamma$. We let $t-t' =: h \in (0,t/3)$ and write

$$-\frac{1}{2}(u(x,t) - u(x,t')) =$$

$$\int_{t-2h}^t K_x(x,t,s)(\chi(s) - \chi(t))ds$$

$$-\int_{t-2h}^{t'} K_x(x,t',s)(\chi(s) - \chi(t'))ds$$

$$+\int_0^{t-2h}(K_x(x,t,s) - K_x(x,t',s))(\chi(s) - \chi(t'))ds$$

$$+\int_0^{2h} K_x(x,t,t-s)(\chi(t) - \chi(t'))ds$$

$$+\int_0^{2h} (K_x(x,t,t-s) - K_x(x,t',t'-s))\chi(t')ds$$

$$+\int_0^{t'-2h}(K_x(x,t,t-2h-s) - K_x(x,t',t'-2h-s))\chi(t')ds$$

$$+\int_0^h K_x(x,t,h-s)\chi(t')ds$$

$$=: \sum_{\nu=1}^7 I_\nu \quad .$$

We set

$$(2.35) \qquad A(t,s) := \frac{x+2b(t^{1/2}-s^{1/2})}{2(t-s)^{1/2}}$$

and with this abbreviation

$$K_x(x,t,s) = -\frac{1}{2\pi^{1/2}}(t-s)^{-1}A(t,s)\exp(-A(t,s)^2)$$

$$=: K_1(x,t,s) + K_2(x,t,s)$$

where

$$K_1(x,t,s) = -\frac{1}{4\pi^{1/2}}\frac{x}{(t-s)^{3/2}}\exp(-A(t,s)^2)$$

$$K_2(x,t,s) = -\frac{1}{2\pi^{1/2}}\frac{b}{(t-s)^{1/2}(t^{1/2}+s^{1/2})}\exp(-A(t,s)^2) \quad .$$

Note that, by (2.2),

$$(2.36) \qquad \exp(-A(t,s)^2) \le c\exp\left(-c\frac{x^2}{t-s}\right) \quad .$$

Using this, $t-t' = h \le t/3$ and (2.27) we estimate the integrals I_ν as follows.

$$|I_1| \le c\int_{t-2h}^{t}(|K_1|+|K_2|)(t-s)^\gamma ds$$

$$\le c(h^\gamma + \int_{t-2h}^{t} t^{-1/2}(t-s)^{\gamma-1/2}ds) \le ch^\gamma \quad .$$

$|I_2|$ is estimated similarly.

$$|I_3| \le ch\int_0^{t-2h}|K_{xt}(x,\xi,s)|\ (t'-s)^\gamma ds$$

$$\le ch\int_0^{t-2h}(t'-s)^{-2+\gamma}dx \le ch^\gamma \quad .$$

$$|I_4| \le ch^\gamma\int_0^{2h}(|K_1|+|K_2|)$$

$$\le ch^\gamma(1+\int_0^{2h} t^{-1/2}s^{-1/2}ds) \le ch^\gamma \quad .$$

Since $\left|\frac{d}{dA}(A\exp(-A^2))\right| < c$ and

$$|A(t,t-s) - A(t',t'-s)| =$$

(2.37) $$cs^{-1/2}\left|\frac{s}{t^{1/2} + (t-s)^{1/2}} - \frac{s}{(t')^{1/2} + (t'-s)^{1/2}}\right| <$$

$$cs^{1/2}t^{-1}h^{1/2}$$

we obtain

$$|I_5| < c(t')^{\gamma} \int_0^{2h} s^{-1}|A(t,t-s)\exp(-A(t,t-s)^2) -$$

$$A(t',t'-s)\exp(-A(t',t'-s)^2)|ds$$

$$< ch^{1/2}t^{\gamma-1} \int_0^{2h} s^{-1/2}ds < ch^{\gamma} \ .$$

Since

$$|A(t,t-2h-s) - A(t',t'-2h-s)| =$$

(2.38) $$c(s+2h)^{-1/2}|t^{1/2} - (t-2h-s)^{1/2} - (t')^{1/2} + (t'-2h-s)^{1/2}| <$$

$$c(s+2h)^{-1/2}h(t'-2h-s)^{-1/2}$$

we obtain, for $\gamma < 1/2$,

$$|I_6| < c(t')^{\gamma} \int_0^{t'-2h}(s+2h)^{-1}|A(t,t-2h-s)\exp(-A(t,t-2h-s)^2) -$$

$$A(t',t'-2h-s)\exp(-A(t',t'-2h-s)^2)|ds$$

$$< ct^{\gamma} \int_0^{t'-2h}(s+2h)^{-3/2}h(t'-2h-s)^{-1/2}ds$$

$$< ct^{\gamma}h^{1/2}t^{-1/2} < ch^{\gamma} \ .$$

$$|I_7| < c(t')^{\gamma} \int_0^{h} (t-h+s)^{-1}ds < ct^{\gamma}\, ht^{-1} < ch^{\gamma} \ .$$

<u>Proposition 2.3</u>. Let u be defined by (2.29) with $\chi \in Q^{\gamma}$, $\gamma < 1/2$, then

(2.39) $$|u_x|_{\gamma,t} < c|t^{1/2}\chi'|_{\gamma} \ .$$

Proof. Assume $|t^{1/2}\chi'|_\gamma < 1$. By (2.28) we have

$$u_x(x,t) = -2\int_0^t K_{xx}(x,t,s)\chi(s)ds =$$

$$- 2\int_0^t K(x,t,s)\chi'(s)ds + 2b\int_0^t K_x(x,t,s)s^{-1/2}\chi(s)ds$$

$$=: v(x,t) + w(x,t) \quad .$$

Since, by Lemma 2.4, $|t^{-1/2}\chi|_\gamma < c$, Proposition 2.2 implies $|w|_{\gamma,t} < c$. Let $t-t' =: h \in (0,t/3)$. As in the proof of Proposition 2.2 we write the difference $-\frac{1}{2}(v(x,t) - v(x,t'))$ in the form $\sum_{\nu=1}^{7} J_\nu$ where the integrals J_ν are defined as I_ν but with K_x replaced by K and χ by χ'. Using $t-t' = h < t/3$, (2.27) and the inequalities (c.f. Lemma 2.3)

$$|\chi'(t)| < ct^{-1/2+\gamma}$$

$$|\chi'(t) - \chi'(s)| < c(t-s)^\gamma s^{-1/2}, \ s < t \quad ,$$

we estimate the integrals J_ν as follows.

$$|J_1| < c\int_{t-2h}^t (t-s)^{-1/2}(t-s)^\gamma s^{-1/2}ds < ch^\gamma \quad .$$

$|J_2|$ is estimated similarly.

$$|J_3| < ch\int_0^{t-2h} |K_t(x,\xi,s)|\ (t'-s)^\gamma s^{-1/2}ds$$

$$< ch\int_0^{t-2h}(t-s)^{-3/2+\gamma}s^{-1/2}ds < ch^\gamma \quad .$$

$$|J_4| < ch^\gamma(t')^{-1/2}\int_0^{2h} s^{-1/2}ds < ch^\gamma \quad .$$

With A defined by (2.35) we have, using (2.37),

$$|J_5| < c(t')^{\gamma-1/2}\int_0^{2h} s^{-1/2}|\exp(-A(t,t-s)^2) - \exp(-A(t',t'-s)^2)|ds$$

$$< ct^{\gamma-1/2}\int_0^{2h} t^{-1}h^{1/2}ds < ch^\gamma \quad .$$

By (2.38),

$$|\exp(-A(t,t-2h-s)^2) - \exp(-A(t',t'-2h-s)^2)|$$

$$< c(s+2h)^{-1/2}h(t'-2h-s)^{-1/2}\ |A(\xi,\xi-2h-s)\exp(-A(\xi,\xi-2h-s)^2)|$$

$$< c(s+2h)^{-1/2}h(t'-2h-s)^{-1/2}\left|\frac{x}{(s+2h)^{1/2}}\exp\left(-c\,\frac{x^2}{s+2h}\right) + (s+2h)^{1/2}(t')^{-1/2}\right| \ .$$

It follows that

$$|J_6| < c(t')^{-1/2+\gamma}\int_0^{t'-2h}(s+2h)^{-1/2}|\exp(-A(t,t-2h-s)^2) -$$

$$\exp(-A(t',t'-2h-s)^2)|ds$$

$$< ct^{-1/2+\gamma}\ h\left|\int_0^{t'-2h}\frac{x}{(s+2h)^{-3/2}}\exp\left(-c\,\frac{x^2}{s+2h}\right)(t'-2h-s)^{-1/2}ds\right.$$

$$\left. + t^{-1/2}\int_0^{t'-2h}(s+2h)^{-1/2}(t'-2h-s)^{-1/2}ds\right|$$

$$< ct^{-1/2+\gamma}ht^{-1/2} < ch^{\gamma} \ .$$

$$|J_7| < c(t')^{-1/2+\gamma}\int_0^h (t-h+s)^{-1/2}dx$$

$$< ct^{-1/2+\gamma}ht^{-1/2} < ch^{\gamma} \ .$$

<u>Proposition 2.4</u>. For $\psi \in Q^{\alpha/2}$, the solution of problem (2.20) (with $f = \phi = 0$) satisfies

$$|u_x|_{\alpha,x} < c|t^{1/2}\psi'|_{\alpha/2} \ . \tag{2.40}$$

<u>Proof</u>. Assume $|t^{1/2}\psi'|_{\alpha/2} < 1$. Since $v := u_x$ satisfies (2.25) and $v(\cdot,0) = 0$, $v(0,\cdot) = u_x(0,\cdot)$ we have

$$v(x,t) = -2\int_0^t K_x(x,t,s)\bar{\chi}(s)ds$$

with $\bar{\chi}$ the solution of

$$\bar{\chi}(t) = u_x(0,t) + 2\int_0^t K_x(0,t,s)\bar{\chi}(s)ds \ .$$

By Proposition 2.3, $|u_x(0,\cdot)|_{\alpha/2} < c$, and the proof of Proposition 2.1, in particular (2.33), shows that this implies $|\bar{\chi}|_{\alpha/2} < c$.

Let $x-x' =: h > 0$ and assume that $t > h^2$. We write

$$-\frac{1}{2}(v(x,t) - v(x',t)) =$$

$$\int_{t-h^2}^{t} K_x(x,t,s)(\overline{\chi}(s) - \overline{\chi}(t))ds$$

$$- \int_{t-h^2}^{t} K_x(x',t,s)(\overline{\chi}(s) - \overline{\chi}(t))ds$$

$$+ \int_0^{t-h^2} (K_x(x,t,s) - K_x(x',t,s))(\overline{\chi}(s) - \overline{\chi}(t))ds$$

$$+ \int_0^t (K_x(x,t,s) - K_x(x',t,s))\overline{\chi}(t)ds$$

$$=: \sum_{\nu=1}^{4} I_\nu \ .$$

Using $h^2 < t$, (2.27) and $|\overline{\chi}(t)| < ct^{\alpha/2}$ we estimate these integrals as follows.

$$|I_1| < c \int_{t-h^2}^{t} (t-s)^{-1}(t-s)^{\alpha/2}ds < ch^\alpha \ .$$

$|I_2|$ is estimated similarly.

$$|I_3| < ch \int_0^{t-h^2} |K_{xx}(\xi,t,s)|(t-s)^{\alpha/2}ds$$

$$< ch \int_0^{t-h^2} (t-s)^{-3/2+\alpha/2}ds < ch^\alpha \ .$$

Set $I(x) := \int_0^t K_x(x,t,s)ds$. By (2.28) we have

$$|I'(x)| = |\int_0^t K_{xx}(x,t,s)ds|$$

$$< |K(x,t,\cdot)|_0^t| + c \int_0^t |K_x(x,t,s)|s^{-1/2}ds$$

$$< c(t^{-1/2} + \int_0^t \frac{x}{(t-s)^{3/2}} \exp\left(-c\frac{x^2}{t-s}\right)s^{-1/2}ds$$

$$+ \int_0^t \frac{1}{(t-s)^{1/2}(t^{1/2}+s^{1/2})} s^{-1/2}ds)$$

$$< ct^{-1/2} \ .$$

It follows that

$$|I_4| \leq ct^{\alpha/2}\, ht^{-1/2} \leq ch^\alpha \quad .$$

Combining the above estimates we have

$$|v(x,t) - v(x',t)| \leq c|x-x'|^\alpha, \ |x-x'|^2 \leq t \quad .$$

Since $v(\cdot,0) = 0$ and, by Proposition 2.3, $|v|_{\alpha/2,t} \leq c$, this inequality holds for $|x-x'|^2 \geq t$ too.

<u>Proof of Theorem 2.2</u>. As we already remarked we may assume $f = \phi = 0$. For $\psi \in Q^{\alpha/2}$ it follows from Propositions 2.1 - 2.4 that

$$\|u,u_x\|_{\alpha,\alpha/2} \leq c|t^{1/2}\psi'|_{\alpha/2} \quad . \tag{2.41}$$

To complete the proof of assertions (2.22), (2.23) we write the solution of problem (2.20) in the form $u = v+w$ where v and w are solutions of the problems

$$\begin{cases} v_t - v_{xx} = t^{-1/2}b\bar{u}_x, \ (x,t) \in \mathbb{R} \times [0,T] \quad , \\ v(\cdot,0) = 0 \end{cases} \tag{2.42}$$

$$\begin{cases} w_t - w_{xx} = 0, \ (x,t) \in \mathbb{R}_+ \times [0,T] \quad , \\ w(\cdot,0) = 0 \\ w(0,\cdot) = \psi - v(0,\cdot) \quad . \end{cases} \tag{2.43}$$

Here, $\bar{u}_x$ denotes a smooth extension of u_x to the domain $\mathbb{R} \times [0,T]$. By (2.41) and Theorem 2.1,

$$\|v,v_x,t^{1/2}v_{xx},w,w_x\|_{\alpha,\alpha/2} \leq c|t^{1/2}\psi'|_{\alpha/2} \quad .$$

Also note that, by (2.10),

$$\lim_{t\to0} t^{1/2}(\psi'(t) - v_t(0,t)) = (\lim_{t\to0} t^{1/2}\psi'(t)) - b\phi_x(0) = 0 \ ,$$

and it follows from $\|t^{1/2}v_t\|_{\alpha,\alpha/2} \leq \|t^{1/2}v_{xx}\|_{\alpha,\alpha/2} + \|b\bar{u}_x\|_{\alpha,\alpha/2}$ that $\tilde{\psi} := \psi - v(0,\cdot) \in Q^{\alpha/2}$. Therefore it remains to show that

$$\|t^{1/2}w_{xx}\|_{\alpha,\alpha/2} \leq c|t^{1/2}\tilde{\psi}'|_{\alpha/2} \tag{2.44}$$

$$\lim_{t\to 0} t^{1/2} w_{xx}(x,t) = 0,\ x \in R_+ \quad . \tag{2.45}$$

Assume $|t^{1/2}\,\tilde{\psi}'|_{\alpha/2} \le 1$. We have

$$|w_{xx}(x,t)| = |w_t(x,t)| = |2 \int_0^t \Gamma_x(x,t-s)\tilde{\psi}'(s)ds|$$

$$\le c \int_0^t \frac{x}{(t-s)^{3/2}} \exp\left(-c\,\frac{x^2}{t-s}\right)s^{-1/2+\alpha/2}ds \le c\, t^{-1/2+\alpha/2} \tag{2.46}$$

which proves (2.45).

Let $x-x' =: h > 0$ and assume that $h^2 \le t,\ h \le x/2$. We have

$$|w_{xx}(x,t) - w_{xx}(x',t)| =$$

$$|2 \int_0^t (\Gamma_x(x,t-s) - \Gamma_x(x',t-s))\tilde{\psi}'(s)ds| \le$$

$$ch \int_0^t |\Gamma_{xx}(\xi,t-s)|s^{-1/2+\alpha/2}ds \le$$

$$ch \int_0^t (t-s)^{-3/2} \exp\left(-c\,\frac{x^2}{t-s}\right)s^{-1/2+\alpha/2}ds \le$$

$$ct^{-1/2}h^{\alpha} \quad .$$

In combination with (2.46) this shows that

$$|t^{1/2}w_{xx}|_{\alpha,x} \quad , \|t^{1/2-\alpha/2}w_{xx}\|_{\infty} \le c \quad .$$

To complete the proof of (2.44) note that

$$|w_{xx}|_{\alpha/2,t,R_+\times[t,T]} \le$$

$$c(|w_{xx}(\cdot,t)|_{\alpha} + |\psi'|_{\alpha/2,[t,T]}) \le ct^{-1/2}$$

and apply Lemma 2.3.

3. Extension to variable coefficients

3.1. Reduction of the problem

We show in this section that it suffices to prove Theorem 1.1 for $f \in \overset{o}{H}{}^{\alpha,\alpha/2}$ and $\phi = \psi_\nu = 0$.

Let $\bar{b}$, $\bar{f}$, $\bar{\phi}$ be smooth extensions of the functions b, f, ϕ to the domains $\mathbb{R} \times [0,T]$ and $\mathbb{R}$ respectively and set $\tilde{\phi}(x,t) := \bar{\phi}(x)$. Let v be the solution of the problem

$$(3.1) \qquad \begin{cases} v_t - v_{xx} = t^{-1/2}(\bar{b}\,\tilde{\phi}_x + \bar{f}), \; (x,t) \in \mathbb{R}\times[0,T] \quad , \\ v(\cdot,0) = \bar{\phi} \end{cases}$$

and set

$$V(x,t) := x(\psi_1(t)-v(1,t)) + (1-x)(\psi_0(t) - v(0,t)) \quad .$$

Then the solution of problem (1.1) can be written in the form

$$(3.2) \qquad u = v + V + w$$

where w is the solution of the problem

$$(3.3) \qquad \begin{cases} w_t - aw_{xx} - t^{-1/2}bw_x = t^{-1/2}g, \quad (x,t) \in \Omega_T := [0,1]\times[0,T] \quad , \\ w(x,0) = 0 \; , \; x \in [0,1] \quad , \\ w(\nu,t) = 0 \; , \; t \in [0,T], \; \nu = 0,1 \quad , \end{cases}$$

with

$$g := t^{1/2}(a-1)v_{xx} + b(v_x - \tilde{\phi}_x) - t^{1/2}V_t + bV_x \quad .$$

By Theorem 2.1 and the definition of V we have

$$(3.4) \qquad \begin{aligned} &\| v, v_x, t^{1/2}v_{xx}, t^{1/2}v_t, V, V_x, t^{1/2}V_t \|_{\alpha,\alpha/2} \le \\ &c(\|f\|_{\alpha,\alpha/2} + \|\phi,\phi'\|_\alpha + \sum_{\nu=0}^{1} \|\psi_\nu\|_{\alpha/2}) \quad , \end{aligned}$$

where c depends on $\|b\|_{\alpha,\alpha/2}$. Moreover,

(3.5) $$\lim_{t\to 0} t^{1/2} v_{xx}(x,t) = 0 \quad , \quad x \in [0,1] \quad .$$

This, together with the compatibility conditions (1.5) and the equation (3.1), implies

$$\lim_{t\to 0} g(x,t) = \lim_{t\to 0} (-t^{1/2} V_t(x,t) + b(x,t)V_x(x,t)) =$$

$$\lim_{t\to 0} (-t^{1/2}(x\psi_1'(t) + (1-x)\psi_0'(t))) +$$

$$x(\bar{b}(1,0)\phi'(1) + \bar{f}(1,0)) + (1-x)(\bar{b}(0,0)\phi'(0) + \bar{f}(0,0))$$

$$+ b(x,0)((\psi_1(0) - v(1,0)) - (\psi_0(0) - v(0,0))) = 0 \quad .$$

By (3.4) and the assumptions (1.3) and (1.4) on a, b, ϕ, ψ_ν it follows that $g \in \overset{\circ}{H}{}^{\alpha,\alpha/2}$.

In view of the above reductions, Theorem 1.1 is a consequence of

<u>Theorem 3.1</u>. For $g \in \overset{\circ}{H}{}^{\alpha,\alpha/2}(\Omega_T)$, the solution of problem (3.3) satisfies

(3.6) $$\|w, w_x t^{1/2} w_{xx}\|_{\alpha,\alpha/2,\Omega_T} < c\|g\|_{\alpha,\alpha/2,\Omega_T}$$

where c depends on α, a_0, $\|a,b\|_{\alpha,\alpha/2,\Omega_T}$ and T. Moreover we have

(3.7) $$\lim_{t\to 0} t^{1/2} w_{xx}(x,t) = 0 \quad .$$

3.2. Auxiliary lemmas

For $\lambda := 1/J$, $J \in \mathbb{N}$, we set $x_j := j\lambda$, $j = 0,\ldots,J$, and choose cut off functions η_j with the following properties

$$(3.8)\qquad \begin{cases} 0 \leq \eta_j(x) \leq 1 \\ \sum\limits_j \eta_j(x)^2 = 1 \quad , \quad x \in [0,1] \quad , \\ x_j \in \operatorname{supp} \eta_j \\ |\operatorname{supp} \eta_j| \leq c\lambda \\ \|D^k \eta_j\|_\infty \leq c\lambda^{-k} \quad . \end{cases}$$

Note that, by the third and fourth property, the supports of at most c (independent of λ) η_j's can have a common intersection. We set

$$M_j(x,t) := \eta_j(x) \quad .$$

and for simplicity of notation

$$\| \ \|_T := \| \ \|_{\alpha,\alpha/2,\Omega_T} \quad .$$

Lemma 3.1. For $f = \sum\limits_j f_j$ with $\operatorname{supp} f_j \subseteq \operatorname{supp} M_j$,

$$(3.9)\qquad \|f\|_T \leq c \sup_j \|f_j\|_T \quad .$$

Proof. Let us prove the estimate, e.g., for $| \ |_{\alpha,x}$. For x,x' set $I := \{j : x \in \operatorname{supp} \eta_j \text{ or } x' \in \operatorname{supp} \eta_j\}$. Since by (3.8) $|I| \leq c$, we have

$$|f(x,t) - f(x',t)| = |\sum_{j\in I} f_j(x,t) - f_j(x',t)|$$

$$\leq c|x-x'|^\alpha \sup_{j\in I} |f_j|_{\alpha,x} \quad .$$

Lemma 3.2. For $f \in \overset{\circ}{H}{}^{\alpha,\alpha/2}(\Omega_T)$ with $T \leq \lambda^2$,

$$(3.10)\qquad \sup_j \|M_j f\|_T \leq c\|f\|_T \quad ,$$

where the constant c does not depend on λ and T.

This follows from (3.8) and the inequality

(3.11) $$|uv|_\gamma \le \|u\|_\infty |v|_\gamma + |u|_\gamma \|v\|_\infty \;.$$

<u>Lemma 3.3</u>. If $w(\cdot,0) = w_x(\cdot,0) = 0$ and $|w|_{1/2+\alpha/2,t,\Omega_T}$, $\|w_x\|_T \le 1$, then

(3.12) $$\|(D_x^k M_j)t^{\ell/2} D_x^m w\|_T \le c\lambda^{-k} T^{(\ell-m+1)/2}, \; m = 0,1 \;,$$

where the constant c does not depend on λ and T.

<u>Proof</u>. Let us prove, e.g., the inequality

$$|w|_{\alpha,x,\Omega_T} \le cT^{1/2} \;.$$

For $|x-x'| \le T^{1/2}$ we have

$$|w(x,t) - w(x',t)| \le |x-x'| \; \|w_x\|_\infty$$

$$\le |x-x'|^\alpha T^{1/2-\alpha/2} T^{\alpha/2} |w_x|_{\alpha/2,t,\Omega_T}$$

and for $|x-x'| > T^{1/2}$,

$$|w(x,t) - w(x',t)| \le 2|x-x'|^\alpha T^{-\alpha/2} \|w\|_\infty$$

$$\le 2|x-x'|^\alpha T^{-\alpha/2} T^{1/2+\alpha/2} |w|_{1/2+\alpha/2,t,\Omega_T} \;.$$

All other inequalities can be easily deduced, using (3.11), (3.8) and the inequality

(3.13) $$|\chi|_{\gamma',[0,T]} \le T^{\gamma-\gamma'} \; |\chi|_{\gamma,[0,T]}$$

for $\chi \in \overset{\circ}{H}{}^\gamma([0,T])$ and $\gamma' < \gamma$.

3.3. <u>Proof of the main theorem</u>

Denote by L the differential operator

$$L : w \mapsto w_t - aw_{xx} - t^{-1/2} bw_x \;.$$

To prove Theorem 3.1 we define an approximate right inverse $\tilde{L}^{-1}$ to L by

(3.14) $$\tilde{L}^{-1}(t^{-1/2} g) = \sum_j M_j L_j^{-1}(M_j t^{-1/2} g)$$

where $w_j := L_j^{-1}(M_j t^{-1/2} g)$ is the solution of the boundary value problem

$$(3.15)\qquad \begin{cases} (w_j)_t - a(x_j,0)(w_j)_{xx} - t^{-1/2} b(x_j,0)(w_j)_x = M_j t^{-1/2} g & \text{on } \Omega_j \\ w_j = 0 & \text{on } \partial\Omega_j \end{cases},$$

with $\Omega_j := \mathbb{R}_+ \times [0,T]$ if $0 \in \operatorname{supp} \eta_j$ and $\Omega_j := (-\infty,1] \times [0,T]$ otherwise.

__Proposition 3.1.__ Set $\tilde{w} := \tilde{L}^{-1}(t^{-1/2} g)$. Then, for $g \in \overset{o}{H}{}^{\alpha,\alpha/2}(\Omega_T)$ with $T < \lambda^2$, we have $\tilde{w}, \tilde{w}_x, t^{1/2}\tilde{w}_{xx} \in \overset{o}{H}{}^{\alpha,\alpha/2}(\Omega_T)$ and

$$(3.16)\qquad \|\tilde{w}, \tilde{w}_x, t^{1/2}\tilde{w}_{xx}\|_T < c\|g\|_T$$

where the constant c does not depend on λ and T.

__Proof.__ Assume that $T < \lambda^2$ and $\|g\|_T < 1$. By Lemma 3.2, $\|M_j g\|_T < c$. Applying Theorem 2.2 to problem (3.1), it follows that $w_j, (w_j)_x$, $(t^{1/2} w_j)_{xx} \in \overset{o}{H}{}^{\alpha,\alpha/2}(\Omega_T)$ and

$$(3.17)\qquad \|w_j, (w_j)_x, (t^{1/2} w_j)_{xx}\|_T < c\ .$$

By Lemma 2.4 and (3.15) this implies in particular

$$(3.18)\qquad |w_j|_{1/2+\alpha/2,t,\Omega_T} < \|t^{1/2}(w_j)_t\|_T < c\ .$$

Since $\tilde{w} = \sum_j M_j w_j$ we obtain, using Lemmas 3.1 - 3.3,

$$\|\tilde{w}\|_T < c \sup_j \|M_j w_j\|_T < c\, T^{1/2}\ .$$

$$\|\tilde{w}_x\|_T < c \sup_j (\|(M_j)_x w_j\|_T + \|M_j (w_j)_x\|_T) < c\ .$$

$$\|(t^{1/2}\tilde{w}_j)_{xx}\|_T <$$

$$c \sup_j (\|(M_j)_{xx} t^{1/2} w_j\|_T + \|(M_j)_x (t^{1/2} w_j)_x\|_T + \|(t^{1/2} w_j)_{xx}\|_T) < c\ .$$

<u>Proposition 3.2</u>. The operator

$$S : g \mapsto t^{1/2} L \tilde{L}^{-1}(t^{-1/2} g)$$

maps $\overset{\circ}{H}^{\alpha,\alpha/2}(\Omega_T)$ into itself. There exist λ_0 and $T_0 < \lambda_0^2$ such that

$$\|Sg - g\|_{T_0} < \frac{1}{2} \|g\|_{T_0} \quad . \tag{3.19}$$

<u>Proof</u>. The first assertion of the Proposition is clear from Proposition 3.1. We write

$$\begin{aligned}
& t^{1/2} L \tilde{L}^{-1}(t^{-1/2} g) - g = \\
& t^{1/2}\Big(\sum_j (L(M_j w_j) - M_j L w_j) + \sum_j M_j (L - L_j) w_j\Big) = \\
& t^{1/2} \sum_j a(M_j)_{xx} w_j + 2\, t^{1/2} \sum_j a(M_j)_x (w_j)_x + \sum_j b(M_j)_x w_j + \\
& t^{1/2} \sum_j M_j (a - a(x_j,0))(w_j)_{xx} + \sum_j M_j (b - b(x_j,0))(w_j)_x \\
& =: \sum_{\nu=1}^{5} I_\nu \quad ,
\end{aligned}$$

where we used that

$$t^{1/2} \sum_j M_j L_j w_j = t^{1/2} \sum_j M_j L_j L_j^{-1}(M_j t^{-1/2} g) = g$$

since $\sum_j (M_j)^2 = 1$. Assuming $T < \lambda^2$, $\|g\|_T < 1$ and using (3.17), (3.18) and Lemmas 3.1 - 3.3 we estimate these terms as follows.

$$\|I_1\|_T < c \sup_j \|a\|_T \|(M_j)_{xx} t^{1/2} w_j\|_T < c\, \lambda^{-2}\, T \quad .$$

$$\|I_2\|_T < c \sup_j \|a\|_T \|(M_j)_x (t^{1/2} w_j)_x\|_T < c\, \lambda^{-1}\, T^{1/2} \quad .$$

$$\|I_3\|_T < c \sup_j \|b\|_T \|(M_j)_x w_j\|_T < c\, \lambda^{-1}\, T^{1/2} \quad .$$

$$\|I_4\|_T \le$$
$$c \sup_j \left(\|a - a(x_j,0)\|_{\infty,\mathrm{supp}\, M_j} \|M_j(t^{1/2}w_j)_{xx}\|_T +\right.$$
$$\left.\|a\|_T \|M_j(t^{1/2}w_j)_{xx}\|_\infty\right) \le$$
$$c\,(\lambda^\alpha + T^{\alpha/2}) \quad .$$

$$\|I_5\|_T \le$$
$$c \sup_j \left(\|b-b(x_j,0)\|_{\infty,\mathrm{supp}\, M_j} \|M_j(w_j)_x\|_T + \|b\|_T \|M_j(w_j)_x\|_\infty\right) \le$$
$$c\,(\lambda^\alpha + T^{\alpha/2}) \quad .$$

Combining these estimates we have

$$\left\|\sum_{\nu=1}^{5} I_\nu\right\|_T \le c(\lambda^{-2}T + \lambda^{-1}T^{1/2} + \lambda^\alpha + T^{\alpha/2})$$

which proves the Proposition with $\lambda_0 := (8c)^{-1/\alpha}$, $T_0 := (\lambda_0/(8c))^2$.

By Proposition 3.2, the operator S is boundedly invertible on $\overset{\circ}{H}{}^{\alpha,\alpha/2}(\Omega_{T_0})$. It follows that, for $g \in \overset{\circ}{H}{}^{\alpha,\alpha/2}(\Omega_{T_0})$, $\tilde{L}^{-1}(t^{-1/2}S^{-1}g)$ is a solution of problem (3.3). Therefore, Proposition 3.1 proves Theorem 3.1 with $T = T_0$. Using the standard results for linear parabolic equations with smooth coefficients [L, Thm. 5.2, p. 320] the estimate (3.6) remains valid for any finite rectangle Ω_T. In view of the reductions made in section 3.1, the proof of Theorem 1.1 is complete.

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REFERENCE

[L] Ladyženskaja, O. A., Solonnikov, V. A., Uralčeva, N. N., Linear and quasilinear equations of parabolic type, Translations of Mathematical monographs, vol. 23, 1968.

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Let a,b belong to the Hölder class $H^{\alpha,\alpha/2}([0,1]\times[0,T])$ with $\alpha \in (0,1)$ and $a \geq a_0 > 0$. It is shown that for the solution of the problem

$$u_t - au_{xx} - t^{-1/2}bu_x = t^{-1/2}f, \quad (x,t) \in [0,1]\times[0,T] \ ,$$
$$u(\cdot,0) = \phi$$
$$u(\nu,\cdot) = \psi_\nu \ , \quad \nu = 0,1 \ ,$$

the estimate

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ABSTRACT (continued)

$$\|u,u_x,t^{1/2}u_{xx}\|_{\alpha,\alpha/2} \le$$

$$c\{\|f\|_{\alpha,\alpha/2} + \|\phi,\phi'\|_{\alpha} + \sum_{\nu=0}^{1} \|\psi_\nu,t^{1/2}\psi'_\nu\|_{\alpha/2}\}$$

holds if the data f, ϕ, ψ_ν satisfy the appropriate compatibility conditions. Here $\|\ \|_{\alpha,\alpha/2}$, $\|\ \|_{\alpha}$, $\|\ \|_{\alpha/2}$ denote the norms of the Hölder classes $H^{\alpha,\alpha/2}([0,1]\times[0,T])$, $H^{\alpha}([0,1])$ and $H^{\alpha/2}([0,T])$ respectively and the constant c depends on α, a_0, $\|a,b\|_{\alpha,\alpha/2}$ and T. The result extends to quasilinear problems with a, b and f depending on x, t and u.